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## BLOCK INTERSECTION NUMBERS OF BLOCK DESIGNS II

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### 1. Introduction

Let  $t, v, k$  and  $\lambda$  be positive integers with  $v \geq k \geq t$ . A  $t$ - $(v, k, \lambda)$  design is a pair consisting of a  $v$ -set  $\Omega$  and a family  $\mathbf{B}$  of  $k$ -subsets of  $\Omega$ , such that each  $t$ -subset of  $\Omega$  is contained in just  $\lambda$  elements of  $\mathbf{B}$ . Elements of  $\Omega$  and  $\mathbf{B}$  are called points and blocks, respectively. A  $t$ - $(v, k, 1)$  design is often called a Steiner system  $S(t, k, v)$ . A  $t$ - $(v, k, \lambda)$  design is called nontrivial provided  $\mathbf{B}$  is a proper subfamily of the family of all  $k$ -subsets of  $\Omega$ , then  $t < k < v$ . In this paper we assume that all designs are nontrivial. For a  $t$ - $(v, k, \lambda)$  design  $\mathbf{D}$  we use  $\lambda_i (0 \leq i \leq t)$  to represent the number of blocks which contain a given set of  $i$  points of  $\mathbf{D}$ . Then we have

$$\lambda_i = \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}} \lambda = \frac{(v-i)(v-i-1)\cdots(v-t-1)}{(k-i)(k-i-1)\cdots(k-t-1)} \lambda \quad (0 \leq i \leq t).$$

A  $t$ - $(v, k, \lambda)$  design  $\mathbf{D}$  is called block-schematic if the blocks of  $\mathbf{D}$  form an association scheme with the relations determined by size of intersection (cf. [3]). Any Steiner system  $S(2, k, v)$  ( $t=2$ ) is block-schematic (cf. [2]). For a block  $B$  of a  $t$ - $(v, k, \lambda)$  design  $\mathbf{D}$  we use  $x_i(B)$  ( $0 \leq i \leq k$ ) to denote the number of blocks each of which has exactly  $i$  points in common with  $B$ . If, for each  $i$  ( $i=0, \dots, k$ ),  $x_i(B)$  is the same for every block  $B$ , we say that  $\mathbf{D}$  is block-regular and we write  $x_i$  instead of  $x_i(B)$ . Any Steiner system  $S(t, k, v)$  is block-regular (cf. [6]), and any block-schematic  $t$ - $(v, k, \lambda)$  design is also block-regular.

Atsumi [1] proved

Result 1. If a Steiner system  $S(t, k, v)$  is block-schematic with  $t \geq 3$ ,

then  $v \leq k^t \binom{k}{2}$  holds.

Yoshizawa [7] extended Result 1 and prove

Result 2. (a) For each  $n \geq 1$  and  $\lambda \geq 1$ , there exist at most finitely many block-schematic  $t$ - $(v, k, \lambda)$  designs with  $k-t=n$  and  $t \geq 3$ .

(b) For each  $n \geq 1$  and  $\lambda \geq 2$ , there exist at most finitely many block-schematic  $t$ -( $v, k, \lambda$ ) designs with  $k-t=n$  and  $t \geq 2$ .

In §2 we first prove the following proposition, and we prove the following theorem related to the above results.

**Proposition.**  $x_{i-1}^2 \geq x_0$  holds for any block-schematic Steiner system  $S(t, k, v)$  with  $k \geq 2(t-1)$ .

**Theorem 1.** Let  $\varepsilon$  be a positive real number. Then for each  $t \geq 3$  there exist at most finitely many block-schematic Steiner systems  $S(t, k, v)$  with  $v < k^{2-\varepsilon}$ , and for each  $t > \frac{2}{\varepsilon} + 2$  there exist at most finitely many block-schematic Steiner systems  $S(t, k, v)$  with  $v > k^{3+\varepsilon}$ .

Yoshizawa [7] proved the following result about block-regular designs.

