Interest-Rate Control Rule and Macroeconomic Stability
（利子率操作ルールとマクロ経済の安定性）

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Contents

Preface and Acknowledgements

This dissertation is prepared for the partial fulfillment of the requirements of the Degree of Doctor of Philosophy in Economics, Osaka University.

The theme of this dissertation is the relation between macroeconomic stability and the interest-rate control rule. Each chapter is based on:

- Overview,

  written for this dissertation,

- Chapter 1,

  Seiya Fujisaki (2008), "Timing, Interest-Rate Control, and Equilibrium Determinacy with Capital Accumulation", mimeo,

- Chapter 2,

  Seiya Fujisaki and Kazuo Mino (2007), "Generalized Taylor Rule and Determinacy of Growth Equilibrium", Economics Bulletin, Vol. 5, No. 11, pp. 1-7,

- Chapter 3,

  Seiya Fujisaki (2008), "Equilibrium Determinacy of Endogenous Growth with Generalized Taylor Rule: A Discrete-Time Analysis", Discussion Papers In Economics And Business, Graduate School of Economics, Osaka University, No. 08-21, pp. 1-25,

- Chapter 4,

  Seiya Fujisaki (2008), "Stabilization Effect of Interest-Rate Control in a Growing Cash-In-Advance Economy", mimeo,

  Seiya Fujisaki (2008), "Velocity and Monetary Expansion in a Growing Economy with Interest-Rate Control", mimeo,

- Chapter 5,

  Seiya Fujisaki and Kazuo Mino (2008), "Income Taxation, Interest-Rate Control and Macroeconomic Stability with Balanced-Budget", Discussion Papers In Economics And Business, Graduate School of Economics, Osaka University, No. 08-20, pp. 1-28.

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From now on, I would like to do the further repayment of favor to all of them by continuing research.

Overview

In this dissertation, we investigate the roles of interest-rate control in dynamic economies with capital accumulation. In the theoretical studies on monetary macrodynamics, it was usually assumed that the growth rate of nominal money supply is kept constant. However, the stance of many central banks have recently the shifted from the base-money targeting to the interest-rate control. Moreover, Taylor (1993) shows that central bank's behavior such as the Federal Reserve System can be assumed that it controls the rate of nominal interest by responding to the rate of inflation and to the income level. Therefore, in the literature, it becomes popular to specify the interest-control rule as monetary policy When the central bank controls nominal interest rate, money supply is endogenously determined, and thus the effects of monetary policy is more complex than the case of constant money growth. If there are multiple equilibria, the rational expectations equilibrium path of the economy is indeterminate so that sunspot-driven changes in expectations can generate economic fluctuations. The interest-control rule may enhance macroeconomic instability if it is not appropriately implemented. The central concern of this dissertation is a relevant relationship between the interest-rate control and aggregate stability of the economy.

There has been a large numbers of studies on the issue mentioned above. The earlier studies such as Benhabib et. al. (2001a) focus on the model without investment and reveal that equilibrium indeterminacy easily emerges under the Taylor-type interest-rate control. The recent studies consider models including capital accumulation and show that destabilization effect of interest-rate control would be smaller in models with investment. For example, Dupor (2001) shows that determinacy does not appear if the policy is active. It is to be noted that many of recent studies on models with interest-rate control employ New Keynesian frameworks with sticky prices and monopolistic competition. In fact, Carlstrom and Fuerst (2005) and Huang and Meng (2007), who analyze a discrete-time version of Dupor (2001), claim that not only the strength of monetary policy rule but also other factors such as the monopolistic distortion affect the uniqueness of equilibrium path. Therefore, the roles of monetary policy and capital accumulation for equilibrium determinacy tend to be rather ambiguous in these sticky-price models.

We use only the flexible-price models in this dissertation to clarify the stabilization effect of interest-rate control in economies with capital accumulation. More specifically, this dissertation focuses on the following issues:

(i) timing and equilibrium determinacy under interest-rate control;

(ii) long-run effects of interest-control rule in endogenously growing economies;

(iii) interaction between fiscal and monetary policy.

Chapters 1 and 3 discuss the issue (i), Chapters 2-4 deal with issue (ii), and Chapter 5 is devoted to issue (iii). Each chapter is summarized as follows.

In Chapter 1, we examine the relation between the types of interest rate rules and equilibrium determinacy. We construct a money-in-the-utility-function model that involves a neo-classical production function as in Meng and Yip (2004). We use a discrete-time model to consider the alternative timings of the real money holdings in the utility and of the inflation rate in monetary policy rule. It is shown that the results of determinacy heavily depends on the timing of the inflation rate in interest-rate control. This is in contrast to endowment economy as in Benhabib et. al (2001).

Chapter 2 constructs a continuous-time AK growth model. Money is introduced via a standard money-in-the-utility formulation so that the balanced-growth path is unique and money is superneutral in the long run. We show that even in this simple environment the interest-rate feedback rule á la Taylor (1993) may produce indeterminacy of equilibrium if the monetary authority adjusts the nominal interest rate in response to the growth rate of real income as well as to the rate of inflation.

In Chapter 3, we re-examine the model in Chapter 2 by use of a discrete-time formulation. Unlike the continuous-time model in Chapter 2, we can demonstrate that equilibrium determinacy depends on the timing of money holding of households as well. The role of timing in the endogenous growth model can be more clearly seen than in the exogenous growth (Chapter 1).

Chapter 4 also assumes that the central bank adjusts the nominal interest rate in response not only to the inflation but also to the growth rate of real income. We analyze an AK model with a cash-in-advance (CIA) constraint in which money balance binds not only consumption but also investment so that money is not superneutral on the balanced-growth path (BGP). We first analyze equilibrium determinacy around the BGP and find that the result is sensitive to the choice of interest-rate control rule. In the latter half, we focus on the long-run relation between velocity of money and the rate of nominal money growth. We again confirm that the relation is closely related to what type of interest-rate control is employed.

In Chapter 5, we study stabilization effects of fiscal and monetary policy rules in the context of a standard real business cycle model with money. We assume that the fiscal authority adjusts the rate of income tax subject to the balanced-budget constraint as in Guo and Lansing (1998), while the monetary authority controls the nominal interest rate by observing inflation. We demonstrate that whether or not policy rules eliminate the possibility of sunspot-driven fluctuations critically depends upon the appropriate combination of progressiveness of taxation and activeness of interest-rate control.

Chapter 1

Timing, Interest-Rate Control, and Equilibrium Determinacy with Capital Accumulation

1.1 Introduction

It has been known that in monetary dynamic models without capital accumulation (investment), the equilibrium path is uniquely determined if monetary policy rule á la Taylor (1993) is active in the sense that the monetary authority adjusts the nominal interest rate in response to the inflation rate more than one for one [1]. However, recent studies have shown that this result may not hold when the economic model includes capital accumulation. Among others, Meng and Yip (2004) examine a continuous-time money-in-the-utility-function (MIUF) model with flexible prices and demonstrate that equilibrium determinacy tends to hold regardless of types of interest rate control rules. Most of the studies on the relationship between the equilibrium determinacy and monetary policy rule with investment assume that price adjustments are sluggish. Using a continuous-time model with monopolistic competition and sluggish prices, Dupor (2001) shows that equilibrium determinacy holds if the monetary policy is passive, while determinacy does not appear if the policy is active. Carlstrom and Fuerst (2005) and Huang and Meng (2007) analyze the

[1] Benhabib et. al. (2001a) show that in a model without capital determinacy may depend not only on the types of the policy rule but also on the substitutability of consumption and real money balances.

discrete-time version of Dupor (2001) [2] . They conclude that not only the strength of monetary policy rule but also other factors such as the monopolistic distortion affect the uniqueness of equilibrium. Although these findings are interesting, the role of capital is rather ambiguous in the sticky-price models, because adding a price adjustment mechanism may alter the dynamic structure of a model economy.

Unlike the mainstream literature mentioned above, this chapter examines the stabilization role of interest-rate control in a standard neoclassical monetary growth model with flexible prices and fixed labor supply. We use a discrete-time monetary growth model in which money is introduced via the MIUF formulation. In a discrete-time MIUF model with an interest-rate control rule, two kinds of timings would be critical. First, in discrete-time MIUF models, we should specify the timing of money holding. The timing of money balance holding can be classified into the cases of cash-in-advance (CIA) and cash-when-I'm-done (CWID). The CIA (resp. CWID) timing means that the money balances held for transactions are the stock of money that household has before entering (resp. after leaving) the goods market trading [3]. Second, in the discrete-time settings, we can easily distinguish the current-looking rule from the forward-looking interest control rules, according to the difference of the timing of the inflation rate which is used as an index of monetary policy [4]. Therefore, we can consider the following cases: (i) CIA timing with a forward-looking rule; (ii) CIA timing with a current-looking rule; (iii) CWID timing with a forward-looking rule; and (iv) CWID timing with a current-looking rule. In this chapter, we analyze the relation between equilibrium determinacy and interest-rate control rules in all of these four patterns of formulations.

The main results of this chapter are as follows. First, we confirm that the equilibrium path tends to be determinate under the forward-looking rule. Second, under the current-looking rule, determinacy holds if the policy is active, while indeterminacy can emerge if the policy is passive. Note that these results also generally hold in the flexible-price economy without capital. Therefore, our finding means that

[2]Dupor (2001) and Huang and Meng (2007) assume that the instantaneous utility function is additively separable between consumption and money balances. On the other hand, Carlstrom and Fuerst (2005) use a non-separable utility function.

[3]The discrete-time MIUF models usually assume CWID timing as in Carlstrom and Fuerst (2005) and Huang and Meng (2007). However, as Carlstrom and Fuerst (2001) claim, it is difficult to justify CWID timing on theoretical grounds, since this assumption means that the money held at the beginning of $t+1$ reduces transaction costs in period t.

[4]Backward-looking rule as in Carlstrom and Fuerst (2000) can be considered, but it is analytically difficult and used not so much.

introducing capital accumulation does not make a remarkable qualitative change in dynamic behaviors of the model economy. Nevertheless, we can show that determinacy may hold more easily in the economy with capital than in the one without capital. This is because adding capital stock as a predetermined variable to the dynamic system reduces the possibility of equilibrium indeterminacy in our flexible-price economy. Our analysis also demonstrates that the presence of capital makes the role of timing of money holdings less relevant in equilibrium determinacy.

It is to be noted that Carlstrom and Fuerst (2001) and Meng and Yip (2004) also discuss stabilization effects of interest-rate control in flexible-price models with capital accumulation. Carlstrom and Fuerst (2001) utilize a discrete-time model and focus on the timings of money holdings in a MIUF, but they investigate only the case of forward-looking interest-rate control [5]. In addition, they assume that labor supply is endogenously determined. Since the assumption of variable labor could be an additional source of equilibrium indeterminacy, our model with fixed labor supply is useful for investigating the effects of capital stock in a clear manner. Meng and Yip (2004) use a continuous-time neoclassical growth model with MIUF and conclude that the equilibrium path is generally determinate regardless of the form of interest-rate control rule [6]. Our study demonstrates that their conclusion depends heavily upon their continuous-time formulation in which timings plays no role.

1.2 The Model

We use the standard Sidrauski-type formulation. The economy consists of a continuum of identical household-firms with a unit mass. The agent maximizes his lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_{t-J}), \quad 0 < \beta < 1, \quad J = 0 \text{ or } 1, \tag{1.1}$$

subject to the flow budget constraint

$$k_{t+1} - (1-\delta)k_t + c_t + m_t + b_t + \tau_t = y_t + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t}. \tag{1.2}$$

[5] The most monetary models that use discrete-time formulation also focus on specific timings of household's money holding and interest-rate control.

[6] Chapter 2 and 3 using an endogenous growth model show that the form of monetary policy rule can be significant for equilibrium determinacy.

The instantaneous utility function $u(c_t, m_{t-J})$ is assumed to satisfy $u_c > 0,\ u_m > 0,$ $u_{cc} < 0,\ u_{mm} < 0,\ u_{cc}u_m - u_{cm}u_c < 0$, and $u_{mm}u_c - u_{cm}u_m < 0$. That is, the utility function is strictly increasing and strictly concave in c and m, and consumption c and real money balances m are both normal goods. We define $J = 1$ as cash-in-advance (CIA) timing, and $J = 0$ as cash-when-I'm-done (CWID) timing [7].

The production function is given by $y_t = f(k_t)$, where $f(k_t)$ satisfies $f''(k_t) < 0 < f'(k_t)$. We assume that labor supply is inelastic. The government budget constraint is the following:

$$m_t + b_t + \tau_t = g_t + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t}, \tag{1.3}$$

where $g_t = g \geq 0$ is the real government spending which is assumed to be fixed. From (1.2) and (1.3), we obtain the goods-market equilibrium condition:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t - g.$$

We assume that the monetary policy rule follows the Taylor principle, in which the central bank controls the nominal interest rate by responding to either the current and expected future inflation rate. The control rule of the gross nominal interest rate is given by

$$R_t = R(\pi_t, \pi_{t+1}),\ \text{where}\ R_1 \equiv \frac{\partial R_t}{\partial \pi_t} \geq 0\ \text{and}\ R_2 \equiv \frac{\partial R_t}{\partial \pi_{t+1}} \geq 0. \tag{1.4}$$

$R_t = R(\pi_t, \pi_{t+1})$ is a continuous, nondecreasing, and strictly positive function of π_t and π_{t+1}. We assume that there exists at least one steady-state inflation rate $\bar{\pi} > \beta$ such that $R(\bar{\pi}, \bar{\pi}) = \dfrac{\bar{\pi}}{\beta} > 1$. If $R_1 > 0$ and $R_2 = 0$, the interest rate rule is current-looking, and it is forward-looking when $R_1 = 0$ and $R_2 > 0$. It means that the monetary policy is active if $\beta\bar{R}_2 > 1$ (resp. $\beta\bar{R}_1 > 1$), and it is passive

[7]Each variable means the following:
β=the time discounting rate;
c_t =real consumption;
m_{t-J}=real money balances at the beginning of period $t-J+1$ $(J = 0, 1)$;
k_t=(per capita) stock of capital;
δ=the depreciation rate $(0 < \delta < 1)$;
P_t=the nominal price level;
b_t=real bonds at the end of period;
τ_t=lump-sum tax;
y_t=net income;
$\pi_t \equiv \dfrac{P_t}{P_{t-1}}$=the gross rate of inflation;
R_{t-1}=the gross nominal interest rate set at period $t-1$.

if $\beta\bar{R}_2 < 1$ (resp. $\beta\bar{R}_1 < 1$) under the forward-looking (resp. the current-looking) rule.

1.3 Cash-in-advance(CIA) Timing

1.3.1 The Dynamic System

The CIA timing implies that the money balances held for transactions are the money balances that household has before entering the goods market, and thus the utility function is given by $u(c_t, m_{t-1})$. To derive the optimality conditions for the household's consumption plan, set up the following Lagrangian function:

$$\mathcal{L}_{CIA} \equiv \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, m_{t-1}) + \lambda_t \left[-k_{t+1} + (1-\delta)k_t - c_t - m_t - b_t - \tau_t + f(k_t) + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t} \right] \right\}. \tag{1.5}$$

The first-order conditions for the household's optimization problem are:

$$\lambda_t = u_c(c_t, m_{t-1}); \tag{1.6}$$

$$u_m(c_t, m_{t-1}) = \frac{\lambda_{t-1}}{\beta} - \frac{\lambda_t}{\pi_t}; \tag{1.7}$$

$$\lambda_{t-1} = \beta\lambda_t[f'(k_t) + 1 - \delta]; \tag{1.8}$$

$$\lambda_t = \frac{\beta\lambda_{t+1}R_t}{\pi_{t+1}}; \tag{1.9}$$

$$\lim_{t\to\infty} \beta^{t+1}\lambda_{t+1}k_{t+1} = 0; \tag{1.10}$$

$$\lim_{t\to\infty} \beta^t\lambda_t m_t = 0; \tag{1.11}$$

$$\lim_{t\to\infty} \beta^t\lambda_t b_t = 0. \tag{1.12}$$

The trasversality conditions are equations (1.10)-(1.12).

The complete dynamic system in the CIA timing is described by following:

$$k_{t+1} - f(k_t) - (1-\delta)k_t + c_t + g = 0; \tag{1.13}$$

$$\beta u_c(c_{t+1}, m_t)[f'(k_{t+1}) + 1 - \delta] - u_c(c_t, m_{t-1}) = 0; \tag{1.14}$$

$$\frac{u_m(c_{t+1}, m_t)}{u_c(c_{t+1}, m_t)} - \frac{R(\pi_t, \pi_{t+1}) - 1}{\pi_{t+1}} = 0; \tag{1.15}$$

$$R(\pi_t, \pi_{t+1}) - \pi_{t+1}[f'(k_{t+1}) + 1 - \delta] = 0. \tag{1.16}$$

Equation (1.13) is the goods-market equilibrium condition. From (1.6) and (1.8), we obtain (1.14), which presents the Euler equation. Equations (1.7), (1.8) and (1.9) yield (1.15) and (1.16), which respectively express the money-demand function (the relation between the marginal rate of substitution and the opportunity cost for holding money) and the Fisher equation (the no-arbitrage relationship between bonds and capital) .

In the steady state, all the real variables remain constant over time, that is, in the case of CIA, from (1.13)-(1.16) we obtain the following conditions:

$$f(\bar{k}) = \delta\bar{k} + \bar{c} + g, \tag{1.17}$$

$$f'(\bar{k}) + 1 - \delta = \frac{1}{\beta}, \tag{1.18}$$

$$\frac{u_m(\bar{c},\bar{m})}{u_c(\bar{c},\bar{m})} = \frac{R(\bar{\pi},\bar{\pi}) - 1}{\bar{\pi}}, \tag{1.19}$$

$$R(\bar{\pi},\bar{\pi}) - \bar{\pi}[f'(\bar{k}) + 1 - \delta] = 0. \tag{1.20}$$

In the above, $\bar{z}$ represents the steady-state value of variable z. From (1.18), the nontrivial steady-state level of capital uniquely exists because of the property of the production function, $f(k)$. Substituting this steady-state value of capital into (1.17) and (1.20), we can derive a unique set of the steady-state levels of consumption and the inflation rate. If we assume that there exists at least one steady-state inflation rate $\bar{\pi} > \beta$ such that$R(\bar{\pi},\bar{\pi}) = \dfrac{\bar{\pi}}{\beta} > 1$, then $\bar{\pi}$ is uniquely determined by (1.20) under a given level of $\bar{k}$. Finally, we obtain a unique steady-state level of real money holdings from (1.19).

The dynamic system linearized at the steady state consists of the following:

$$\hat{k}_{t+1} - \frac{1}{\beta}\hat{k}_t + \hat{c}_t = 0, \tag{1.21}$$

$$\beta\bar{u}_c\bar{f}''\hat{k}_{t+1} + \bar{u}_{cc}(\hat{c}_{t+1} - \hat{c}_t) + \bar{u}_{cm}(\hat{m}_t - \hat{m}_{t-1}) = 0, \tag{1.22}$$

$$\bar{\pi}\bar{S}_c\hat{c}_{t+1} + \bar{\pi}\bar{S}_m\hat{m}_t - \bar{R}_1\hat{\pi}_t - \left(\bar{R}_2 - \frac{1}{\beta} + \frac{1}{\bar{\pi}}\right)\hat{\pi}_{t+1} = 0, \text{ and} \tag{1.23}$$

$$-\bar{\pi}\bar{f}''\hat{k}_{t+1} + \left(\bar{R}_2 - \frac{1}{\beta}\right)\hat{\pi}_{t+1} + \bar{R}_1\hat{\pi}_t = 0. \tag{1.24}$$

In the above, $\hat{z}_t \equiv z_t - \bar{z}$ and $S(c,m) \equiv \dfrac{u_m(c,m)}{u_c(c,m)}$, where $S_c(c,m) \equiv \dfrac{\partial S}{\partial c} = \dfrac{u_{cm}u_c - u_m u_{cc}}{(u_c)^2} > 0$ and $S_m(c,m) \equiv \dfrac{\partial S}{\partial m} = -\dfrac{u_{cm}u_m - u_c u_{mm}}{(u_c)^2} < 0$.

1.3.2 Forward-looking Rule

Under the forward-looking interest rate rule where $\bar{R}_1 = 0$ and $\bar{R}_2 > 0$, (1.23) and (1.24) are respectively replaced with (1.25) and (1.26) below:

$$\bar{\pi}\bar{S}_c\hat{c}_{t+1} + \bar{\pi}\bar{S}_m\hat{m}_t - \left(\bar{R}_2 - \frac{1}{\beta} + \frac{1}{\bar{\pi}}\right)\hat{\pi}_{t+1} = 0, \tag{1.25}$$

$$-\bar{\pi}\bar{f}''\hat{k}_{t+1} + \left(\bar{R}_2 - \frac{1}{\beta}\right)\hat{\pi}_{t+1} = 0. \tag{1.26}$$

From (1.21), (1.22), (1.25) and (1.26), we can eliminate $\hat{m}$ and $\hat{\pi}$ to derive a reduced form dynamic system in the following manner:

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -1 \\ -\frac{\bar{S}_m}{s}\left(\bar{u}_c\bar{f}'' + \left(\frac{1}{\beta} - 1\right)\frac{\bar{f}''X_1\bar{u}_{cm}}{\bar{S}_m}\right) & \frac{\bar{S}_m}{s}\left(\beta\bar{u}_c\bar{f}'' + \frac{\bar{f}''X_1\bar{u}_{cm}}{\bar{S}_m}\right) + 1 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}, \tag{1.27}$$

where $X_1 \equiv 1 + \frac{1}{\bar{R}(\beta\bar{R}_2 - 1)}$, and $s \equiv \frac{\bar{u}_{cc}\bar{u}_{mm} - (\bar{u}_{cm})^2}{\bar{u}_c} > 0$.

Since there is one predetermined variable $\hat{k}_t$ and one jump variable $\hat{c}_t$, in the system (1.27), determinacy of equilibrium in this system, which means that there is a unique equilibrium path under a given initial capital stock, is satisfied if one eigenvalue is outside the unit circle and the other is inside the unit circle. There is more detailed explanation for analytical results and drawing figures in the Appendix 1.A. Then, we have shown the following proposition and Figure 1.1:

Proposition 1.1 *In the economy with the CIA timing under the forward-looking interest rate rule, the equilibrium path is determinate if $\bar{u}_{cm}(\beta\bar{R}_2 - 1) \leq 0$. When $\bar{u}_{cm}(\beta\bar{R}_2 - 1) > 0$, the equilibrium path may be determinate or non-stationary (unstable).*

1.3.3 Current-looking Rule

Under the current-looking rule where $\bar{R}_1 > 0$ and $\bar{R}_2 = 0$, the dynamic system around the steady state consists (1.21), (1.22), and

$$\bar{\pi}\bar{S}_c\hat{c}_{t+1} + \bar{\pi}\bar{S}_m\hat{m}_t - \bar{R}_1\hat{\pi}_t - \left(-\frac{1}{\beta} + \frac{1}{\bar{\pi}}\right)\hat{\pi}_{t+1} = 0, \tag{1.28}$$

$$-\bar{\pi}\bar{f}''\hat{k}_{t+1} - \frac{1}{\beta}\hat{\pi}_{t+1} + \bar{R}_1\hat{\pi}_t = 0. \tag{1.29}$$

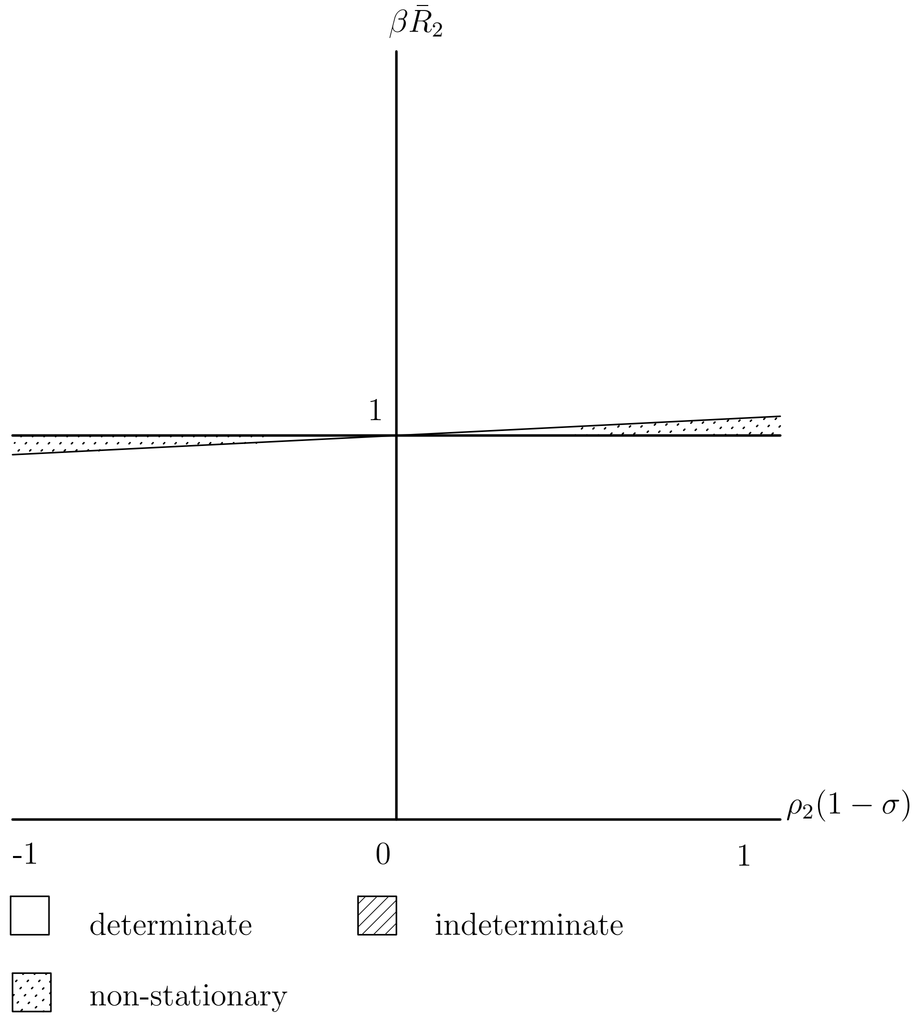

Figure 1.1: Determinacy of the CIA timing with forward-looking rule

These equations are summarized as

$$\begin{bmatrix}\hat{k}_{t+1}\\ \hat{c}_{t+1}\\ \hat{\pi}_{t+1}\end{bmatrix} = \begin{bmatrix}\dfrac{1}{\beta} & -1 & 0\\ -\dfrac{\bar{f}''}{s}(\bar{u}_{mm}-\bar{u}_{cm}) & \dfrac{\beta\bar{f}''\bar{u}_{mm}}{s}+1 & -\dfrac{\bar{u}_{cm}}{\bar{\pi}^2 s}(\beta\bar{R}_1-1)\\ -\bar{\pi}\bar{f}'' & \bar{\pi}\bar{f}''\beta & \beta\bar{R}_1\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\\ \hat{\pi}_t\end{bmatrix}. \qquad (1.30)$$

Since the system (1.30) consists of one predetermined variable, $\hat{k}_t$, and two jump variables, $\hat{c}_t$ and $\hat{\pi}_t$, the equilibrium is determinate if there are two eigenvalues outside the unit circle and one eigenvalue inside the unit circle. We explain the more detailed calculation in Appendix 1.A. The graphical result is displayed in Figure 1.2, and the argument is summarized as the following:

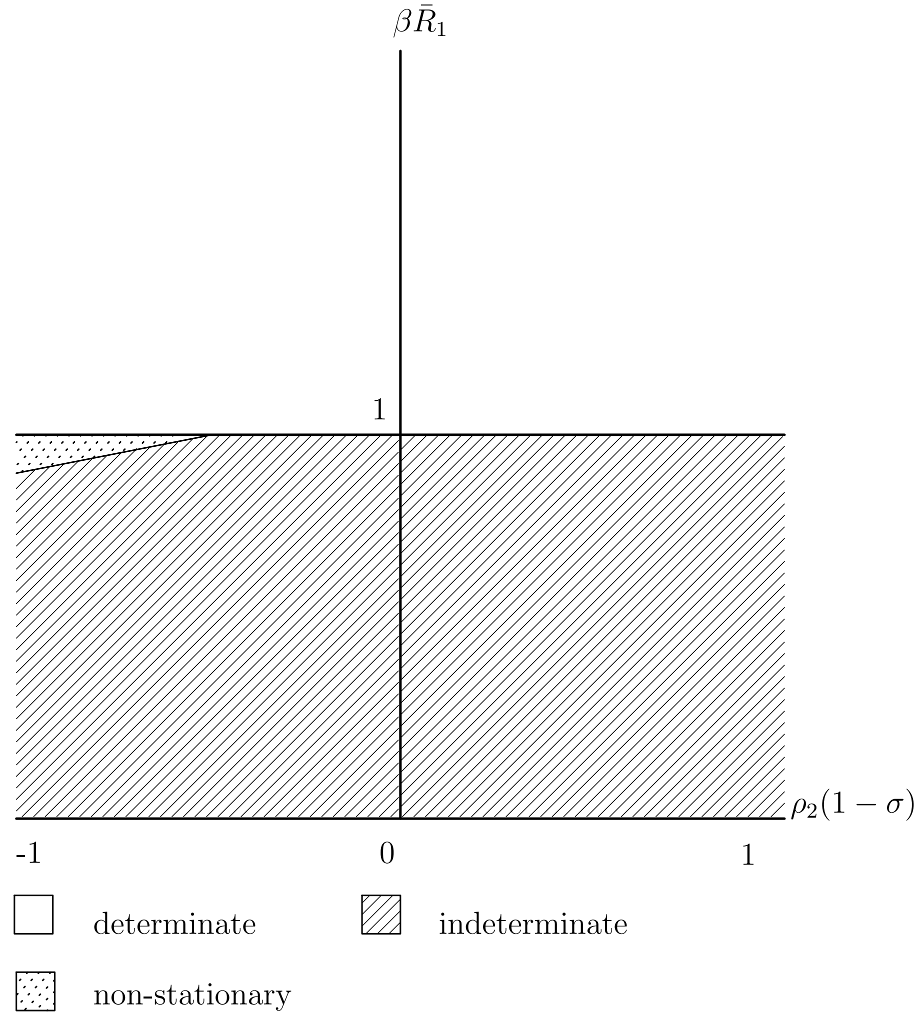

Figure 1.2: Determinacy of the CIA timing with current-looking rule

Proposition 1.2 *In the economy with the CIA timing under the current-looking interest rate rule, there generally exists a unique equilibrium path if the policy is active, while indeterminacy can emerge if the policy is passive. Non-stationarity might arise when the policy rule is slightly passive and the consumption and real money balances are highly Edgeworth substitute.*

1.4 Cash-when-I'm-done(CWID) Timing

The CWID timing assumes that real money balances aided in transactions are ones that household holds after leaving the store. In the CWID timing model, the Lagrangian function is rewritten as the following:

$$\mathcal{L}_{CWID} \equiv \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, m_t) + \lambda_t \left[-k_{t+1} + (1-\delta)k_t - c_t - m_t - b_t - \tau_t + f(k_t) + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t} \right] \right\}. \tag{1.31}$$

The first-order conditions for the household's optimization problem are:

$$\lambda_t = u_c(c_t, m_t); \tag{1.32}$$

$$u_m(c_t, m_t) = \lambda_t - \frac{\beta \lambda_{t+1}}{\pi_{t+1}}; \tag{1.33}$$

(1.8), and (1.9). Equations (1.10)-(1.12) are the transversality conditions.

