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Osaka University
THE SPACE OF LOOPS ON THE EXCEPTIONAL LIE GROUP $E_6$

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1. Introduction

Let $G$ be a compact 1-connected simple Lie group and $\Omega G$ the space of loops on $G$. As is well known $\Omega G$ is a homotopy commutative H-space and its integral homology $H_*(\Omega G)$ has no torsion and no odd dimensional part ([2]). Therefore $H_*(\Omega G)$ becomes a commutative Hopf algebra over the integers $\mathbb{Z}$. In [3] R. Bott introduced a “generating variety” and determined the Hopf algebra structure of $H_*(\Omega G)$ explicitly for $G = SU(n)$, Spin($n$) and $G_2$. In [11] T. Watanabe determined $H_*(\Omega F_4)$ in a similar way. On the other hand A. Kono and K. Kozima determined $H_*(\Omega Sp(n))$ by different method using the Bott periodicity ([16]).

In this paper we carry out the Bott’s program for $G = E_6$, where $E_6$ is the compact 1-connected exceptional Lie group of rank 6 and determine the Hopf algebra structure of $H_*(\Omega E_6)$ explicitly.

Let $\psi$ be the coproduct of $H_*(\Omega G)$ induced by the diagonal map $\Omega G \rightarrow \Omega G \times \Omega G$. To avoid the cumbersome notation, following [11] we introduce a map $\tilde{\psi}: H_*(\Omega G) \rightarrow H_*(\Omega G) \otimes H_*(\Omega G)$ satisfying

$$\psi(\sigma) = (\sigma \otimes 1 + 1 \otimes \sigma) = \tilde{\psi}(\sigma) + T \tilde{\psi}(\sigma) \quad \text{for} \quad \sigma \in H_*(\Omega G)$$

where $T: H_*(\Omega G) \otimes H_*(\Omega G) \rightarrow H_*(\Omega G) \otimes H_*(\Omega G)$ is defined by

$$T(\sigma \otimes \tau) = \begin{cases} \tau \otimes \sigma & \text{for} \quad \sigma \neq \tau, \\ 0 & \text{for} \quad \sigma = \tau. \end{cases}$$

Note that $\tilde{\psi}(\sigma) = 0$ if and only if $\sigma \in PH_*(\Omega G)$, where $PH_*(\Omega G)$ denotes the primitive module of the Hopf algebra $H_*(\Omega G)$.

Then our main results are stated as follows:

**Theorem 1.1.** The Hopf algebra structure of $H_*(\Omega E_6)$ is given as follows:

(i) As an algebra

$$H_*(\Omega E_6) = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8, \sigma_{11}] / (\sigma_1^2 - 2\sigma_2, \sigma_1\sigma_2 - 3\sigma_3)$$

where $\deg(\sigma_i) = 2i$. 

(ii) With suitably chosen generators $\sigma_i$, $i = 1, 2, 3, 4, 5, 7, 8, 11$, the coproduct is given by

$$
\psi(\sigma_k) = \sum_{i+j=k} \sigma_i \otimes \sigma_j \quad \text{for} \quad k = 1, 2, 3, 4 \quad (\sigma_0 = 1),
$$

$$
\tilde{\psi}(\sigma_5) = \tau_4 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_2,
$$

$$
\tilde{\psi}(\sigma_7) = (-\tau_6 + \sigma_1\sigma_5) \otimes \sigma_1 + \sigma_5 \otimes \sigma_2 + \tau_4 \otimes \sigma_3,
$$

$$
\tilde{\psi}(\sigma_8) = (-\sigma_3\sigma_4 + \sigma_2\sigma_5) \otimes \sigma_1 + (\tau_6 - \sigma_2\sigma_4 + \sigma_1\sigma_5) \otimes \sigma_2
$$

$$
+ (\sigma_1\tau_4 + \sigma_1\sigma_4 + \sigma_3) \otimes \sigma_3 + 2\tau_4 \otimes \tau_4 - \sigma_4 \otimes \sigma_4,
$$