Result 3. Let  $c$  be a real number with  $c > 2$ . Then for each  $n \geq 1$  and  $l \geq 0$ , there exist at most finitely many block-regular  $t$ -( $v, k, \lambda$ ) designs with  $k-t=n$ ,  $v \geq ct$  and  $x_i \leq l$  for some  $i$  ( $0 \leq i \leq t-1$ ).

In §3 we notice that the block-regularity of Result 3 is essentially unnecessary, and we prove

**Theorem 2.** Let  $c$  be a real number with  $c > 2$ , and  $n, l$  be integers with  $n \geq 1, l \geq 0$ . Then there exist at most finitely many  $t$ -( $v, k, \lambda$ ) designs each of which satisfies the following conditions: (i)  $k-t=n$ , (ii)  $v \geq ct$ , (iii) there exist a block  $B$  and an integer  $i$  ( $0 \leq i \leq t-1$ ) with  $x_i(B) \leq l$ .

## 2. Proof of Theorem 1

Let  $D$  be a  $t$ -( $v, k, \lambda$ ) design. Let  $B_1, \dots, B_{\lambda_0}$  be the blocks of  $D$ , and  $A_h$  ( $0 \leq h \leq k$ ) be the  $h$ -adjacency matrix of  $D$  of degree  $\lambda_0$  defined by

$$A_h(i, j) = \begin{cases} 1 & \text{if } |B_i \cap B_j| = h, \\ 0 & \text{otherwise.} \end{cases}$$

If  $D$  is block-schematic, then

$A_i A_j = \sum_{h=0}^k \mu(i, j, h) A_h$  ( $0 \leq i, j \leq k$ ) where  $\mu(i, j, h)$  is a non-negative integer defined by the following: When there exist blocks  $B_p$  and  $B_q$  with  $|B_p \cap B_q| = h$ ,

$$\mu(i, j, h) = |\{B_r: |B_p \cap B_r| = i, |B_q \cap B_r| = j, 1 \leq r \leq \lambda_0\}|,$$

and when there exist no blocks  $B_p$  and  $B_q$  with  $|B_p \cap B_q| = h$ ,  $\mu(i, j, h) = 0$ . Let  $\mathbf{a}$  be the all -1 column vector of degree  $\lambda_0$ . Then

$$A_i A_j \mathbf{a} = \sum_{h=0}^k \mu(i, j, h) A_h \mathbf{a}.$$

Hence we have

**Lemma 1.** For a block-schematic  $t$ - $(v, k, \lambda)$  design,  $x_i x_j = \sum_{h=0}^k \mu(i, j, h) x_h$  holds ( $0 \leq i, j \leq k$ ).

REMARK. Lemma 1 is essentially well-known (cf. [1], [7]).

**Lemma 2.** Let  $D$  be a Steiner system  $S(t, k, v)$  with  $t \geq 2$  and  $k \geq 2(t-1)$ . If  $(t, k, v) \neq (4, 7, 23), (2, n+1, n^2+n+1)$  ( $n \geq 2$ ), then there exist three blocks  $B_1, B_2$  and  $B_3$  of  $D$  such that  $|B_1 \cap B_2| = |B_1 \cap B_3| = t-1$  and  $|B_2 \cap B_3| = 0$ .

Proof. By [6] we have

$$x_{t-1} = (\lambda_{t-1} - 1) \binom{k}{t-1} = \frac{v-k}{k-t+1} \binom{k}{t-1} > 0.$$

Hence we may assume that there exist two blocks  $B_1$  and  $B_2$  with  $|B_1 \cap B_2| = t-1$ . Since  $k \geq 2(t-1)$ ,  $B_1 - B_2$  has (distinct)  $t-1$  points  $\alpha_1, \dots, \alpha_{t-1}$ . Let  $M_1 (=B_1), M_2, \dots, M_{\lambda_{t-1}}$  be the blocks which contain  $\alpha_1, \dots, \alpha_{t-1}$ . If  $M_i \cap B_2 = \phi$  for some  $i$  ( $2 \leq i \leq \lambda_{t-1}$ ), then  $|B_1 \cap M_i| = t-1$  and  $|B_2 \cap M_i| = 0$  hold. Let us suppose  $M_i \cap B_2 \neq \phi$  for  $i=2, \dots, \lambda_{t-1}$ . Then we have