The complete dynamic system in the CWID timing is described by (1.13), (1.16), and

$$\beta u_c(c_{t+1}, m_{t+1})[f'(k_{t+1}) + 1 - \delta] - u_c(c_t, m_t) = 0, \tag{1.34}$$

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} - \frac{R(\pi_t, \pi_{t+1}) - 1}{R(\pi_t, \pi_{t+1})} = 0. \tag{1.35}$$

The meaning and derivation of each equation are similar as those in Section 1.3.1. Real money balances which effect the marginal utility of consumption in the current period are ones held at the end of that period in the CWID timing, while they are ones held at the end of the previous period in the CIA timing. The opportunity cost of holding money is discounted by the real rate of interest. The steady-state conditions are almost the same as those in the CIA model except for

$$\frac{u_m(\bar{c}, \bar{m})}{u_c(\bar{c}, \bar{m})} = 1 - \frac{1}{R(\bar{\pi}, \bar{\pi})}, \tag{1.36}$$

instead of (1.19), which is obtained from (1.35). As well as in the CIA model, it is easy to show that the nontrivial steady state uniquely exists in the CWID model.

We linearize the complete dynamic system around the steady state. The resulting linearized system consists of (1.21), (1.24), and the following two equations:

$$\beta \bar{u}_c \bar{f}'' \hat{k}_{t+1} + \bar{u}_{cc}(\hat{c}_{t+1} - \hat{c}_t) + \bar{u}_{cm}(\hat{m}_{t+1} - \hat{m}_t) = 0, \tag{1.37}$$

$$\bar{R}^2 \bar{S}_c \hat{c}_t + \bar{R}^2 \bar{S}_m \hat{m}_t - \bar{R}_1 \hat{\pi}_t - \bar{R}_2 \hat{\pi}_{t+1} = 0. \tag{1.38}$$

In the case of the forward-looking rule ($\bar{R}_1 = 0$ and $\bar{R}_2 > 0$), Figure 1.3 and the following proposition summarize our finding:

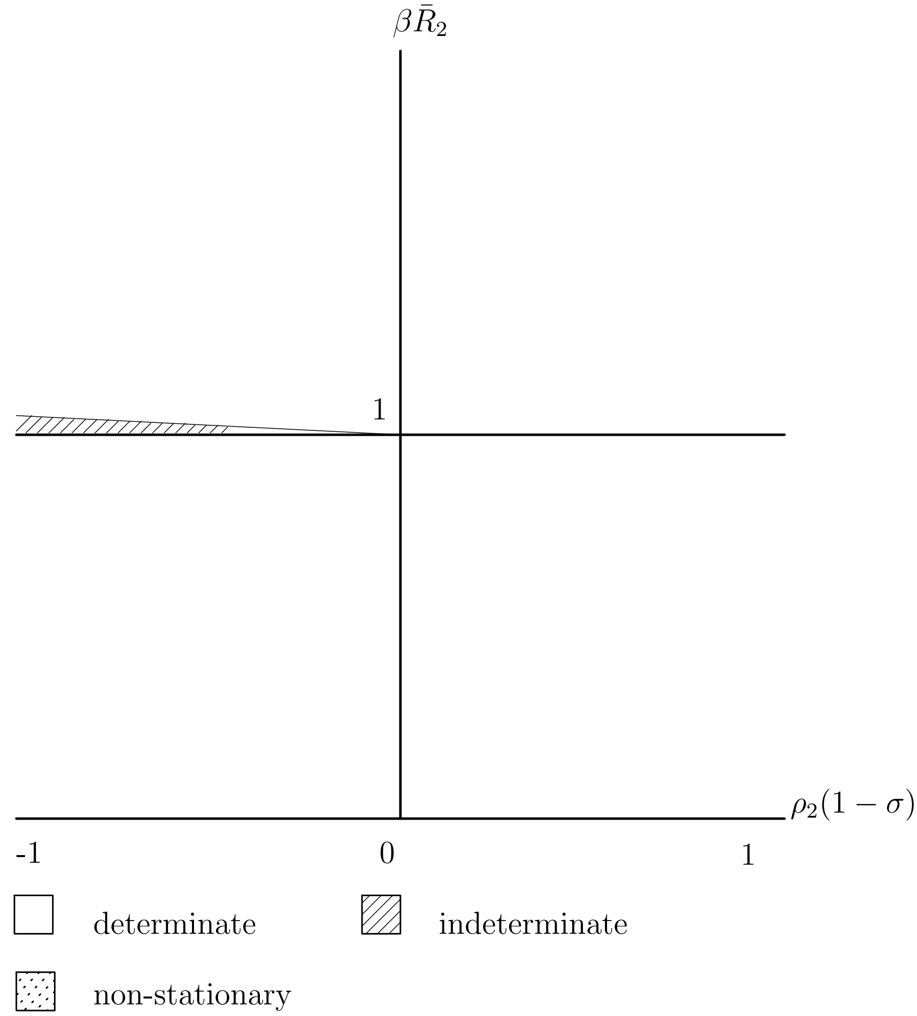

Figure 1.3: Determinacy of the CWID timing with forward-looking rule

Proposition 1.3 *In the economy with the CWID timing and the forward-looking interest rate rule, the determinacy of equilibrium generally holds. Indeterminacy could emerge, if the policy rule is slightly active and consumption and real money balances are highly Edgeworth substitute.*

The result under the current-looking rule ($\bar{R}_1 > 0$ and $\bar{R}_2 = 0$) is described by Figure 1.4 and it is summarized as:

Proposition 1.4 *In the economy with the CWID timing under the current-looking interest rate rule, active policy rule generates equilibrium determinacy, while indeterminacy can be produced under passive policy rule. When consumption and real*

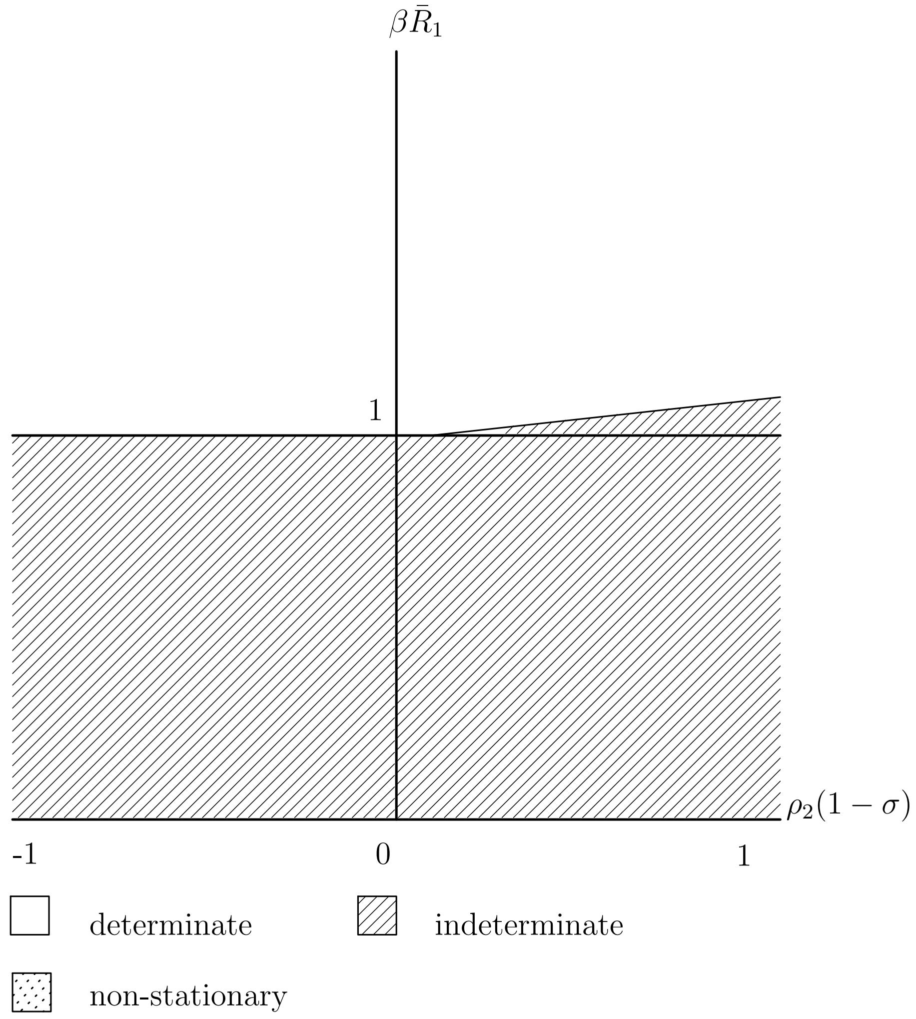

Figure 1.4: Determinacy of the CWID timing with current-looking rule

money balances are highly Edgeworth complement, the slightly active policy rule might emerge indeterminacy.

We show the detailed manipulation in Appendix 1.B.

1.5 Discussion

1.5.1 A General Consideration

We claim the general points which may be critical to capture the differences in results under alternative formulations. To begin with, we consider the role of capital. As a

Table 1.1: Equilibrium determinacy under forward-looking rule

forward-looking rule	CIA			CWID		
with capital	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$
active	D	D	D(, NS)	D	D	D
passive	D(, NS)	D	D	D(, I)	D	D
without capital	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$
active	I	D	D	I	D	D
passive	D	D	I	D	D	I

*D=determinate, I=indeterminate, NS=non-stationary(unstable)

Table 1.2: Equilibrium determinacy under current-looking rule

current-looking rule	CIA			CWID		
with capital	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$
active	D	D	D	D	D	D(, I)
passive	I(, NS)	I	I	I	I	I
without capital	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$	$u_{cm} < 0$	$u_{cm} = 0$	$u_{cm} > 0$
active	I	D	D	D	D	D
passive	D	I	I	I	I	I

*D=determinate, I=indeterminate, NS=non-stationary(unstable)

rule, in the model without capital, production (and thus consumption) is determined by a given endowment. In this case, the dynamic system consists of two jump variables, that is, real money holdings and the inflation rate. Tables 1.1 and 1.2 summarize the equilibrium determining results under alternative specifications of policy rules and the timing of real money balances. As for the model without capital, we classify the case of CWID according to Benhabib et. al. (2001a), and we calculate the case of CIA by ourselves. There are some differences between the results of the models with and without capital. First, the possibility of non-stationary equilibrium may exist in the model with capital, while it does not exist without capital. Second, while the property of MIUF such as the timing of money holdings and the Edgeworth complementarity between consumption and real money balances may affect equilibrium determinacy in the absence of capital, these effects are mostly negligible in the models with capital. The main reason for such a difference is that capital as a state (non-jumpable) variable works like an anchor. Though this role is the same as that in the sticky-price models, these models contain other factors such as monopolistic pricing behavior of firms, so that the meaning of including capital into the models is less clear.

Next, we examine the difference between the continuous and discrete time formulations. As mentioned in Section 1.1, Meng and Yip (2004) show that in the basic continuous-time model the equilibrium path uniquely exists regardless whether the monetary policy is active or passive. Within the discrete-time setting, the result close to them is derived in the case of the CIA timing under the forward-looking rule (Section 1.3.2). It generally holds that the discrete-time CIA model converges to the continuous-time counterpart as the time interval approaches to zero. We understand the similarity between the continuous-time model and the discrete-time model with the forward-looking rule by observing the no-arbitrage condition between capital and bonds. It shows the relation between the current capital and the current inflation rate in the continuous-time model, while it describes the relation between these two variables in period $t + 1$ in the discrete-time model under the forward-looking rule. The implication is not the same, but both present the intratemporal relation between capital and the rate of inflation. This point is also mentioned by Carlstrom and Fuerst (2005) who analyze a discrete-time model with sticky prices. However, examining numerical examples with plausible parameter values, they show that the range of the monetary policy rule that generates determinacy is very narrow. This is in contrast to the flexible-price discrete-time model in this chapter. Note that in the sticky price model with monopolistic competition, the marginal cost (or the markup ratio) is an additional jump variable. We may conjecture that this enhances the possibility of equilibrium indeterminacy in the model discussed by Carlstrom and Fuerst (2005).

1.5.2 Intuitive Implications

Now, we consider the mechanism of equilibrium determinacy in this model. We assume that the instantaneous utility function is additively separable. Then, the dynamic system consisting of capital and consumption is the same as the standard Ramsey model, which has a unique equilibrium path under a given capital stock. In this case, therefore, equilibrium indeterminacy is generated by monetary factors, that is, the inflation rate and the real money holdings. As a benchmark, we clarify the mechanism of determinacy (indeterminacy) under the assumption of separable utility function. When the utility function is not additively separable between consumption and real balances, behavior of the real money balances affect dynamics of consumption and capital. However, equilibrium determinacy in the model with capital is almost unaffected by the property of MIUF, and therefore the analysis of the separable MIUF can be useful.

Suppose that the economy in period T-1 stays at the steady state, and that the capital in period T is above (and thus the rate of return in period T is below) its steady-state level. Under this situation, agents anticipate that consumption in period T is smaller than its steady-state level. It gives a path such that capital increases and consumption decreases over time, which is not equilibrium in the normal model since it violates the transversality condition. We examine whether the path is equilibrium or not. If it is not, determinacy holds. When it is an equilibrium path, indeterminacy occurs since another path under this given capital stock in which capital stock and consumption converge to the steady state is equilibrium.

Under the active forward-looking rule, the inflation in period T is below its steady-state value for lowering the real interest rate according to the arbitrage condition, $\dfrac{R(\pi_T)}{\pi_T} = f'(k_T) + 1 - \delta$. Capital in period T+1 is larger than in period T for the goods equilibrium condition, and thus the inflation and consumption in period T+1 are smaller than in period T. The path is still violating the transversality condition since the inflation does not converge to the steady state. Therefore, determinacy holds. On the contrary, the rate of inflation in period T is above its steady-state level for the arbitrage condition when the central bank adopts the passive forward-looking rule. The inflation rate is rising over time, and the path violates the transversality condition as above. Determinacy still holds.

Under the current-looking rule, the nominal rate of interest does not change in period T-1. Corresponding to lowering the real rate of return, the inflation rate in period T increases according to the arbitrage condition $\dfrac{R(\pi_{T-1})}{\pi_T} = f'(k_T) + 1 - \delta$. The rate of return is falling over time. When the monetary policy rule is active, the inflation rate is increasing over time since the nominal interest rate is highly rising. Then, equilibrium determinacy occurs as before. If the passive rule is carried out, the nominal rate of interest is not so increasing, the inflation is not needed to increase over time and is diminishing to the steady-state level. Consumption and capital also converge to their steady-state values. We can show another equilibrium path with a standard Ramsey model and thus indeterminacy.

The timing of money holding may play some role, especially under the non-separable utility. The opportunity cost of holding money is discounted by the real rate of interest in the CWID timing. Under the assumption that the all variables are at their steady-state levels in period T-1, the real money balance at the ending of period T-1 has already been fixed as a steady-state value in the instantaneous utility in period T in the CIA timing. On the other hand, the real money balance at the ending of period T may change the instantaneous utility in period T under the CWID timing. This may bring a difference of equilibrium determinacy, but as shown above, the difference is not so large in this flexible-price model with capital.

We can roughly say that the Taylor rule satisfying one at least of the forward-looking or active property brings a unique path for equilibrium. In this chapter, we focus on the extreme monetary policy rules in which the central bank respond to either the forward or current inflation rate. In reality, the monetary authority adopts a "mixed" rule in which the nominal interest rate responds to both the forward and current inflation rates. Under the mixed rule, the result in this chapter may be overturned. We do not mention a detailed calculation, but we can consider one example by applying the above-mentioned intuition. We assume monetary policy such that a reaction to the forward inflation rate is much stronger than the current rate. When the inflation rate in period T is anticipated to increase, the inflation rate in period T+1 is not needed to increase so much since the respond to the forward inflation rate is very strong. As a result, the inflation rate converges to its steady-state level, and indeterminacy may generate.

1.6 Conclusion

In this chapter, we analyze the relation between the interest-rate control rules and equilibrium determinacy in the discrete-time, flexible-price economic model with capital. We have shown that the equilibrium path tends to be determinate under the forward-looking rule. Under the current-looking rule, determinacy holds if the policy is active, while indeterminacy may emerge if the policy is passive one. As examined in Section 1.5.1, this depends on the structural difference and similarity between the continuous-time model and the discrete-time one, and thus indeterminacy is generated in contrast to Meng and Yip (2004). In the flexible-price economy, these results are generally robust. However, we have confirmed that determinacy may easily (indeterminacy may hardly) hold in the economy with capital in which capital stock works as a state variable. In this context, we see that neither the timing of money holdings nor complementarity between consumption and real money holdings, is significant for equilibrium determinacy. It is the monetary policy rule that is critical for equilibrium determinacy in our setting.

As for examining the robustness of our results, we may extend our discussion in several ways. For instance, we can introduce fiscal rules into the model. As discussed by Leeper (1991), it would be interesting to consider the interaction between fiscal and monetary policy rules. Some authors have discussed this issue in dynamic settings, but most of the existing studies have analyzed models without capital formation: see, for example, Evans and Honkapohja (2007). It is to be noted that Benhabib and Eusepi (2005) and Lubik (2003) consider fiscal rules in models with sticky prices and capital accumulation. Their results can be reconsidered in the context of our flexible-price setting.

Appendix 1.A: Manipulation of the CIA Timing Model

Forward-looking Rule

The characteristic equation of (1.27) is

$$p_1(\mu) = \mu^2 - \left(1 + \frac{1}{\beta} - \frac{\bar{S}_m}{s}\left(-\beta \bar{u}_c \bar{f}' - \frac{\bar{f}'' X_1 \bar{u}_{cm}}{\bar{S}_m}\right)\right)\mu + \frac{1}{\beta} + \frac{\bar{f}'' X_1 \bar{u}_{cm}}{s}. \qquad (1.39)$$

Table 1.3 displays the relation between the equilibrium determinacy and the characteristic equation. Determinacy of equilibrium in this system, which means that there is a unique equilibrium path under a given initial capital stock, is satisfied if one eigenvalue is outside the unit circle and the other is inside the unit circle. It can be easily checked that two roots are real. Since $p_1(1) = (\mu_1 - 1)(\mu_2 - 1) = -\dfrac{\bar{S}_m \beta \bar{u}_c \bar{f}''}{s} < 0$, determinacy holds if $p_1(-1) > 0$, and non-stationary equilibria, in which there are no equilibrium path under a given initial capital stock, exist if $p_1(-1) < 0$, where $p_1(-1) = (\mu_1 + 1)(\mu_2 + 1) = 2\left(1 + \dfrac{1}{\beta} + \dfrac{\bar{f}'' X_1 \bar{u}_{cm}}{s}\right) + \dfrac{\bar{S}_m \beta \bar{u}_c \bar{f}''}{s}$. It means that we can theoretically reject the possibility of indeterminacy in this case.

We find that $X_1 u_{cm} < 0$ is a sufficient condition for $p_1(-1) > 0$, that is, for equilibrium determinacy, but is not a necessary condition. Similarly, in other cases, it is difficult to find analytically the necessary and sufficient conditions for

Table 1.3: Determinacy and the characteristic equation under the forward-looking rule

	$p(-1) > 0$	$p(-1) < 0$
$p(1) > 0$	1)Non-stationary: $p(0) > 1$ 2)Indeterminate: $p(0) < 1$	Determinate
$p(1) < 0$	Determinate	Non-stationary

equilibrium determinacy. We now specify the following example:

$$u(c_t, m_{t-J}) = \frac{(c_t^{\rho_1} m_{t-J}^{\rho_2})^{1-\sigma}}{1-\sigma}, \rho_1, \rho_2 > 0, \sigma > 0; \tag{1.40}$$

$$f(k_t) = Ak_t^\gamma, 0 < \gamma < 1; \tag{1.41}$$

$$R_t = R(\pi_{t+j}) = \bar{R}\left(\frac{\pi_{t+j}}{\bar{\pi}}\right)^\alpha, \alpha > 0, \text{ and } j = 0, 1. \tag{1.42}$$

Note that in the above it holds that

$$\text{sign}(1-\sigma) = \text{sign}(u_{cm})$$

and that

$$\bar{R}_{j+1} \equiv \left.\frac{\partial R_t}{\partial \pi_{t+j}}\right|_{ss} = \frac{\alpha}{\beta}.$$

Then, $p_1(-1)$ is rewritten as

$$p_1(-1) = 2\left(1 + \frac{1}{\beta}\right) + \bar{c}\frac{A\gamma(1-\gamma)}{\bar{k}^{2-\gamma}}\left\{-2\left(1 + \frac{1}{\bar{R}(\alpha-1)}\right)\rho_2(1-\sigma)\frac{\bar{\pi}}{\bar{R}-1} + \beta\right\},$$

which is a function of $\rho_2(1-\sigma)$ and α. We account the reason why the parameter $\rho_2(1-\sigma)$ is important in Appendix 1.A. Suppose that $A = 1, \gamma = 0.35, \beta = 0.99 = \frac{1}{\bar{R}}(\because \bar{\pi} = 1), \delta = 0.02$, and that $g = 0.5$. These functional forms and values are frequently assumed in the literature, and are coordinated to depict an epitome of the actual economy [8]. Using these typical parameter values, the classification of equilibrium determining in $(\beta R_2, \rho_2(1-\sigma))$ space is depicted by Figure [9] 1.1.

[8]The steady-state values are

$$(\bar{k}, \bar{c}, \bar{R}, \bar{y}) = \left(\left[\frac{A\gamma}{\frac{1}{\beta} - 1 + \delta}\right]^{\frac{1}{1-\gamma}}, A\bar{k}^\gamma - \delta\bar{k} - g, \frac{\bar{\pi}}{\beta}, A\bar{k}^\gamma\right) = (43.572, 2.376, 1.010, 3.747).$$

These values are the same in the two models except for real money balances, $\bar{m} = \frac{\rho_2}{\rho_1}\bar{c}\frac{\bar{\pi}}{\bar{R}-1}$ in the CIA timing and $\bar{m} = \frac{\rho_2}{\rho_1}\bar{c}\frac{\bar{R}}{\bar{R}-1}$ in the CWID timing.

[9]Figure 1.2-1.4 shown below also use those parameter values.

Table 1.4: Determinacy and the characteristic equation under the current-looking rule

	$p(-1) > 0$	$p(-1) < 0$
$p(1) > 0$	1)Non-stationary : $p(0) > 1$ and $q > 0$ 2)Indeterminate	Determinate
$p(1) < 0$	1)Determinate :(i)$q > 0$, (ii)$q < 0, \|A_2\| > 3$ 2)Indeterminate	1)Non-stationary : $p(0) < -1$ and $q > 0$ 2)Indeterminate

Current-looking Rule

The characteristic equation of (1.30) is

$$p_2(\mu) = -\mu^3 + \left(1 + \frac{1}{\beta} + \beta\bar{R}_1 + \frac{\beta\bar{f}''\bar{u}_{mm}}{s}\right)\mu^2$$
$$- \left[\beta\bar{R}_1\left(1 + \frac{1}{\beta} + \frac{\beta\bar{f}''\bar{u}_{mm}}{s} + \frac{\beta\bar{f}''\bar{u}_{cm}}{\bar{\pi}s}\right) + \frac{\bar{f}''\bar{u}_{cm}}{s}\left(\frac{\bar{\pi}-\beta}{\bar{\pi}} + \frac{1}{\beta}\right)\right]\mu$$
$$+ \beta\bar{R}_1\left(\frac{1}{\beta} + \frac{\bar{f}''\bar{u}_{cm}}{s}\right). \quad (1.43)$$

The relation between the equilibrium determinacy and the characteristic equation is shown by Table 1.4. Expressing (1.43) as $p(\mu) = -\mu^3 - A_2\mu^2 - A_1\mu - A_0$, we find that determinacy holds if one of the following conditions is satisfied:

1. $p(1) > 0$ and $p(-1) < 0$
2. $p(1) < 0$, $p(-1) > 0$ and $q \equiv (A_0)^2 - A_0A_2 + A_1 - 1 > 0$
3. $p(1) < 0$, $p(-1) > 0$, $q < 0$ and $|A_2| > 3$.

We will check each condition. Note that in this system $p_2(1) = -\dfrac{\beta\bar{u}_c\bar{f}''\bar{S}_m}{s}(\beta\bar{R}_1 - 1)$, and therefore $p_2(1) < 0$ (resp. $p_2(1) > 0$) if the monetary policy is active (resp. passive). The other key values are:

$$p_2(-1) = \left[2\left(1 + \frac{1}{\beta}\right) + \frac{\bar{f}''\bar{u}_{cm}}{s}\left(1 + \frac{\beta}{\bar{\pi}}\right) + \frac{\beta\bar{f}''\bar{u}_{mm}}{s}\right](\beta\bar{R}_1 + 1) - \frac{2\beta\bar{f}''\bar{u}_{cm}}{\bar{\pi}s},$$

$$p_2(0) = \beta\bar{R}_1\left(\frac{1}{\beta} + \frac{\bar{f}''\bar{u}_{cm}}{s}\right),$$

$$q = \left[\beta\bar{R}_1\left(\frac{1}{\beta} + \frac{\bar{f}''\bar{u}_{cm}}{s}\right)\right]^2 - \beta\bar{R}_1\left(\frac{1}{\beta} + \frac{\bar{f}''\bar{u}_{cm}}{s}\right)\left(1 + \frac{1}{\beta} + \beta R_1 + \frac{\beta\bar{f}''\bar{u}_{mm}}{s}\right)$$
$$+ \left[\beta\bar{R}_1\left(1 + \frac{1}{\beta} + \frac{\beta\bar{f}''\bar{u}_{mm}}{s} + \frac{\beta\bar{f}''\bar{u}_{cm}}{\bar{\pi}s}\right) + \frac{\bar{f}''\bar{u}_{cm}}{s}\left(\frac{\bar{\pi}-\beta}{\bar{\pi}}\right) + \frac{1}{\beta}\right] - 1, \text{and}$$

$$|A_2| = \left|1 + \frac{1}{\beta} + \beta\bar{R}_1 + \frac{\beta\bar{f}''\bar{u}_{mm}}{s}\right|.$$

Appendix 1.B: Manipulation of the CWID Timing Model

Forward-looking Rule

From (1.21), (1.24), (1.37) and (1.38) with $\bar{R}_1 = 0$ and $\bar{R}_2 > 0$, the reduced dynamic system is thus given by

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -1 \\ -\frac{\bar{u}_c\bar{f}''\bar{R}\bar{S}_m + \bar{u}_{cm}\bar{f}''X_2\left(\frac{1}{\beta}-1\right)}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2} & \frac{\beta\bar{u}_c\bar{f}''\bar{R}\bar{S}_m + \bar{u}_{cm}\bar{f}''X_2}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2} + 1 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}, \tag{1.44}$$

where $X_2 = \dfrac{\beta\bar{R}_2}{\beta\bar{R}_2 - 1}$. The characteristic equation is

$$p_3(\mu) = \mu^2 - \left(1 + \frac{1}{\beta} + \frac{\beta\bar{u}_c\bar{f}''\bar{R}\bar{S}_m + \bar{u}_{cm}\bar{f}''X_2}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2}\right)\mu$$
$$+ \frac{1}{\beta}\left(\frac{\beta\bar{u}_c\bar{f}''\bar{R}\bar{S}_m + \bar{u}_{cm}\bar{f}''X_2}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2}\right) - \frac{\bar{u}_c\bar{f}''\bar{R}\bar{S}_m + \bar{u}_{cm}\bar{f}''X_2\left(\frac{1}{\beta}-1\right)}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2}. \tag{1.45}$$

The system (1.44) is similar to (1.27) in the case of CIA timing model, so that Table 1.3 presents the relation between the equilibrium determinacy and the characteristic equation again. It can be easily checked that two roots are real. We obtain the following values from the polynomial equation (1.45) :

$$p_3(1) = -\frac{\beta\bar{u}_c\bar{f}''\bar{R}\bar{S}_m + \bar{u}_{cm}\bar{f}''X_2}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2};$$

$$p_3(-1) = 2\left(1 + \frac{1}{\beta} + \frac{\bar{u}_{cm}\bar{f}''X_2}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2}\right) + \frac{\beta\bar{u}_c\bar{f}''\bar{R}\bar{S}_m}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2};$$

$$p_3(0) = \frac{1}{\beta} + \frac{\bar{u}_{cm}\bar{f}''X_2}{s\bar{R} - \beta\bar{u}_{cm}\bar{f}''X_2}.$$

In contrast to the CIA timing with the forward-looking rule, we cannot reject the possibility of indeterminacy. From the example as above, Figure 1.3 and Proposition 1.3 show the result in this case.