$$
\tilde{\psi}(\sigma_{11}) = (-2\tau_4\tau_6 + \sigma_4\tau_6 + 2\sigma_1\tau_4\sigma_5 - \sigma_3\sigma_7 - \sigma_1\sigma_4\sigma_5 - \sigma_5^2) \otimes \sigma_1
$$

$$
+ (-2\sigma_3\tau_6 + 4\tau_4\sigma_5 - \sigma_2\sigma_7 - \sigma_4\sigma_5) \otimes \sigma_2
$$

$$
+ (-\sigma_2\tau_6 - \tau_4\tau_4 + 4\sigma_3\sigma_5 - \sigma_1\sigma_7) \otimes \sigma_3
$$

$$
- \sigma_3\tau_4 \otimes \tau_4 + \sigma_2\sigma_5 \otimes \tau_4 - \sigma_7 \otimes \sigma_4 - \tau_6 \otimes \sigma_1\tau_4 + 4\tau_6 \otimes \sigma_5
$$

$$
+ \tau_6 \otimes \sigma_1\sigma_4 - \sigma_2\sigma_4 \otimes \sigma_5 + \sigma_1\sigma_5 \otimes \sigma_1\tau_4 - \sigma_1\sigma_5 \otimes \sigma_1\sigma_4 - 2\sigma_1\sigma_5 \otimes \sigma_5
$$

where $\tau_4 = \sigma_2^3 - \sigma_1\sigma_3$, $\tau_6 = \sigma_2^3 - 4\sigma_3^2$.

(iii) $PH_*\Omega E_6 = \langle \sigma_1, \tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_7, \tilde{\sigma}_8, \tilde{\sigma}_{11} \rangle$

as a free module, where

$$
\tilde{\sigma}_4 = \tau_4 - 2\sigma_4,
$$

$$
\tilde{\sigma}_5 = \sigma_1\tau_4 - 5\sigma_5,
$$

$$
\tilde{\sigma}_7 = 4\sigma_1\tau_6 - 7\sigma_2\sigma_5 + 7\sigma_7,
$$

$$
\tilde{\sigma}_8 = \sigma_2\tau_6 - 4\sigma_3\sigma_5 + 2\sigma_4^2 + 4\sigma_8,
$$

$$
\tilde{\sigma}_{11} = 6\sigma_1\tau_4\tau_6 - 44\sigma_5\tau_6 + 11\sigma_1\sigma_5^2 + 11\sigma_4\sigma_7 + 11\sigma_{11}.
$$

The paper is organized as follows: In §2 we prepare various results which are needed in later sections. In §3 we carry out the Bott’s program for $E_6$ and determine the primitive elements of $H^*(\Omega E_6)$. In §4 we determine the algebra structure of $H_*\Omega E_6$ depending on the results of mod $p$ homology for all primes $p$. In the last section, §5 we determine the Hopf algebra structure of $H_*\Omega E_6$ explicitly using the “generating variety”. Throughout this paper $\sigma_i(x_1, \ldots, x_\eta)$ denotes the $i$-th elementary symmetric function in the variables $x_1, \ldots, x_\eta$.

I would like to thank Professors Takashi Watanabe and Akira Kono for much helpful advice and encouragement.
2. Preliminaries

Let $T$ be a maximal torus of $E_6$ and $\{\alpha_i\}_{1 \leq i \leq 6}$ the root system given in [4], where the corresponding fundamental weights are given as follows:

\[
\begin{align*}
    w_1 &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6), \\
    w_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\
    w_3 &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6), \\
    w_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6, \\
    w_5 &= \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6), \\
    w_6 &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6).
\end{align*}
\]

(2.1)

