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On an Isomorphism between Specht Module and Left Cell of $\mathfrak{S}_n$

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(Communicated by Y. Kawada)

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

Introduction.

The irreducible representations of the symmetric group $\mathfrak{S}_n$ are classically known and they are parametrized by the partitions of n. One of their realizations is so called Specht module which is defined over Z and has a natural base (Young's natural base). On the other hand, Kazhdan and Lusztig [3] constructed another realization, the W-graph representation which is obtained from a left cell of $\mathfrak{S}_n$ and it also has a natural Z-base (the vertices of the W-graph). We give in this paper an explicit isomorphism between the above two modules and show that the base change matrix is uni-triangular for some ordering of the base. Using this isomorphism, if a partition λ satisfies certain condition, we can construct the W-graph of the left cell of $\mathfrak{S}_n$ (without some edges which connect vertices having the same I-set) corresponding to λ not using the Kazhdan-Lusztig polynomials $P_{y,w}$ but using the relations in Specht module (the Garnir relations) inductively.

§1. The λ-diagram.

1.1. Notations. Let $P(n)$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$, where $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_r > 0$ and $\sum_{i=1}^r \lambda_i = n$. For a partition λ, the Young frame of λ is the arrangement of n squares; the first row λ_1, the second row $\lambda_2, \cdots$, the last row λ_r parts, and line up to the left. Young tableau of shape λ has the frame λ and each square is numbered from 1 to n. A Young tableau of shape λ is called standard if its numbering is increasing from left to right in each row and from top to bottom in

each column. We write Tab_λ the set of all Young tableaux of shape λ and STab_λ the set of all standard Young tableaux of shape λ. The transposition of the partition λ is denoted by ${}^t\lambda$. For a Young tableau D of shape λ, ${}^tD$ denotes the transposition of D whose shape is ${}^t\lambda$. We denote by $D_{\lambda,\text{Top}}$ the standard Young tableau of shape λ, whose numbering is 1 to λ_1 for the first row, λ_1+1 to $\lambda_1+\lambda_2$ for the second row and so on. We set $D_{\lambda,\text{Bot}}={}^t(D_{{}^t\lambda,\text{Top}})$. For a Young tableau D, we put

$I(D)=\{i \mid 1\leq i\leq n-1,\ i+1 \text{ is in a lower position than } i \text{ in } D\}$,

$I_0(D)=\{i \in I(D) \mid i+1 \text{ is in the left side of } i \text{ in } D\}$,

$I_1(D)=\{i \in I(D) \mid i+1 \text{ is directly below } i \text{ in } D\}$.

Then we have

LEMMA 1. *For a standard Young tableau D of shape λ, $\lambda \in P(n)$,*
(1) $I(D)=I_0(D) \amalg I_1(D)$ *(disjoint union)*.
(2) $I(D) \amalg I({}^tD)=\{1, 2, \cdots, n-1\}$.
(3) $I_0(D)=\varnothing$ *if and only if* $D=D_{\lambda,\text{Bot}}$.
(4) $I_0({}^tD)=\varnothing$ *if and only if* $D=D_{\lambda,\text{Top}}$.

PROOF. (1) The equality follows immediately from the definitions.
(2) We see easily that $i+1$ is either in A or in B (see Fig. 1). Hence $i+1\in A$ iff $i\in I({}^tD)$ and $i+1\in B$ iff $i\in I(D)$.
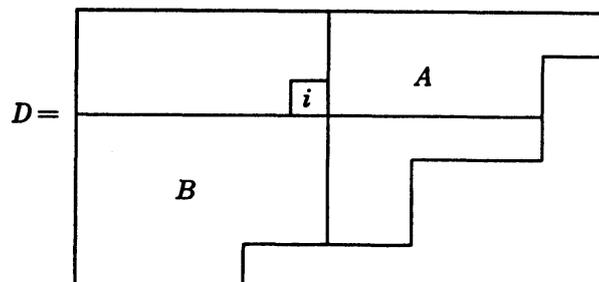

FIGURE 1

(3) The if part clearly follows. On the other hand, if $I_0(D)=\varnothing$, $i+1$ must be directly below i or in the right side of i, hence D must have the form $D_{\lambda,\text{Bot}}$.
(4) Note that ${}^tD_{\lambda,\text{Top}}=D_{{}^t\lambda,\text{Bot}}$. Then (4) follows from (3).

1.2. The Robinson-Schensted correspondence. The following fact is well known.

PROPOSITION 1 (cf. [4]). *There is a bijection from $\mathfrak{S}_n$ to $\amalg_{\lambda\in P(n)}$ $(\text{STab}_\lambda\times\text{STab}_\lambda)$ which is called Robinson-Schensted's correspondence. By this, if x corresponds to $(P(x),\ Q(x))$, then $P(x^{-1})=Q(x)$.*

We denote by s_i the transposition $(i, i+1) \in \mathfrak{S}_n$. The set $S = \{s_i\}_{1 \leq i \leq n-1}$ generates $\mathfrak{S}_n$ as a Coxeter group. Let $l(x)$ be the length of $x \in \mathfrak{S}_n$ with respect to this generator set S.

LEMMA 2. *For x in $\mathfrak{S}_n$, we have*
(1) $l(s_i x) = l(x) - 1$ *if and only if* $i \in I(P(x))$,
(2) $l(x s_i) = l(x) - 1$ *if and only if* $i \in I(Q(x))$.

PROOF. (1) Suppose $x \in \mathfrak{S}_n$ has a form $x = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$. Then $l(s_i x) = l(x) - 1$ iff i appears after $i+1$ in $a_1, a_2, \cdots, a_n$, which is equivalent to the condition $i \in I(P(x))$ by the construction of $P(x)$.
   (2)   As $l(x^{-1}) = l(x)$, (2) follows from (1).

1.3.   For $D \in \mathrm{STab}_\lambda$, we define $x(D) \in \mathfrak{S}_n$ by

$$x(D) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} \quad \text{if} \quad D = \begin{array}{|ccc} a_s & & \\ \vdots & & \\ a_2 & \vdots & \\ a_1 & a_{s+1} & \end{array}$$

LEMMA 3. *For $D \in \mathrm{STab}_\lambda$, we have*
(1) *By the Robinson-Schensted correspondence, $x(D)$ corresponds to $(D, D_{\lambda, \mathrm{Bot}})$.*
   (2) $l(s_i x(D)) = l(x(D)) - 1$ *if and only if* $i \in I(D)$.

PROOF. (1) It is clear from the definition of $x(D)$ and the construction of the Robinson-Schensted correspondence.
   (2)   Clear from (1) and Lemma 2.

LEMMA 4.
(1) $l(x(D_{\lambda, \mathrm{Top}})) = \dfrac{n^2 - \sum \lambda_i^2}{2} - \dfrac{\sum (i-1)\lambda_i(\lambda_i - 1)}{2}$.

(2) $l(x(D_{\lambda, \mathrm{Bot}})) = \sum (i-1)\lambda_i$.

1.4.   For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in P(n)$, we put

$$N = l(x(D_{\lambda, \mathrm{Top}})) - l(x(D_{\lambda, \mathrm{Bot}})) = \frac{n(n+1)}{2} - \frac{\sum i\lambda_i(\lambda_i + 1)}{2}.$$

For $0 \leq k \leq N$, we define a subset $\mathscr{D}_k^\lambda$ of STab_λ inductively from N to 0. We put

$$\mathscr{D}_N^\lambda = \{D_{\lambda,\mathrm{Top}}\} \, ,$$

and for $0 \leq k < N$, we put

$$\mathscr{D}_k^\lambda = \{D \in \mathrm{STab}_\lambda \mid D = s_j D' \text{ for some } D' \in \mathscr{D}_{k+1}^\lambda, \ j \in I_0(D')\} \, ,$$

where $s_j D'$ is the standard tableau interchanging j and $j+1$ in D.

PROPOSITION 2.
(1) $\displaystyle\coprod_{0 \leq k \leq N} \mathscr{D}_k^\lambda = \mathrm{STab}_\lambda$ (*disjoint union*).
(2) $\mathscr{D}_0^\lambda = \{D_{\lambda,\mathrm{Bot}}\}$.

PROOF. For $D \in \mathscr{D}_k^\lambda$, we can show inductively that

$$l(x(D)) = l(x(D_{\lambda,\mathrm{Top}})) - (N - k) \, .$$

Hence if $k \neq l$, $\mathscr{D}_k^\lambda \cap \mathscr{D}_l^\lambda = \varnothing$ i.e. disjoint. For $D \in \mathrm{STab}_\lambda$, compare it with $D_{\lambda,\mathrm{Top}}$ for the top row from left to right then for the second row in the same manner and so on. If the first difference is a in D, we must have $a - 1 \in I_0({}^t D)$ and $D' = s_{a-1} D \in \mathrm{STab}_\lambda$. For this D', instead of D, do the same operation. Continuing this, we finally get $D_{\lambda,\mathrm{Top}}$. If it takes m times, then $D \in \mathscr{D}_{N-m}^\lambda$. If we do the same operation on ${}^t D$ (compare with ${}^t D_{\lambda,\mathrm{Bot}}$), we get ${}^t D_{\lambda,\mathrm{Bot}}$ after k times interchange. Transposing all of them, if $D \in \mathscr{D}_l^\lambda$ then $D_{\lambda,\mathrm{Bot}} \in \mathscr{D}_{l-k}^\lambda$. Therefore we get (1) and (2).