Current-looking Rule

From (1.21), (1.24), (1.37) and (1.38) with $\bar{R}_1 > 0$ and $\bar{R}_2 = 0$, we obtain [10]

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{\pi}_t \end{bmatrix}. \tag{1.46}$$

The characteristic equation is

$$\begin{aligned} p_4(\mu) = -\mu^3 + \left(1 + \frac{1}{\beta} + \beta\bar{R}_1 - \frac{\bar{f}''\bar{u}_{cm}[\beta(\bar{R}-1) + \beta^2\bar{R}_1] - \beta\bar{f}''\bar{R}\bar{u}_{mm}}{\bar{R}s}\right)\mu^2 \\ - \left[\frac{1}{\beta} + \beta\bar{R}_1\left(1 + \frac{1}{\beta} + \frac{\beta\bar{f}''}{s}(\bar{u}_{mm} - \bar{u}_{cm})\right)\right]\mu + \bar{R}_1. \end{aligned} \tag{1.47}$$

The system (1.46) is analogous to (1.30) in the case of CIA with current-looking rule so that Table 1.4 shows the relation between the equilibrium determinacy and the characteristic equation. $p_4(1) = -\frac{\beta\bar{u}_c\bar{f}''\bar{S}_m}{s}(\beta\bar{R}_1 - 1) < 0$(resp. > 0) if the monetary policy is active (resp. passive). Since $p_4(0) = \bar{R}_1 > 0$, there are no or two eigenvalues which are negative, so that there is indeterminacy if $p_4(1) < 0$ and $p_4(-1) < 0$, where $p_4(-1) = -\frac{\bar{f}''\bar{u}_{cm}[\beta\bar{R}_1(\beta(\bar{R}+1)) + \beta(\bar{R}-1)]}{\bar{R}s} + (1+\beta\bar{R}_1)\left[2\left(1 + \frac{1}{\beta}\right) + \frac{\beta\bar{f}''\bar{u}_{mm}}{s}\right]$. Again, we check the conditions for determinacy of such a system

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[10] $e_1 \equiv \begin{bmatrix} \frac{1}{\beta} \\ \frac{\bar{f}''\bar{u}_{cm}[(\bar{R}-1) + \beta\bar{R}_1] - \bar{f}''\bar{R}\bar{u}_{mm}}{\bar{R}s} \\ -\bar{\pi}\bar{f}'' \end{bmatrix}$, $e_2 \equiv \begin{bmatrix} -1 \\ \frac{\bar{f}''\bar{u}_{cm}[\beta(\bar{R}-1) + \beta^2\bar{R}_1] - \beta\bar{f}''\bar{R}\bar{u}_{mm}}{\bar{R}s} + 1 \\ \bar{\pi}\bar{f}''\beta \end{bmatrix}$,

and $e_3 \equiv \begin{bmatrix} 0 \\ -\frac{\bar{u}_{cm}\bar{R}_1}{\bar{R}^2 s}(\beta\bar{R}_1 - 1) \\ \beta\bar{R}_1 \end{bmatrix}$.

described in Appendix 1.A.

$$q = \left(1-\beta+\frac{\beta^2\bar{f}''\bar{u}_{cm}}{\bar{R}s}\right)(\bar{R}_1)^2+\left(-\frac{1}{\beta}+\beta+\frac{\beta\bar{f}''}{s}\left[\left(1-\beta-\frac{1}{\bar{R}}\right)\bar{u}_{cm}-(1-\beta)\bar{u}_{mm}\right]\right)\bar{R}_1 + \frac{1}{\beta} - 1$$

$$|A_2| = \left|1 + \frac{1}{\beta} + \beta\bar{R}_1 - \frac{\bar{f}''\bar{u}_{cm}[\beta^2\bar{R}_1 + \beta(\bar{R}-1)] - \beta\bar{f}''\bar{R}\bar{u}_{mm}}{\bar{R}s}\right|$$

The result based on the example as above is shown in Figure 1.4 and Proposition 1.4.

Appendix 1.C: The Euler Equation for a Specified Model

In the example (1.40)-(1.42), the Euler equation in the CIA timing is

$$\begin{aligned}\beta\left[\frac{A\gamma}{k_{t+1}^{1-\gamma}} + 1 - \delta\right] &= \left(\frac{c_{t+1}}{c_t}\right)^{1-\rho_1(1-\sigma)}\left(\frac{m_t}{m_{t-1}}\right)^{-\rho_2(1-\sigma)} \\ &= \left(\frac{c_{t+1}}{c_t}\right)^{1-(\rho_1+\rho_2)(1-\sigma)}\left(\frac{(R(\pi_{t+j})-1)/\pi_{t+1}}{(R(\pi_{t+j-1})-1)/\pi_t}\right)^{\rho_2(1-\sigma)}.\end{aligned}$$

Similarly, the Euler equation under the CWID timing is given by

$$\begin{aligned}\beta\left[\frac{A\gamma}{k_{t+1}^{1-\gamma}} + 1 - \delta\right] &= \left(\frac{c_{t+1}}{c_t}\right)^{1-\rho_1(1-\sigma)}\left(\frac{m_{t+1}}{m_t}\right)^{-\rho_2(1-\sigma)} \\ &= \left(\frac{c_{t+1}}{c_t}\right)^{1-(\rho_1+\rho_2)(1-\sigma)}\left(\frac{(R(\pi_{t+j+1})-1)/R(\pi_{t+j+1})}{(R(\pi_{t+j})-1)/R(\pi_{t+j})}\right)^{\rho_2(1-\sigma)}.\end{aligned}$$

When $(\rho_1+\rho_2)(1-\sigma) > 1$, the utility displays increasing returns-to-scale in consumption and real money balances. It may effect equilibrium determinacy. This is pointed out by Guo and Harrison (2008), in which the utility function involves government expenditure instead of real money balances. They conclude that indeterminacy holds if and only if $(\rho_1+\rho_2)(1-\sigma) > 1$. We suppose that ρ_1 and ρ_2 are free parameters to consider this argument. In this example, we can express the conditions for equilibrium determinacy, such as q, $|A_2|$, and so on, as the functions of $\rho_2(1-\sigma)$ and α. We can eliminate ρ_1 in the process of calculation. If $\rho_1(1-\sigma) < 1$ and $\rho_2(1-\sigma) < 1$, that is, $1-\sigma < \min\left\{\frac{1}{\rho_1}, \frac{1}{\rho_2}\right\}$, then the utility function satisfies strict concavity.

When we see the figures, we must be careful of the fact that the higher ρ_2 is, the broader the plausible range of $\rho_2(1-\sigma)$ is. For example, in Figure 1.1, which draws the result of the CIA timing with forward-looking rule, if $\alpha = 1.05$, non-stationary happens when $0.92 < \rho_2(1-\sigma) < 1$. Assuming $\rho_1 = 0.98$ and $\rho_2 = 0.02$, $\rho_2(1-\sigma) < \min\left\{\frac{\rho_2}{\rho_1}, 1\right\} = 0.021$, must be satisfied for concavity of utility function, so that non-stationary does not happen under $\alpha = 1.05$. Now, we suppose that only ρ_2 is changed, $\rho_2 = 0.92$. The area for concave utility function becomes $\rho_2(1-\sigma) < 0.939$, and therefore non-stationary under $\alpha = 1.05$ happens if $0.92 < \rho_2(1-\sigma) < 0.939$. However, this instance and Figure 1.1-1.4 shows that whether $(\rho_1+\rho_2)(1-\sigma)$ is higher than 1 or not is little important for equilibrium determinacy unless we consider the extreme values of ρ_1, ρ_2, and $1-\sigma$. It is contrast to Guo and Harrison (2008).

In Guo and Harrison (2008), the Euler equation is rewritten as [11]

$$\beta\left[\frac{A\gamma}{k_{t+1}^{1-\gamma}} + 1 - \delta\right] = \left(\frac{c_{t+1}}{c_t}\right)^{1-\rho_1(1-\sigma)} \left(\frac{g_{t+1}}{g_t}\right)^{-\rho_2(1-\sigma)} = \left(\frac{c_{t+1}}{c_t}\right)^{1-(\rho_1+\rho_2)(1-\sigma)},$$

since the marginal utility of government spending is equal to the one of consumption and thus $g_t = \frac{\rho_2}{\rho_1} c_t$. Their Euler equation is the relation between capital and consumption. Therefore, increasing returns-to-scale of the utility directly effects equilibrium determinacy in their model. On the other hand, the effect of increasing returns-to-scale of the utility for equilibrium determinacy is indefinite because our Euler equation includes the inflation rate, i.e., the opportunity cost of holding money.

[11] Tax is included in their model. However, this is not critical for the Euler equation. We describe their Euler equation such that we can compare with that of our model.

Chapter 2

Generalized Taylor Rule and Endogenous Growth I: A Continuous-Time Analysis

2.1 Introduction

Many authors have explored whether the interest-rate control rule based on Taylor's (1993) idea contributes to reducing equilibrium indeterminacy which generates expectations-driven economic fluctuations. In the literature, it has been well known that an economy following Taylor's rule may easily produce multiple equilibria, if the model economy does not consider capital accumulation. For example, Benhabib et. al. (2001b) reveal that an active interest-rate control under which the nominal interest rate is adjusted more than one-for-one with the rate of inflation, the competitive equilibrium is determinate. Conversely, under a passive interest-rate feedback rule which controls the nominal interest rate less than one-for-one with inflation, the competitive equilibrium tends to be indeterminate. At the same time, Benhabib et. al. (2001a) demonstrate that those results would be reversed if the production function contains the stock of real money balances as an input.

In contrast to the models without capital, Meng and Yip (2004) confirm that the possibility of equilibrium indeterminacy under the Taylor rule is significantly reduced, if the economy allows capital accumulation.[1] Technically speaking, intro-

[1] Meng and Yip (2004) use a neoclassical monetary growth model based on the money-in-the-utility function formulation.Li and Yip (2004), on the other hand, show that if a cash-in-advance constraint applies to both investment and consumption so that money is not superneutral in the

ducing capital adds a non-jumpable state variable to the model, which generally contributes to eliminating multiple converging paths. Meng and Yip (2004) also show that such a conclusion still holds, even if monetary authority changes the nominal interest rate by observing the level of real income as well as inflation.[2]

This chapter reconsiders the issue of equilibrium determinacy under interest-rate control rules in the context of a simple growth model. We use a standard money-in-the-utility function model with an AK technology and exogenous labor supply. In this setting, regardless of interest-rate control rules, money is superneutral on the balanced-growth path and the long-term growth rate of income is uniquely determined by the technology and preference parameters alone. In addition, if the monetary authority adjusts the nominal interest rate by observing the rate of inflation alone, such a monetary policy only affects the steady-state rate of inflation, and hence behaviors of consumption and capital will not respond to the monetary authority's behavior. However, if the monetary authority adopts Taylor's (1993) original proposal and controls nominal interest in response not only to inflation but also to the growth rate of income, then the balanced-growth path may exhibit indeterminacy: there is a continuum of equilibrium paths converging to the balanced-growth equilibrium. In this case, although the balanced-growth path satisfies superneutrality of money, the transition process is affected by the monetary policy. We reveal that, in addition to activeness of interest-rate control, the intertemporal substitutability in felicity also plays a key role for the presence of equilibrium indeterminacy.[3]

steady state, the interest-rate control rule may generate indeterminacy. See also Dupor (2001).

[2]Indeterminacy may emerge if the model introduces labor-leisure choice. As pointed out by Meng and Yip (2004), this possibility, however, requires that labor supply curve has a positive slope.

[3]When the nominal interest rate responds to inflation alone in an AK growth model, intermediacy would emerge either if labor supply is endogenous or if a cash-in- advance constraint applies to investment as well: see Mino and Itaya (2004 and 2007) and Suen and Yip (2005). In those cases, money is not superneutral on the balanced- growth path, which is different from our present formulation where monetary policy cannot affect long-term economic growth.

2.2 The Model

We employ a standard money-in-the-utility-function modelling with an AK technology. The representative household maximizes a discounted sum of utilities

$$U = \int_0^{\infty} e^{-\rho t} u\left(c, m\right) dt, \quad \rho > 0$$

subject to the flow budget and wealth constraints:

$$\dot{a} = ra - c - Rm,$$

$$a = k + m,$$

where c is consumption, m real money balances, k capital stock, a total wealth, r real interest rate and R denotes nominal interest rate. The initial holding of a is exogenously given. Here, we specify the instantaneous utility function in the following manner:

$$u\left(c, m\right) = \frac{\left(c^{\gamma} m^{1-\gamma}\right)^{1-\sigma}}{1-\sigma}, \qquad 0 < \gamma < 1, \quad \sigma > 0, \quad \sigma \neq 1.$$

Denoting the shadow value of a as q, we find that the optimization conditions include the following:

$$\frac{\left(1-\gamma\right) c}{\gamma m} = R, \tag{2.1}$$

$$\gamma c^{\gamma(1-\sigma)-1} m^{(1-\sigma)(1-\gamma)} = q, \tag{2.2}$$

$$\dot{q} = q\left(\rho - r\right), \tag{2.3}$$

together with the transversality condition: $\lim_{t\to\infty} e^{-\rho t} aq = 0$. Equation (2.1) means that the marginal rate of substitution between consumption and real money balances equal the nominal interest rate.

We assume that the production function is specified as

$$y = Ak, \tag{2.4}$$

where y denotes aggregate output. The commodity market is assumed to be competitive so that the real interest rate is determined by

$$r = A. \tag{2.5}$$

We ignore capital depreciation and thus the market equilibrium condition for commodity is $y = \dot{k} + c$, which yields

$$\frac{\dot{k}}{k} = A - z, \tag{2.6}$$

where $z = c/k$.

Following Taylor (1993), we assume that the monetary authority adjusts the nominal interest rate by observing the level of real income as well as the rate of inflation. Since we deal with a growing economy in which real income continues expanding, we consider that the monetary authority changes the nominal interest rate in response not to the level of income but to the growth rate of income.[4] The monetary policy rule is thus specified as

$$R = \phi(\pi) + \eta(g). \quad \phi' > 0, \quad \eta' > 0, \tag{2.7}$$

where g denotes the growth rate of income. From (4.28) and (2.6), g is given by

$$g = \frac{\dot{y}}{y} = \frac{\dot{k}}{k} = A - z.$$

In view of the Fisher condition, the relation and nominal and real interest rates is described by

$$r + \pi = R. \tag{2.8}$$

From (2.7) we obtain:

$$A + \pi = \phi(\pi) + \eta(A - z), \tag{2.9}$$

which yields

$$\frac{d\pi}{dz} = \frac{\eta'(A - z)}{\phi'(\pi) - 1}.$$

As a result, the relation between π and z is expressed as

$$\pi = \pi(z), \tag{2.10}$$

where

$$\text{sign } \pi'(z) = \text{sign } [\phi'(\pi) - 1].$$

Namely, the equilibrium rate of inflation is positively (resp. negatively) related to the consumption-capital ratio, z, if the monetary authority actively (resp. passively) responds to a change in the rate of inflation.

[4] In our notation, Taylor's principle is expressed as $R = 1.5(\pi - \pi^*) + 0.5y$ (or $R = 1.5(\pi - \pi^*) + 1.0y$), where π^* denotes the target rate of inflation.

2.3 Policy Rules and Aggregate Stability

To derive a complete dynamic system, first note that from (2.1), (2.8) and (2.9) we obtain

$$\frac{c}{m} = \frac{\gamma}{1-\gamma}[A + \pi(z)].$$

Taking the time derivatives of the both sides of the above, we obtain

$$\frac{\dot{c}}{c} - \frac{\dot{m}}{m} = \frac{\pi'(z)\,\dot{z}}{A + \pi(z)}. \tag{2.11}$$

Using (2.2) and (2.3) , we derive:

$$[\gamma(1-\sigma) - 1]\frac{\dot{c}}{c} + (1-\sigma)(1-\gamma)\frac{\dot{m}}{m} = \rho - A. \tag{2.12}$$

Eliminating $\dot{m}/m$ from (2.11) and (2.12) yields

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(A-\rho) - \left(\frac{1}{\sigma} - 1\right)(1-\gamma)\frac{\pi'(z)\,\dot{z}}{A + \pi(z)}. \tag{2.13}$$

Since it holds that $\dot{z}/z = \dot{c}/c - \dot{k}/k$, equations (2.6) and (2.13) present the following:

$$\frac{\dot{z}}{z} = \frac{1}{\sigma}(A-\rho) - \left(\frac{1}{\sigma} - 1\right)(1-\gamma)\frac{\pi'(z)\,\dot{z}}{A + \pi(z)} - A + z.$$

The above is rewritten as

$$\frac{\dot{z}}{z} = \frac{\frac{1}{\sigma}(A-\rho) - A + z}{\Gamma(z)}, \tag{2.14}$$

where

$$\Gamma(z) = 1 + \left(\frac{1}{\sigma} - 1\right)(1-\gamma)\frac{\pi'(z)\,z}{A + \pi(z)}.$$

Equation (2.14) gives a complete dynamic equation that summarizes the dynamic behavior of our economy.

It is easy to see that either if $0 < \sigma < 1$ and $\pi'(z) > 0$ or if $\sigma > 1$ and $\pi'(z) < 0$, then

$$\Gamma(z) > 0,$$

so that a unique balanced-growth path in which z is determined by

$$\frac{1}{\sigma}(A-\rho) - A + z^{*} = 0 \tag{2.15}$$

is unstable. This means that the economy always stays on the balanced-growth path, which implies that the economy exhibits global determinacy. Notice that both active control ($\phi' > 1$ so that $\pi'(z)$ is positive) and passive control ($\phi' < 1$ so that $\pi'(z)$ is negative) may yield determinacy depending on the magnitude of σ.

In contrast, either if $\sigma > 1$ and $\pi'(z) > 0$ or if $\sigma < 1$ and $\pi'(z) < 0$, then it is possible to hold $\Gamma(z) < 0$ and thus $d(\dot{z}/z)/dz < 0$ on the balanced-growth path. In this case, we see that the balanced-growth path is stable and it exhibits local indeterminacy.

To sum up, we have shown:

Proposition 2.1 *Suppose that the interest rate control rule is given by* (2.7) *. Then either if* $\phi'(\pi) > 0$ *and* $0 < \sigma < 1$ *or if* $\phi'(\pi) < 1$ *and* $\sigma > 1$, *the balanced-growth path satisfies local determinacy.*

Proposition 2.2 *The necessary and sufficient condition for local indeterminacy is:*

$$1 + \left(\frac{1}{\sigma} - 1\right) \frac{\eta'(A - z)(1 - \gamma) z}{[\phi'(\pi) - 1][A + \pi(z)]} < 0, \tag{2.16}$$

where z^* *and* π^* *are their steady-state values.*[5]

Intuitive implication of the above results is as follows. Suppose that the economy is initially in the balanced-growth equilibrium where capital, consumption and real money balances grow at a common rate of $g^* = (1/\sigma)(A - \rho)$. Suppose further that, due to a change of sunspot-driven expectations, households anticipate a rise in the rate of capital accumulation and that the consumption-capital ratio, z, will decline. Then, for example, if $0 < \sigma < 1$ and $\phi' > 1$, equation (2.13) indicates that the growth rate of consumption will decrease.[6] This means that consumption growth is insufficient to meet the output expansion caused by the expected acceleration of capital formation. Hence, the initial expectations are not self fulfilled, implying that the balanced-growth path itself is a unique competitive equilibrium and the economy has no transition process. Conversely, if (2.16) is satisfied, (2.13) indicates that consumption growth is enhanced. Therefore, there would be enough consumption demand for the expected increase in production, so that the initial expectations are self-fulfilled. If this is the case, there exists a infinite number of converting trajectories at least around the balanced-growth equilibrium: the economy can be out

[5]Global indeterminacy emerges if (2.16) is satisfied for all $z \in (0, A)$, which imposes further restrictions on $\phi(\pi)$ and $\eta(g)$ functions.

[6]In this situation the substitution effect of a change in the nominal interest rate dominates the income effect, which depresses growth of consumption demand.

of the balanced-growth equilibrium and monetary disturbances affect the dynamic behavior of the economy.

To be more concrete, let us specify the policy-rule function in such a way that

$$R=\pi^{*}\left(\frac{\pi}{\pi^{*}}\right)^{\phi}+A\left(\frac{g}{g^{*}}\right)^{\eta},\quad \phi>0,\quad \eta>0, \tag{2.17}$$

where π^* is the target rate of inflation and g^* denotes the balanced-growth rate determined by $g^* = (1/\sigma)(A-\rho)$. In this specification, the target rate of inflation is set by the monetary authority and (2.8) is satisfied on the balanced-growth path where $g = g^*$ and $\pi = \pi^*$. Given this specification, equation (2.9) becomes

$$A+\pi=\pi^{*}\left(\frac{\pi}{\pi^{*}}\right)^{\phi}+A\left(\frac{A-z}{A-z^{*}}\right)^{\eta},$$

which yields

$$\frac{d\pi}{dz}=\frac{\frac{A\eta}{A-z^{*}}\left(\frac{A-z}{A-z^{*}}\right)^{\eta-1}}{\phi\left(\frac{\pi}{\pi^{*}}\right)^{\phi-1}-1}.$$

When we evaluate the above on the balanced growth path where $z = z^*$ and $\pi = \pi^*$, we obtain

$$\left.\frac{d\pi}{dz}\right|_{z=z^{*}}\equiv\pi'(z^{*})=\frac{1}{\phi-1}\left[\frac{\sigma A\eta}{A-\rho}\right].$$

Using the above, we find that

$$\begin{aligned}\Gamma(z^{*})&=1+\left(\frac{1}{\sigma}-1\right)(1-\gamma)\frac{\pi'(z^{*})z^{*}}{A+\pi(z^{*})}\\&=1+\frac{(1-\sigma)(1-\gamma)A\eta}{\sigma(A+\pi^{*})(\phi-1)(A-\rho)}\left[A-\frac{1}{\sigma}(A-\rho)\right].\end{aligned} \tag{2.18}$$

Therefore, $\Gamma(z^*)$ is strictly negative, if and only if

$$\frac{\eta(1-\sigma)}{\phi-1}<-\frac{\sigma(A+\pi^{*})(A-\rho)}{A(1-\gamma)\left[A-\frac{1}{\sigma}(A-\rho)\right]}\ (<0) \tag{2.19}$$

The necessary conditions to hold this inequality are (i) $\sigma < 1$ and $\phi < 1$ or (ii) $\sigma > 1$ and $\phi > 1$. If one of these conditions are met, the possibility of indeterminacy increases as η has a larger value, that is, the monetary authority is more sensitive to a divergence between the actual growth rate and the long-run target rate of income expansion.

As an numerical example, let us set:

$$A=0.07,\quad \rho=0.04,\quad \gamma=0.7,\quad \pi^{*}=0.02.$$

Then the relation between ϕ, σ and η that satisfies $\Gamma(z^*) = 0$ in (2.18) is given by

$$\phi = 1 + \frac{7.77(\sigma - 1)[0.07(\sigma - 1) + 0.04]}{\sigma^2}\eta. \tag{2.20}$$

Panels (a) and (b) in Figure 2.1 depict the graphs between ϕ and η under given levels of σ. Figures 2.1 (a) assumes that $\sigma = 2.0$ so that the balanced growth rate is $g^* = (1/\sigma)(A - \rho) = 0.015$, while Figure (b) sets $\sigma = 0.5$ and thus $g^* = 0.06$. As these figures demonstrate, in both cases the region of the value of ϕ under which indeterminacy emerges is enhanced as η increases. Figure 2.2 shows the graph of (2.20) with a given η. Since in this figure z^* has a negative value for $0 < \sigma < 0.428$, we focus on the region where $\sigma > 0.428$. Again, the graph means that an increase in η enhances the region of indeterminacy in the (ϕ, σ) space.

2.4 Conclusion

In this chapter, we re-examine whether the interest-rate feedback rule according to Taylor (1993) eliminates expectations-driven fluctuations in an endogenously growing economy. To focus on the role of monetary policy rule, we have used an AK model with fixed labor supply in which money is superneutral on the balanced-growth path. Even in such a simple setting, the interest-control rule may generate indeterminacy of equilibrium, if the monetary authority adjusts the nominal interest rate in response to the growth rate of income as well as to the rate of inflation. It is shown that the key elements for indeterminacy conditions are the sensitivity of nominal interest to inflation and the intertemporal rate of substitution in felicity.

For expositional simplicity, this chapter examines the issue in a continuous-time model. As is well known, in discrete-time settings, both the timing of money holding and the time perspective of the monetary authority (for example, forward-looking vs. current-looking rules) are also relevant for determinacy of equilibrium.[7] Examining the role of generalized Taylor rule in alternative formulations of discrete-time monetary growth models deserves further investigation. We approach this problem in the next chapter.

[7] Chapter 1 explores equilibrium determinacy in a discrete-time neoclassical growth model under alterative formulations of money holding and interest-rate feedback rule.

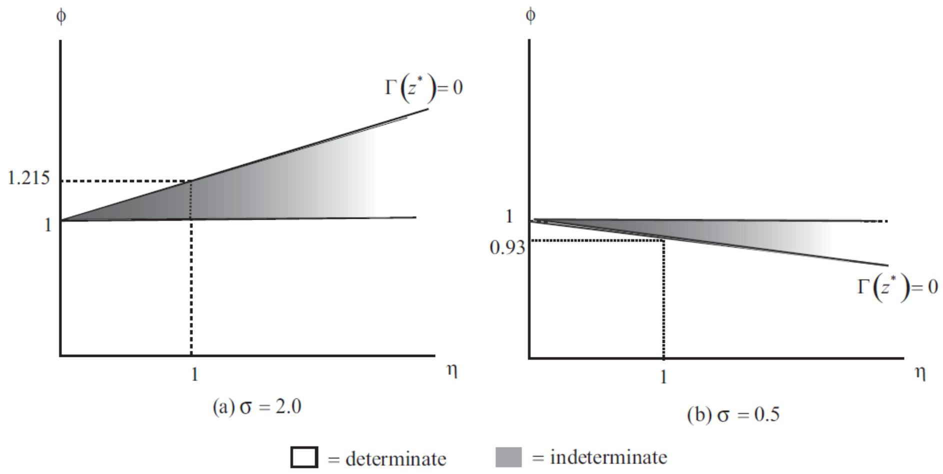

Figure 2.1: (2.20) given σ

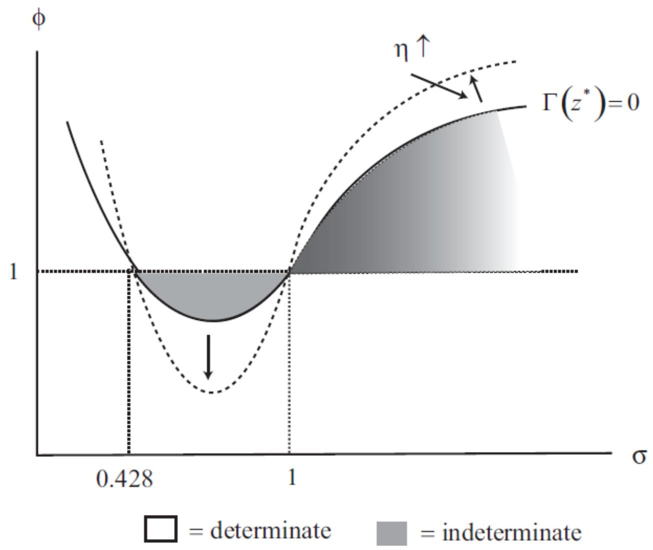

Figure 2.2: (2.20) given η

Chapter 3

Generalized Taylor Rule and Endogenous Growth II: A Discrete-Time Analysis

In Chapter 2, we use an AK growth model with a generalized Taylor rule to demonstrate that equilibrium indeterminacy may emerge more easily than in the exogenous growth models. The discrete-time setting in this chapter can provide us with a richer set of results concerning equilibrium determinacy.