As usual we regard roots and weights as elements of $H^1(T) \cong H^2(BT)$. Then $\{w_i\}_{1 \leq i \leq 6}$ forms a basis of $H^2(BT)$ and $H^*(BT) = \mathbb{Z}[w_1, w_2, \ldots, w_6]$. Let $R_i$ denote the reflection to the hyperplane $\alpha_i = 0$, then the Weyl group of $E_6$, $W(E_6)$, is a finite group generated by $R_i$ ($1 \leq i \leq 6$) and the action on $\{w_i\}_{1 \leq i \leq 6}$ is given as follows:

\[
R_i(w_j) = w_j - \sum_{j=1}^{6} \frac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)} w_j, \quad R_j(w_i) = w_i \quad \text{for} \quad j \neq i.
\]

(2.2)

Following [10], we introduce the elements

\[
\begin{align*}
    t_6 &= w_6, \\
    t_5 &= R_6(t_6) = w_5 - w_6, \\
    t_4 &= R_5(t_5) = w_4 - w_5, \\
    t_3 &= R_4(t_4) = w_2 + w_3 - w_4, \\
    t_2 &= R_3(t_3) = w_1 + w_2 - w_3, \\
    t_1 &= R_1(t_2) = -w_1 + w_2, \\
    c_i &= \sigma_i(t_1, \ldots, t_6), \quad t = \frac{1}{3}c_1 = w_2.
\end{align*}
\]

Then we have the following isomorphism:

\[
H^*(BT) = \mathbb{Z}[t_1, t_2, \ldots, t_6, t]/(c_1 - 3t).
\]

From (2.2) the action of $W(E_6)$ on these elements is given by Table 1, where blanks indicate the trivial action.
Next as in [10] we put

\[ x_i = 2t_i - t \quad (1 \leq i \leq 6). \]

From Table 1 we see easily that

\[ S = \{ x_i + x_j \ (1 \leq i < j \leq 6), \ t - x_i, -t - x_i \ (1 \leq i \leq 6) \} \subset H^2(BT; \mathbb{Q}) \]

is invariant under the action of \( W(E_6) \). (In fact \( S \) is an orbit of \( 2w_1 \) under the action of \( W(E_6) \).) Thus we have \( W(E_6) \)-invariant forms

\[ I_n = \sum_{y \in S} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_6)}. \]

As in [10] \( I_n \) is computed by the formula:

\[ I_n = \frac{1}{2} \sum_{i=2}^{n-2} \binom{n}{i} s_is_{n-i} + (6 - 2^{n-1})s_n + 2(-1)^n \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} s_{n-2j}t^{2j} \]

where \( s_n = x_1^n + \cdots + x_6^n \) and \( s_n \) is written as a polynomial in \( d_i \)'s, \( d_i = \sigma_i(x_1, \ldots, x_6) \), by use of the Newton formula:

\[ s_n = \sum_{i=1}^{n-1} (-1)^{i-1}s_{n-i}d_i + (-1)^{n-1}nd_n \quad (d_n = 0 \text{ for } n > 6). \]

Moreover since \( x_i = 2t_i - t \) for \( 1 \leq i \leq 6 \)

\[ \sum_{n=0}^{6} d_n = \prod_{i=1}^{6} (1 + x_j) = \prod_{i=1}^{6} (1 - t + 2t_i) = \sum_{j=0}^{6} (1 - t)^{6-i}2^j c_i \]

and we have

\[ d_n = \sum_{i=0}^{n} \binom{6-i}{n-i} (-t)^{n-i}2^j c_i. \]
By using (2.3), (2.4), (2.5) $I_n$ can be written as a polynomial in $t$ and $c_i$'s (for details see [10], §5).

Then the next lemma is proved in [10], Lemma 5.2:

**Lemma 2.1.** The rational invariant subalgebra of the Weyl group $W(E_6)$ is given as follows:

$$H^*(BT; \mathbb{Q})^{W(E_6)} = \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}].$$

Let $C_1$ be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ ($i \neq 1$). Then as is well known

$$C_1 = T^1 \cdot \text{Spin}(10), \quad T^1 \cap \text{Spin}(10) \cong \mathbb{Z}/4\mathbb{Z}$$

and the Weyl group of $C_1$, $W(C_1)$, is generated by $R_i$ ($i \neq 1$).