For $D \in \mathrm{STab}_\lambda$, we define $\mathrm{ht}(D) = k$ if $D \in \mathscr{D}_k^\lambda$.

1.5. Partial orders on STab_λ. We define three partial orders on STab_λ.

(1) For $D, D' \in \mathrm{STab}_\lambda$, if $\mathrm{ht}(D') = \mathrm{ht}(D) + 1$ and $D' = (i, j)D$, for some transposition (i, j), we define $D < D'$. Let $\leq$ be the transitive closure of this relation, thus $\leq$ is a partial order on STab_λ.

(2) We define a partial order $\unlhd$ on STab_λ as follows. For D, $D' \in \mathrm{STab}_\lambda$,

$$D \unlhd D' \text{ if and only if } m_{ir}(D) \leq m_{ir}(D') \text{ for all } i \text{ and all } r \, ,$$

where $m_{ir}(D) = \#\{j \mid j \leq i \text{ and } j \text{ appears in the first } r \text{ rows of } D\}$ (cf. James [2; 3.10]). This order can be interpreted as follows. For $D \in \mathrm{STab}_\lambda$ and $1 \leq i \leq n$, we define $D_{\leq i}$ the sub-diagram of D whose entries are less than or equal to i. Then this is also a standard Young tableau of size i. We also define for a Young tableau D, $\lambda(D) = (\lambda_1(D), \lambda_2(D), \cdots, \lambda_r(D))$ where $\lambda_i(D)$ is the length of i-th row of D, hence D has a Young frame of shape $\lambda(D)$. For two partitions λ, $\mu \in P(n)$, we define $\lambda \leq \mu$ iff

$\lambda_1 \leqq \mu_1, \quad \lambda_1 + \lambda_2 \leqq \mu_1 + \mu_2, \cdots, \lambda_1 + \lambda_2 + \cdots + \lambda_s \leqq \mu_1 + \mu_2 + \cdots + \mu_s$ and $r \geqq s$ $(\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r), \quad \mu = (\mu_1, \mu_2, \cdots, \mu_s))$. Then $D \leqq D'$ iff $\lambda(D_{\leq i}) \leqq \lambda(D'_{\leq i})$ for all $1 \leqq i \leqq n$.

(3) We define the third partial order $\leqq$ on STab_λ. $D \leqq D'$ if and only if $x(D) \leqq x(D')$, where $\leqq$ is the Bruhat order on $\mathfrak{S}_n$. Then we have

PROPOSITION 3. *For D, $D' \in \mathrm{STab}_\lambda$,*

$$D \leqq D' \text{ if and only if } D \unlhd D', \text{ and these imply } D \lessapprox D'.$$

PROOF. Clearly $D \leqq D'$ implies $D \lessapprox D'$. The partial order $\leqq$ is generated by the relation $D_1 < D_2$, $D_2 = (p, q)D_1$. If p is in the k-th row and q is in the l-th row in D_1, and if $p > q$, then $k < l$. It is easy to see that

$$m_{ir}(D_1) = m_{ir}(D_2) \qquad \text{for} \quad i < q \text{ or } i \geqq p \text{ or } r < k \text{ or } r \geqq l,$$

and

$$m_{ir}(D_1) + 1 = m_{ir}(D_2) \qquad \text{for} \quad q \leqq i < p \text{ and } k \leqq r < l,$$

therefore $D_1 \lhd D_2$. Hence $D \leqq D'$ implies $D \unlhd D'$. On the other hand, using the lemma below inductively, we can prove that $D \unlhd D'$ implies $D \leqq D'$.

LEMMA 5. *For D, $D' \in \mathrm{STab}_\lambda$ such that $D \lhd D'$, there exists $D'' \in \mathrm{STab}_\lambda$ such that $\mathrm{ht}(D'') = \mathrm{ht}(D) + 1$ and $D < D'' \unlhd D'$.*

PROOF. Compare D and D' for the first row from left to right then for the second row in the same manner and so on. If the first difference is a in D and b in D' in the p-th row, then $a > b$ by $D \lhd D'$. A corner of a Young tableau is a square that there is no square attached to the right of and below it. We put

$$C_D = \{j \mid b \leqq j < a, \ j \text{ is in a corner of } D_{\leq a-1},$$

$$\text{and } j \text{ is a lower square than } a \text{ in } D\}.$$

As b is in the lower position than a in D, $C_D \neq \varnothing$. Let $c \in C_D$ be in the heighest position among C_D. Then $b \leqq c < a$, and if we put $D'' = (a, c)D$ then $D'' \in \mathrm{STab}_\lambda$ and $\mathrm{ht}(D'') = \mathrm{ht}(D) + 1$ by the definition of c, therefore $D < D''$. We must show $D'' \unlhd D'$. If c is in the q-th row of D,

$$m_{ir}(D'') = m_{ir}(D) \leqq m_{ir}(D') \qquad \text{if} \quad i < c \text{ or } i \geqq a \text{ or } r < p \text{ or } r \geqq q,$$

and

$$m_{ir}(D'') = m_{ir}(D) + 1 \quad \text{if} \quad c \leq i < a \text{ and } p \leq r < q .$$

Therefore it is enough to show that

(1.5.1) $\qquad m_{ir}(D) < m_{ir}(D') \qquad \text{if} \quad c \leq i < a \text{ and } p \leq r < q .$

We fix i, $c \leq i < a$, and move r from p to $q-1$. If the first negation of (1.5.1) occurs for $r = r_0$, then $m_{ir_0}(D) = m_{ir_0}(D')$. We define s to be the smallest j such that $\lambda_j(D_{\leq a-1}) = \lambda_q(D_{\leq a-1})$. For $p \leq k < s$,

$$m_{ik}(D) = m_{b-1,k}(D) \leq m_{b-1,k}(D') < m_{b,k}(D') \leq m_{ik}(D') .$$

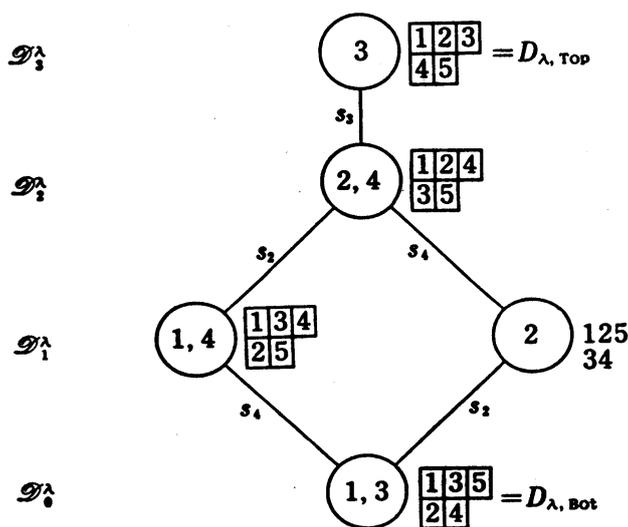Therefore r_0 must satisfy $s \leq r_0 < q$. But then,

$$\lambda_{r_0+1}(D_{\leq i}) = \lambda_{r_0}(D_{\leq i}) > \lambda_{r_0}(D'_{\leq i}) \geq \lambda_{r_0+1}(D'_{\leq i}) ,$$

hence $m_{i,r_0+1}(D) > m_{i,r_0+1}(D')$ which contradicts the fact $D \lhd D'$. Therefore we have (1.5.1) and finally we get $D'' \trianglelefteq D'$.

1.6. We define the λ-diagram to be the Hasse diagram of $(\mathrm{STab}_\lambda, \leq)$ whose vertices have two data D and $I(D)$. If $D \in \mathscr{D}_k^\lambda$, $D' \in \mathscr{D}_{k+1}^\lambda$ and $s_j D' = D$ then we label the edge $\{D, D'\}$ as s_j. An edge may have no label (cf. Example 5.1).

EXAMPLE. $\lambda = (3, 2)$.

§2. The left cells of $\mathfrak{S}_n$.

By Kazhdan and Lusztig [3], the left cells of $\mathfrak{S}_n$ are determined as follows. For $D \in \mathrm{STab}_\lambda$, we define

$X_D = \{x \in \mathfrak{S}_n \mid Q(x) = D \text{ for the Robinson-Schensted correspondence}\}$.

We denote by Γ_D the W-graph corresponding to X_D with edge and multiplicity defined by $\mu(x, y)$ (cf. [3]). Then $\{\Gamma_D \mid D \in \text{STab}_\lambda, \lambda \in P(n)\}$ is the set of all left cells of $\mathfrak{S}_n$ and if $D, D' \in \text{STab}_\lambda$, then Γ_D and $\Gamma_{D'}$ are isomorphic as W-graphs ([3; Theorem 1.4]). We particularly consider $\Gamma^\lambda = \Gamma_{D_{\lambda,\text{Bot}}}$ on $X^\lambda = X_{D_{\lambda,\text{Bot}}}$.