$$\frac{v-t+1}{k-t+1} - 1 \leq k-t+1. \tag{1}$$

On the other hand by Theorems 3A. 3 and 4 in [4], we have  $v-t+1 \geq (k-t+2)(k-t+1)$ , with equality only when  $(t, k, v) = (2, n+1, n^2+n+1)$  ( $n \geq 2$ ),  $(3, 4, 8), (3, 6, 22), (3, 12, 112), (4, 7, 23)$  or  $(5, 8, 24)$ . Hence by (1) and the assumption of Lemma 2, we have  $(t, k, v) = (3, 4, 8), (3, 6, 22), (3, 12, 112)$  or  $(5, 8, 24)$ . But we can easily check that  $S(3, 4, 8), S(3, 6, 22), S(3, 12, 112)$  and  $S(5, 8, 24)$  satisfy the conclusion of Lemma 2 if  $S(3, 12, 112)$  exists (cf. [5, Corollary 1]).

Proof of Proposition. Let us suppose that  $D$  is a block-schematic Steiner system  $S(t, k, v)$  with  $k \geq 2(t-1)$ . Then by Lemma 1, we have

$$x_{t-1}^2 = \sum_{h=0}^k \mu(t-1, t-1, h) x_h.$$

Now by Lemma 2,  $\mu(t-1, t-1, 0) > 0$  or  $x_0 = 0$  holds when  $k \geq 2(t-1)$  and  $t \geq 2$  hold. Hence we have  $x_{t-1}^2 \geq x_0$ .

Proof of Theorem 1. First let us suppose that  $D$  is a block-schematic Steiner system  $S(t, k, v)$  with  $t \geq 3$  and  $v < k^2 - e$ . By Theorems 3A. 3 and 4 in [4], we have  $v \geq (k-t+2)(k-t+1) + t - 1$ , where the right hand of this

inequality is a polynomial in  $k$  of degree two. Hence there exists a positive number  $N(\varepsilon, t)$  with  $k < N(\varepsilon, t)$ , where  $N(\varepsilon, t)$  depends only on  $\varepsilon$  and  $t$ . Hence by Result 1,  $v$  is bounded above by a function of  $\varepsilon$  and  $t$ .

Next suppose that  $\mathcal{D}$  is a block-schematic Steiner system  $S(t, k, v)$  with  $t > \frac{2}{\varepsilon} + 2$  and  $v > k^{3+t}$ .

By [6] we have

$$x_{t-1}^2 = (\lambda_{t-1} - 1)^2 \binom{k}{t-1}^2 = \frac{(v-k)^2}{(k-t+1)^2} \binom{k}{t-1}^2 \quad (2)$$

By [5, Lemma 6] (or [7, Lemma 5]) we have

$$\begin{aligned} x_0 &= \left\{ \binom{v-k}{k} + (-1)^{t+1} \sum_{q=0}^{k-t-1} \binom{t-1+q}{q} \binom{v-k+q}{k-t} \right\} / \binom{v-t}{k-t}, \\ x_0 &\geq \frac{\binom{v-k}{k} (k-t) \binom{k-2}{k-t-1} \binom{v-t-1}{k-t}}{\binom{v-t}{k-t}}. \end{aligned} \quad (3)$$

Hence by (2) and (3) we have

$$\begin{aligned} x_0 - x_{t-1}^2 &\geq \frac{(v-k) \cdots (v-2k+1)}{(v-t) \cdots (v-k+1) k \cdots (k-t+1)} - (k-t)(k-2)^{t-1} - \frac{(v-k)^2}{(k-t)^2} \binom{k}{t-1}^2, \\ x_0 - x_{t-1}^2 &\geq \frac{(v-2k)^k}{v^{k-t} k^t} - k^t - \frac{v^2 k^{2t-2}}{(k-t)^2}. \end{aligned}$$

Since we may assume  $k \geq 2t$  by Result 1, we have

$$x_0 - x_{t-1}^2 \geq \frac{(v-2k)^k}{v^{k-t} k^t} - k^t - 4v^2 k^{2t-4} \geq \frac{(v-2k)^k}{v^{k-t} k^t} - 5v^2 k^{2t-4}.$$