3.1 Introduction

Taylor (1993) proposes a monetary policy rule for economic stabilization under which the central bank adjusts the nominal interest rate in response to real income as well as to the rate of inflation. However, the existing theoretical studies on the interest control rules often assume that the interest rate responds to inflation alone [1]. The purpose of this chapter is to explore the efficacy of the original Taylor rule in the context of a model of endogenous growth. We introduce money into the basic AK growth model via the money-in-the-utility-function formulation. In such a simple environment, money is superneutral on the balanced growth path. In our setting, however, money is not superneutral in the transition process and, hence, the selection of monetary policy rule may have relevant effects on determinacy of equilibrium path leading to the balanced-growth equilibrium.

[1] In models of endowment economy as in Leeper (1991) or Benhabib et. al. (2001a), real income cannot be used as an index of monetary policy.

We construct our model in a discrete-time setting, which enables us to consider alternative timings of households' money holdings and of the inflation rate used for controlling nominal interest rate. As for money holding of the household, we can distinguish the cash-in-advance (CIA) timing from the cash-when-I'm-done (CWID) timing. The CIA (resp. CWID) timing means that real money balances in the utility function is the stock of money the household holds before entering (resp. after leaving) the final goods market [2]. Moreover, in our discrete-time model we find that the main results are also sensitive to the assumption whether the central bank's control rule is current-looking or forward-looking. Therefore, in a discrete-time modelling, we can analyze four patterns of formulations: (i) CWID timing with a forward-looking rule, (ii) CIA timing with a forward-looking rule, (iii) CWID timing with a current-looking rule, and (iv) CIA timing with a current-looking rule.

We obtain two main findings. First, the response of the interest rate to the growth rate of income may play a significant role for equilibrium determinacy. In fact, if the monetary authority controls interest rate in response to inflation alone, we obtain the standard results: equilibrium determinacy holds under the forward-looking and active current-looking monetary rule, while the passive current-looking interest-control rule generates equilibrium indeterminacy. If the interest rate responds to the growth rate of income as well, the possibility of emergence of equilibrium indeterminacy may be enhanced. Second, the efficacy of the generalized Taylor rule for macroeconomic stability depends upon the timings of money holding of the households. We can easily show that the timing of households' money does not affect equilibrium determinacy in an AK growth economy when the central bank does not responds to the rate of inflation alone. The discrete-time analysis in an AK model becomes significant due to the generalization of the interest-rate control. These findings demonstrate that the monetary authority should carefully select a specific interest rate control rule in order to attain stability even if the economic environment is simple enough to hold superneutrality of money in the long run.

Several studies are closely related to this chapter. As for the equilibrium determinacy in monetary growth model with an AK technology, Suen and Yip (2005) and Chen and Guo (2008a) introduce money into the model in the form of cash-in-advance (CIA) constraint. Those authors show that the balanced-growth path may

[2] The discrete-time monetary models usually assume the CWID timing of the money holdings. However, as Carlstrom and Fuerst (2001) claim, it is difficult to justify CWID timing on theoretical grounds, because this assumption means that the money held at the beginning of $t+1$ reduces transaction costs in period t.

be indeterminate under a constant money growth rule if the CIA constraint applies not only to consumption but also to investment so that money is not superneutral on the balanced growth path [3]. Indeterminacy is generated by this form of the CIA constraint rather than by monetary policy rule.

Li and Yip (2004) and Meng and Yip (2004) investigate the effect of Taylor-type interest rate control in the neoclassical growth (i.e. exogenous growth) models. The main message of these studies is that in the neoclassical growth models equilibrium is mostly determinate regardless of the form of interest rate control rules. In contrast, the sticky-price models with capital utilize exogenous growth settings and conclude that forward-looking interest rate controls to generate equilibrium indeterminacy: see, for example, Dupor (2001) and Huang and Meng (2007). In this chapter, using a discrete-time endogenous growth model, we demonstrate that the stabilization effect of interest-rate rules with capital formation shown by the existing literature critically depends on their assumption under which continuing growth is not sustained in the long-run equilibrium.

3.2 The Model

3.2.1 Households

The economy consists of a continuum of identical households with a unit mass. The agent maximizes her lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_{t-J}), \quad 0 < \beta < 1, \quad J = 0, 1 \tag{3.1}$$

subject to the flow budget constraint such that

$$k_{t+1} - (1-\delta)k_t + c_t + m_t + b_t + \tau_t = y_t + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t}, \quad 0 < \delta < 1. \tag{3.2}$$

Each variable means the following: β=time discounting rate; δ=capital depreciation rate; c_t =real consumption; m_{t-J}=real money balances at the beginning of period $t-J+1$; k_t=(per capita) stock of capital; b_t=real stock of bonds at the end of period; τ_t=lump-sum tax; y_t=real income; $\pi_t \equiv P_t/P_{t-1}$=gross rate of inflation;

[3] Chen and Guo (2008a) generalize Suen and Yip (2005) in a way that the CIA constraint applies to consumption and to a certain fraction of gross investment.

P_t=nominal price level; R_{t-1}=gross nominal interest rate in period $t-1$. In this chapter, we specify the utility function as follows:

$$u(c_t, m_{t-J}) = \frac{(c_t^{\rho_1} m_{t-J}^{\rho_2})^{1-\sigma}}{1-\sigma}, \quad \rho_1 + \rho_2 = 1, \quad \sigma > 0,$$

where σ is the inverse of intertemporal elasticity of substitution [4]. This felicity function satisfies $\mathrm{sign}(u_{cm}) = \mathrm{sign}(1-\sigma)$, so that consumption and real money balances are Edgeworth complements if $0 < \sigma < 1$, while they are Edgeworth substitutes if $\sigma > 1$. We define $J = 1$ as cash-in-advance (CIA) timing, and $J = 0$ as cash-when-I'm-done (CWID) timing.

We assume that the production function of the representative firm is given by a simple AK technology, $y_t = Ak_t$. Thus the competitive rate of return on capital is fixed at A.

To derive the optimality conditions for the household's consumption plan, set up the following Lagrangian function:

$$\mathcal{L} \equiv \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, m_{t-J}) + \lambda_t \left[-k_{t+1} + (1-\delta)k_t - c_t - m_t - b_t - \tau_t + Ak_t + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t} \right] \right\}.$$

The first-order conditions for the household's optimization problem are:

$$\lambda_t = u_c(c_t, m_{t-J}) = (c_t^{\rho_1} m_{t-J}^{\rho_2})^{(1-\sigma)} \frac{\rho_1}{c_t}; \tag{3.3}$$

$$u_m(c_t, m_t) = (c_t^{\rho_1} m_t^{\rho_2})^{(1-\sigma)} \frac{\rho_2}{m_t} = \lambda_t - \frac{\beta\lambda_{t+1}}{\pi_{t+1}} \text{ when } J = 0; \tag{3.4}$$

$$u_m(c_{t+1}, m_t) = (c_{t+1}^{\rho_1} m_t^{\rho_2})^{(1-\sigma)} \frac{\rho_2}{m_t} = \frac{\lambda_t}{\beta} - \frac{\lambda_{t+1}}{\pi_{t+1}} \text{ when } J = 1; \tag{3.5}$$

$$\lambda_{t-1} = \beta\lambda_t(A + 1 - \delta); \tag{3.6}$$

$$\lambda_t = \frac{\beta\lambda_{t+1}R_t}{\pi_{t+1}}; \tag{3.7}$$

$$\lim_{t\to\infty} \beta^{t+1}\lambda_{t+1}k_{t+1} = 0; \tag{3.8}$$

$$\lim_{t\to\infty} \beta^t\lambda_t m_t = 0; \tag{3.9}$$

$$\lim_{t\to\infty} \beta^t\lambda_t b_t = 0. \tag{3.10}$$

[4] This instantaneous utility function satisfies $u_c > 0$, $u_m > 0$, $u_{cc} < 0$, $u_{mm} < 0$, $u_{cc}u_m - u_{cm}u_c < 0$, and $u_{mm}u_c - u_{cm}u_m < 0$. That is, the utility function is strictly increasing and strictly concave in c and m, and consumption c and real money balances m are both normal goods.

Equations (3.8)-(3.10) are the transversality conditions.

From (3.6) and (3.7), we obtain the following Fisher equation:

$$\frac{R_t}{\pi_{t+1}} = A + 1 - \delta. \tag{3.11}$$

This represents the non-arbitrage condition, under which the real interest rate of bond is equal to the net real rate of return on capital. Moreover, we acquire the following equations showing that the marginal rate of substitution between consumption and real money holdings is equal to the opportunity cost of holding money:

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{\rho_2}{\rho_1}\frac{c_t}{m_t} = \frac{1}{A+1-\delta}\frac{R_t - 1}{\pi_{t+1}} \quad \text{when } J = 0; \tag{3.12}$$

$$\frac{u_m(c_{t+1}, m_t)}{u_c(c_{t+1}, m_t)} = \frac{\rho_2}{\rho_1}\frac{c_{t+1}}{m_t} = \frac{R_t - 1}{\pi_{t+1}} \quad \text{when } J = 1. \tag{3.13}$$

3.2.2 Capital Formation

The government budget constraint is

$$m_t + b_t + \tau_t = \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t}. \tag{3.14}$$

From (3.2), (3.14), and the production function $y_t = Ak_t$, we obtain the goods-market equilibrium condition:

$$k_{t+1} = Ak_t + (1-\delta)k_t - c_t. \tag{3.15}$$

Denoting $z_t \equiv \frac{c_t}{k_t}$, we can rewrite the condition (3.15) as

$$\frac{k_{t+1}}{k_t} = A + 1 - \delta - z_t. \tag{3.16}$$

3.2.3 Policy Rules

We consider the Taylor-type monetary policy rule under which the central bank controls the nominal interest rate in response to the growth rate of income as well as to the rate of either current or expected inflation. Formally, we assume that

$$R_t = R(\pi_{t+i}, g_{t+i}), \quad \frac{\partial R_t}{\partial \pi_{t+i}} \geq 0, \quad \frac{\partial R_t}{\partial g_{t+i}} \geq 0, \quad i = 0 \text{ or } 1, \tag{3.17}$$

where $g_{t+i} \equiv \frac{y_{t+1+i}}{y_{t+i}} = \frac{k_{t+1+i}}{k_{t+i}} = A+1-\delta-z_{t+i}$ is the gross rate of real income growth. If $i = 0$ (resp. $i = 1$), the interest rate rule is said to be current-looking (resp.

forward-looking), in which monetary authority uses the current (resp. expected) values of economic variables as indices to stabilize economy. Since we deal with a growing economy in which real income continues expanding, our formulation of interest-rate control rule is a natural extension of Taylor's (1993) original proposal.

For analytical simplicity, we specify (3.17) as

$$R_t = \pi^* \left(\frac{\pi_{t+i}}{\pi^*}\right)^{\phi} (A+1-\delta) \left(\frac{g_{t+i}}{g^*}\right)^{\eta}, \quad \phi \geq 0, \quad \phi \neq 1, \quad \eta \geq 0. \tag{3.18}$$

In the above, x^* is the balanced-growth value of a variable x_t, and π^* is the target rate of inflation. If $\phi > 1$, the nominal interest rate rises more than one for one in response to a change in the rate of inflation. Then, the interest control rule is said to be active as to inflation. Conversely, the rule (4.12) with $\phi < 1$ is defined as passive monetary policy.

From (3.11), (3.18) and $g_t = 1 + A - \delta - z_t$, the equilibrium rate of inflation is

$$\pi_{t+1} = \pi_F(z_{t+1}) = \pi^* \left(\frac{1+A-\delta-z_{t+1}}{1+A-\delta-z^*}\right)^{-\frac{\eta}{\phi-1}} \text{ for } i = 1,$$

$$\pi_{t+1} = \pi_C(\pi_t, z_t) = (\pi^*)^{-(\phi-1)}(\pi_t)^{\phi} \left(\frac{1+A-\delta-z_t}{1+A-\delta-z^*}\right)^{\eta} \text{ for } i = 0,$$

where

$$\text{sign}[\pi_F{}'(z_{t+1})] = \text{sign}\left[\frac{\eta}{\phi-1}\right],$$

$$\frac{\partial \pi_C(\pi_t, z_t)}{\partial \pi_t} > 0, \text{sign}\left[\frac{\partial \pi_C(\pi_t, z_t)/\pi_t}{\partial \pi_t}\right] = \text{sign}(\phi - 1), \text{ and}$$
$$\text{sign}\left[\frac{\partial \pi_C(\pi_t, z_t)}{\partial z_t}\right] = \text{sign}\left[\frac{\partial \pi_C(\pi_t, z_t)/\pi_t}{\partial z_t}\right] = -\text{sign}(\eta).$$

Let us consider these properties to understand the role of the interest-rate control in the AK growth economy. When the growth rate of income in which monetary policy rule targets increases, the central bank raises the nominal interest rate to stabilize economy. Since the net real rate of return on capital is constant due to the assumption of AK technology, the real interest rate also should be kept constant by controlling the rate of inflation to satisfy non-arbitrage condition. This process added by a generalization of Taylor rule is important for macroeconomic stability as shown in the following sections. Due to the absence of the process, the timing of money in the felicity does not have an impact on equilibrium determinacy in the AK model with Taylor rule which responds only to the rate of inflation.

If monetary policy is forward-looking and active (resp. passive) , this is achieved by lowering (resp. increasing) the rate of inflation. Under the active (resp. passive) current-looking interest-rate control, the growth rate of inflation is usually positive (resp. negative) when the rate of inflation today rises. However, the nominal interest rate becomes much higher for a positive response to the growth rate of income so that the growth rate of inflation may be positive, even if the interest control rule is passive.

Using the functions of the inflation rate, we obtain the following:

$$\frac{R_t - 1}{\pi_{t+1}} = A + 1 - \delta - \frac{1}{\pi_F(z_{t+1})} = o_F(z_{t+1}) \quad \text{for } i = 1, \tag{3.19}$$

$$\frac{R_t - 1}{\pi_{t+1}} = A + 1 - \delta - \frac{1}{\pi_C(\pi_t, z_t)} = o_C(\pi_t, z_t) \quad \text{for } i = 0, \tag{3.20}$$

where

$$o_F{}'(z_{t+1}) = \frac{\pi_F{}'(z_{t+1})}{[\pi_F(z_{t+1})]^2} : \text{sign}[o_F{}'(z_{t+1})] = \text{sign}\left[\frac{\eta}{\phi - 1}\right],$$

$$\frac{\partial o_C(\pi_t, z_t)}{\partial \pi_t} = \frac{\partial \pi_C(\pi_t, z_t)}{\partial \pi_t} \frac{1}{[\pi_C(\pi_t, z_t)]^2} > 0,$$

$$\frac{\partial o_C(\pi_t, z_t)}{\partial z_t} = \frac{\partial \pi_C(\pi_t, z_t)}{\partial z_t} \frac{1}{[\pi_C(\pi_t, z_t)]^2} < 0.$$

Hence, the opportunity cost of holding money is positively related to the equilibrium rate of inflation [5].

3.3 Forward-looking Rule

3.3.1 CWID Timing

When we assume CWID timing of money holding and forward-looking monetary policy rule, a complete dynamic equation is given by the following :

$$z_{t+1} = [\theta^* \theta^{FW}(z_{t+2}, z_{t+1}) - A + \delta + z_t] z_t, \tag{3.21}$$

[5] We consider the special case in which the nominal interest rate is pegged ($\phi = \eta = 0$). From the non-arbitrage condition (3.11), the rate of inflation is also fixed in the case of AK technology. Therefore, the dynamics of z_t is the same as in the standard AK model regardless of the timing of money in the utility and, hence, equilibrium determinacy around the balanced-growth path always holds.

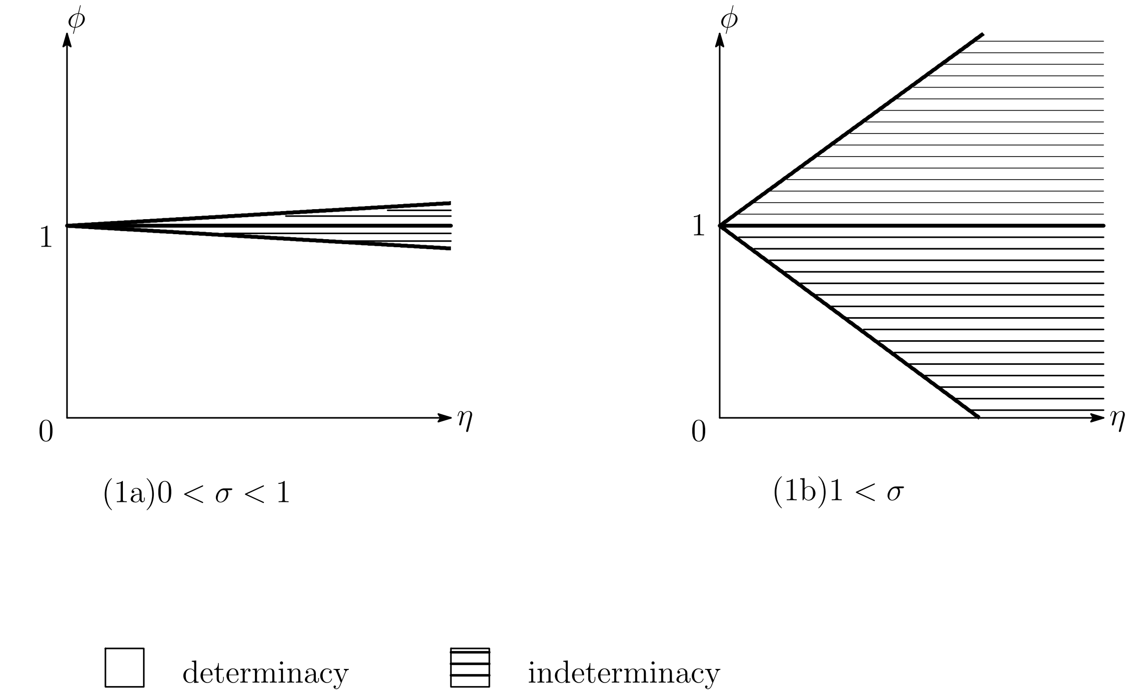

Figure 3.1: The CWID timing with forward-looking rule

where $\theta^* \equiv \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} = 1+A-\delta-z^*$ and $\theta^{FW}(z_{t+2}, z_{t+1}) = \left(\frac{o_F(z_{t+2})}{o_F(z_{t+1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$. In the following, we focus on equilibrium determinacy around the balanced-growth path with a positive growth rate, assuming that $0 < z^* < A - \delta$. We linearize a dynamic system in each case [6]. We summarize the result in the following proposition and Figure 3.1.

Proposition 3.1 *Consider the economy with the CWID timing under the forward-looking interest rate rule. Then, regardless of the sign of* $(1-\sigma)$*, equilibrium indeterminacy tends to hold if* $\frac{\eta}{|\phi - 1|}$ *is high.*

[6]A derivation of the dynamics of z_t and the linearized system around the balanced-growth in each case are shown in Appendix 3.A. This is also the basis for drawing Figures 3.1-3.4.

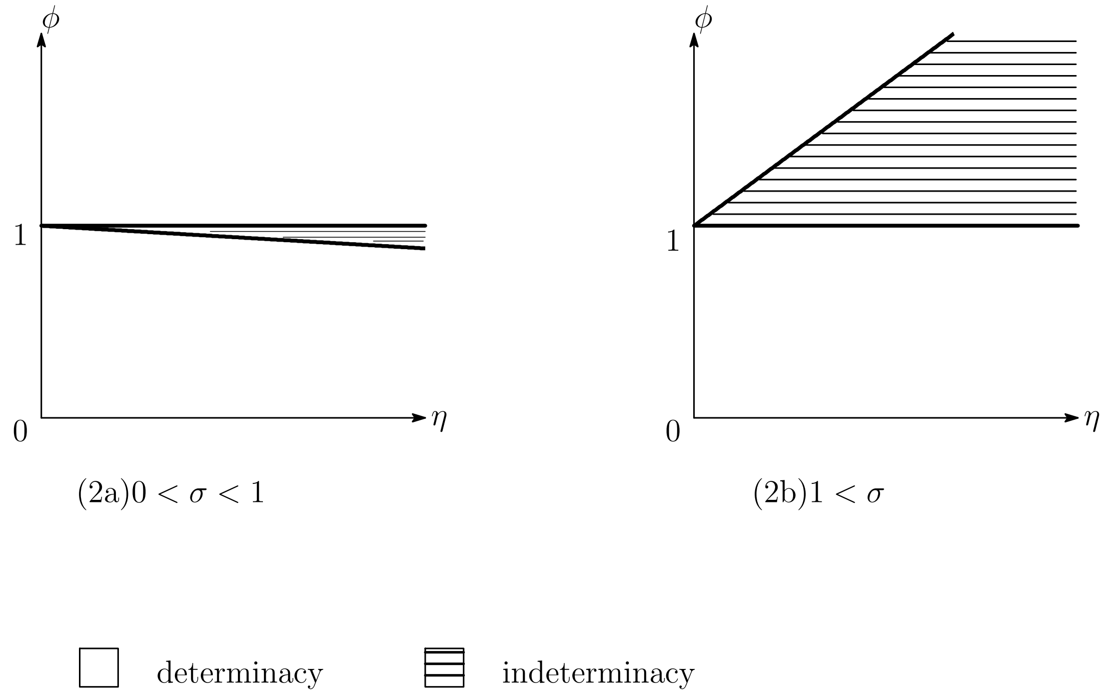

Figure 3.2: The CIA timing with forward-looking rule

3.3.2 CIA Timing

In the case of CIA timing, we can derive a complete dynamic system as a single equation such that

$$z_{t+1} = [\theta^* \theta^{FI}(z_{t+1}, z_t) - A + \delta + z_t] z_t, \tag{3.22}$$

where $\theta^{FI}(z_{t+1}, z_t) = \left(\frac{o_F(z_{t+1})}{o_F(z_t)} \right)^{-\frac{1-\sigma}{\sigma}\rho_2}$. The following proposition and Figure 3.2 summarize the result in this case.

Proposition 3.2 *In the economy with the CIA timing under the forward-looking interest rate rule, equilibrium path is determinate if* $\frac{(1-\sigma)\rho_2\eta}{\phi - 1} \geq 0$. *Otherwise, equilibrium indeterminacy may emerge.*

3.3.3 Intuitive Implication

Under the forward-looking interest control rules, $\frac{(1-\sigma)\rho_2\eta}{\phi - 1} > 0$ is a sufficient condition for equilibrium determinacy in the case of CIA timing, while it is not in the case of CWID timing. We investigate the role of the timing of households' money.

For example, we consider the case under which agents have a preference with $\sigma > 1$ and forward-looking monetary policy rule is passive ($\phi < 1$) so that $\dfrac{(1-\sigma)\rho_2\eta}{\phi - 1} > 0$ is satisfied. Suppose that the economy initially stays in the balanced-growth equilibrium and that a rise of the growth rate of economy is anticipated. According to this anticipation, each agent increases capital accumulation and thus the ratio of consumption to capital z becomes lower ($z_t < z^*$). If z can be higher again, it is also equilibrium path and thus equilibrium indeterminacy holds. We assume $z_t < z_{t+1}$. Then, the rate of inflation falls ($\pi_t > \pi_{t+1}$) over time under the passive interest control rule. As shown in Section 3.2.3, this effect results from the generalization of the interest control rule.

When the timing of money holdings is CIA, the growth rate of consumption corresponds to that of the opportunity cost of holding money at the same periods. This is the reason why the result shown in Proposition 3.2 is close to the finding in Chapter 2 which constructs a continuous-time formulation. If consumption and real money balances are substitutes, decreasing the opportunity cost means a fall of consumption, which contradicts to a rise of z. Therefore, z should diminish over time so that determinacy holds. In the CWID timing, this mechanism is not effective, because the timing of the growth rate of the opportunity cost of holding money which affects the rate of consumption growth is different from the case of CIA. Therefore, a higher z can be realized and indeterminacy may be generated.

3.4 Current-looking Rule

3.4.1 CWID Timing

A complete dynamic system in the case of CWID consists of the following difference equations:

$$\pi_{t+1} = (\pi^*)^{-(\phi-1)}(\pi_t)^{\phi}\left(\frac{1+A-\delta-z_t}{1+A-\delta-z^*}\right)^{\eta}, \tag{3.23}$$

$$z_{t+1} = [\theta^*\theta^{CW}(\pi_t, z_{t+1}, z_t) - A + \delta + z_t]z_t, \tag{3.24}$$

where $\theta^{CW}(\pi_t, z_{t+1}, z_t) = \left(\dfrac{o_C(\pi_{t+1}, z_{t+1})}{o_C(\pi_t, z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$, because $\pi_{t+1} = \pi^C(\pi_t, z_t)$. The results for equilibrium determinacy around the balanced-growth path are summarized in the propositions below and Figure 3.3.

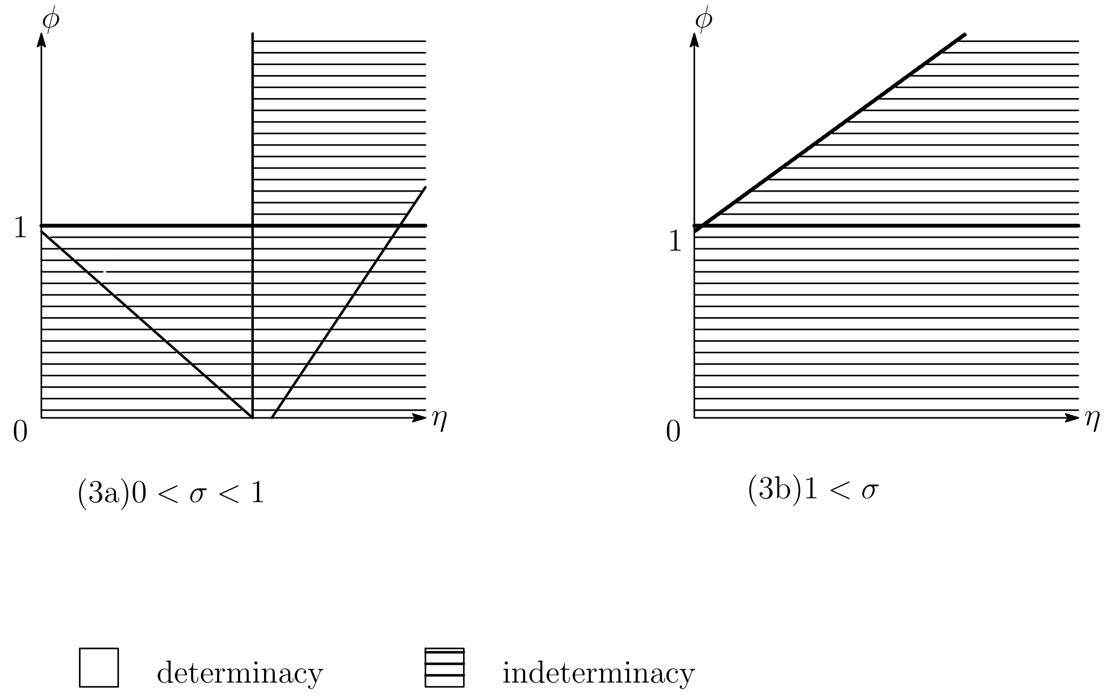

Figure 3.3: The CWID timing with current-looking rule

Proposition 3.3 *Suppose that money holding satisfies the CWID timing and that the interest-rate control is active ($\phi > 1$) and current-looking. Then, the equilibrium path is determinate either if η is small or if $(1-\sigma)\rho_2\eta = 0$. If η is sufficiently large, indeterminacy may emerge.*

Proposition 3.4 *In the case of the CWID timing and the passive current-looking monetary policy rule ($\phi < 1$), equilibrium indeterminacy is generated.*

3.4.2 CIA Timing

Since $o_C(\pi_{t-1}, z_{t-1}) = A+1-\delta-\dfrac{1}{\pi_t}$, a complete dynamic system in this case consists of (3.23) and

$$z_{t+1} = [\theta^*\theta^{CI}(\pi_t, z_t) - A + \delta + z_t]z_t, \tag{3.25}$$

where $\theta^*\theta^{CI}(\pi_t, z_t) = \left(\dfrac{o_C(\pi_t, z_t)}{o_C(\pi_{t-1}, z_{t-1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$. The main results obtained in this system are summarized as the following propositions and Figure 3.4.