PROPOSITION 4. *The λ-diagram can be identified with the induced subgraph which is obtained from the Hasse diagram of the Bruhat order of $\mathfrak{S}_n$ by restricting vertices to X^λ.*

PROOF. For $D \in \text{STab}_\lambda$, $x(D)$ corresponds to $(D, D_{\lambda,\text{Bot}})$. For D, $D' \in \text{STab}_\lambda$, $l(x(D')) = l(x(D)) + 1$ if and only if $\text{ht}(D') = \text{ht}(D) + 1$, and in this case $x(D') > x(D)$ if and only if $D' > D$ (cf. Hiller [1; Definition 6.0]).

LEMMA 6. *If $x \leqq y$ and $l(y) - l(x) \geqq 2$, then*

$$\mu(x, y) \neq 0 \text{ implies } \mathscr{L}(y) \subset \mathscr{L}(x) ,$$

where $\mathscr{L}(x) = \{s \in S \mid sx < x\}$.

PROOF. By [3; (2.3.g)], for $s \in S$,

$$P_{x,y} = P_{sx,y} \quad \text{if} \quad x < y, \; sy < y, \; sx > x ,$$

where $P_{x,y}$ is the Kazhdan-Lusztig polynomial.

The degree of $P_{sx,y} \leqq \dfrac{1}{2}(l(y) - l(sx) - 1)$

$$= \dfrac{1}{2}(l(y) - l(x) - 2)$$

$$< \dfrac{1}{2}(l(y) - l(x) - 1) .$$

Therefore if $\mu(x, y) \neq 0$ and $sy < y$ then we must have $sx < x$. Hence $\mathscr{L}(y) \subset \mathscr{L}(x)$.

COROLLARY 1. *Γ^λ is obtained from λ-diagram by adding some edges $\{D, D'\}$ for $D \in \mathscr{D}_k^\lambda$, $D' \in \mathscr{D}_l^\lambda$, $l - k = \text{odd} \geqq 3$, $I(D') \subset I(D)$.*

PROOF. For $D \in \text{STab}_\lambda$, $I(D) = \{i \mid s_i \in \mathscr{L}(x(D))\}$, thus it is clear from the lemma above.

For $x \in X^\lambda$ we denote by g_D the base element of Γ^λ corresponding

to x, where $D=P(x)$. Thus $\{g_D\}_{D \in \mathrm{STab}_\lambda}$ is the natural Z-base for Γ^λ.

§3. Specht modules.

In this section we will quote some results of James [2].

3.1. We define an equivalence relation $\sim$ on Tab_λ by $D \sim D'$ if and only if i-th rows of D and D' are equal as sets for all i. We define

$$\mathrm{Tabloid}_\lambda = \mathrm{Tab}_\lambda/\sim \quad \text{(the set of equivalence classes under } \sim\text{)}.$$

We denote by $[D]$ the equivalence class containing D. And we put

$$M_\lambda = \text{the free } Z\text{-module on } \mathrm{Tabloid}_\lambda.$$

Then $\mathfrak{S}_n$ acts naturally on M_λ. We also define for $D \in \mathrm{Tab}_\lambda$

$$V_D = \{x \in \mathfrak{S}_n \mid x \text{ stabilizes each column of } D\},$$

$$k_D = \sum_{x \in V_D} \mathrm{sgn}(x)x \in Z[\mathfrak{S}_n],$$

$$e_D = k_D[D] \in M_\lambda,$$

$$S^\lambda = \langle e_D \rangle_{Z, D \in \mathrm{Tab}_\lambda} \subset M_\lambda.$$

S^λ is called the Specht module corresponding to the partition λ.

LEMMA 7. *The $Z[\mathfrak{S}_n]$ module S^λ is a cyclic module generated by any* e_D, $D \in \mathrm{Tab}_\lambda$.

It is also known that

PROPOSITION 5 ([2; Theorem 8.4]). $\{e_D\}_{D \in \mathrm{STab}_\lambda}$ *is a Z-base of S^λ.*

The action of $\mathfrak{S}_n$ on S^λ with respect to this base $\{e_D\}_{D \in \mathrm{STab}_\lambda}$ is determined by the action of $s_i = (i, i+1)$ on e_D for $D \in \mathrm{STab}_\lambda$.

PROPOSITION 6 (James [2; 25.1]). *The action of s_i on e_D is calculated as follows* $(D \in \mathrm{STab}_\lambda)$:

(1) *If $i \in I_1(D)$, then $s_i e_D = -e_D$.*

(2) *If $i \in I_0(D) \cup I_0({}^t D)$, then $s_i e_D = e_{s_i D}$.*

(3) *If $i \in I_1({}^t D)$, then $s_i e_D = e_D + \sum_{D' < D} a_{D'} D'$.*

The case (3) is calculated by the Garnir relations.

3.2. The Garnir relations. For $D \in \mathrm{Tab}_\lambda$, let X be a subset of the i-th column of D and Y be a subset of the $(i+1)$-th column of D. We define, for $Z \subset \{1, 2, \cdots, n\}$, $\mathfrak{S}_Z$ to be the permutation group on the set Z. We fix a coset representative of $\mathfrak{S}_{X \cup Y}/\mathfrak{S}_X \times \mathfrak{S}_Y$ and denote it by

$\{\sigma_i\}_{1 \le i \le r}$. Then we define

$$G_{X,Y} = \sum_{1 \le i \le r} \mathrm{sgn}(\sigma_i)\sigma_i \in Z[\mathfrak{S}_n] \ .$$

PROPOSITION 7 (James [2; Theorem 7.2]). *If $|X \cup Y| >$ the number of squares in the i-th column of D, then*

$$G_{X,Y}e_D = 0$$

which is called the Garnir relation.

§ 4. The isomorphism.

4.1. Now we can state the main result of this paper. We consider two $Z[\mathfrak{S}_n]$ modules with natural bases, $S^\lambda = \langle e_D \rangle_{Z, D \in \mathrm{STab}_\lambda}$ and $\Gamma^\lambda = \langle g_D \rangle_{Z, D \in \mathrm{STab}_\lambda}$.

THEOREM. *There exists a unique isomorphism φ from S^λ to Γ^λ such that*

$$\varphi(e_D) = g_D + \sum_{D' < D} a_{D',D} g_{D'} \qquad (a_{D',D} \in Z) \ .$$

PROOF. Let W_I be the subgroup of $\mathfrak{S}_n$ generated by a subset $I \subset S$. For $\mu \in P(n)$, we set $I(\mu) = I(D_{t_{\mu, \mathrm{Bot}}})$ and regard it as a subset of S by identifying i with s_i. Then $W_{I(\mu)}$ is the horizontal subgroup of $D_{\mu, \mathrm{Top}}$. We also set $S_Q^\lambda = S^\lambda \otimes Q$ and $\Gamma_Q^\lambda = \Gamma^\lambda \otimes Q$. By Stanley [5; 4.1]

$$\langle \mathrm{Ind}_{W_{I(\mu)}}^{\mathfrak{S}_n} 1, \ S_Q^\lambda \rangle = \#\{D \in \mathrm{STab}_\lambda \mid I(D) \subset \widehat{I(\mu)}\}$$

where $\widehat{I(\mu)} = \{1, 2, \cdots, n-1\} - I(\mu)$. This multiplicity is the Kostka number $K_{\lambda, \mu}$. On the other hand

$$\begin{aligned}
\langle \mathrm{Ind}_{W_{I(\mu)}}^{\mathfrak{S}_n} 1, \ \Gamma_Q^\lambda \rangle &= \langle 1, \ \Gamma_Q^\lambda|_{W_{I(\mu)}} \rangle \\
&= \#\{D \in \mathrm{STab}_\lambda \mid I(D) \cap I(\mu) = \varnothing\} \\
&= \#\{D \in \mathrm{STab}_\lambda \mid I(D) \subset \widehat{I(\mu)}\} \ .
\end{aligned}$$

As the matrix $(K_{\lambda, \mu})_{\lambda, \mu \in P(n)}$ is invertible, we get $S_Q^\lambda \simeq \Gamma_Q^\lambda$ as $Q[\mathfrak{S}_n]$ modules. The -1 eigenspace of s_i in Γ_Q^λ is $\sum_{i \in I(D)} Q g_D$, and it is easy to see that if $I(D) \supset I(D_{\lambda, \mathrm{Bot}})$ for $D \in \mathrm{STab}_\lambda$, then $D = D_{\lambda, \mathrm{Bot}}$. Therefore the common -1 eigenspace of $I(D_{\lambda, \mathrm{Bot}})$ in Γ_Q^λ is $Q g_{D_{\lambda, \mathrm{Bot}}}$. But since $e_{D_{\lambda, \mathrm{Bot}}}$ is a common -1 eigenvector of $I(D_{\lambda, \mathrm{Bot}})$ in S_Q^λ, there is an isomorphism $\tilde{\varphi}: S_Q^\lambda \xrightarrow{\sim} \Gamma_Q^\lambda$ such that $\tilde{\varphi}(e_{D_{\lambda, \mathrm{Bot}}}) = g_{D_{\lambda, \mathrm{Bot}}}$.