On the other hand by Proposition,  $x_{t-1}^2 \geq x_0$  holds because of  $k \geq 2t$ . Thus we get

$$(v-2k)^k - 5v^{k-t+2} k^{3t-4} \leq 0. \quad (4)$$

Since  $v > k^3$ , we have

$$\frac{(v-2k)^k}{5v^{k-t+2} k^{3t-4}} = \frac{\left(1 - \frac{2k}{v}\right)^{k-t+2} (v-2k)^{t-2}}{5k^{3t-4}} \geq \frac{\left(1 - \frac{1}{k}\right)^k (v-2k)^{t-2}}{5\left(1 - \frac{1}{k}\right)^{t-2} k^{3t-4}}. \quad (5)$$

Since  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$ , where  $e$  is the Napier number, there is a positive num-

ber  $M_1(t)$  which depends only on  $t$ , such that  $\left(1 - \frac{1}{n}\right)^n / \left\{5 \left(1 - \frac{1}{n}\right)^{t-2}\right\} > M_1(t)$  holds for all integer  $n \geq 2$ . On the other hand,  $(v-2k)^{t-2} / k^{3t-4} \geq v^{t-2} / \{2^{t-2} k^{3t-4}\}$  holds. Hence by (5), we have

$$\frac{(v-2k)^k}{5v^{k-t+2} k^{3t-4}} \geq M_2(t) \frac{v^{t-2}}{k^{3t-4}},$$

where  $M_2(t)$  is a positive number which depends only on  $t$ . Since  $v > k^{3+\epsilon}$  and  $t > \frac{2}{\epsilon} + 2$ , there exists a positive number  $M_3(\epsilon, t)$  which depends only on  $\epsilon$  and  $t$ , such that

$$\frac{(v-2k)^k}{5v^{k-t+2} k^{3t-4}} > 1 \text{ holds for any } k \geq M_3(\epsilon, t).$$

Hence by (4), we must have  $k < M_3(\epsilon, t)$ . Hence by Result 1,  $v$  is bounded above by a function of  $\epsilon$  and  $t$ .

**3. Proof of Theorem 2**

The proof of Theorem 2 is essentially similar to that of Theorem 2 in [7]. So we give its outline.

Let  $D$  be a  $t$ -( $v, k, \lambda$ ) design, and  $B$  be a block of  $D$ . Counting in two ways the number of the following set

$\{(B', \{\alpha_1, \dots, \alpha_i\}) : B' \text{ a block } (\neq B), B' \cap B \ni \alpha_1, \dots, \alpha_i, \alpha_j \neq \alpha_{j'} \text{ if } j \neq j'\}$  gives

$$x_i(B) + \binom{i+1}{i} x_{i+1}(B) + \dots + \binom{t}{i} x_t(B) + \dots + \binom{k-1}{i} x_{k-1}(B) = (\lambda_i - 1) \binom{k}{i} \tag{6}$$

for  $i=0, \dots, t-1$ , and

$$x_i(B) + \binom{i+1}{i} x_{i+1}(B) + \dots + \binom{k-1}{i} x_{k-1}(B) \leq (\lambda - 1) \binom{k}{i} \tag{7}$$

for  $i=t, \dots, k-1$ . Let  $w_i(B)$  ( $t \leq i \leq k-1$ ) be the left hand of the above inequality, where  $w_i(B) = (\lambda - 1) \binom{k}{i}$ .