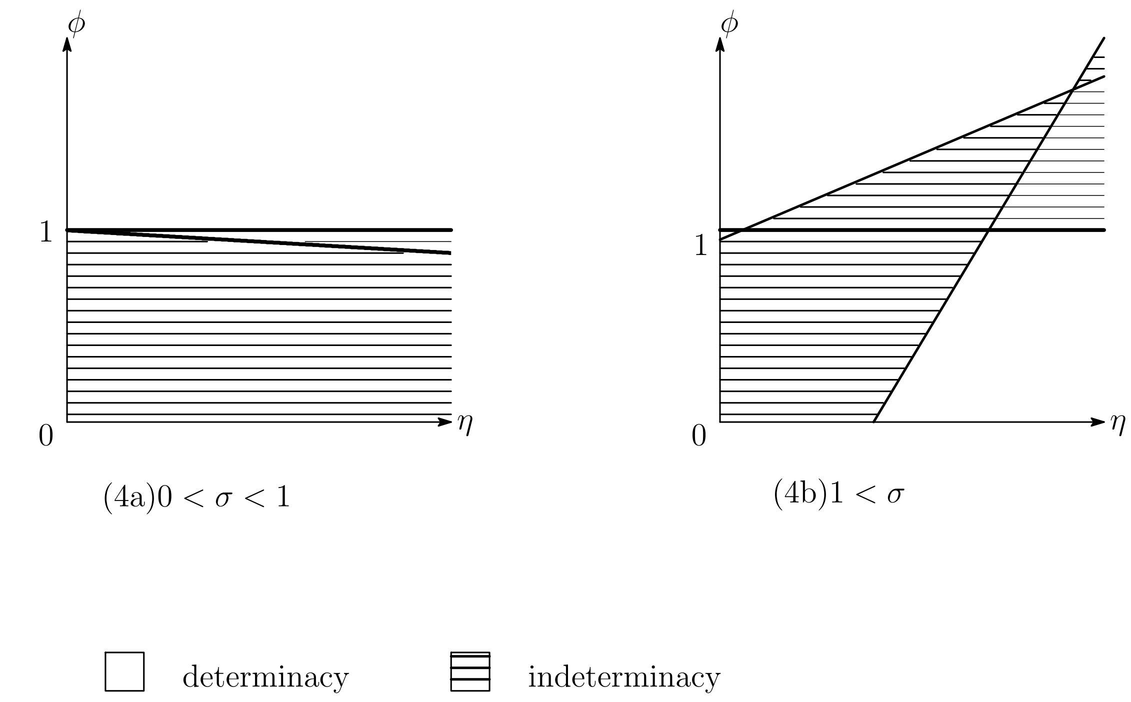

Figure 3.4: The CIA timing with current-looking rule

Proposition 3.5 *Assume that the money holdings satisfies the CIA timing and that the interest-rate control is active current-looking* ($\phi > 1$)*. Then equilibrium determinacy holds if* $(1-\sigma)\rho_2\eta \geq 0$*. Otherwise, the equilibrium path can be indeterminate when* η *is large.*

Proposition 3.6 *Assume that the economy with the CIA timing under the passive current-looking interest-rate control rule* ($\phi < 1$)*. If* $(1-\sigma)\rho_2\eta \geq 0$*, balanced growth path is a saddlepoint so that indeterminacy emerges. Otherwise, the equilibrium path is determinate when* η *is large.*

3.4.3 Intuitive Implication

Under the current-looking monetary policy rule, we also should consider the dynamics of inflation as well as that of z. The result for equilibrium determinacy seems to be more complex than under the forward-looking rule. We roughly discuss intuition.

Suppose the rates of both inflation and income growth rise. If monetary policy is passive ($\phi < 1$), the inflation converges to the target rate. However, when nominal interest rate responds to the growth rate of income much strongly, inflation can deviate from the target rate, as well as the active monetary policy [7]. Therefore, generalization of the Taylor rule might be a source for equilibrium determinacy in this case.

However, this change affects the dynamics of z, and the timing of the money-in-the-utility may play a significant role as shown in the previous section. For instance, if inflation and thus the opportunity cost of holding money are higher over time, the growth rate of consumption rises when consumption and real money balances are substitutes ($\sigma > 1$), that is, z can be larger. Under the CWID (resp. CIA), passive interest control and $\sigma > 1$ does not (resp. could) generate equilibrium determinacy, due to the difference as shown in Section 3.3.3. Macroeconomic stability depends on the total effects.

3.5 Endogenous vs Exogenous Growth

We summarize the results for equilibrium determinacy in Table 3.1. We have shown that the generalized Taylor rule has a pivotal effect on economic stability in the AK growth model. Furthermore, the generalization gives the discrete-time AK model the role of the timing of the money in the utility. In contrast, Meng and Yip (2004) claim that a generalized Taylor rule may not yield indeterminacy in the standard neoclassical growth model.

To see the reason for the presence of such a difference, we consider a continuous-time model for simplicity [8]. Substituting the interest-rate control rule $R = \mathcal{R}(\pi, f(k), g)$ into the non-arbitrage condition, $R - \pi = f'(k) - \delta$, and linearizing it around the steady state, we obtain:

$$(\mathcal{R}_1 - 1)\hat{\pi} + \mathcal{R}_2 f'\hat{k} + \mathcal{R}_3\hat{g} = f''\hat{k}. \tag{3.26}$$

Suppose that $\mathcal{R}_1 > 1$, that is, monetary policy rule is active.

In an exogenous growth model, it holds that $\mathcal{R}_3 = 0$ and $f'' < 0 < f'$, so that (3.26) becomes $\hat{\pi} = \dfrac{f'' - \mathcal{R}_2 f'}{\mathcal{R}_1 - 1}\hat{k}$, which satisfies $\dfrac{d\hat{\pi}}{d\hat{k}} < 0$, regardless whether $\mathcal{R}_2$ is zero or positive. We describe this result more intuitively. Assume that capital is

[7] Of course, the rate of inflation always deviates from the target rate under the generalized active monetary policy. Such an asymmetry between active interest-rate control and passive one implies that of the result concerning equilibrium determinacy.

[8] Notations are the same in the model of Section 3.2, and time index is omitted. $\hat{x}$ means a deviation from the steady state as in Appendix 3.A.

Table 3.1: Equilibrium Determinacy

CWID, FL (3.3.1)	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D, I	D	D, I
$\phi < 1$	D, I	D	D, I

FL with $\eta = 0$	CWID	CIA
$\phi > 1$	D	D
$\phi < 1$	D	D

CIA, FL (3.3.2)	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D	D	D, I
$\phi < 1$	D, I	D	D

CL with $\eta = 0$	CWID	CIA
$\phi > 1$	D	D
$\phi < 1$	I	I

CWID, CL (3.4.1)	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D, I	D	D, I
$\phi < 1$	I	I	I

$\phi = \eta = 0$	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
CWID	D	D	D
CIA	D	D	D

CIA, CL (3.4.2)	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D	D	D, I
$\phi < 1$	I	I	I, D

FL=forward-looking rule, CL=current-looking rule
D=determinate, I=indeterminate
ex)"D, I"=determinate for low η, and indeterminate for high η

increasing from the steady state. In the case of neoclassical production, it lowers the real rate of return on capital. Therefore, the real interest rate should fall for the non-arbitrage condition. If the interest-rate control rule is active, the rate of inflation must be lower. Fall width of the inflation rate becomes bigger as much as nominal interest rate rises for the generalization of the Taylor rule, but there is not a qualitative change so that equilibrium determinacy still always holds.

On the other hand, when the AK technology is assumed, the property of equilibrium rate of inflation is dramatically changed for the generalization of the Taylor rule. With $\mathcal{R}_2 = 0$ and $f'' = 0 < f'$, (3.26) is rewritten as $\hat{\pi} = \dfrac{-\mathcal{R}_3}{\mathcal{R}_1 - 1}\hat{g}$. Hence, $\hat{\pi} = 0$ if $\mathcal{R}_3 = 0$ and $\dfrac{d\hat{\pi}}{d\hat{g}} < 0$ if $\mathcal{R}_3 > 0$. Intuition is similar as in Section 3.3. Therefore, the generalization of interest-rate control may affect macroeconomic stability. A difference of the structure in the real rate of return on capital is the main reason for a stark contrast in equilibrium determinacy conditions between the neoclassical and AK growth models.

This difference may affect the analysis of the discrete-time models. Chapter 1, a discrete-time version of Meng and Yip (2004) [9], shows that the timing of households' money could be a little more significant even though the interest-rate control responds only to the inflation rate, in contrast to the AK model in which it is not effective at all if the Taylor rule is not generalized. However, when the monetary policy rule is generalized, while the result in Chapter 1 cannot be expected to change drastically due to the structure of the equilibrium rate of inflation as shown above, the importance of the timing of money in the utility becomes clear in this chapter.

3.6 Conclusion

By use of a discrete-time AK growth model with money, we have investigated the stabilization effect of a generalized Taylor rule under which the nominal interest rate responds to the growth rate of income as well as to the rate of inflation. The central messages of our study are as follows. First, if the interest-rate control is sensitive to the growth rate of income, monetary policy rule may play a pivotal role for economic stability even in a simple environment in which money is superneutral in the balanced growth equilibrium. Second, our discrete-time modelling clearly demonstrates that the timings of money holding of the households and the time perspective of the monetary authority critically affect the efficacy of interest control rules. This aspect cannot be considered in the foregoing studies on equilibrium determinacy of monetary AK growth models in continuous-time settings and on the interest-rate control in which the nominal interest rate responds only to the rate of inflation.

Appendix 3.A: Calculation

In all four cases of Sections 3.3 and 3.4, we use the same step for obtaining the reduced dynamic system. First, using (3.12), (3.13), (3.19) and (3.20), we derive the demand for real money balances in each case. Second, we substitute this money

[9] For example, Carlstrom and Fuerst (2001) also analyze the discrete-time model, but the timing of the nominal interest rate is different from the models in Chapter 1 and this chapter.

demand function into (3.3), which gives the Euler equation. The growth rate of consumption in each case consists of two parts: a common balanced growth rate of consumption obtained in the standard AK growth model, $\theta^* \equiv \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}}$, and the part related to the growth rate of the opportunity cost of holding money. Note that the timing of the growth rate of the opportunity cost of holding money is one period ahead in the case of CIA than that of CWID. Using these Euler equations and the capital dynamics (3.16), we obtain the dynamics of z_t in each case.

We linearize the equations of the system in each case around the balanced-growth path to examine local equilibrium determinacy.

Forward-looking Rule and CWID Timing (Section 3.3.1)

We show the step formally in the case of CWID with forward-looking rule (Section 3.3.1). From (3.12) and (3.19),

$$m_t = \frac{\rho_2}{\rho_1}(1+A-\delta)\frac{c_t}{o_F(z_{t+1})}. \tag{3.27}$$

Substituting this into (3.3), we obtain

$$\lambda_t = \rho_1 \left\{ \frac{\rho_2}{\rho_1}(1+A-\delta) \right\}^{\rho_2(1-\sigma)} \frac{o_F(z_{t+1})^{-\rho_2(1-\sigma)}}{c_t^\sigma}. \tag{3.28}$$

Thus the Euler equation can be expressed as

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_F(z_{t+2})}{o_F(z_{t+1})} \right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{FW}(z_{t+2}, z_{t+1}). \tag{3.29}$$

From (3.21), we obtain the linearized equation

$$\hat{z}_{t+2} = \left(1 - \frac{1}{z^* \theta^* \bar{\theta}_z^{FW}} \right) \hat{z}_{t+1} + \frac{1+z^*}{z^* \theta^* \bar{\theta}_z^{FW}} \hat{z}_t, \tag{3.30}$$

where $\hat{z}_t \equiv z_t - z^*$ and

$$\bar{\theta}_z^{FW} \equiv \left. \frac{\partial \theta^{FW}}{\partial z_{t+1}} \right|_{ss} = - \left. \frac{\partial \theta^{FW}}{\partial z_{t+2}} \right|_{ss} = \frac{1-\sigma}{\sigma} \frac{\rho_2 \eta}{[\pi^*(1+A-\delta)-1](\phi-1)\theta^*}.$$

Equation (3.30) is derived from $\hat{z}_{t+1} = z^* \theta^* \bar{\theta}_z^{FW}(\hat{z}_{t+1} - \hat{z}_{t+2}) + (1+z^*)\hat{z}_t$. In this dynamic system, there are two jump variables, z_{t+1} and z_t. Thus equilibrium determinacy holds if the two roots of the characteristic equation of (3.30) are out of the unit circle. When $\bar{\theta}_z^{FW} = 0$, equation (3.30) becomes $\hat{z}_{t+1} = (1+z^*)\hat{z}_t$, which implies that there is a unique equilibrium path [10].

[10]The generalization of the Taylor rule is not effective alone. If $(1-\sigma)\rho_2 = 0$ is satisfied, we

Forward-looking Rule and CIA Timing (Section 3.3.2)

The Euler equation in this system is

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_F(z_{t+1})}{o_F(z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{FI}(z_{t+1}, z_t). \tag{3.31}$$

Linearizing the system (3.22), we obtain

$$\hat{z}_{t+1} = \left(1 + \frac{z^*}{1 + z^*\theta^*\bar{\theta}_z^{FW}}\right)\hat{z}_t. \tag{3.32}$$

To derive (3.32), we use $\hat{z}_{t+1} = (1+z^*)\hat{z}_t - z^*\theta^*\bar{\theta}_z^{FW}(\hat{z}_{t+1} - \hat{z}_t)$. Since z_t is a jump variable, the condition for indeterminacy is $\left(1 + \frac{z^*}{1 + z^*\theta^*\bar{\theta}_z^{FW}}\right)^2 < 1$, that is,

$$\frac{z^*(z^* + 2 + 2z^*\theta^*\bar{\theta}_z^{FW})}{(1 + z^*\theta^*\bar{\theta}_z^{FW})^2} < 0.$$

Since $z^* > 0$, the condition can be rewritten such that $z^* + 2 + 2z^*\theta^*\bar{\theta}_z^{FW} < 0$. This can be satisfied when $\bar{\theta}_z^{FW} < 0$.

Current-looking Rule and CWID Timing (Section 3.4.1)

Using $\pi_{t+1} = \pi^C(\pi_t, z_t)$ in (3.33), we obtain the Euler equation in this system:

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_C(\pi_{t+1}, z_{t+1})}{o_C(\pi_t, z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{CW}(\pi_t, z_{t+1}, z_t). \tag{3.33}$$

The system (3.23)-(3.24) linearized at the balanced-growth path is

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \phi & -\frac{\eta\pi^*}{\theta^*} \\ X_\pi & X_z \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{z}_t \end{bmatrix}, \tag{3.34}$$

where

$$X_\pi = -\frac{\phi}{\eta}\frac{z^*}{\pi^*}\frac{(\theta^*)^2\bar{\theta}_z^{CW}(\phi-1)}{(\phi-1) - z^*\theta^*\bar{\theta}_z^{CW}} \text{ and } X_z = \frac{(1 + z^* + z^*\theta^*\bar{\theta}_z^{CW})(\phi-1)}{(\phi-1) - z^*\theta^*\bar{\theta}_z^{CW}}.$$

obtain the standard results in the AK growth model with the Taylor rule under which the nominal interest rate responds to inflation alone, even though $\eta > 0$. Two factors neutralizing the effect of the opportunity cost of holding money eliminate the efficacy of the generalized Taylor rule. The first is $\rho_2 = 0$, which means no need for money. Secondly, when the utility is additively separable ($\sigma = 1$), the optimal consumption is independent from the demand for real money holdings.

The linearized dynamic equation of z_t in (3.34) is derived from

$$\left(1 - \frac{z^*}{\phi - 1}\theta^*\bar{\theta}_z^{CW}\right)\hat{z}_{t+1} = (1 + z^* + z^*\theta^*\bar{\theta}_z^{CW})\hat{z}_t - \frac{\phi}{\eta}\frac{z^*}{\pi^*}(\theta^*)^2\bar{\theta}_z^{CW}\hat{\pi}_t,$$

where

$$\begin{aligned}\bar{\theta}_z^{CW} \equiv \left.\frac{\partial\theta^{CW}}{\partial z_t}\right|_{ss} &= \frac{1-\sigma}{\sigma}\frac{\eta(\phi-1)}{(\pi^*(1+A-\delta)-1)\theta^*} \\ &= (\phi-1)\left.\frac{\partial\theta^{CW}}{\partial z_{t+1}}\right|_{ss} = -\frac{\rho_2\eta}{\phi}\frac{\pi^*}{\theta^*}\left.\frac{\partial\theta^{CW}}{\partial\pi_t}\right|_{ss} \\ &= (\phi-1)^2\bar{\theta}_z^{FW}.\end{aligned}$$

There are two jump variables, π_t and z_t, in this system so that equilibrium determinacy emerges when two roots of the characteristic equation of (3.34) are out of the unit circle.

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The Euler equation in this case is given by

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}}\left(\frac{o_C(\pi_t, z_t)}{o_C(\pi_{t-1}, z_{t-1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^*\theta^{CI}(\pi_t, z_t). \tag{3.35}$$

We use $o_C(\pi_{t-1}, z_{t-1}) = A + 1 - \delta - \dfrac{1}{\pi_t}$.

Linearizing (3.23) and (3.25) yields

$$\begin{bmatrix}\hat{\pi}_{t+1} \\ \hat{z}_{t+1}\end{bmatrix} = \begin{bmatrix}\phi & -\dfrac{\eta\pi^*}{\theta^*} \\ -\dfrac{\phi-1}{\eta\pi^*}(\theta^*)^2\bar{\theta}_z^{CI}z^* & 1 + z^* + z^*\theta^*\bar{\theta}_z^{CI}\end{bmatrix}\begin{bmatrix}\hat{\pi}_t \\ \hat{z}_t\end{bmatrix}, \tag{3.36}$$

where

$$\begin{aligned}\bar{\theta}_z^{CI} \equiv \left.\frac{\partial\theta^{CI}}{\partial z_t}\right|_{ss} &= \frac{1-\sigma}{\sigma}\rho_2\frac{\eta}{(\pi^*(1+A-\delta)-1)\theta^*} \\ &= -\frac{\eta}{\phi-1}\frac{\pi^*}{\theta^*}\left.\frac{\partial\theta^{CI}}{\partial\pi_t}\right|_{ss} = \frac{\bar{\theta}_z^{CW}}{\phi-1} \\ &= (\phi-1)\bar{\theta}_z^{FW}.\end{aligned}$$

There are two jump variables π_t and z_t in this system so local equilibrium determinacy requires that the balanced-growth equilibrium is a source.

Chapter 4

Growth, Velocity, and Equilibrium Determinacy in a Cash-In-Advance Economy

In this chapter, we introduce money as the cash-in-advance (CIA) constraint and analyze a role of generalized Taylor rule in an AK growth model. We assume that the CIA constraint applies to investment as well as to consumption, and thus money is not superneutral even on the balanced-growth path. This is in marked contrast to the monetary endogenous growth models discussed in the previous two chapters where money fails to affect long-term growth. In what follows, we focus on the two issues. In Part A of this chapter, we examine equilibrium determinacy of the balanced-growth path. In Part B, we consider a general CIA constraint and discuss the long-run relation between velocity of money and monetary expansion.

Part A: Growth and Determinacy

4.1 Introduction to Part A

This part examines the stabilization role of interest-rate control in an AK model of endogenous growth with a cash-in-advance (CIA) constraint. We assume that the liquidity constraint applies not only to consumption spending but also to a part of investment expenditure, so that money is not superneutral on the balanced growth path (BGP). It is also assumed that the monetary authority may adjust the nominal interest rate in response to the growth rate of income control as well as to the rate of inflation [1].

Using a rather simple setting mentioned above, we obtain two main findings. First, if the monetary authority controls the nominal interest rate in response to the rate of inflation alone, then the BGP is uniquely given and it satisfies global determinacy. This conclusion is in contrast to the result claimed by Suen and Yip (2005) and Chen and Guo (2008a) who show that in an AK growth model with CIA constraint on investment there may exist dual BGPs and one of them is locally indeterminate, if the intertemporal elasticity in consumption of the representative household is less than one [2]. Since they assume that the money growth rate is kept constant, our discussion reveals that indeterminacy found by Suen and Yip (2005) and Chen and Guo (2008a) critically depends on their assumption of monetary supply rule.

Our second finding is that if investment is subject to the CIA constraint and if the nominal interest rate responds to the growth rate of income as well, then the unique BGP may be indeterminate if the interest rate responds either more or less than one for one to inflation. This result is substantially different from Meng and Yip's (2004) result claiming that the generalized Taylor rule we use does not produce indeterminacy in a neoclassical (exogenous) growth model [3]. Our study suggests that, as well as in short-run models (e.g. Benhabib et. al. (2001a)), the form of Taylor-type interest control rule should be carefully selected in an endogenous growth setting.

4.2 The Base Model

The economy is populated by a continuum of identical infinitely-lived households with a unit mass. Each household has perfect foresight and maximizes a discounted stream of utilities

$$\int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad \sigma > 0, \quad 0 < \rho < 1, \tag{4.1}$$

[1] This assumption is according to Taylor (1993), but the response to the growth rate of income is often ignored in economic models.

[2] Suen and Yip (2005) consider a CIA constraint in which all consumption purchases and all gross investment are financed by real money holdings.

[3] Using a neoclassical growth model with a generalized CIA constraint, Li and Yip (2004) show that a passive interest control may generate indeterminacy even if the nominal interest rate does not respond to the level of income. The difference between their conclusion and our finding relies on the difference in the assumption on production technology.

where c is consumption, ρ denotes the time discount rate, and σ is the inverse of the intertemporal elasticity of substitution in consumption. The budget constraint for the representative household is

$$\dot{m} = y - \pi m - c - \nu + \tau, \tag{4.2}$$

where ν is gross investment, y is output, $\pi \equiv \dfrac{\dot{P}}{P}$ is the rate of inflation, P the price level, and m denotes the real money balances that equal the nominal money supply M divided P. The seigniorage is returned to households from the government as a lump-sum transfer so that $\tau = \dot{m} + \pi m$.

The production function is given by

$$y = Ak, \quad A > 0, \tag{4.3}$$

where k is the household's capital stock. Capital stock changes according to

$$\dot{k} = \nu - \delta k, \quad 0 < \delta < 1, \quad k_0 : \text{given} \tag{4.4}$$

where δ is the capital depreciation rate.

The representative household also faces the following generalized CIA constraint:

$$c + \psi\nu \leq m, \psi \in [0, 1]. \tag{4.5}$$

Namely, all consumption purchases and a fraction ψ of gross investment must be financed by the household's real money balances; and the remaining fraction $(1 - \psi)$ of investment goods are credit goods.

The representative household maximizes (4.1) subject to (4.2)-(4.5). The first-order conditions are

$$c^{-\sigma} = \lambda + \zeta, \tag{4.6}$$

$$\mu - \lambda = \psi\zeta, \tag{4.7}$$

$$\dot{\mu} = (\rho + \delta)\mu - A\lambda, \tag{4.8}$$

$$\dot{\lambda} = (\rho + \pi)\lambda - \zeta, \tag{4.9}$$

$$\zeta(m - c - \psi\nu) = 0, \quad \zeta \geq 0, \quad m \geq c + \psi\nu, \tag{4.10}$$

together with the transversality conditions $\lim_{t\to\infty} e^{-\rho t}\lambda_t m_t = 0$ and $\lim_{t\to\infty} e^{-\rho t}\mu_t k_t = 0$, where λ and μ are the utility values of real money balances and capital, respectively, and ζ represents the Lagrange multiplier for the CIA constraint (4.5). In the following, we assume that the CIA constraint (4.5) is strictly binding in equilibrium, and thus $\zeta > 0$ for all t.

The market equilibrium condition for commodity is $y = \dot{k} + \delta k + c$, which yields

$$\frac{\dot{k}}{k} = A - \delta - z, \tag{4.11}$$

where $z \equiv \dfrac{c}{k}$.

Following Taylor (1993), we assume that the monetary authority adjusts the nominal interest rate by observing the level of real income as well as the rate of inflation. Since we deal with a growing economy in which real income continuously expands, we assume that the monetary authority changes the nominal interest rate in response not to the level of income but to the growth rate of income. The monetary policy rule is specified as

$$R = R(\pi, g) = \pi^* \left(\frac{\pi}{\pi^*}\right)^{\phi} + (A - \delta)\left(\frac{g}{g^*}\right)^{\eta}, \quad \phi \geq 0, \quad \phi \neq 1, \quad \eta \geq 0, \quad A > \delta, \tag{4.12}$$

where R is the nominal interest rate, $g = \dfrac{\dot{y}}{y} = \dfrac{\dot{k}}{k} = A - \delta - z$ is the growth rate of income, g^* is the balanced-growth rate of income and $\pi^* > 0$ is the target rate of inflation. If $\phi > 1$, the nominal interest rate rises more than one for one in response to a change in the rate of inflation. In this case, the interest control rule is said to be active as to inflation. Conversely, the rule (4.12) with $\phi < 1$ is defined as a passive monetary policy.

Combining (4.12) with the Fisher equation, $R - \pi = A - \delta$, which implies that the real interest rate equals to the real rate of return to capital, we see that the equilibrium rate of inflation depends on z and thus it is expressed as $\pi = \pi(z)$. This function satisfies that sign[$\pi'(z)$]=sign($\phi - 1$) if $\eta > 0$ around the BGP, because

$$\left.\frac{d\pi}{dz}\right|_{BGP} = \frac{A - \delta}{A - \delta - z^*}\frac{\eta}{\phi - 1}. \tag{4.13}$$

When the growth rate of income increases, the central bank should raise the nominal interest rate to stabilize economy. However, since the net real rate of return to capital is constant due to the assumption of AK technology, the real interest rate also should be kept constant by controlling the rate of inflation to satisfy the Fisher equation. This is achieved by reducing inflation if the monetary policy is active, because the fall in nominal interest rate is larger than that in the inflation rate.

4.3 Balanced Growth Path and Equilibrium Determinacy

We focus on the economy's BGP on which output, consumption and capital grow at a common, positive constant rate. Denoting $p \equiv \dfrac{\mu}{\lambda}$ and using (4.6) through (4.9), we obtain the following dynamic equations:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[l(p)\frac{\dot{p}}{p} + \frac{A}{p} - \rho - \delta\right], \tag{4.14}$$

$$\frac{\dot{\mu}}{\mu} = \rho + \delta - \frac{A}{p}, \tag{4.15}$$

$$\frac{\dot{\lambda}}{\lambda} = \left(\rho + \pi(z) + \frac{1}{\psi}\right) - \frac{p}{\psi}, \tag{4.16}$$

where $l(p) = -\dfrac{1-\psi}{p-(1-\psi)}$. Equations (4.11), (4.14), (4.15) and (4.16) give (4.17) and (4.18) as below:

$$\dot{z} = \left\{\frac{1}{\sigma}\left[l(p)\frac{\dot{p}}{p} + \frac{A}{p} - \rho - \delta\right] - A + \delta\right\}z + z^2, \tag{4.17}$$

$$\dot{p} = \frac{p^2}{\psi} + \left(\delta - \pi(z) - \frac{1}{\psi}\right)p - A. \tag{4.18}$$

It is obvious that the BGP is realized when both z and p stay constant over time so that the balanced growth rate is

$$\theta = \frac{1}{\sigma}\left[\frac{A}{p^*} - \rho - \delta\right] = A - \delta - z^*, \tag{4.19}$$

where p^* and z^* are the steady-state values of p and $z\left(= \dfrac{c}{k}\right)$.

We first consider the case of $\psi = 0$ in which money is required only for real purchases of the consumption goods. Then, from (4.7), a relative price of capital to money p always equals one. Therefore, the dynamic equation is

$$\dot{z} = \left[\frac{A-\rho-\delta}{\sigma} - A + \delta\right]z + z^2. \tag{4.20}$$

As we can see from this equation, the interest-rate control rule is not functional even if it is generalized. Assuming that $\sigma > \dfrac{A-\rho-\delta}{A-\delta}$ to satisfy $z^* > 0$, we find that the

BGP is uniquely determined. As usual, in this case money is superneutral on the BGP and the feasible BGP with a positive z satisfies global determinacy [4].

Next, consider the generalized case in which $\psi \neq 0$. In this case, the BGP is uniquely determined and it holds that $p^* > 1 - \psi$. To show this, note that the following facts:

$$\dot{p}(p = 0; z = z^*) = -A < 0,$$

$$\dot{p}(p = 1 - \psi; z = z^*) = -A - (1 - \psi)[1 + \pi^* - \delta] < 0, \text{ and}$$

$$\dot{p}(p = 1; z = z^*) = -(A - \delta) - \pi^* < 0.$$

Hence, the number of nontrivial BGP that satisfy $p^* > 1 > 1 - \psi > 0$ is only one, implying that the corresponding $z^*(> 0)$ is also uniquely given. Moreover, we can prove that $\dfrac{\partial p^*}{\partial \psi} > 0$, $\dfrac{\partial z^*}{\partial \psi} > 0$ and $\dfrac{\partial \theta}{\partial \psi} < 0$.