By Lemma 7, we can define $\varphi = \tilde{\varphi}|_{S^\lambda}: S^\lambda \to \Gamma^\lambda$, which is a $Z[\mathfrak{S}_n]$ homo-

morphism. We will show that this homomorphism φ has the required property by induction on $\mathrm{ht}(D)$. If $\mathrm{ht}(D)=0$, then $D=D_{\lambda,\mathrm{Bot}}$ and by the construction $\varphi(e_D)$ has the required form. If $\mathrm{ht}(D)>0$, take $j\in I_0(D)$ and define $D'=s_jD$. Then by induction

$$\varphi(e_{D'})=g_{D'}+\sum_{D''<D'}a_{D'',D'}g_{D''}\ .$$

As φ is a $Z[\mathfrak{S}_n]$ homomorphism,

$$\varphi(e_D)=\varphi(s_je_{D'})=s_j\varphi(e_{D'})\ .$$

On the other hand

$$s_jg_{D'}=g_D+g_{D'}+\sum_{D^*<D'}b_{D^*}g_{D^*}$$

and

$$s_jg_{D''}=\begin{cases}-g_{D''} & \text{if}\quad j\in I(D'')\\ g_{D''}+\text{lower} & \text{if}\quad j\in I_1({}^tD'')\\ g_{D^\circ}+g_{D''}+\text{lower} & \text{if}\quad j\in I_0({}^tD'')\end{cases}$$

where $D^\circ=s_jD''$ and $D''<D^\circ<D$. Therefore $\varphi(e_D)$ also has the required form. φ is an isomorphism by this form, and uniqueness is clear.

4.2. Construction of left cells. For $D\in\mathrm{STab}_\lambda$ we define

$$B_D=\{D'\in\mathrm{STab}_\lambda\mid \mathrm{ht}(D)-\mathrm{ht}(D')=\mathrm{odd}\geqq 3\ \text{and}\ I(D')\supset I(D)\}$$

and

$$C(D)=\Big(\bigcup_{D'\in B_D}I(D')\Big)-I(D)\subset\{1,\,2,\,\cdots,\,n-1\}\ .$$

For $\lambda\in P(n)$, we define a condition (C_λ) as follows:

(C_λ) For all $D\in\mathrm{STab}_\lambda-\{D_{\lambda,\mathrm{Bot}}\}$, there exists $j\in I_0(D)$ such that $j\notin C(s_jD)$.

REMARK. For $n\leqq 7$, all partitions $\lambda\in P(n)$ satisfy the condition (C_λ). But there are partitions which do not satisfy the condition for $n\geqq 8$, e.g. $\lambda=(2,\,2,\,2,\,1,\,1)$.

COROLLARY 2. *If a partition λ satisfies the condition (C_λ), we can construct sub-W-graph of the left cell Γ^λ from the λ-diagram using the Garnir relations.*

PROOF. We construct Γ^λ inductively on $\mathrm{ht}(D)$, $D \in \mathrm{STab}_\lambda$. For $\mathrm{ht}(D) < 3$, λ-diagram is equal to Γ^λ. For $k \geq 3$, assume that Γ^λ is already constructed for $\mathrm{ht}(D) < k$, so the action of s_i on g_D is already defined for $\mathrm{ht}(D) < k$ and all i. Take $D \in \mathscr{D}_k^\lambda$. If $C(D) = \varnothing$, then no edges are added and take another $D \in \mathscr{D}_k^\lambda$. For $i \in C(D)$, we have two cases.

(1) If $i \in I_0({}^t D)$, put $D' = s_i D$. By the condition (C_λ) for D', there exists $j \in I_0(D')$ such that $j \notin C(s_j D')$. Then $\varphi(e_{D'}) = s_j \varphi(e_{s_j D'})$ is already determined in terms of g_{D°, $D^\circ \leq D'$. On the other hand $\varphi(e_D) = g_D +$ lower is also determined. As $\varphi(e_{D'}) = s_i \varphi(e_D)$, we get $s_i g_D = g_{D'} + g_D + \sum_{D'' < D} \alpha_{D''} g_{D''}$. Then add all the edges $\{D, D''\}$ for $\alpha_{D''} \neq 0$ and define $\mu(D'', D) = \alpha_{D''}$.

(2) If $i \in I_1({}^t D)$, then $s_i e_D = e_D + \sum_{D^* \lhd D} \beta_{D^*} e_{D^*}$ (calculated by the Garnir relations). $\varphi(e_D)$ is already determined and as $s_i \varphi(e_D) = \varphi(s_i e_D)$, we get $s_i g_D = g_D + \sum_{D'' < D} \gamma_{D''} g_{D''}$. Then add all the edges $\{D, D''\}$ for $\gamma_{D''} \neq 0$ and define $\mu(D'', D) = \gamma_{D''}$.

Do this operation for all $i \in C(D)$. For all $D \in \mathscr{D}_k^\lambda$ do the same operation and we get Γ^λ for $\mathrm{ht}(D) \leq k$. Thus inductively we get Γ^λ. (If there are vertices with the same I-set, some edges may be added to become the left cell.)
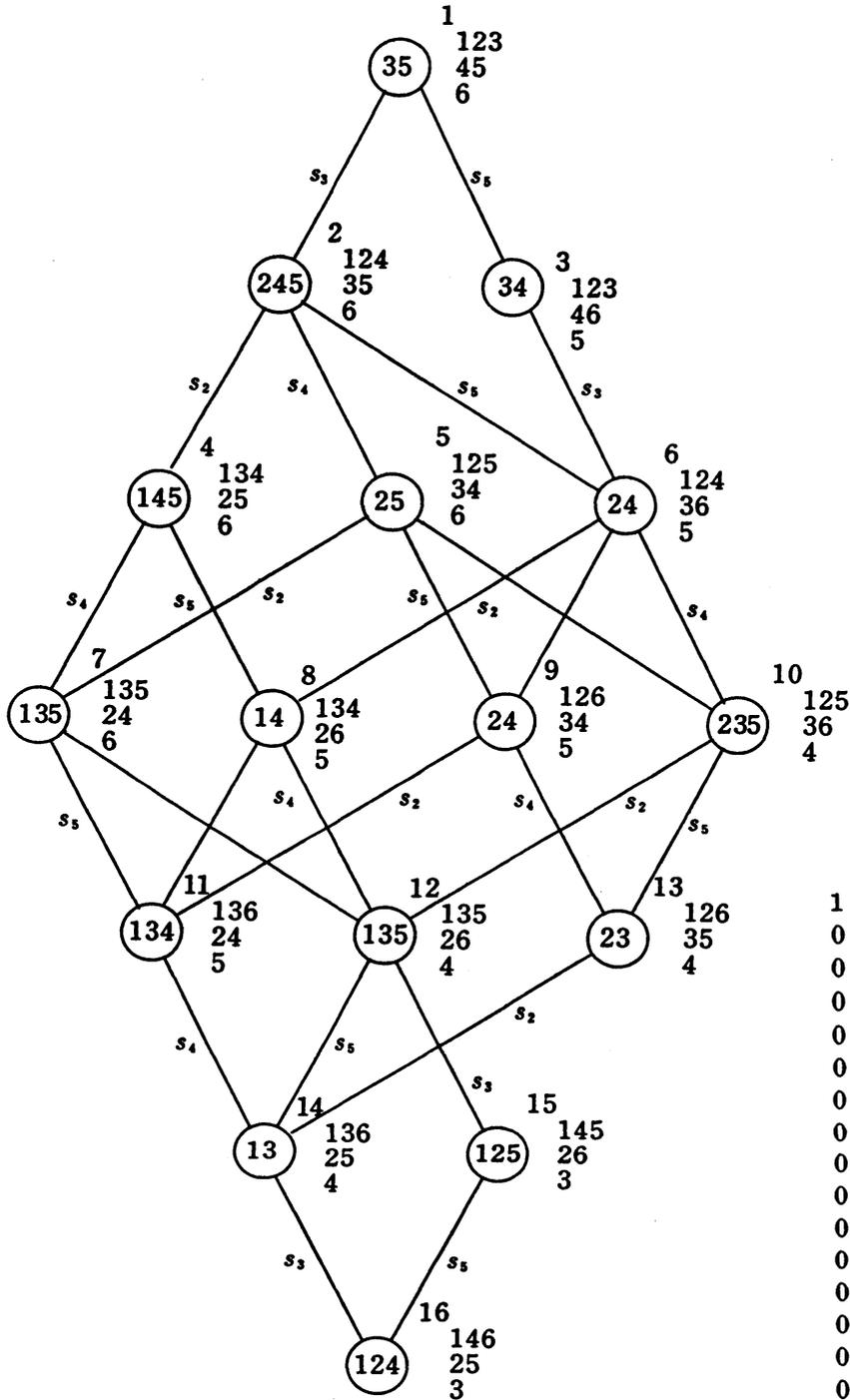
§5. Examples.