By (6) and (7) we have

$$x_i(B) = \sum_{j=0}^{i-1} \binom{j}{i} (\lambda_j - 1) \binom{k}{j} (-1)^{i+j} + \sum_{j=i}^{k-1} \binom{j}{i} w_j(B) (-1)^{1+j}, \tag{8}$$

for  $i=0, \dots, t-1$  (cf. [7, Proof of Lemma 1]). By (8) we have that there exists

a positive number  $C(k, l, t, i)$  which depends only on  $k, l, t, i$ , such that  $x_i(B) - l > 0$  holds if  $v \geq C(k, l, t, i)$  (cf. [7, Proof of Lemma 6]). Namely,  $v < C(k, l, t, i)$  holds if  $x_i(B) \leq l$ . Hence we get

**Lemma 3.** *For each  $k \geq 2$  and  $l \geq 0$ , there exist at most finitely many  $t$ - $(v, k, \lambda)$  designs each of which satisfies that there exists a block  $B$  and an integer  $i$  ( $0 \leq i \leq t-1$ ) with  $x_i(B) \leq l$ .*

**Proof of Theorem 2.** By Lemma 3 we may assume that  $t \geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n$ . Let  $D$  be a  $t$ - $(v, k, \lambda)$  design satisfying  $v \geq ct$  and  $t \geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n$ . Set  $v = mt$  ( $m \geq c$ ), where  $m$  is not always integral. By (8) we have

$$x_i(B) = \frac{\lambda \binom{k}{i}}{\binom{v-t}{k-t}} \left\{ \binom{v-k}{k-i} + (-1)^{t+i+1} \sum_{q=0}^{t-i-1} \binom{t-i-1+q}{q} \binom{v-k+q}{k-i} \right\} \\ + (\lambda-1) \sum_{j=i}^{t-1} \binom{j}{i} \binom{k}{j} (-1)^{i+j} + \sum_{j=i}^{t-1} \binom{j}{i} w_j(B) (-1)^{i+j},$$

where  $x_j(B) \leq w_j(B) \leq (\lambda-1) \binom{k}{j} = (\lambda-1) \binom{t+n}{j}$  ( $t \leq j \leq k-1$ ) (cf. [7, Proof of Lemma 5]). Hence we get

$$x_i(B) = \frac{\lambda \binom{t+n}{i}}{\binom{(m-1)t}{n}} \left\{ \binom{(m-1)t-n}{t+n-i} + (-1)^{t+i+1} \sum_{q=0}^{t-i-1} \binom{t-i-1+q}{q} \binom{(m-1)t-n+q}{n} \right\} \\ + (\lambda-1) \sum_{j=i}^{t-1} \binom{j}{i} \binom{t+n}{j} (-1)^{i+j} + \sum_{j=i}^{t-1} \binom{j}{i} w_j(B) (-1)^{i+j},$$

for  $i=0, \dots, t-1$ . By the above equality and the condition on  $t$ , we have

$$x_i(B) > \frac{((c-1)t-n)}{((n+1)!)^2} \binom{c-2}{c-1}^n - 5n \quad (\text{cf. [7, pp. 797, 798]}).$$

We remark that the right hand of the above inequality does not depend on  $i$ . Hence there exists a positive number  $N(c, n, l)$  ( $\geq \frac{2n + ((2n+2)!)^2}{c-2} + 2n$ ) which depends only on  $c, n, l$ , such that  $x_i(B) - l > 0$  holds for  $i=0, \dots, t-1$  if  $t \geq N(c, n, l)$ . Namely,  $t < N(c, n, l)$  holds if  $x_i(B) \leq l$  holds for some  $i$  ( $0 \leq i \leq t-1$ ). Hence by Lemma 3, we complete the proof.

## References

- [1] T. Atsumi: *An extension of Cameron's result on block schematic Steiner systems*, J. Combin. Theory Ser. A **27** (1979), 388–391.
- [2] R.C. Bose: *Strongly regular graphs, partial geometries, and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.
- [3] P.J. Cameron: *Two remarks on Steiner systems*, Geom. Dedicata **4** (1975), 403–418.
- [4] P.J. Cameron: *Parallelisms of complete designs*, London Math. Soc. Lecture note series No. 23, Cambridge Univ. Press, Cambridge 1976.
- [5] B.H. Gross: *Intersection triangles and block intersection numbers for Steiner systems*, Math. Z. **139** (1974), 87–104.
- [6] N.S. Mendelsohn: *A theorem on Steiner systems*, Canad. J. Math. **22** (1970), 1010–1015.
- [7] M. Yoshizawa: *Block intersection numbers of block designs*, Osaka J. Math. **18** (1981), 787–799.

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