We linearize the dynamic system (4.17) and (4.18) around the BGP to obtain:

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \dot{z}_z & \dot{z}_p \\ \dot{p}_z & \dot{p}_p \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{p} \end{bmatrix} = J \begin{bmatrix} \hat{z} \\ \hat{p} \end{bmatrix}, \tag{4.21}$$

where $\hat{z} \equiv z - z^*$ and $\hat{p} \equiv p - p^*$. The elements of matrix J are:

$$\dot{z}_z = \left.\frac{\partial \dot{z}}{\partial z}\right|_{BGP} = \left[\frac{l(p^*)\dot{p}_z}{\sigma p^*} + 1\right] z^* = \left(-\frac{\pi'(z^*)l(p^*)}{\sigma} + 1\right) z^*,$$

$$\dot{z}_p = \left.\frac{\partial \dot{z}}{\partial p}\right|_{BGP} = \frac{z^*}{\sigma (p^*)^2}[l(p^*)\dot{p}_p p^* - A] = -\frac{z^*}{\sigma p^*}\frac{p^*(1 - \psi) + A\psi}{\psi[p^* - (1 - \psi)]},$$

$$\dot{p}_z = \left.\frac{\partial \dot{p}}{\partial z}\right|_{BGP} = -\pi'(z^*)p^*,$$

$$\dot{p}_p = \left.\frac{\partial \dot{p}}{\partial p}\right|_{BGP} = \frac{A}{p^*} + \frac{p^*}{\psi} > 0.$$

Thus, the trace and determinant of J are respectively given by:

$$\text{tr}J = \dot{z}_z + \dot{p}_p = \left(-\frac{\pi'(z^*)l(p^*)}{\sigma} + 1\right) z^* + \frac{A}{p^*} + \frac{p^*}{\psi_\nu}, \tag{4.22}$$

$$\det J = \dot{z}_z \dot{p}_p - \dot{z}_p \dot{p}_z = \frac{z^*}{p^*}\left[\frac{A\psi_\nu + (p^*)^2}{\psi_\nu} - \frac{A\pi'(z^*)}{\sigma}\right]. \tag{4.23}$$

[4] By use of a money-in-the-utility-function model of endogenous growth in which money is superneutral on the BGP, Chapters 2 and 3 reveal that indeterminacy may hold if the interest rate responds to the growth rate of income. Thus, superneutrality of money may not always establish determinacy of equilibrium in endogenous growth models.

Since z and p are jump variables, if $\mathrm{tr}J > 0$ and $\det J > 0$, then the BGP is totally unstable so that the economy always stays on the BGP, that is, the equilibrium path is determinate. Otherwise, the equilibrium path is indeterminate. Note that if $\pi'(z^*) = 0$, that is, if the interest-control is controlled by the rate of inflation alone, then the equilibrium path is determinate since both $\mathrm{tr}J$ and $\det J$ are positive even if money is not superneutral. Note that if $\eta = 0$, then $\det J > 0$, implying that the equilibrium path is determinate even if money is not superneutral. If $\dfrac{\eta}{\phi - 1} > 0$, $\det J$ may be negative from (4.23), so that equilibrium path might be indeterminate. On the contrary, $\det J > 0$ when $\dfrac{\eta}{\phi - 1} < 0$, but $\mathrm{tr}J$ may be negative.

To sum up, we have obtained the following results:

Proposition 4.1 *Either if $\eta = 0$ or if $\psi = 0$, the equilibrium path is determinate.*

Proposition 4.2 *In the case of $0 < \psi \leq 1$ and $\eta > 0$, equilibrium path may be indeterminate under low $|\phi - 1|$.*

In the above, we have shown that the generalization of the CIA constraint has a pivotal effect on equilibrium determinacy. If the CIA constraint is not effective to investment, the utility value of real money balances equals that of capital due to the monetary superneutrality and thus the relative price of capital is one over time. Therefore, even if the Taylor rule is generalized, it fails to affect dynamic behavior of the economy.

When real purchases of both consumption and investment are subject to the CIA constraint, economic stability depends on two forces. The first is the portfolio substitution effect. Suppose that agents expect a rise of the rate of economic growth without fundamentals and thus they anticipate that the consumption-capital ratio z decreases. As shown in Section 4.2, the rate of inflation falls if monetary policy is active. This diminishes the relative value of capital p due to the decrease in the need of capital. Then, the growth rate of income increases. When the interest-rate control is not generalized, the rate of inflation is fixed at the target rate. Therefore, the portfolio substitution effect described above is not effective.

The intertemporal substitution effect is the second one. Under active policy, nominal growth rate may be negative and it can make investment lower, which has a negative effect on economic growth.

If the interest-rate control is passive, converse discussion holds. Totally, optimistic expectations may be self-fulfilling under active interest-rate control, and thus equilibrium indeterminacy can emerge. When $\left|\dfrac{\eta}{\phi - 1}\right|$ is high, the equilibrium rate of inflation should be more larger so that instability may emerge easily.

Part B: Growth and Velocity of Money

4.4 Introduction to Part B

In this part, we examine the long-run effects of an endogenous monetary expansion via the interest-rate control on income growth and on the velocity of money in the context of an AK growth model with a cash-in-advance (CIA) constraint.

In the existing literature, Suen and Yip (2005) and Chen and Guo (2008a) also study the AK growth model with a CIA constraint. The main finding of those studies can be summarized as follows. First, if the interetemporal elasticity of substitution in consumption is less than one, the balanced growth path is unique and determinate, while there may exist dual balanced growth paths if the interetermporal elasticity of substitution in consumption is higher than one. In the latter case, the balance-growth path (BGP) with a higher growth rate is locally indeterminate and there is a positive relation between the growth rate of nominal money supply and the velocity of money. Such a positive relationship is, however, empirically implausible. Chen and Guo (2008b) overcome this problem by assuming that the CIA constraint is more effective on investment than on consumption. They justify this assumption based on the recent increases in the consumer credit and in the cash holdings of firms.

These foregoing studies mentioned above assume that the central bank keeps the growth rate of nominal money supply constant. However, many central banks have shifted their policy stance from the base-money targeting to the interest-rate control, we re-examine the long-run relation between monetary growth and velocity of money under interest-rate control rules. Except for the monetary policy rule, we employ the same analytical framework as Chen and Guo (2008b) use.

The present part reveals that the relation between money growth and velocity of money around the BGP depends on the stance of the monetary policy as well as on the form of CIA constraint. When the nominal interest rate responds to the current rate of inflation alone, the velocity of money is negatively related to the growth rate of nominal money supply around the unique BGP, if the CIA constraint is more binding for investment than consumption. However, if the interest rate responds to the growth rate of income as well and it responds to inflation more than one for one, then a lower velocity of money may be associated with a higher money growth under the normal CIA constraint which is more effective for consumption spending than for investment expenditure.

4.5 An Extension of the Base Model

The representative household's problem is to maximize a discounted stream of utilities

$$\int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad \sigma > 0, \quad \rho > 0, \tag{4.24}$$

subject to

$$\dot{m} = y - \pi m - c - \nu + \tau, \tag{4.25}$$

$$\dot{k} = \nu - \delta k, \quad 0 \leq \delta \leq 1, \quad k_0 : \text{given}, \tag{4.26}$$

$$\psi_c c + \psi_\nu \nu \leq m, \quad 0 < \psi_c \leq 1, \quad 0 \leq \psi_\nu \leq 1, \tag{4.27}$$

where c is consumption, ρ denotes the time discount rate, k is the household's capital stock, δ denotes the capital depreciation rate and σ is the inverse of the intertemporal elasticity of substitution in consumption. Moreover, ν is gross investment, y is output, $\pi \equiv \dfrac{\dot{P}}{P}$ is the rate of inflation, P the price level, and m denotes the real money balances that equal the nominal money supply M divided by P. The household follows the budget constraint (4.25) and the dynamics of capital stock (4.26). The generalized CIA constraint (4.27) means that parts of consumption and gross investment must be financed by the household's real money balances. The seigniorage is returned to households from the government as a lump-sum transfer so that the government's budget constraint is $\tau = \dot{m} + \pi m$.

The production function is given by

$$y = Ak, \quad A > 0, \tag{4.28}$$

and thus the market equilibrium condition for commodity, $y = \dot{k} + \delta k + c$, yields

$$\frac{\dot{k}}{k} = A - \delta - z, \tag{4.29}$$

where $z \equiv \dfrac{c}{k}$.

Following Taylor (1993), we assume that the monetary authority controls the nominal interest rate by observing the real income as well as inflation. Since we deal with a growing economy in which real income continuously expands, we assume that the monetary authority changes the nominal interest rate in response not to the level of income but to the growth rate of income. Specifically, we assume the following control rule:

$$R = R(\pi, g) = \pi^* \left(\frac{\pi}{\pi^*}\right)^{\phi} + (A - \delta)\left(\frac{g}{g^*}\right)^{\eta}, \quad \phi \geq 0, \quad \phi \neq 1, \quad \eta \geq 0, \quad A > \delta, \tag{4.30}$$

where R is the nominal interest rate, and g denotes the growth rate of real income given by

$$g = \frac{\dot{y}}{y} = \frac{\dot{k}}{k} = A - \delta - z. \tag{4.31}$$

In addition, $g^* > 0$ represents the balanced-growth rate of income and $\pi^* > 0$ denotes the target rate of inflation. If $\phi > 1$, the nominal interest rate rises more than one for one in response to a change in the rate of inflation. In this case, the interest control rule is said to be active as to inflation. Conversely, the rule (4.30) with $\phi < 1$ is defined as a passive monetary policy.

We focus on the economy's BGP on which income, capital, consumption and real money balances grow at a common rate. Combining (4.30) and (4.31) with the Fisher equation,

$$R - \pi = A - \delta, \tag{4.32}$$

which implies that the real interest rate equals to the real rate of return to capital, we obtain

$$A - \delta + \pi = \pi^* \left(\frac{\pi}{\pi^*}\right)^{\phi} + (A - \delta)\left(\frac{A - \delta - z}{g^*}\right)^{\eta}. \tag{4.33}$$

Therefore, we see that the equilibrium rate of inflation depends on z, that is, $\pi = \pi(z)$.

We consider the case in which ψ_ν is non-zero [5]. Denoting λ_m and λ_k as the shadow prices of real money balances and capital, we define $p \equiv \frac{\lambda_k}{\lambda_m}$. As shown in Appendix 4.A, we obtain the following dynamic system:

$$\dot{z} = \left\{\frac{1}{\sigma}\left[l(p)\frac{\dot{p}}{p} + \frac{A}{p} - \rho - \delta\right] - A + \delta\right\} z + z^2, \tag{4.34}$$

$$\dot{p} = \frac{p^2}{\psi_\nu} + \left(\delta - \pi(z) - \frac{1}{\psi_\nu}\right) p - A. \tag{4.35}$$

[5]Since money is superneutral when $\psi_\nu = 0$, interesting result cannot be obtained even if $\psi_c > 0$. Technically, $p = 1$ for all t so that the dynamic system is consisted by z alone and it does not depend on ψ_c.

where $l(p) = -\dfrac{\psi_c - \psi_\nu}{\psi_c p - (\psi_c - \psi_\nu)}$. Note that $\mathrm{sign}[l(p)] = -\mathrm{sign}[\psi_c - \psi_\nu]$, since $p > 1$ from the assumption that the CIA constraint is strictly binding in equilibrium.

4.6 Velocity of Money

In the BGP where $\dot{z} = \dot{p} = 0$ in (4.34) and (4.35), we obtain

$$g^* = \frac{1}{\sigma}\left[\frac{A}{p^*} - \rho - \delta\right] = A - \delta - z^* > 0, \tag{4.36}$$

$$\frac{p^*}{\psi_\nu} + \delta - \pi^* - \frac{1}{\psi_\nu} - \frac{A}{p^*} = 0. \tag{4.37}$$

As above conditions show, the BGP is uniquely determined [6], regardless of the magnitude of σ, which is in a marked contrast to the model with a fixed growth rate of nominal money supply. Moreover, from (4.33),

$$\pi'(z^*) = \left.\frac{d\pi}{dz}\right|_{BGP} = \frac{A-\delta}{A-\delta-z^*}\frac{\eta}{\phi-1}, \tag{4.38}$$

and therefore $\mathrm{sign}[\pi'(z^*)]=\mathrm{sign}\left[\dfrac{\eta}{\phi-1}\right]$ is satisfied around the BGP.

From now on, we investigate the relation between the velocity and the monetary expansion rate. Defining $\dfrac{\dot{M}}{M} = \mu$, and using (4.27), (4.28) and $\nu = y - c$, we can represent the growth rate of money as following [7]:

$$\mu = \mu(z, p) = q(z)\frac{\dot{z}}{z} + A - \delta - z + \pi(z), \tag{4.39}$$

where $q(z) \equiv \dfrac{(\psi_c - \psi_\nu)z}{(\psi_c - \psi_\nu)z + \psi_\nu A}$ is positive (resp. negative) if $\psi_c > \psi_\nu$ (resp. $\psi_c < \psi_\nu$). Under the interest-control rule, the nominal growth rate of money supply is endogenously determined. Since the growth rate of real money supply equals that of real income around the BGP, that is, $\dfrac{\dot{m}}{m} = g^*$, we derive

$$\mu^* = A - \delta - z^* + \pi(z^*). \tag{4.40}$$

[6] From (4.35), we have two p^*s, but one of them is negative, while another is positive and satisfies $p^* > 1$ since $\dot{p}(p = 1; z = z^*) = -(A - \delta) - \pi^* < 0$. Therefore, the number of plausible p^* is only one. Under the unique p^*, we can give the nontrivial unique z^*. Determinacy of this unique BGP is detailed in Appendix 4.A.

[7] Substituting $\dfrac{\dot{m}}{m} = \mu - \pi$ and $\dfrac{\dot{z}}{z} = \dfrac{\dot{c}}{c} - \dfrac{\dot{k}}{k}$ into $m = (\psi_c - \psi_\nu)c + \psi_\nu Ak$, we can obtain (4.39).

Table 4.1: The Relation between Velocity and Money Supply

	$\psi_c > \psi_\nu$	$\psi_c < \psi_\nu$	$\psi_c = \psi_\nu$
$\eta > 0$, $\phi > 1$, high $\frac{A-\delta}{g^*}$	$-$	$+$	0
$\eta > 0$, $\phi > 1$, low $\frac{A-\delta}{g^*}$	$+$	$-$	0
$\eta > 0$, $0 < \phi < 1$	$+$	$-$	0
$\eta = 0$	$+$	$-$	0

As a result, we can obtain

$$\frac{d\mu^*}{dz^*} = \pi'(z^*) - 1 = \frac{A-\delta}{A-\delta-z^*}\frac{\eta}{\phi-1} - 1, \tag{4.41}$$

and thus

$$\text{sign}\left(\frac{d\mu^*}{dz^*}\right) = \text{sign}\left(\frac{\eta}{\phi-1} - \frac{A-\delta-z^*}{A-\delta}\right). \tag{4.42}$$

Note that $g^* = A - \delta - z^*$. The income velocity of money around the BGP is represented by

$$V^* = \frac{y^*}{m^*} = \frac{A}{(\psi_c - \psi_\nu)z^* + \psi_\nu A}, \tag{4.43}$$

implying that

$$\text{sign}\left(\frac{dV^*}{dz^*}\right) = -\text{sign}[\psi_c - \psi_\nu]. \tag{4.44}$$

Combining (4.42) and (4.44), we can describe the effect of money supply on velocity around the BGP as in Table 4.1. In this table, + means a positive relation, − means a negative relation and 0 indicates that there is no relation.

Now, we consider intuitive implication of the results. From (4.41), two effects of economic growth on the nominal expansion rate of money supply can be seen. First, a decrease z^* yields a higher balanced-growth rate g^*. The second is the change of the inflation rate via the interest-control rule. When the growth rate of income increases, the central bank should raise the nominal interest rate to stabilize economy. However, since the net real rate of return to capital is constant due to the assumption of AK technology, the real interest rate also should be kept constant by adjusting the rate of inflation to satisfy the Fisher equation [8]. This is achieved by depressing inflation if the monetary policy is generalized and active, because the decline in nominal interest rate is larger than that in the inflation rate. Therefore, the expansion rate of nominal money supply may fall. Otherwise, the

[8] When the interest rate is controlled by the rate of inflation alone, this channel is not effective.

rise of economic growth does not generate the fall of the inflation rate so that the nominal growth rate of money supply increases.

We consider the result in (4.44). In light of AK technology, decreasing z^* generates the same magnitude of positive movement in the investment-capital ratio $\frac{\nu^*}{k^*}$. Therefore, if $\psi_c > \psi_\nu$, velocity of money which means the ratio of capital to real money balances becomes larger. Conversely, when $\psi_c < \psi_\nu$, the negative relation between the velocity and the growth rate of income is produced.

Taylor (1993) suggests that the observable policy stance of the Federal Reserve may be described by setting $\phi = 1.5$ and $\eta = 0.5$. Moreover, it is plausible to consider that real investment expenditures for machines, factories and housing are less constrained by cash holdings than consumption spending. Therefore, we focus on the case under which $\eta > 0$, $\phi > 1$, and $\psi_c > \psi_\nu$. When the ratio of the net real rate of return on capital $A - \delta$ to the balanced-growth rate $g^* = A - \delta - z^*$ is higher enough to satisfy $0 < \frac{\phi - 1}{\eta} < \frac{A - \delta}{g^*}$, the negative relation between the nominal money expansion and the income velocity of money holds, regardless of whether or not the economy displays sunspot-driven fluctuations around the BGP. Since $A - \delta > g^*$, the condition $0 < 1 = \frac{\phi - 1}{\eta} < \frac{A - \delta}{g^*}$ holds for $\phi = 1.5$ and $\eta = 0.5$. This result is different from Chen and Guo (2008b), who show that the negative relation emerges only on the determinate BGP if $\psi_c > \psi_\nu$. Consequently, we can conclude that both the form of the CIA constraint and the central bank's policy stance are important for the relation between and velocity and money expansion.

Appendix 4.A

The first-order conditions of the household's maximization problem are

$$c^{-\sigma} = \lambda_m + \psi_c \zeta, \tag{4.45}$$

$$\lambda_k - \lambda_m = \psi_\nu \zeta, \tag{4.46}$$

$$\dot{\lambda}_k = (\rho + \delta)\lambda_k - A\lambda_m, \tag{4.47}$$

$$\dot{\lambda}_m = (\rho + \pi)\lambda_m - \zeta, \tag{4.48}$$

$$\zeta(m - \psi_c c - \psi_\nu \nu) = 0, \quad \zeta \geq 0, \quad m \geq \psi_c c + \psi_\nu \nu, \tag{4.49}$$

together with the transversality conditions $\lim_{t \to \infty} e^{-\rho t} \lambda_{mt} m_t = 0$ and $\lim_{t \to \infty} e^{-\rho t} \lambda_{kt} k_t = 0$, where λ_m and λ_k are the shadow prices of real money balances and capital, respectively, and ζ represents the Lagrange multiplier for the CIA constraint (4.27). In the following, we assume that the CIA constraint (4.27) is strictly binding in equilibrium, and thus $\zeta > 0$ for all t. Using (4.45) through (4.48), we obtain the following dynamic equations:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[l(p)\frac{\dot{p}}{p} + \frac{A}{p} - \rho - \delta\right],$$

$$\frac{\dot{\lambda}_k}{\lambda_k} = \rho + \delta - \frac{A}{p},$$

$$\frac{\dot{\lambda}_m}{\lambda_m} = \left(\rho + \pi(z) + \frac{1}{\psi_\nu}\right) - \frac{p}{\psi_\nu}.$$

From these equations and (4.29), the dynamics equations (4.34) and (4.35) are derived.

We linearize the dynamic system (4.34) and (4.35) around the BGP to obtain:

$$\begin{bmatrix}\dot{z}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}\dot{z}_z & \dot{z}_p\\ \dot{p}_z & \dot{p}_p\end{bmatrix}\begin{bmatrix}\hat{z}\\ \hat{p}\end{bmatrix} = J\begin{bmatrix}\hat{z}\\ \hat{p}\end{bmatrix}, \tag{4.50}$$

where $\hat{z} \equiv z - z^*$ and $\hat{p} \equiv p - p^*$. The elements of matrix J (4.50) are the following:

$$\dot{z}_z = \left.\frac{\partial \dot{z}}{\partial z}\right|_{BGP} = \left[\frac{l(p^*)\dot{p}_z}{\sigma p^*} + 1\right]z^* = \left(-\frac{\pi'(z^*)l(p^*)}{\sigma} + 1\right)z^*,$$

$$\dot{z}_p = \left.\frac{\partial \dot{z}}{\partial p}\right|_{BGP} = \frac{z^*}{\sigma(p^*)^2}[l(p^*)\dot{p}_p p^* - A] = -\frac{z^*}{\sigma p^*}\frac{p^*(\psi_c - \psi_\nu) + A\psi_c\psi_\nu}{\psi_\nu[\psi_c p^* - (\psi_c - \psi_\nu)]},$$

$$\dot{p}_z = \left.\frac{\partial \dot{p}}{\partial z}\right|_{BGP} = -\pi'(z^*)p^*,$$

$$\dot{p}_p = \left.\frac{\partial \dot{p}}{\partial p}\right|_{BGP} = \frac{A}{p^*} + \frac{p^*}{\psi_\nu} > 0.$$

The trace and determinant of J are respectively given by:

$$\mathrm{tr}J = \dot{z}_z + \dot{p}_p = \left(-\frac{\pi'(z^*)l(p^*)}{\sigma} + 1\right)z^* + \frac{A}{p^*} + \frac{p^*}{\psi_\nu}, \tag{4.51}$$

$$\det J = \dot{z}_z\dot{p}_p - \dot{z}_p\dot{p}_z = \frac{z^*}{p^*}\left[\frac{A\psi_\nu + (p^*)^2}{\psi_\nu} - \frac{A\pi'(z^*)}{\sigma}\right]. \tag{4.52}$$

Since z and p are jump variables, if $\mathrm{tr}J > 0$ and $\det J > 0$, then the BGP is totally unstable so that the economy always stays on the BGP, that is, the equilibrium path is determinate. Otherwise, the equilibrium path is indeterminate. That is, the economy always stays on the BGP, or endogenous income fluctuations driven by sunspots are generated. Inspecting (4.51) and (4.52), we can find the following proposition, which shows that the generalization of the Taylor rule and the most generalized CIA constraint play a significant role in macroeconomic stability.

Proposition 4.3 *In the case of* $0 < \psi_c, \psi_\nu \leq 1$*, equilibrium determinacy holds either if (i) monetary policy rule responds only to the rate of inflation. or if (ii)* $\psi_c \leq \psi_\nu$ *and monetary policy is passive. Otherwise, BGP could be locally indeterminate.*

Chapter 5

Income Taxation, Interest-Rate Control and Macroeconomic Stability with Balanced-Budget

5.1 Introduction

Income taxation under balanced-budget rule and interest-rate control have been considered most effective tools for establishing macroeconomic stability. If it is appropriately selected, each policy rule may stabilize the economy by mitigating income fluctuations. It is, however, rather unclear whether or not those fiscal and monetary policy rules strengthen their stabilizing effects each other, if the fiscal authority and the central bank adopts specific actions simultaneously. Although stabilization effects of policy rules have been discussed extensively, the main stream literature has investigated the stabilization roles of income taxation and interest control rules separately. Therefore, these studies fail to answer the relevant question mentioned above.

The purpose of this chapter is to explore the interactions between income taxation and interest rate control rules under the balanced-budget discipline in a prototype model of real business cycle theory. Unlike most of the foregoing studies, this chapter treats both fiscal and monetary policy rules in a single model. More specifically, we introduce money into the baseline real business cycle model with flexible price via a cash-in-advance constraint. We assume that the fiscal authority adjusts income tax endogenously in each moment subject to the balanced-budget rule. In the main part of the chapter, we follow the taxation scheme assumed by

Guo and Lansing (1998) in which the rate of income tax depends on the individual income relative to the average income in the economy at large. The key distinction in this policy rule is whether taxation on individual income relative to the average income is progressive or regressive. Given such a fiscal action, the monetary authority adopts an interest-control rule under which the nominal interest rate responds to the current rate of inflation relative to the target level of inflation. As usual, the effect of interest rate control on macroeconomic stability depends on the sensitivity of interest rate to a change in inflation. Introducing those fiscal and monetary actions into the baseline model, we examine the dynamic behavior of the model economy.

Our study presents three main findings. First, if progressive income taxation is combined with active interest rate control rule (i.e. nominal interest rate responds to inflation more than one for one), then the economy exhibits equilibrium determinacy, so that we will not observe expectations-driven fluctuations. Second, if the interest rate control is passive, equilibrium indeterminacy could emerge even under progressive income taxation. Third, if income taxation is regressive, then the interest rate control rule may play a pivotal role for establishing macroeconomic stability. In this case, indeterminacy may emerge under both active and passive interest rate control rules. However, if the interest rate is relatively insensitive to inflation, then equilibrium indeterminacy can be eliminated even in the presence of strong regressiveness of income taxation. Those findings claim that in the general equilibrium settings with money and capital, it is critically relevant to find appropriate combinations of fiscal and monetary policy rules. Even though the balanced-budget rule and progressive income taxation may contribute to establishing aggregate stability, it is still important to select a suitable monetary policy rule to avoid depressing stabilization power of fiscal actions.

In the existing literature, Guo and Lansing (1998) show that progressive tax may eliminate the possibility of equilibrium indeterminacy even in the presence of strong degree of external increasing returns. Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004) examine the interrelationship between balanced-budget rule and determinacy of equilibrium. While Schmitt-Grohé and Uribe (1997) emphasize that the balanced-budget with a fixed government spending and endogenous taxation may generate sunspot-driven fluctuations, Guo and Harrison (2004) claim that such an unstable behavior can be eliminated if the balanced budget is maintained by adjusting government expenditure under fixed rates of income tax. These studies utilize the baseline real business cycle models without money. As for monetary policy rules, there has been a large body of literature that investigates stabilization effect of interest-rate control rule à la Taylor (1993). Although many authors (e.g. Benhabib et. al. (2001a)) point out that the interest control rule may easily produce expectations-driven fluctuations in the economies without capital, more recent studies show that the role of interest rate rules for aggregate stability is less relevant in an economy with capital formation: see, for example, Carlstrom and Fuerst (2005) and Meng and Yip (2004). These studies, however, ignore the role of fiscal policy. The present chapter integrates these two lines of research on the stabilization roles of taxation and interest control.

It is to be noted that several authors have examined interactions between fiscal and monetary policy rules in the context of new Keynesian models with sticky price adjustment. Following Leeper's (1991) modelling, Kurozumi (2005), Linnenmann (2006) and Lubik (2003) consider the effects of interest rate rule when the fiscal authority adjusts the rate of income tax to maintain a target level of the government debt. Those studies, therefore, do not assume the balanced-budget rule in its strict sense. Edge and Rudd (2007) explore how the presence of distortionary taxation on interest income affects the sensitivity of interest rate control to inflation and income necessary for avoiding equilibrium indeterminacy. Although Edge and Rudd (2007) utilize a sticky price model with fixed rates of income tax, the primary concern of their study is close to ours.

5.2 The Base Model

5.2.1 Households

There is a continuum of identical, infinitely lived households with a unit mass. The flow budget constraint for the household is

$$\dot{M} = (1-\tau)\, py + pT - pc - pv,$$

where M nominal stocks of money, p price level, y real income per capita, c consumption, v gross investment for capital, τ rate of factor income tax, and T is the real transfer from the government (or lump-sum tax if it has a negative value). Since we have normalized the number of household to unity, M, y, T, c and v represent their aggregate values as well. Real income consists of rent from capital and wage revenue:

$$y = rk + wl,$$

where r is real rate of return to capital, w is real wage rate and l denotes labor supply. Denoting real money balances $m \equiv M/p$ and the rate if inflation $\pi \equiv \dot{p}/p$, we rewrite the household's flow budget constraint as

$$\dot{m} = (1-\tau)(rk + wl) + T - c - v - \pi m. \tag{5.1}$$

The stock of capital changes according to

$$\dot{k} = v - \delta k, \tag{5.2}$$

where $\delta \in (0,1)$ denotes the rate of capital depreciation. In addition, a cash-in-advance constraint applies to consumption spending so that $pc \leq M$ or

$$c \leq m \tag{5.3}$$

in each moment of time. In this chapter, we assume that investment spending is not subject to the cash-in-advance constraint.

The instantaneous utility of the representative family depends on consumption and labor supply. Following the standard specification, we assume that the objective function of the household is

$$U = \int_0^\infty e^{-\rho t}\left(\log c - B\frac{l^{1+\gamma}}{1+\gamma}\right) dt, \quad \gamma > 0, \ \rho > 0, \quad B > 0.$$

Given the initial holdings of k_0 and m_0, the household maximizes U subject to (5.1), (5.2) and (5.3) under given trajectories of $\{r_t, w_t, \tau_t, T_t\}_{t=0}^\infty$.