In this section we calculate W-graphs (the left cells) and transition matrices for $\lambda = (3, 2, 1)$ and for all partitions of $n = 7$ using Corollary 2.

5.1. $\lambda = (3, 2, 1)$. The λ-diagram is indicated below. We label the Young tableaux $D_1, D_2, \cdots, D_{16}$ from top to bottom. We also write $e_i = e_{D_i}$ and $g_j = g_{D_j}$. As the action of s_i on g_D is already known for $\mathrm{ht}(D) \leq 2$, we can calculate $\varphi(e_D)$ for $\mathrm{ht}(D) \leq 3$ inductively:

$$\varphi(e_{16}) = g_{16} \qquad \text{by definition of } \varphi ,$$
$$\varphi(e_{15}) = s_5 \varphi(e_{16}) = s_5 g_{16} = g_{15} + g_{16} ,$$
$$\varphi(e_{14}) = s_3 \varphi(e_{16}) = s_3 g_{16} = g_{14} + g_{16} ,$$
$$\varphi(e_{13}) = s_2 \varphi(e_{14}) = s_2(g_{14} + g_{16}) = g_{13} + g_{14} ,$$
$$\vdots$$
$$\varphi(e_7) = s_5 \varphi(e_{11}) = g_7 + g_{11} + g_{12} + g_{14} .$$

We define the action of s_i on g_D for $\mathrm{ht}(D) = 3$. $C(D_9) = \{1\}$ and this is the case (2) of Corollary 2.

$$s_1 e_9 = e_9 - e_{11} - e_{16} \qquad \text{by Garnir relation} ,$$

1 35   123 45 6

s_3 s_5

2 245   124 35 6   **3** 34   123 46 5

s_2 s_4 s_5 s_3

4 145   134 25 6   **5** 25   125 34 6   **6** 24   124 36 5

s_4 s_5 s_2 s_5 s_2 s_4

7 135   135 24 6   **8** 14   134 26 5   **9** 24   126 34 5   **10** 235   125 36 4

s_5 s_4 s_2 s_4 s_2 s_5

11 134   136 24 5   **12** 135   135 26 4   **13** 23   126 35 4

s_4 s_5 s_3 s_2

14 13   136 25 4   **15** 125   145 26 3

s_3 s_5

16 124   146 25 3

λ-diagram for λ=(3, 2, 1)

```
1 11 111 1111 110 01 1
0 10 111 1111 111 11 1
0 01 001 0110 100 00 1
0 00 100 1100 110 10 1
0 00 010 1011 111 11 1
0 00 001 0111 111 10 1
0 00 000 1000 110 10 0
0 00 000 0100 110 11 1
0 00 000 0010 101 10 1
0 00 000 0001 011 10 0
0 00 000 0000 100 10 0
0 00 000 0000 010 11 1
0 00 000 0000 001 10 0
0 00 000 0000 000 10 1
0 00 000 0000 000 01 1
0 00 000 0000 000 00 1
```

transition matrix for λ=(3, 2, 1)

and

$$s_1 \varphi(e_9) = \varphi(e_9 - e_{11} - e_{16}) = g_9 + g_{13} \ .$$

On the other hand

$$s_1 \varphi(e_9) = s_1 g_9 + s_1(g_{11} + e_{13} + g_{14} + g_{16}) = s_1 g_9 - g_{11} + g_{13} - g_{16} \ .$$

Therefore $s_1 g_9 = g_9 + g_{11} + g_{16}$ and we get $\mu(D_{16}, D_9) = 1$. $C(D_8) = \{2\}$ and this is the case (1) of Corollary 2. $s_2 e_8 = e_6 = s_4 e_{10}$ and the action of s_4 on g_{10} is known:

$$s_4 \varphi(e_{10}) = s_4(g_{10} + g_{12} + g_{13} + g_{14})$$
$$= g_6 + g_8 + g_9 + g_{10} + g_{11} + g_{12} + g_{13} + g_{14} + g_{16} \ .$$

On the other hand

$$s_2 \varphi(e_8) = s_2 g_8 + s_2(g_{11} + g_{12} + g_{14} + g_{15} + g_{16})$$
$$= s_2 g_8 + g_9 + g_{10} + g_{11} + g_{12} + g_{13} + g_{14} \ ,$$

therefore $s_2 g_8 = g_6 + g_8 + g_{16}$ and we get $\mu(D_{16}, D_8) = 1$. As we defined all the actions of s_i on g_D for $\mathrm{ht}(D) \leq 3$, we can calculate $\varphi(e_D)$ for $\mathrm{ht}(D) = 4$. In this way we get transition matrix and W-graph simultaneously. The added edges are
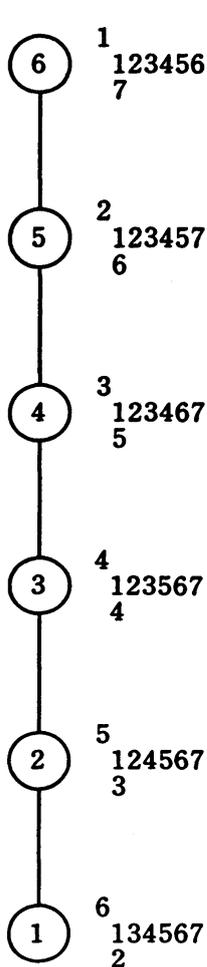
$$D_8 \text{——} D_{16} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{case (1),}$$
$$D_9 \text{——} D_{16}, \quad D_5 \text{——} D_{15}, \quad D_3 \text{——} D_{11}, \quad D_1 \text{——} D_7, \quad D_1 \text{——} D_{10} \qquad \text{case (2),}$$

with multiplicities 1. This W-graph is also the left cell.

5.2. $n = 7$. We indicate only half of the cases. For transposed partition ${}^t\lambda$, the Young tableaux are ${}^tD$ with the same numbering. The edges are the same and the I-sets are $I({}^tD) = \{1, 2, \cdots, 6\} - I(D)$. The transition matrix is indicated in lower triangle, e.g. for ${}^t\lambda = (2, 2, 1, 1, 1)$, $\varphi(e_8) = g_8 + g_6 + g_5 + g_3$. Dotted edges are added for the W-graph to become the left cell.

$$\oslash \ {}^1_{\ 1234567} \qquad\qquad\qquad \begin{array}{c} \lambda = (7) \\ 1 \\ {}^t\lambda = (1, 1, 1, 1, 1, 1, 1) \end{array}$$

1
(6)   123456
        7

2
(5)   123457
        6

3
(4)   123467
        5
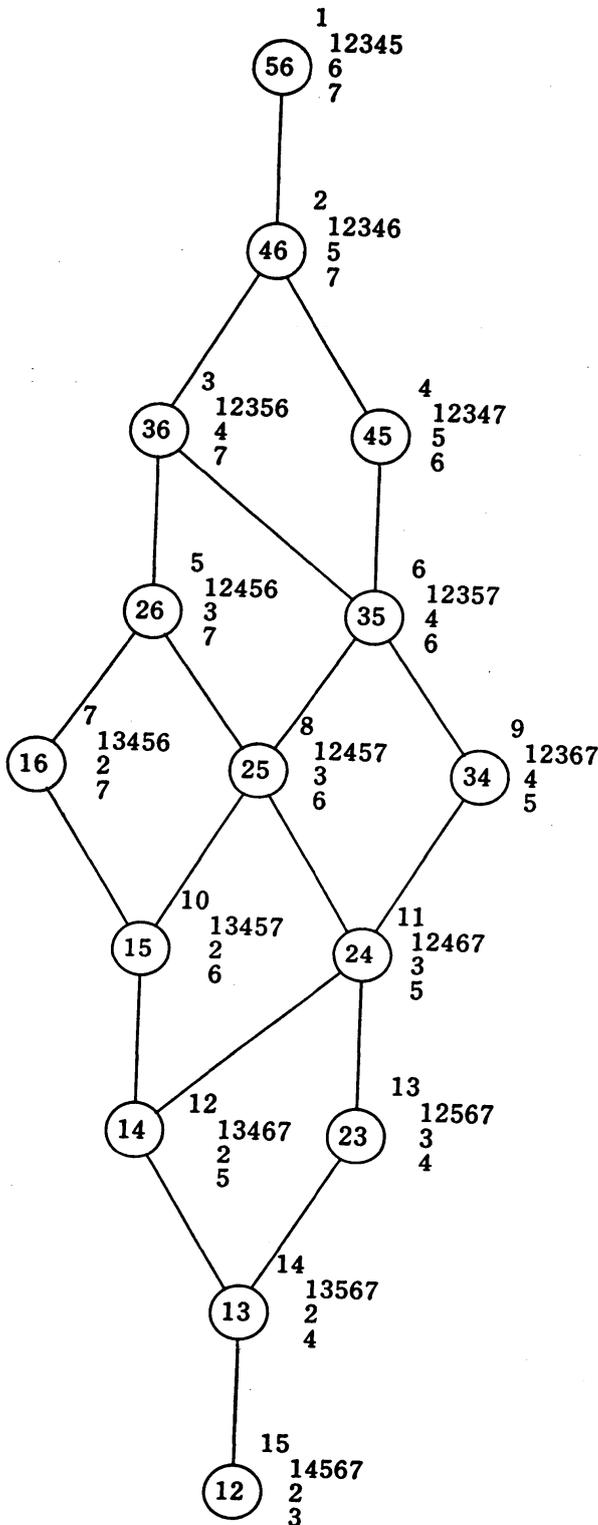
4
(3)   123567
        4

5
(2)   124567
        3

6
(1)   134567
        2

$\lambda=(6,1)$

```
1 1 1 1 1 1
1 1 1 1 1 1
0 1 1 1 1 1
0 0 1 1 1 1
0 0 0 1 1 1
0 0 0 0 1 1
```

$^t\lambda=(2,1,1,1,1,1)$

1
(56) 12345
        6
        7

2
(46) 12346
        5
        7

3
(36) 12356
        4
        7

4
(45) 12347
        5
        6

5
(26) 12456
        3
        7

6
(35) 12357
        4
        6

7
(16) 13456
        2
        7

8
(25) 12457
        3
        6

9
(34) 12367
        4
        5

10
(15) 13457
        2
        6

11
(24) 12467
        3
        5

13
(23) 12567
        3
        4

12
(14) 13467
        2
        5

14
(13) 13567
        2
        4

15
(12) 14567
        2
        3
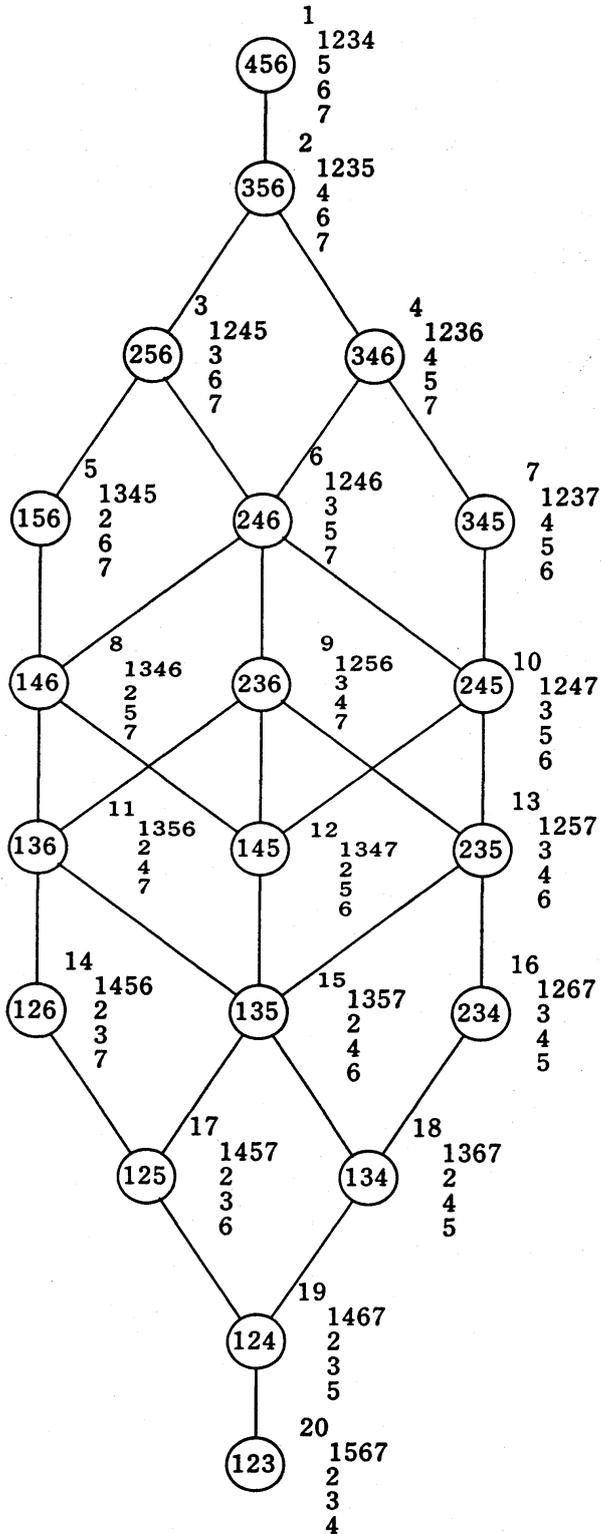
$\lambda=(5,1,1)$

```
1 1 10 10 100 00 00 0 0
1 1 11 11 110 10 00 0 0
0 1 10 11 111 11 10 0 0
1 1 01 01 010 10 00 0 0
0 0 10 10 110 11 11 1 0
0 1 11 01 011 11 10 0 0
0 0 00 10 100 10 10 1 1
0 0 10 11 010 11 11 1 0
0 0 01 01 001 01 10 0 0
0 0 00 10 110 10 10 1 1
0 0 00 01 011 01 11 1 0
0 0 00 00 010 11 10 1 1
0 0 00 00 001 01 01 1 0
0 0 00 00 000 01 11 1 1
0 0 00 00 000 00 01 1 1
```
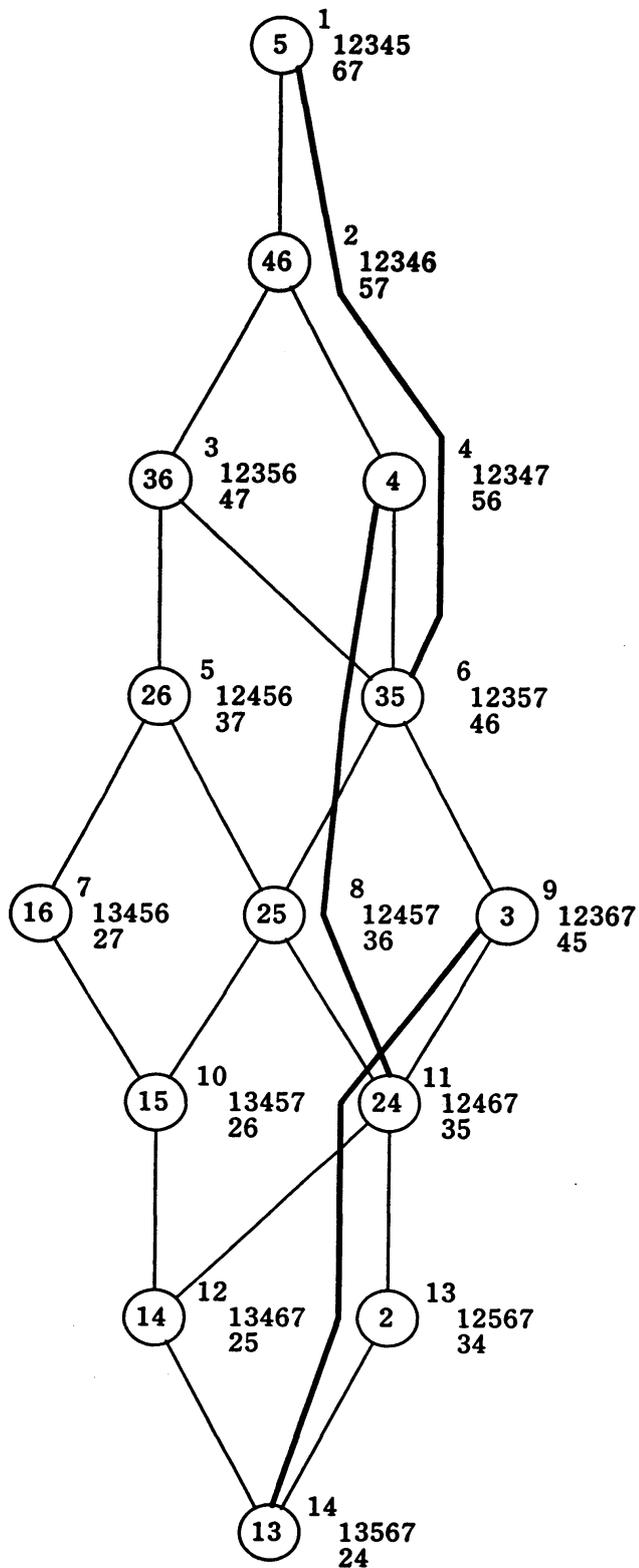
$^t\lambda=(3,1,1,1,1)$