To derive the optimization conditions for the household, we set up the current-value Hamiltonian function:

$$\begin{aligned}\mathcal{H} = &\log c - B\frac{l^{1+\gamma}}{1+\gamma} + \lambda\left[(1-\tau)(rk+wl) + T - c - v - \pi m\right] \\ &+ \mu(v - \delta k) + \zeta(m - c),\end{aligned}$$

where λ and μ respectively denote the costate variables of m and k, and ζ is a Lagrange multiplier corresponding to the cash-in-advance constraint on consumption spending. In what follows, we assume that the rate of tax, τ, depends on the level of individual income. The rate of income tax is thus given by

$$\tau = \tau(y) = \tau(rk + wl).$$

Considering such a taxation rule, we find that the necessary conditions for an optimum involve the following:

$$\partial\mathcal{H}/\partial c = 1/c - (\lambda + \zeta) = 0, \tag{5.4}$$

$$\partial\mathcal{H}/\partial l = -Bl^{\gamma} + \lambda\left[1 - \tau(y) - \tau'(y)\,y\right]w = 0, \tag{5.5}$$

$$\partial\mathcal{H}/\partial v = -\lambda + \mu = 0, \tag{5.6}$$

$$\zeta\,(m - c) = 0, \quad m - c \geq 0, \quad \zeta \geq 0, \tag{5.7}$$

$$\dot{\lambda} = \lambda\,(\rho + \pi) - \zeta, \tag{5.8}$$

$$\dot{\mu} = (\rho + \delta)\,\mu - \lambda\left[1 - \tau(y) - \tau'(y)\,y\right)]\,r, \tag{5.9}$$

together with the transversality conditions, $\lim_{t\to\infty} k_t \mu_t e^{-\rho t} = 0$ and $\lim_{t\to\infty} m_t \lambda_t e^{-\rho t} = 0$ as well as the initial conditions on m and k. In conditions (5.4) and (5.5), $\tau(y) + \tau'(y)\,y$ represents the marginal tax rate perceived by the household. As in Guo and Lansing (1998), we assume that each household takes the proportional tax rule into account when deciding their optimal consumption plan.

In this chapter we focus on the situation where the cash-in-advance constraint is always effective, so that $c = m$ holds for all $t \geq 0$. First, (5.6) means that $\mu = \lambda$ so that from (5.8) and (5.9) we obtain:

$$\zeta = \lambda\left\{\left[1 - \tau(y) - \tau'(y)\,y\right]r + \pi - \delta\right\}. \tag{5.10}$$

Thus (5.4) is written as

$$\frac{1}{c} = \lambda\left\{1 + \left[1 - \tau(y) - \tau'(y)\,y\right]r + \pi - \delta\right\}. \tag{5.11}$$

As a result, (5.5) and (5.10) yields:

$$cl^{\gamma}B = \frac{\left[1 - \tau(y) - \tau'(y)\,y\right]w}{1 + \left[1 - \tau(y) - \tau'(y)\,y\right]r + \pi - \delta}. \tag{5.12}$$

The left-hand side of the above is the marginal rate of substitution between consumption and the labor and the right-hand side expresses the effective, after-tax rate of real wage rate. Since we assume that the cash-in-advance constraint always binds, additional consumption generates an additional opportunity cost of holding money, which is given by the after-tax, net rate of return to capital plus the rate of inflation. Thus the right-hand side of (5.12) expresses the real wage rate in terms of the effective price including the cost of money holding.

5.2.2 Firms

The production side of the model economy follows the standard formulation. There are identical, infinitely many firms and the total number of firms is normalized to one. The production function of an individual firms is given by

$$y = Ak^{\alpha}l^{1-\alpha}, \qquad 0 < \alpha < 1, \quad A > 0. \tag{5.13}$$

In a competitive economy we consider, α represents the income share of capital. We focus on the case where α has an empirically plausible value, so that in what follows, we assume that α is less than 0.5. The commodity market is assumed to be competitive and thus the rate of return to capital and the real wage equal the marginal products of capital and labor, respectively:

$$r = \alpha A k f^{\alpha-1} l^{1-\alpha} = \alpha \frac{y}{k}, \tag{5.14}$$

$$w = (1-\alpha) A k^{\alpha} l^{-\alpha} = (1-\alpha) \frac{y}{l}. \tag{5.15}$$

5.2.3 Policy Rules

The fiscal and monetary authorities respectively control the rate of income tax, τ, and the nominal interest rate, R, according to their own policy rules. As assumed by Schmitt-Grohé and Uribe (1997) and Guo and Lansing (1998), the fiscal authority follows the balanced budget discipline. To emphasize this assumption, we assume away government debt. The flow budget constraint for the government is thus given by

$$\tau y + \dot{m} + \pi m = g + T,$$

where g denotes the government's consumption spending. A key assumption of our analysis is that under the balanced-budget rule the fiscal authority cannot use seigniorage income to finance the government consumption.[1] This means that

$$g = \tau y \tag{5.16}$$

holds in each moment. As a consequence, the real seigniorage income, $\dot{M}/p$, is transferred back to the households, so that $\dot{m} + \pi m = T$.

Given the general principle mentioned above, the monetary authority is assumed to follow an interest rate control rule such that

$$R(\pi) = \pi + r^* \left(\frac{\pi}{\pi^*}\right)^{\eta}, \quad r^* > 0, \quad \pi^* \geq 0, \tag{5.17}$$

where $r^* > 0$ is the steady-state level of net rate of return to capital and π^* expresses the target rate of inflation. We assume that the target rate of inflation is positive so that π^* is a positive constant set by the monetary authority. Under given r^* and π^*, we see that $R'(\pi) > 1$ (resp. $R'(\pi) < 1$) according to $\eta > 0$ (resp. $\eta < 0$). Hence, if $\eta > 0$, then the monetary authority adopts an active control rule under which it

[1]Hence, fiscal policy is 'passive' in the sense of Leeper (1991).

adjusts the nominal interest rate more than one for one with inflation. Conversely, when when $\eta < 0$, the interest rate control is passive in the sense that the monetary authority changes the nominal interest rate less than one for one with inflation. When $\eta = 0$, the monetary authority controls the nominal interest rate to keep the real interest rate at the rate of r^*. Notice that the Fisher equation gives the relation between the nominal and real interest in such a way that

$$R = r + \pi. \tag{5.18}$$

Therefore, (5.17) and (5.18) yield:

$$\pi = \left(\frac{r}{r^*}\right)^{\frac{1}{\eta}} \pi^*, \tag{5.19}$$

which gives the relation between the equilibrium rate of inflation and the real rate of return to capital. This means that in our setting the nominal interest rate control is to adjust the rate of inflation tax according to a specified rule.[2]

As for the fiscal rule under balanced budget, we consider two alternative regimes. One is the taxation rule use the formulation by Guo and Lansing (1998). In this regime, the government consumption is adjusted to keep the balanced budget and the rate of income is determined by the following taxation rule:

$$\tau(y) = 1 - (1 - \tau_0)\left(\frac{y^*}{y}\right)^{\phi}, \quad -\frac{1-\alpha}{\alpha} < \phi < 1, \quad 0 < \tau_0 < 1, \tag{5.20}$$

where y^* denotes the steady-state level of per capita income.[3] Given this taxation rule, the after-tax income is written as

$$[1 - \tau(y)]\, y = (1 - \tau_0)\, y^{*\phi} y^{1-\phi}.$$

As a result, if we denote the after-tax real income by $I(y) \equiv [1 - \tau(y)]\, y$, we obtain

$$\frac{I'(y)}{I(y)/y} = 1 - \phi.$$

[2]In the presence of distortional income taxation, the Fisher equation may be modified: see, for example, Feldstein (1976). For example, the non-arbitrage condition (5.18) may be replaced with $(1-\tau)R = (1-\tau)r + \pi$ so that $R = r + \pi/(1-\tau)$. If this is the case, we assume that the central bank adopts an interest control rule such that

$$R = \frac{\pi}{1-\tau} + r^*\left(\frac{\pi}{\pi^*}\right)^{\eta},$$

which is compatible with the modified Fisher equation in the long-run equilibrium where $\pi = \pi^*$. As a result, we obtain (5.19) even in the case of modified Fisher condition.

[3]As shown in Section 5.3.2. the restriction $\phi > -(1-\alpha)/\alpha$ ensures that the steady state level of consumption has a positive value.

Since $I(0) = 0$, the above equation means that if $0 < \phi < 1$, then $I'(y) < I(y)/y$ so that the after-tax income is strictly concave in taxable income y, that is, income taxation is progressive. Conversely, when $\phi < 0$, function $I(y)$ is strictly convex and hence, income taxation is regressive. When $\phi = 0$, we obtain the linear taxation rule under which the rate of income tax is fixed as τ_0. It is also to be noted that in the steady state where $y = y^*$, the rate of tax is also fixed at τ_0.[4]

The alternative fiscal rule, which is assumed by Schmitt-Grohé and Uribe (1997), is to keep the government spending fixed and the rate of income tax is adjusted to balance the budget. In this case, the rate of income tax is determined by

$$\tau = \frac{g}{y},$$

where g is fixed at a certain level. Obviously, income taxation in this regime is strongly regressive, because a higher income reduces the tax rate. While this chapter mostly focus on the first rule, we briefly discuss this second rule in Section 5.4.

5.2.4 Capital Accumulation

Combining the flow budget constraints for the household and the government yields the commodity-market equilibrium condition: $y = \dot{k} + \delta k + c + g$. Under the first fiscal rule the government consumption is endogenously determined, and thereby the market equilibrium is written as

$$\dot{k} = (1 - \tau) y - c - \delta k. \tag{5.21}$$

5.3 Policy Rules and Macroeconomic Stability

In this section we assume that the fiscal authority uses the taxation rule given by (5.20). We first derive the dynamical system that describes the equilibrium dynamics of the model economy and explore the stability condition around the steady state equilibrium.

[4] Individual tax payment is $T(y) = \tau(y)y$. Given (5.20), we have $T'(y) = 1 - (1-\tau_0)(1-\phi) y^{*\phi} y^{-\theta}$. In the steady state where $y = y^*$, we see that $T'(y) = \phi(1-\tau_0) + \tau_0$. Hence, if

$$\phi > -\frac{\tau_0}{1-\tau_0},$$

then the total tax payment increases with income y. Since income share of capital, α, is less than 0.5 in reality, when ϕ satisfies $\phi > -\alpha/(1-\alpha)$, it in general holds that $\phi > -\tau_0/(1-\tau_0)$.

5.3.1 Dynamic System

In order to derive a complete dynamic system that summarizes the model displayed above, we focus on the behaviors of capital stock, k, and the shadow value of real money balances, λ. First observe that (5.20) gives

$$1-\tau(y)-\tau'(y)\,y=(1-\tau_0)(1-\phi)\left(\frac{y^*}{y}\right)^{\phi}.$$

Using the above equation, together with (5.5) and (5.15), we may express the equilibrium level of employment in the following way:

$$l=\left[\frac{(1-\tau_0)(1-\phi)(1-\alpha)}{B}\right]^{\frac{1}{1+\gamma}} y^{*\frac{\phi}{1+\gamma}}y^{\frac{1-\phi}{1+\gamma}}\lambda^{\frac{1}{1+\gamma}}.$$

Inserting the above into the production function (5.13) and solving it with respect to y, we obtain

$$y=\hat{A}k^{\frac{\alpha(1+\gamma)}{\Delta}}\lambda^{\frac{1-\alpha}{\Delta}}\equiv y(k,\lambda), \tag{5.22}$$

where

$$\Delta=\alpha+\gamma+\phi(1-\alpha),$$
$$\hat{A}=A^{\frac{1+\gamma}{\Delta}}\left[\frac{(1-\tau_0)(1-\phi)(1-\alpha)}{B}\right]^{\frac{1-\alpha}{\Delta}} y^{*\frac{\phi(1-\alpha)}{\Delta}}.$$

Equation (5.22) represents the short-run production function under a given level of y^*. Similarly, the real interest rate is expressed as

$$r=\alpha\frac{y}{k}=\alpha\hat{A}k^{-\frac{(1-\alpha)(\gamma+\phi)}{\Delta}}\lambda^{\frac{1-\alpha}{\Delta}},$$

implying that the after-tax marginal rate of return to capital is

$$\begin{aligned}(1-\tau-\tau'y)\,r&=\alpha(1-\tau_0)(1-\phi)\,y^{*\phi}\hat{A}^{1-\phi}k^{\frac{\alpha\gamma(1-\phi)-(\gamma+\phi)}{\Delta}}\lambda^{\frac{(1-\phi)(1-\alpha)}{\Delta}}\\&\equiv\hat{r}(k,\lambda).\end{aligned}$$

For determining the equilibrium rate of inflation, in view of (5.14), (5.19) and (5.22), we may express π as a function of k and λ in such a way that

$$\pi=\pi^*\left(\frac{\alpha\hat{A}}{r^*}\right)^{\frac{1}{\eta}}k^{-\frac{(1-\alpha)(\gamma+\phi)}{\eta\Delta}}\lambda^{\frac{1-\alpha}{\eta\Delta}}\equiv\pi(k,\lambda). \tag{5.23}$$

Hence, using (5.23) , we see that the optimal consumption depends on k and λ in the following manner:

$$c = \frac{1}{\lambda[1 + \hat{r}(k,\lambda) + \pi(k,\lambda) - \delta]} \equiv c(k,\lambda). \tag{5.24}$$

Summing up the above manipulation, we find that the dynamic equation of capital stock is expressed as

$$\dot{k} = (1-\tau_0)y^{*\phi}y(k,\lambda)^{1-\phi} - c(k,\lambda) - \delta k, \tag{5.25}$$

and the shadow value of real money balances changes according to

$$\dot{\lambda} = \lambda[\rho + \delta - \hat{r}(k,\lambda)]. \tag{5.26}$$

A pair of differential equations, (5.25) and (5.26) , constitutes a complete dynamic system under the interest rate control and the taxation rule with endogenous government expenditure. Note that $y(k,\lambda)$ and $\hat{r}(k,\lambda)$ satisfy

$$\hat{r}(k,\lambda) = \alpha(1-\tau_0)(1-\phi)\frac{(y^*)^{\phi}y(k,\lambda)^{1-\phi}}{k}, \tag{5.27}$$

$$\pi(k,\lambda) = \pi^*\left(\frac{r}{r^*}\right)^{\frac{1}{\eta}} = \pi^* r^{*-\frac{1}{\eta}}\left(\frac{\alpha y(k,\lambda)}{k}\right)^{\frac{1}{\eta}}. \tag{5.28}$$

5.3.2 Steady-State Equilibrium

In the steady state where k and λ stay constant over time, it should hold that $\pi = \pi^*$, $r = r^*$ and $y = y^*$. It is to be noted that in the steady state, we obtain:

$$1 - \tau(y^*) - \tau'(y^*)y^* = (1-\tau_0)(1-\phi).$$

We should also notice that from (5.22) the production function in the steady state is given by

$$y^* = \bar{A}^{\frac{\Delta}{\alpha+\gamma}} k^{*\frac{\alpha(1+\gamma)}{\alpha+\gamma}} \lambda^{*\frac{1-\alpha}{\alpha+\gamma}}, \tag{5.29}$$

where

$$\bar{A} = A^{\frac{1+\gamma}{\Delta}}\left[\frac{(1-\tau_0)(1-\phi)(1-\alpha)}{B}\right]^{\frac{1-\alpha}{\Delta}}, \tag{5.30}$$

and the values of k, c and λ satisfy the following conditions:

$$\frac{y^*}{k^*} = \frac{\rho + \delta}{\alpha(1-\tau_0)(1-\phi)}, \tag{5.31}$$

$$\frac{c^*}{k^*} = (1-\tau_0)\frac{y^*}{k^*} - \delta = \frac{\rho + \delta\left[1 - \alpha\left(1-\phi\right)\right]}{\alpha\left(1-\phi\right)}, \tag{5.32}$$

$$\lambda^* k^* = \frac{k^*}{c^*\left(1+\rho+\pi^*\right)}, \tag{5.33}$$

In the above, k^*, c^* and λ^* denote their steady-state values. Equation (5.31) is the modified Golden-rule condition corresponding to $\dot{\lambda} = 0$, while (5.32) comes from the long-rum market equilibrium condition: $\dot{k} = 0$. The modified golden-rule condition (5.31) determines the income-capital ratio, y^*/k^*, which gives the consumption-capital ratio, c^*/k^* by (5.32). Then the steady-state implicit value of capital, $k^*\lambda^*$, is given by (5.33). The last condition yields

$$\lambda^* = \frac{\alpha\left(1-\phi\right)}{\left(1+\rho+\pi^*\right)\left[\rho+\delta(1-\alpha\left(1-\phi\right))\right]k^*} = \frac{\beta}{k^*},$$

where β denotes the coefficient of $1/k^*$. Using the above relation, together (5.29) and (5.31), we find that the steady-state level of capital is uniquely determined such that:

$$k^* = A^{\frac{1}{1-\alpha}}\left[\frac{\alpha\left(1-\tau_0\right)\left(1-\phi\right)}{\rho+\delta}\right]^{\frac{\alpha+\gamma}{(1-\alpha)(1+\gamma)}}\left[\frac{\left(1-\alpha\right)\left(1-\tau_0\right)\left(1-\phi\right)}{B}\right]^{\frac{1}{1+\gamma}}\beta^{\frac{1}{1+\gamma}}. \tag{5.34}$$

Therefore, the steady-state value of k and λ are uniquely expressed by all the parameters involved in the model. Once k^* and λ^* are given, the steady-state levels of $c\,(=m)$ and l are determined uniquely as well.

The steady-state value of capital given by (5.34) demonstrates that policy parameters, τ_0, ϕ, η and π^* affect the long-run levels of capital, income, employment and consumption in a complex manner. However, it is rather easy to derive intuitive implications of the effects of a change in policy parameters. First, observe that the degree of activeness of interest-rate control, η, fails to affect the steady-state levels of capital, employment and income. Second, regardless of progressiveness of income tax (i.e. the sign of ϕ), the steady-state capital decreases with τ_0, ϕ and π^*. Third, (5.31) and (5.32) show that a change in π^* will not affect y^*/k^* and c^*/k^*, so that it alters k^*, y^* and c^* proportionally. Additionally, (5.32) also shows that a rise in ϕ increases c^*/k^*, while τ_0 does not affect c^*/k^*.

Finally, by use of (5.12), (5.14), (5.15), (5.31) and (5.32), the steady-state rete of employment satisfies the following relation:

$$l^{*\gamma+1} = \frac{\left(1-\alpha\right)\left(\rho+\delta\right)\left(1-\phi\right)}{\left(1+\rho+\pi^*\right)\left\{\rho+\delta\left[1-\alpha\left(1-\phi\right)\right]\right\}},$$

Table 5.1: Policy Impacts on the Steady-State Values of Key Variables

	k^*	l^*	y^*	y^*/k^*	c^*/k^*	c^*/y^*
τ_0	-	0	-	+	0	-
ϕ	-	-	-	+	+	+
π^*	-	-	-	0	0	0

implying that l^* decreases with ϕ and π^*, while τ_0 does not affect l^*.

Inspecting the steady-state conditions derived above, we may summarize the comparative statics results in the steady state equilibrium as follows:

Proposition 5.1 *The degree of activeness of interest rate control, η, does not affect the steady state levels of capital, income, consumption and employment. The impacts of changes in policy parameters τ_0, ϕ and π^* are shown as Table 5.1.*

5.3.3 Equilibrium Determinacy

In order to examine the equilibrium dynamics near the steady state, let us conduct linear approximation of (5.25) and (5.26) at the steady-state equilibrium. The coefficient matrix of the approximated system is given by

$$J = \begin{bmatrix} (1-\tau_0)(1-\phi)\, y_k(k^*, \lambda^*) - c_k(k^*, \lambda^*) - \delta & (1-\tau_0)(1-\phi)\, y_\lambda(k^*, \lambda^*) - c_\lambda(k^*, \lambda^*) \\ -\lambda^* \hat{r}_k(k^*, \lambda^*) & -\lambda^* \hat{r}_\lambda(k^*, \lambda^*) \end{bmatrix}.$$

Since the shadow value of capital, λ, is an unpredetermined variable, if J has one stable root, the converging path under perfect foresight is at least locally unique. Thus determinacy of equilibrium is established when the determinacy of J has a negative value. In contrast, when $\det J > 0$ and the trace of J is negative, there exists a continuum of equilibria around the steady state. As shown in Appendix 5.A, using the steady-state conditions, we find that the partial derivatives in J can be expressed by the given parameter values. The trace and determinant of J are respectively written as:

$$\begin{aligned} \text{trace } J = \rho - & \frac{1}{(1+\pi^*+\rho)\Delta} \frac{\rho + \delta[1-\alpha(1-\phi)]}{\alpha(1-\phi)} \\ & \times \left\{ [\phi(\alpha\gamma+1) + \gamma(1-\alpha)](\rho+\delta) + (1-\alpha)(\gamma+\phi)\frac{\pi^*}{\eta} \right\}, \end{aligned} \tag{5.35}$$

$$\det J = -\frac{\rho + \delta\left[1 - \alpha\left(1 - \phi\right)\right]}{\alpha\left(1 - \phi\right)} \frac{\rho + \delta}{\Delta} \left[(\gamma + 1)(1 - \alpha(1 - \phi)) + \frac{\phi(1 - \alpha)\pi^*}{\eta(1 + \pi^* + \rho)}\right]. \tag{5.36}$$

where $\Delta = \alpha + \gamma + \phi\left(1 - \alpha\right).$

First of all, it is easy to see that if the target rate of inflation, π^*, is non negative, income taxation is progressive $(\phi > 0)$ and the interest rate control is active $(\eta > 0)$, then $\det J$ has a negative value, so that the steady-state equilibrium is locally determinate. Similarly, if $-\frac{\alpha+\gamma}{1-\alpha} < \phi < 0$ (so that $\Delta > 0$) and $\eta < 0$, then det $J < 0$. Thus in this case indeterminacy of equilibrium will not emerge either. In addition, if $\phi = 0$ and the rate of tax is fixed at τ_0, then

$$\det J = -\frac{\rho + \delta(1 - \alpha)}{\alpha^2} \left(\rho + \delta\right)(\gamma + 1)(1 - \alpha) < 0,$$

implying that, regardless of the monetary policy rules, the dynamic system exhibits equilibrium determinacy. To sum up, a set of sufficient conditions for equilibrium determinacy are the following:

Proposition 5.2 *(i) Given a positive rate of target inflation, either if income taxation is progressive and interest-rate control is active or if income taxation is regressive to satisfy* $-\frac{\alpha+\gamma}{1-\alpha} < \phi < 0$ *and interest-rate control is passive, then the steady-state equilibrium is locally determinate. (ii) If income tax is flat* $(\phi = 0)$, *local determinacy holds regardless of monetary policy rules.*

To focus on the other possibilities of equilibrium (in)determinacy, as clear as possible, let us assume that the elasticity of labor supply is zero: $\gamma = 0$. This case corresponds to the real business cycle model with indivisible labor analyzed by Hansen (1985). Given this assumption, we obtain

$$\text{trace } J = \rho - \frac{\rho + \delta\left[1 - \alpha(1 - \phi)\right]}{\alpha(1 - \phi)(1 + \pi^* + \rho)} \left[\rho + \delta + (1 - \alpha)\frac{\pi^*}{\eta}\right] \frac{\phi}{\Delta}, \tag{5.37}$$

$$\det J = -\frac{\left\{\rho + \delta\left[1 - \alpha\left(1 - \phi\right)\right]\right\}\left(\rho + \delta\right)}{\alpha\left(1 - \phi\right)} \left[1 - \alpha\left(1 - \phi\right) + \frac{\phi(1 - \alpha)\pi^*}{\eta(1 + \pi^* + \rho)}\right] \frac{1}{\Delta}. \tag{5.38}$$

where

$$\Delta \equiv \alpha + \phi\left(1 - \alpha\right).$$

First, assume that income taxation is progressive $(\phi > 0)$. In this case $\Delta > 0$ and, hence, the necessary and sufficient condition for determinacy is

$$1 - \alpha\left(1 - \phi\right) + \frac{\phi(1 - \alpha)\pi^*}{\eta(1 + \pi^* + \rho)} > 0.$$

Table 5.2: Stability Properties under Progressive Taxation

	$\phi > 0$
$\eta > 0$	D
$\hat{\eta} < \eta < 0$	U
$\eta < \hat{\eta}$	D

D:determinacy, I:indeterminacy, U:unstable

$\hat{\eta} = -\frac{(1-\alpha)\pi^*\phi}{[1-\alpha(1-\phi)](1+\rho+\pi^*)}\ (< 0)$

The above condition implies that if $\pi^* \geq 0$, equilibrium determinacy is established under the following conditions:

$$\eta > 0 \quad \text{or} \quad \eta < -\frac{(1-\alpha)\pi^*\phi}{[1-\alpha(1-\phi)](1+\rho+\pi^*)}. \tag{5.39}$$

If η satisfies

$$-\frac{(1-\alpha)\pi^*\phi}{[1-\alpha(1-\phi)](1+\rho+\pi^*)} < \eta < 0, \tag{5.40}$$

then we see that $\det J > 0$. It is easy to see that in this case we obtain $\rho + \delta + (1-\alpha)\frac{\pi^*}{\eta} < 0$, and, hence, from (5.37) the trace of J has a positive value. Therefore, if η satisfies (5.40), the steady state is a source and there is no converging path around it. Table 5.2 summarizes the patterns of dynamics under progressive tax.

It is to be pointed out that, as shown by numerical examples presented in Section 5.3.5, when $0 < \phi < 1$, condition (5.40) may not be satisfied for plausible parameter values. Therefore, the steady state is mostly unstable for the case of $-\hat{\eta} < \eta < 0$. To sum up, in the case of progressive taxation we obtain:

Proposition 5.3 *If income taxation is progressive and the target rate of inflation is non-negative, the perfect-foresight competitive equilibrium is locally determinate, either if the interest-rate control is active or it is sufficiently passive. Equilibrium indeterminacy may not emerge in this regime.*

Next, consider the case of regressive taxation $(\phi < 0)$. In this case the necessary and sufficient condition for local determinacy is

$$\left[1-\alpha(1-\phi)+\frac{(1-\alpha)\pi^*\phi}{\eta(1+\pi^*+\rho)}\right]\frac{1}{\Delta} > 0. \tag{5.41}$$

Table 5.3: Stability Properties under Regressive Taxation

	$-\dfrac{\alpha}{1-\alpha}<\phi<0$	$-\dfrac{1-\alpha}{\alpha}<\phi<-\dfrac{\alpha}{1-\alpha}$
$-\hat{\eta}<\eta$	D	I or U
$0<\eta<-\hat{\eta}$	U	D
$\bar{\eta}<\eta<0$	D	I or U
$\eta<\bar{\eta}$	D	I

D: determinate, I: indeterminate, U: unstable
$\hat{\eta}\equiv-\frac{(1-\alpha)\pi^*\phi}{[1-\alpha(1-\phi)](1+\rho+\pi^*)}(>0),\quad \bar{\eta}=-\frac{(1-\alpha)\pi^*}{\rho+\delta}\,(<0)\,,$

This condition is fulfilled, either if

$$-\frac{\alpha}{1-\alpha}<\phi<0\quad(\Longleftrightarrow\Delta>0)\ \text{ and }\eta>-\frac{(1-\alpha)\pi^*\phi}{[1-\alpha\,(1-\phi)]\,(1+\rho+\pi^*)}\equiv\hat{\eta}\,(>0)$$

or if

$$\phi<-\frac{\alpha}{1-\alpha}\quad(\Longleftrightarrow\Delta<0)\quad\text{and }0<\eta<-\frac{(1-\alpha)\pi^*\phi}{[1-\alpha\,(1-\phi)]\,(1+\rho+\pi^*)}\equiv\hat{\eta}\,(>0)\,.$$

In words, if a relatively low degree of regressiveness taxation, coupled with a high degree of passive interest-rate control, may produce indeterminacy.

In contrast, the necessary conditions for equilibrium indeterminacy are the following:

$$\left[1-\alpha\,(1-\phi)+\frac{(1-\alpha)\pi^*\phi}{\eta(1+\pi^*+\rho)}\right]\frac{1}{\Delta}<0, \tag{5.42}$$

$$\left[\rho+\delta+(1-\alpha)\,\frac{\pi^*}{\eta}\right]\frac{\phi}{\Delta}>0. \tag{5.43}$$

When $-\alpha/\,(1-\alpha)<\phi<0$ (so $\Delta>0)$, then both (5.42) and (5.43) are satisfied, if and only if

$$-\frac{(1-\alpha)\,\pi^*}{\rho+\delta}<\eta<-\frac{(1-\alpha)\,\pi^*\phi}{1+\pi^*+\rho}.$$

Note that the above condition is necessary but not sufficient for establishing equilibrium indeterminacy: if trace $J>0$ in (5.37), the steady state is a source (unstable). In contrast, if $-\frac{1-\alpha}{\alpha}<\phi<-\frac{\alpha}{1-\alpha}$ (so $\Delta<0)$, we find that indeterminacy may emerge more easily. Table 5.3 gives a classification of dynamic patterns in the case of regressive taxation.

The following proposition summarizes our finding.