$\lambda = (4, 1, 1, 1)$

```
1 1 10 100 000 000 000 00 0 0
1 1 11 110 100 000 000 00 0 0
0 1 10 110 110 100 000 00 0 0
1 1 01 011 101 010 000 00 0 0
0 0 10 100 100 100 100 00 0 0
0 1 11 010 111 111 010 00 0 0
1 1 01 001 001 010 000 00 0 0
0 0 10 110 100 110 110 10 0 0
0 0 01 010 010 101 011 01 0 0
0 1 11 011 001 011 010 00 0 0
0 0 00 010 110 100 110 11 1 0
0 0 10 110 101 010 010 10 0 0
0 0 01 011 011 001 011 01 0 0
0 0 00 000 010 100 100 10 1 1
0 0 00 010 111 111 010 11 1 0
0 0 00 001 001 001 001 01 0 0
0 0 00 000 010 101 110 10 1 1
0 0 00 000 001 011 011 01 1 0
0 0 00 000 000 001 011 11 1 1
0 0 00 000 000 000 001 01 1 1
```

$^t\lambda = (4, 1, 1, 1)$

1 (5) 12345 67

2 (46) 12346 57

3 (36) 12356 47

4 (4) 12347 56

5 (26) 12456 37

6 (35) 12357 46

7 (16) 13456 27

8 (25) 12457 36

9 (3) 12367 45

10 (15) 13457 26

11 (24) 12467 35

12 (14) 13467 25

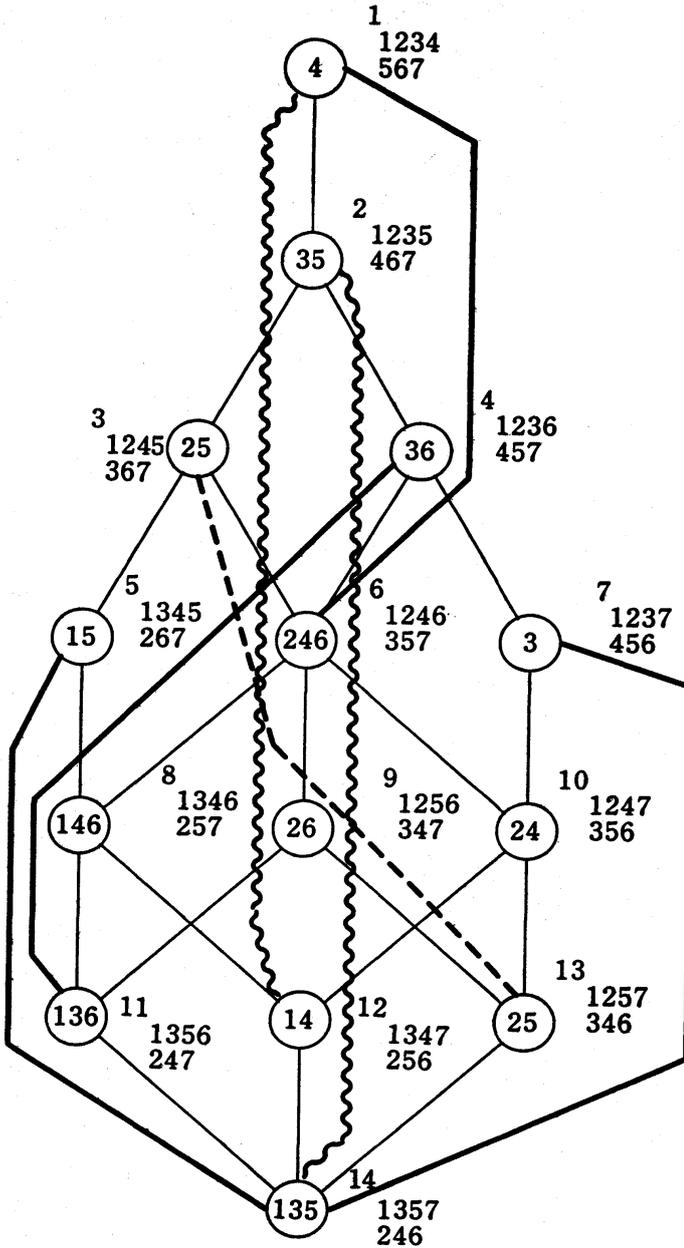13 (2) 12567 34

14 (13) 13567 24

$\lambda = (5, 2)$

```
1 1 11 12 121 12 11 1
1 1 11 11 111 12 11 1
0 1 10 11 111 11 11 1
0 1 01 01 011 12 11 1
0 0 10 10 110 11 11 1
1 1 11 01 011 11 11 1
0 0 00 10 100 10 10 1
0 0 10 11 010 11 11 1
1 0 00 01 001 01 11 1
0 0 00 10 110 10 10 1
0 0 01 01 011 01 11 1
0 0 00 00 010 11 10 1
0 0 01 00 000 01 01 1
0 0 00 00 001 01 11 1
```
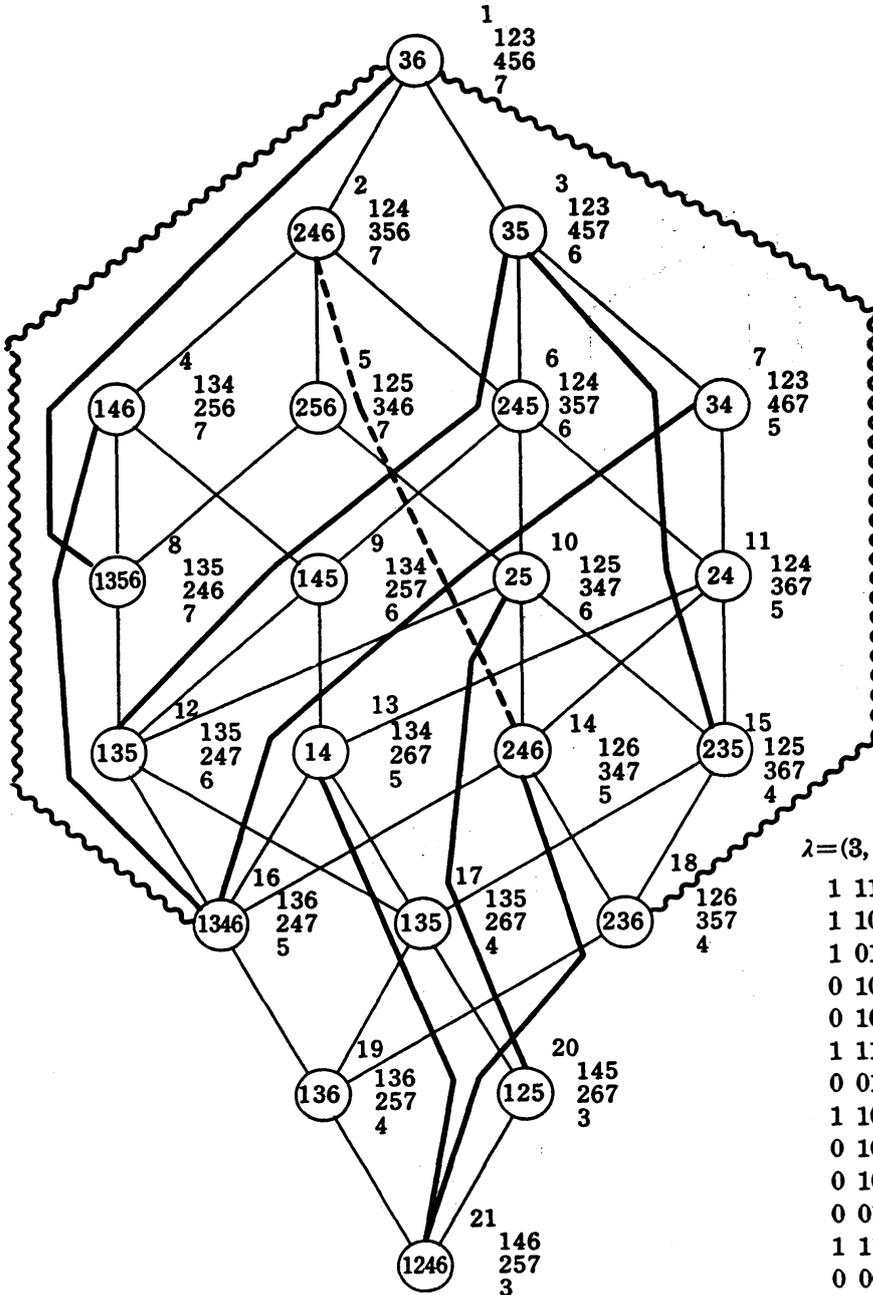
$^{t}\lambda = (2, 2, 1, 1, 1)$

$\lambda=(4,3)$

1	1	11	121	111	111	2
1	1	11	111	111	111	2
0	1	10	110	111	111	1
0	1	01	011	111	111	1
0	0	10	100	100	110	1
1	1	11	010	111	111	1
0	0	01	001	001	011	1
0	0	10	110	100	110	1
1	0	00	010	010	101	1
0	0	01	011	001	011	1
0	0	01	010	110	100	1
1	0	00	010	101	010	1
0	0	10	010	011	001	1
1	1	11	111	111	111	1

$^t\lambda=(2,2,2,1)$

1 123 456 7 — 36

2 124 356 7 — 246

3 123 457 6 — 35

4 134 256 7 — 146

5 125 346 7 — 256

6 124 357 6 — 245

7 123 467 5 — 34

8 135 246 7 — 1356

9 134 257 6 — 145

10 125 347 6 — 25

11 124 367 5 — 24

12 135 247 6 — 135

13 134 267 5 — 14

14 126 347 5 — 246

15 125 367 4 — 235

16 136 247 5 — 1346

17 135 267 4 — 135

18 126 357 4 — 236

19 136 257 4 — 136
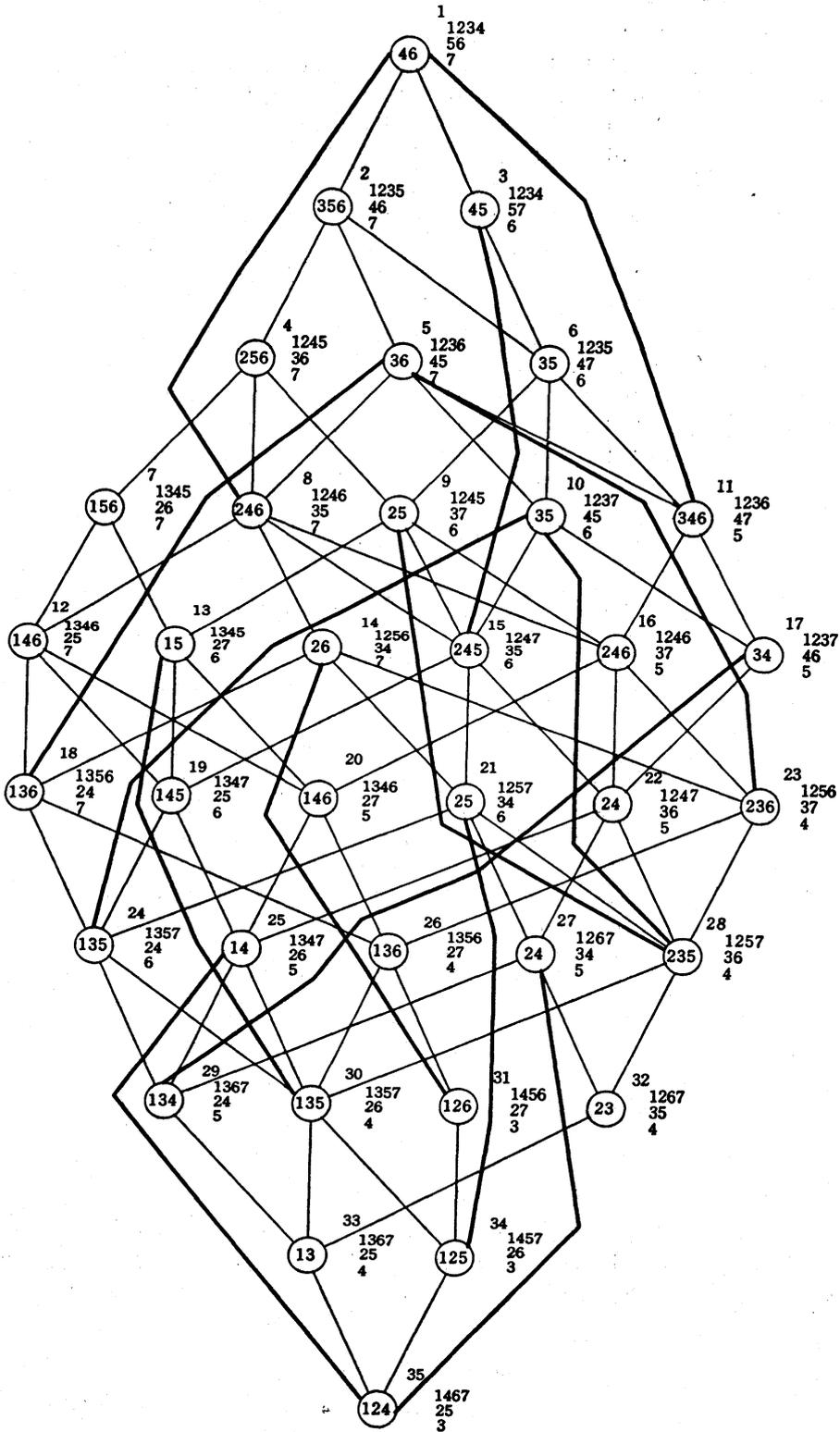
20 145 267 3 — 125

21 146 257 3 — 1246

$\lambda = (3, 3, 1)$

```
1 11 1111 1111 1111 211 11 1
1 10 1110 1111 1111 110 01 1
1 01 0011 0111 1111 110 01 1
0 10 1000 1100 1100 110 00 0
0 10 0100 1010 1001 010 01 0
1 11 0010 0111 1111 111 11 1
0 01 0001 0001 0110 100 00 1
1 10 1100 1000 1000 010 00 0
0 10 1010 0100 1100 110 10 1
0 10 0110 0010 1011 111 11 1
0 01 0011 0001 0111 111 10 1
1 11 1110 1110 1000 110 10 0
0 00 0010 0101 0100 110 11 1
0 10 0110 0011 0010 101 10 1
0 01 0011 0011 0001 011 10 0
1 11 1111 1111 1110 100 10 0
0 01 0010 0111 1101 010 11 1
1 11 0010 0011 0011 001 10 0
1 11 1011 0111 1111 111 10 1
0 01 0000 0010 1001 010 01 1
1 01 0001 0011 1111 111 11 1
```

$^t\lambda = (3, 2, 2)$

1 1234
56
7 (46)

2 1235
46
7 (356)

3 1234
57
6 (45)

4 1245
36
7 (256)

5 1236
45
7 (36)

6 1235
47
6 (35)

7 1345
26
7 (156) (246)

8 1246
35
7 (25)

9 1245
37
6 (35)

10 1237
45
6 (346)

11 1236
47
5

12 1346
25
7 (146)

13 1345
27
6 (15)

14 1256
34
7 (26) (245)

15 1247
35
6 (246)

16 1246
37
5

17 1237
46
5 (34)

18 1356
24
7 (136)

19 1347
25
6 (145)

20 1346
27
5 (146)

21 1257
34
6 (25)

22 1247
36
5 (24)

23 1256
37
4 (236)

24 1357
24
6 (135)

25 1347
26
5 (14) (136)

26 1356
27
4 (24)

27 1267
34
5 (235)

28 1257
36
4

29 1367
24
5 (134) (135)

30 1357
26
4 (126)

31 1456
27
3 (23)

32 1267
35
4

33 1367
25
4 (13)

34 1457
26
3 (125)

35 1467
25
3 (124)

$\lambda=(4,2,1)$

```
1 11 111 12111 111210 111101 10101 0110 01 0
1 10 111 11111 111111 111111 11111 1110 01 1
1 01 001 00110 010200 010100 10001 0100 01 0
0 10 100 11100 111110 111111 11112 1211 11 1
0 10 010 01011 101111 111111 11111 1110 01 1
1 11 001 00111 010111 011110 11011 1100 01 1
0 00 100 10000 110000 111000 11100 1200 11 1
1 10 110 01000 101110 111111 11111 1111 11 1
0 10 101 00100 010110 011111 11112 1201 11 1
0 10 011 00010 000101 010110 11011 1100 01 1
1 11 011 00001 000011 001010 01010 1000 00 1
0 00 100 11000 100000 111000 11100 1100 10 1
0 00 100 10100 010000 011000 11100 1210 11 1
1 00 000 01000 001000 100101 10111 1111 11 1
1 11 111 01110 000100 010110 11011 1101 11 1
1 10 111 01101 000010 001011 01111 1101 10 1
1 10 011 00011 000001 000010 01010 1000 00 1
0 00 010 01000 101000 100000 10100 1100 10 0
0 00 100 11100 110100 010000 11000 1100 10 1
0 00 100 11100 110010 001000 01100 1110 11 1
1 01 000 01000 001100 000100 10011 1101 11 1
1 11 111 01111 000111 000010 01011 1101 10 1
1 00 010 01001 001010 000001 00101 0101 10 0
0 00 010 01010 101100 110100 10000 1100 10 0
0 00 100 11100 110110 011010 01000 1100 11 1
0 00 010 01000 101010 101001 00100 0110 11 1
0 01 000 00000 000100 000110 00010 1001 10 1
1 01 011 01111 001111 000111 00001 0101 10 0
0 00 000 00010 000101 010110 11010 1000 10 0
0 00 010 01110 111110 111111 11101 0100 11 1
0 00 010 00000 001000 100001 00100 0010 01 1
0 01 001 00110 000100 000110 00011 0001 10 0
0 00 000 00110 010101 010110 11011 1101 10 1
0 00 010 00010 001000 100101 10101 0110 01 1
0 00 000 00010 000001 000110 11011 1101 11 1
```

$^t\lambda=(3,2,1,1)$

5.3. Remark. The algorithm of Corollary 2 can be ameliorated and we get W-graphs for all partitions of $n \leqq 9$. We have a microcomputer program to calculate these W-graphs. In these cases, the transition matrices are all nonnegative.

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