Proposition 5.4 *(i) Suppose that income taxation is mildly regressive and the target rate of inflation is positive. Then the steady state is locally determinate, either if interest rate control is sufficiently active or if it is passive. (ii) Suppose that income taxation is sufficiently regressiveness and the target rate of inflation is positive. Then the steady state holds equilibrium determinacy only when interest-rate control is mildly active.*

5.3.4 Discussion

To obtain intuitive implication of determinacy/indeterminacy conditions displayed in Propositions 5.2-5.4, let us inspect the optimization condition (5.12) in detail. Using $w = (1-\alpha)y/l$, this condition is rewritten as

$$cl^{\gamma}B = \frac{(1-\tau_0)\,(1-\phi)\,(1-\alpha)\,y^{*\phi}A^{1-\phi}k^{\alpha(1-\phi)}l^{(1-\alpha)(1-\phi)-1}}{1+\hat{r}+\pi}. \tag{5.44}$$

Under a given level of consumption, c, the left-hand side of (5.44) represents the labor supply curve and the right hand side is considered the labor demand curve. Given c, $\hat{r}$ and π. If we assume that $\gamma = 0$ for expositional simplicity, the labor supply curve becomes a horizontal line. Hence, if $(1-\phi)\,(1-\alpha) - 1 = -\Delta < 0$, the labor demand curve has a negative slope and thus less steep than the labor supply curve. In contrast, if $\Delta < 0$, then the labor demand has a positive slope and is steeper than the labor supply curve.

Now suppose that the economy initially stays at the steady-state equilibrium. Suppose further that a sunspot-driven shock makes agents optimistic and households anticipate that the output and employment will expand. This raises consumption demand and the labor supply curve shifts upward and, hence, the equilibrium employment decreases, as long as the labor demand curve does not shift. Remember that the after-tax rate of return, $\hat{r}$, and the rate of inflation, π, are respectively expressed in the following manner:

$$\hat{r} = \alpha\,(1-\tau_0)\,(1-\phi)\,y^{*\phi}A^{1-\phi}k^{\alpha(1-\phi)-1}l^{(1-\alpha)(1-\phi)},$$

$$\pi = r^{*-\frac{1}{\eta}}\pi^{*}y^{*\phi}A^{\frac{1}{\eta}}k^{\frac{\alpha}{\eta}-1}l^{\frac{1-\alpha}{\eta}}.$$

These expressions show that an expected increase in l raises $\hat{r}$. It also increases inflation, if $\eta > 0$. As a result, the anticipated increase in employment shifts the labor demand curve downwards. Then the equilibrium employment decrease further, implying that the initial, optimistic expectation will not be self-fulfilled, because product will contract rather than expand. Consequently, such an expectation driven fluctuations cannot be realized so that equilibrium is determinate.

Notice that if η is negative and its absolute value is small, a rise in employment yields a sufficiently large decrease in inflation. This may reduces the after-tax nominal interest rate, $\hat{r} + \pi$, and therefore, the labor demand curve may shift up. If such a shift large enough to enhance the equilibrium level of employment, the initial optimistic expectations can be self-fulfilled. Hence, as Table 5.2 shows, if the interest rate control is mildly passive, there may exist multiple converging paths. In contrast, if the interest-rate control is strongly passive so that the absolute value of η is large enough, a decrease in the after-tax nominal interest rate, $\hat{r} + \pi$, is small. As a result, an upward shift of the labor demand curve cannot cancel a reduction of employment due to an upward shift of labor supply curve. Consequently, if η is small enough to fulfill $\eta < -\hat{\eta}\phi$, the possibility of equilibrium indeterminacy is eliminated.

Next, assume that $\Delta < 0$ so that the labor demand curve has a positive slope. If the after-tax nominal interest rate, $\hat{r} + \pi$, is constant, then the initial shift of labor supply curve due to an increase in consumption raises the equilibrium level of employment. If we consider the effective real wage, labor demand curve may have negative slope even if $\Delta < 0$. In particular, when η has a small positive value, a rise in l yields a large increase in π so that the labor demand curve may have a negative slope. In this case, the initial expectation cannot be self fulfilled. At the same time, if $\eta > 0$, both $\hat{r}$ and π are increased by an expansion of employment, which yields a downward shift of the labor demand curve. Since the labor demand curve is steeper than the labor supply curve, this shift produces a further enhancement of the employment level. As a consequence, the initial expectation can be self-fulfilled and equilibrium indeterminacy emerges. In contrast, if $\eta < 0$ and η is close to zero, a higher employment may lower the after-tax nominal interest rate, implying that the labor demand curve shifts upward. If this is the case, the equilibrium level of employment may not increase, which means that the initial expected change in economic condition cannot be realized. Thus indeterminacy may not emerge in this situation.

5.3.5 A Numerical Example

To focus on the roles of key policy parameters, ϕ and η, more clearly, let us inspect a numerical example. In so doing, we first depict the graphs of the conditions for

$\det J = 0$ and trace $J = 0$ in (ϕ, η) space. In what follows, we still focus on the case of indivisible labor ($\gamma = 0$).

Remembering that $\Delta \equiv \alpha + \phi(1-\alpha)$ and defining

$$\tilde{\phi} \equiv -\frac{\alpha}{1-\alpha},$$

we see that sign $\Delta =$ sign $\left(\phi - \tilde{\phi}\right)$. Note that

$$\text{sign } \det J = \text{sign}\left\{(1-\alpha+\alpha\phi)\eta + \frac{\phi(1-\alpha)\pi^*}{(1+\pi^*+\rho)}\right\} \quad \text{if } \Delta\eta > 0,$$
$$\text{sign } \det J = -\text{sign}\left\{(1-\alpha+\alpha\phi)\eta + \frac{\phi(1-\alpha)\pi^*}{(1+\pi^*+\rho)}\right\} \quad \text{if } \Delta\eta < 0.$$

and that $\det J = 0$ holds when the following condition is fulfilled:

$$\eta = -\frac{\phi(1-\alpha)\pi^*}{(1+\pi^*+\rho)(1-\alpha+\alpha\phi)} \equiv \eta_d\left(\phi;\ \pi^*\right). \tag{5.45}$$

Equation (5.45) shows that, given a positive rate of the target inflation, π^* (> 0), η decreases as ϕ rises. In addition η decreases (resp. increases) with π^*, if ϕ is positive (resp. negative).

To consider the sign of trace J, define

$$\mathcal{C}(\phi;\pi^*) \equiv \frac{\rho\alpha(1-\phi)(1+\pi^*+\rho)\Delta}{\rho+\delta[1-\alpha(1-\phi)]} - \phi(\rho+\delta).$$

Then it holds that

$$\text{sign }\{\text{trace } J\} = \text{sign}\left\{\frac{\mathcal{C}(\phi;\pi^*)\eta - (1-\alpha)\phi\pi^*}{\Delta\eta}\right\}.$$

This condition is rewritten as

$$\text{sign }\ \{\text{trace } J\} = \text{sign }\ \{\eta - \eta_{tr}(\phi;\pi^*)\} \quad \text{if } \frac{\mathcal{C}(\phi;\pi^*)}{\Delta\eta} > 0,$$
$$\text{sign }\ \{\text{trace } J\} = -\text{sign }\ \{\eta - \eta_{tr}(\phi;\pi^*)\} \quad \text{if } \frac{\mathcal{C}(\phi;\pi^*)}{\Delta\eta} < 0,$$

where the locus of trace $J = 0$ is given by

$$\eta_{tr}(\phi;\pi^*) = \frac{(1-\alpha)\phi\pi^*}{\mathcal{C}(\phi;\pi^*)}.$$

We set the conventional magnitude for each parameter:

time discount rate $(\rho) = 0.04$, income share of capital $(\alpha) = 0.4$,
capital depreciation rate $(\delta) = 0.05$.

Given those parameter values[5], $\phi \in \left(-\dfrac{1-\alpha}{\alpha}, 1\right) = (-1.5, 1)$. Assuming that the target rate of inflation is $\pi^* = 0.03$, we obtain the following:

$$\tilde{\phi} = -0.67,$$

$$\eta_d(\phi; 0.03) = -\frac{18\phi}{107(6+4\phi)}.$$

$$\eta_{tr}(\phi; 0.03) = \frac{180\phi(7+2\phi)}{-12072\phi^2 - 2876\phi + 6848}.$$

Using the numerical results displayed above, we depict the graphical results in Figure [6] 5.1. This figure first reveals that in our numerical example equilibrium indeterminacy may not emerge as long as ϕ exceeds -0.67 and, hence, progressive taxation ensures determinacy regardless of interest control. Second, in the case of progressive taxation ($\phi > 0$), even though there is a region in which the steady state is totally unstable (so the equilibrium path is nonstationary), such a region in (ϕ, η) space is considerably small. Third, the steady state would be totally unstable if η is positive and sufficiently small for the case of $\phi \in [-0.67, 0]$. Finally when income taxation is regressive enough to satisfy $\phi < -0.67$, the interest control, i.e. the magnitude of η, critically affects the stability property of the economy.

5.4 Alternative Policy Rules

In this section, we briefly discuss alternative fiscal and monetary policy rules that would modify our main findings shown in the previous sections.

5.4.1 Fixed Government Spending

So far, we have assumed that the government consumption is endogenously determined to satisfy the balanced-budget rule. The second scheme of fiscal rule is that the fiscal authority fixes the government expenditure, g, by adjusting the rate

[5] The parameters A and B are not needed to derive the steady-state ratios, but needed to obtain the steady-state value of each variable.

[6] When depicting graphs in Figure 5.1, we use the following facts. First, note that $\mathcal{C}(\cdot) > 0$ for $\phi \in (\phi_L, \phi_H)$, where ϕ_L and ϕ_H respectively satisfy $-\dfrac{1-\alpha}{\alpha} < \phi_L < \tilde{\phi}$ and $0 < \phi_H < 1$. Second, $\eta_{tr}(\cdot)$ is an increasing function of ϕ with a positive value of $\pi^* > 0$. Moreover, $\eta_{tr}(\cdot)$ may move around the origin in the (ϕ, η) plane in the clockwise (resp. counterclockwise) as π^* rises when $-\dfrac{1-\alpha}{\alpha} < \phi < \phi_L$ (resp. $\phi_L < \phi < 1$).

of tax, τ, to balance its budget. Schmitt-Grohé and Uribe (1997) assume such a balanced-budget rule. If this is the case, the rate of average tax is determined as

$$\tau(y) = \frac{g}{y}. \tag{5.46}$$

Unlike the first rule, when deciding its optimal plan, the household takes the tax rate τ as given, because y in (5.46) represents the aggregate income rather than an individual income. In equilibrium, the after tax income is simply given by $I(y) = [1 - \tau(y)]\, y = y - g$ and

$$\frac{I'(y)}{I(y)/y} = \frac{y}{y-g} > 1,$$

implying that income tax is regressive. In this case the after-tax factor prices are given by

$$\hat{r} = (1 - \tau(y))\, \alpha \frac{y}{k} = \alpha \frac{y-g}{k}, \tag{5.47}$$

$$\hat{w} = (1 - \tau(y))(1-\alpha) \frac{y}{l} = (1-\alpha) \frac{y-g}{l}. \tag{5.48}$$

Since the household considers that τ is exogenously determined, two of the the first-order conditions for an optimum shown in Section 5.2.1 are replaced with the following:

$$\partial \mathcal{H}/\partial l = -Bl^{\gamma} + \lambda [1 - \tau(y)]\, w = 0, \tag{5.49}$$

$$\dot{\mu} = (\rho + \delta)\, \mu - \lambda [1 - \tau(y)]\, r, \tag{5.50}$$

where $\tau(y) = g/y$ and g (> 0) is given. From (5.48) and (5.49) ,the instantaneous equilibrium level of employment satisfies

$$\frac{Bl^{\gamma+1}}{(1-\alpha)\lambda} + g = Ak^{\alpha} l^{1-\alpha}.$$

There are at most two values of l satisfying the above. In the following we ignore the smaller level of l because it produces unconventional results (for example, a higher government consumption increases employment). The higher equilibrium level of l can be written as

$$l = l(k, \lambda; g), \quad l_k > 0, \quad l_{\lambda} > 0, \quad l_g < 0. \tag{5.51}$$

Using (5.51) , we find that the equilibrium level of output, the after-tax rate of return are written as

$$Ak^{\alpha} l(k, \lambda; g)^{1-\alpha} = y(k, \lambda; g), \quad y_k > 0, \quad y_{\lambda} > 0, \quad y_g < 0,$$

$$\left(1-\frac{g}{y}\right)\alpha\frac{y}{k}=\alpha\left(\frac{y\left(k,\lambda,g\right)-g}{k}\right)=\hat{r}\left(k,\lambda;g\right),\quad \hat{r}_{\lambda}>0,\quad \hat{r}_{g}<0.$$

Note that the after-tax rate of return to capital may increase with capital if a higher k sufficiently reduces g/k. Consequently, the reduced dynamic system is given by

$$\begin{aligned}\dot{k}&=y\left(k,\lambda;g\right)-c\left(k,\lambda;g\right)-\delta k-g,\\ \dot{\lambda}&=\lambda\left[\rho+\delta-\hat{r}\left(k,\lambda;g\right)\right],\end{aligned}$$

where

$$\begin{aligned}c\left(k,\lambda;g\right)&=\frac{1}{\lambda\left[1+\hat{r}\left(k,\lambda;g\right)+\pi\left(k,\lambda;g\right)\right]},\\ \pi\left(k,\lambda;g\right)&=\pi^{*}r^{*-\frac{1}{\eta}}\left(\frac{y\left(k,\lambda;g\right)}{k}\right)^{\frac{1}{\eta}}.\end{aligned}$$

The following discussion is essentially the same as that in Sections 5.3.2 and 5.3.3. The key for the analysis is the behavior of the after-tax levels of rate of return and real wage. In this fiscal policy regime, equation (5.12) is written as

$$cl^{\gamma}B=\frac{\left(1-\alpha\right)\left(Ak^{\alpha}l^{-\alpha}-g/l\right)}{1+\hat{r}+\pi}, \tag{5.52}$$

where

$$\begin{aligned}\hat{r}&=\alpha\left(Ak^{\alpha-1}l^{1-\alpha}-\frac{g}{k}\right),\\ \pi&=\pi^{*}r^{*-\frac{1}{\eta}}\left(\alpha Ak^{\alpha-1}l^{1-\alpha}\right)^{\frac{1}{\eta}}.\end{aligned}$$

Notice that

$$\frac{\partial\left(\left(1-\tau\right)w\right)}{\partial l}=l^{2}\left(g-\alpha y\right)$$

so that the government consumption is large enough to satisfy $g>\alpha y$, the labor demand function represented by the right-hand side of (5.52) increases with l. Again, assume that $\gamma=0$. According to the discussion in Section 5.3.4, if $g>\alpha y$, then indeterminacy of equilibrium is easy to be observed. In addition, if $\eta>0$, a higher employment caused by a sunspot driven disturbance increases the after-tax nominal interest rate, $\hat{r}+\pi$. Hence, the labor demand curve shifts downward, which enhances the possibility of indeterminacy. If $g<\alpha y$, then the labor demand curve is negatively sloped. Even in this case, if η is negative and its absolute value is small, a higher employment reduces the after-tax nominal interest rate. As a consequence, the labor demand curve shifts up, under which emergence of multiple equilibrium can remain.

5.4.2 Factor Specific Taxation

If capital and labor income are taxed separately, we may set the following tax functions:

$$\tau_k(rk) = 1 - \left(1 - \tau_0^k\right)\left(\frac{r^*k^*}{rk}\right)^{\phi_r}, \qquad \tau_w(wl) = 1 - \left(1 - \tau_0^w\right)\left(\frac{w^*l^*}{wl}\right)^{\phi_w},$$

where $0 < \tau_0^k, \tau_0^r < 1$. Notice that in the case of Cobb-Douglas production function, we obtain

$$\frac{r^*k^*}{rk} = \frac{y^*}{y}, \quad \frac{w^*l^*}{wl} = \frac{y^*}{y}$$

Again, the equivalence between labor demand and supply is described by

$$cl^\gamma B = \frac{\hat{w}}{1 + \hat{r} + \pi},$$

where

$$\begin{aligned} \hat{w} &= \left(1 - \tau_0^w\right)\left(1 - \phi_w\right)\left(1 - \alpha\right) y^{*\phi_w} l^{*-\phi_w} A^{1-\phi_w} k^{\alpha(1-\phi_w)} l^{(1-\alpha)(1-\phi_w)} \\ \hat{r} &= \alpha\left(1 - \tau_0^r\right)\left(1 - \phi_r\right) y^{*\phi} A^{1-\phi} k^{\alpha(1-\phi)-1} l^{(1-\alpha)(1-\phi_r)} \end{aligned}$$

Therefore, it is easy to see that indeterminacy tends to emerge more easily, if wage income taxation is regressive $(\phi_w < 0)$ and capital income taxation is progressive $(\phi_r > 0)$.

5.4.3 Taylor Rule

Taylor (1993) originally proposes the interest rate control rule under which the nominal interest rate responds to real income as well as inflation. In our notation, the original Taylor rule can be formulated as

$$R(\pi) = \pi + r^*\left(\frac{\pi}{\pi^*}\right)^\eta \left(\frac{y}{y^*}\right)^\xi, \quad \xi < 1. \tag{5.53}$$

In this case, the Fisher equation, $R = r + \pi$, gives

$$\pi = \pi^*\left(\frac{r}{r^*}\right)^{\frac{1}{\eta}}\left(\frac{y}{y^*}\right)^{-\frac{\xi}{\eta}}.$$

Since $r = \alpha y/k$, the above is rewritten as

$$\pi = \alpha^{\frac{1}{\eta}} A^{\frac{1-\varepsilon}{\eta}} \pi^* r^{*-\frac{1}{\eta}} y^{*\frac{\xi}{\eta}} k^{\frac{\alpha-1-\varepsilon}{\eta}} l^{\frac{(1-\alpha)(1-\varepsilon)}{\eta}}. \tag{5.54}$$

When $\xi = 0$, the equilibrium rate of inflation is $\pi = \pi = A^{\frac{1}{\eta}} r^{*-\frac{1}{\eta}} \pi^{*} y^{*\phi} k^{\frac{\alpha}{\eta}-1} l^{\frac{1-\alpha}{\eta}}$. Therefore, if the interest-rate control is active with respective to real income, i.e. ξ has a positive value, then a change in labor employment, l, has a smaller impact on the rate of inflation under (5.53) than under (5.17). Hence, when $\eta < 0$, a rise in l yields a smaller decrease in π in the case of $\xi > 0$. Therefore, in view of the discussion in Section 5.3.4, the Taylor type control rule given by (5.53) may contribute to reducing the possibility of equilibrium indeterminacy.

5.5 Conclusion

We have analyzed the stabilization roles of fiscal and monetary policy rules in a monetary real business cycle model with flexible price adjustment. In this chapter, we have assumed that the rate of income tax is endogenously adjusted to balance the government budget, while the monetary authority uses the Taylor-type interest-rate control scheme. Our investigation reveals that in the context of a simple real business cycle model we use, equilibrium determinacy depends heavily on the taxation rule rather than on monetary policy rule. In particular, as suggested by our numerical example, progressive taxation under balanced-budget rule tends to eliminate the possibility of equilibrium indeterminacy regardless of activeness of interest-rate control. On the contrary, if income taxation is regressive, whether interest-rate rule is active or passive may be pivotal to hold equilibrium determinacy. Since the effects of regressive income tax are close to those generated by increasing returns to scale, our finding suggests that the role of interest-rate control would be more relevant in the non-standard situation like regressive taxation or increasing return to scale.

It is, however, to be noticed that our main results emphasized above may partly come from the simplicity of our model. Our conclusion would be modified, if we assume more general settings. Possible generalization of the model includes non-separable utility between consumption and labor as well as a more general form of money demand (for example, distinction between cash goods and credit goods or cash-in-advance constraint on investment),.and more general form of interest rate rule in which the nominal interest rate responds to real income as well as to inflation. Re-examining our discussion in those extended frameworks deserves further research.

Appendix 5.A: Detailed Calculation of the Base Model

The coefficient matrix is expressed as

$$J = \begin{bmatrix} (1-\tau_0)\,(1-\phi)\,y_k\,(k^*,\lambda^*) - c_k\,(k^*,\lambda^*) - \delta & (1-\tau_0)\,(1-\phi)\,y_\lambda\,(k^*,\lambda^*) - c_\lambda\,(k^*,\lambda^*) \\ -\lambda^*\hat{r}_k\,(k^*,\lambda^*) & -\lambda^*\hat{r}_\lambda\,(k^*,\lambda^*) \end{bmatrix}.$$

Note that from (5.22) , (5.27) and (5.28) all of the functions $y\,(.)$, $\hat{r}\,(.)$ and $\pi\,(.)$ are of Cobb-Douglas forms. Thus we find that the partial derivatives in J are respectively given by the following:

$$\begin{aligned} y_k\,(k^*,\lambda^*) &= \frac{\alpha\,(1+\gamma)}{\Delta}\frac{y^*}{k^*}, \\ y_\lambda\,(k^*,\lambda^*) &= \frac{1-\alpha}{\Delta}\frac{y^*}{\lambda^*}, \\ \hat{r}_k\,(k^*,\lambda^*) &= -\frac{\phi(\alpha\gamma+1)+\gamma(1-\alpha)}{\Delta}\frac{\hat{r}^*}{k^*}, \\ \pi_k\,(k^*,\lambda^*) &= -\frac{(1-\alpha)\,(\gamma+\phi)}{\eta\Delta}\frac{\pi^*}{k^*}, \\ \pi_\lambda\,(k^*,\lambda^*) &= \frac{1-\alpha}{\eta\Delta}\frac{\pi^*}{\lambda^*}, \\ \hat{r}_\lambda\,(k^*,\lambda^*) &= \frac{(1-\phi)\,(1-\alpha)}{\Delta}\frac{\hat{r}^*}{\lambda^*}, \\ c_k\,(k^*,\lambda^*) &= -c^*\frac{\hat{r}_k\,(k^*,\lambda^*)+\pi_k\,(k^*,\lambda^*)}{1+\rho+\pi^*}, \\ c_\lambda\,(k^*,\lambda^*) &= -(c^*)^2[1+\pi^*+\rho+\lambda^*(\hat{r}_\lambda(k^*,\lambda^*)+\pi_\lambda(k^*,\lambda^*))]. \end{aligned}$$

As a result, using

$$\frac{y^*}{k^*} = \frac{\rho+\delta}{\alpha\,(1-\tau_0)\,(1-\phi)}, \quad \frac{c^*}{k^*} = \frac{\rho+\delta\,[1-\alpha\,(1-\phi)]}{\alpha\,(1-\phi)} \quad \text{and} \quad \hat{r}^* = \rho+\delta,$$

we express the trace and determinant of J in the following way:

$$\begin{aligned} \text{trace } J &= \frac{\rho+\delta}{\Delta}((1+\gamma)-(1-\alpha)(1-\phi))-\delta \\ &\quad - \frac{c}{1+\pi^*+\rho}\left(\frac{\phi(\alpha\gamma+1)+\gamma(1-\alpha)}{\Delta}\frac{\rho+\delta}{k^*}+\frac{(1-\alpha)\,(\gamma+\phi)}{\eta\Delta}\frac{\pi^*}{k^*}\right) \\ &= \Bigg[\rho\eta - \frac{1}{(1+\pi^*+\rho)\Delta}\frac{\rho+\delta[1-\alpha(1-\phi)]}{\alpha(1-\phi)} \\ &\quad \times \{[\phi(\alpha\gamma+1)+\gamma(1-\alpha)](\rho+\delta)\eta+(1-\alpha)\,(\gamma+\phi)\pi^*\}\Bigg]\frac{1}{\eta}. \end{aligned}$$

$$
\begin{aligned}
\det J = &- \left[(1-\tau_0)\,(1-\phi)\, y_k\,(k^*,\lambda^*) - c_k\,(k^*,\lambda^*) - \delta \right] \left[\lambda^* \hat{r}_\lambda\,(k^*,\lambda^*) \right] \\
&+ \left[(1-\tau_0)\,(1-\phi)\, y_\lambda\,(k^*,\lambda^*) - c_\lambda\,(k^*,\lambda^*) \right] \left[\lambda^* \hat{r}_k\,(k^*,\lambda^*) \right] \\
&= \lambda^* [-c^* \frac{\hat{r}_k\,(k^*,\lambda^*) + \pi_k\,(k^*,\lambda^*)}{1+\rho+\pi^*} \left(\frac{(1-\phi)\,(1-\alpha)}{\Delta} \frac{\hat{r}^*}{\lambda^*} \right) + (c^*)^2 [1 + \pi^* + \rho + \lambda^* (\hat{r}_\lambda (k^*,\lambda^*) \\
&+ \pi_k (k^*,\lambda^*)] \left[- \frac{\phi(\alpha\gamma+1) + \gamma(1-\alpha)}{\Delta} \frac{\hat{r}^*}{k^*} \right] - \delta\lambda^* \frac{(1-\phi)\,(1-\alpha)}{\Delta} \frac{\hat{r}^*}{\lambda^*} \\
&= - \frac{(1-\phi)(1-\alpha)}{\Delta} (\rho+\delta) \frac{c^*}{k^*} - \frac{c^*}{k^*} \frac{\rho+\delta}{\Delta} \left(\frac{\phi(1-\alpha)\pi^*}{(1+\pi^*+\rho)\eta} + \phi(\alpha\gamma+1) + \gamma(1-\alpha) \right) \\
&= - \frac{\rho+\delta\,[1-\alpha\,(1-\phi)]}{\alpha\,(1-\phi)} \frac{\rho+\delta}{\Delta} \left[(\gamma+1)(1-\alpha+\alpha\phi) + \frac{\phi(1-\alpha)\pi^*}{(1+\pi^*+\rho)\eta} \right],
\end{aligned}
$$

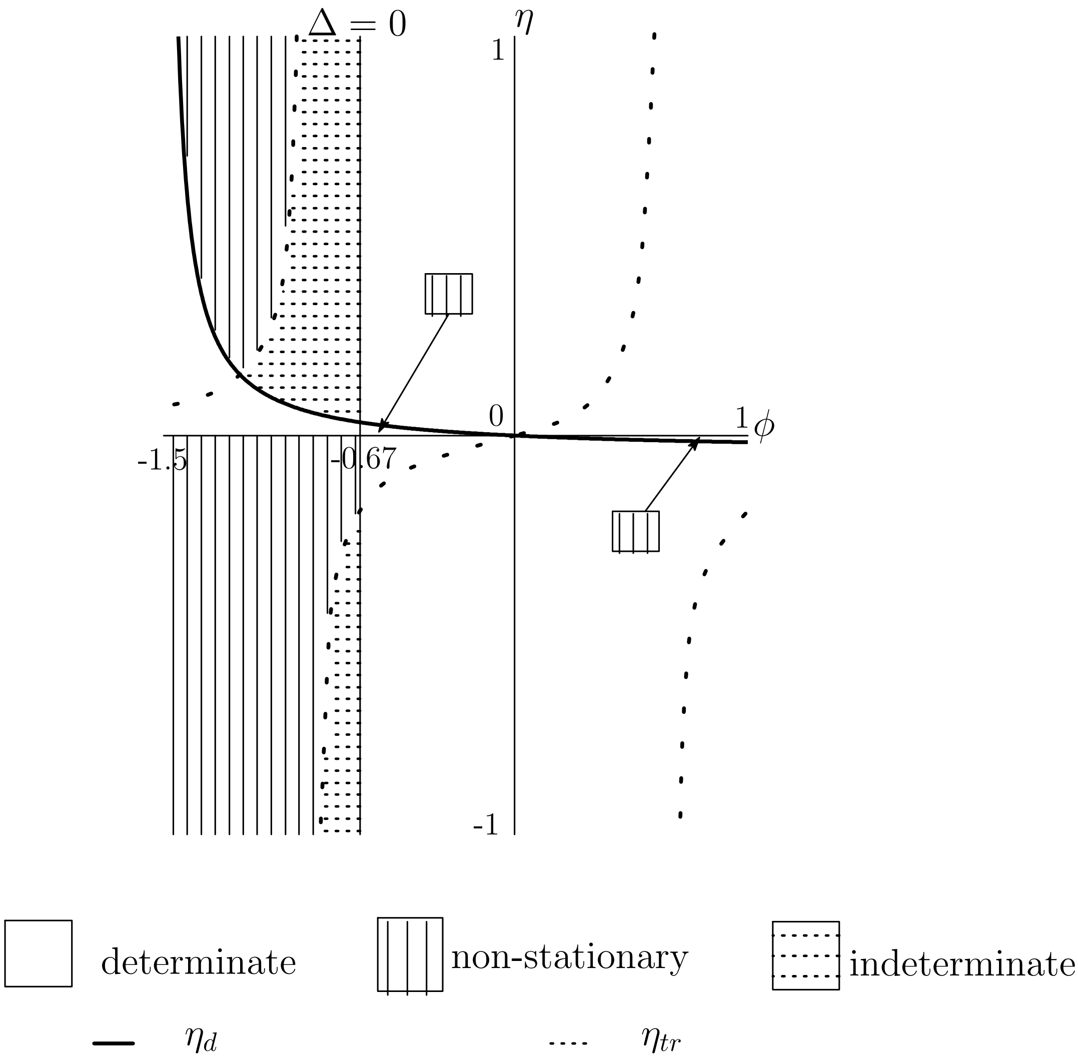

Figure 5.1: Equilibrium Determinacy

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