**Remark 2.2.** The homogeneous space $E_6/C_1$ is the irreducible Hermitian symmetric space and denoted by $EIII$ in $E$. Cartan’s notation.

We put

$$t_0 = t - t_1 = w_1, \quad \epsilon_i = t_{i+1} - \frac{1}{2}t_0 \quad (1 \leq i \leq 5).$$

Then we have

$$H^*(BT; \mathbb{Q}) = \mathbb{Q}[t_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_5]$$

and the action of $W(C_1)$ is given by Table 2.

From Table 2 we have

**Lemma 2.3.** The rational invariant subalgebra of the Weyl group $W(C_1)$ is given as follows:

$$H^*(BT; \mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$$
where

\[ p_i = \sigma_i(\epsilon_1^2, \ldots, \epsilon_5^2), \quad \epsilon = \prod_{i=1}^{5} \epsilon_i. \]

We shall find the relations between \( p_i \)'s and \( I_n \)'s. Put

\[ b_i = \sigma_i(\epsilon_1, \ldots, \epsilon_5) \]

so that

\[
\sum_{i \geq 0} (-1)^i p_i = \prod_{j=1}^{5} (1 - \epsilon_j^2) = \prod_{j=1}^{5} (1 + \epsilon_j) \prod_{j=1}^{5} (1 - \epsilon_j) = \left( \sum_{k \geq 0} b_k \right) \left( \sum_{l \geq 0} (-1)^l b_l \right) = \sum_{k, l \geq 0} (-1)^{k+l} b_k b_l.
\]

Therefore

\[(2.7) \quad p_i = \sum_{k+l=2i} (-1)^{k+l} b_k b_l. \]

On the other hand since \( t_i = \epsilon_i - 1 + (1/2) t_0 \) for \( 2 \leq i \leq 6 \)

\[
\sum_{n=0}^{6} c_n = \prod_{i=1}^{6} (1 + t_i) = (1 + t_1) \prod_{i=2}^{6} (1 + t_i)
\]

\[= (1 + t_1) \prod_{i=1}^{5} \left( 1 + \frac{1}{2} t_0 + \epsilon_i \right) = (1 + t_1) \sum_{i=0}^{5} \left( 1 + \frac{1}{2} t_0 \right)^{5-i} b_i
\]

and we have

\[(2.8) \quad c_n = \sum_{i=0}^{n} \binom{5-i}{n-i} \left( \frac{1}{2} t_0 \right)^{n-i} b_i + \sum_{i=0}^{n-1} \binom{5-i}{n-i-1} \left( \frac{1}{2} t_0 \right)^{n-i-1} t_1 b_i.
\]

Since \( I_n \in H^+ (BT; \mathbb{Q}^{W(E_6)}) \subset H^+ (BT; \mathbb{Q}^{W(C_1)}) = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]. \) \( I_n \) can be written as a polynomial in \( t_0, p_1, p_2, e, p_3, p_4. \) Direct computation using (2.3), (2.4), (2.5), (2.7) and (2.8) then yields the following results:

**Lemma 2.4.** In \( H^+ (BT; \mathbb{Q}^{W(C_1)}) = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4] \) we have

\[ I_2 = 2^4 \cdot 3 \left( \frac{1}{2} p_1 + \frac{3}{8} t_0^2 \right). \]
\[I_5 = -2^7 \cdot 3 \cdot 5 \left\{ e + \frac{1}{4} \left( -p_2 + \frac{1}{4} p_1^2 \right) t_0 + \frac{1}{32} p_1 t_0^3 - \frac{3}{256} t_0^5 \right\},\]

\[I_6 = 2^9 \cdot 3^2 \left\{ \frac{1}{4} p_3 - \frac{1}{24} p_1 p_2 + \frac{1}{32} p_1^3 - \frac{5}{4} e t_0 + \frac{5}{32} \left( -p_2 + \frac{3}{4} p_1^2 \right) t_0^3 + \frac{15}{512} p_1 t_0^4 \right\},\]

\[I_8 = -2^{12} \cdot 3 \cdot 5 \left\{ -\frac{1}{4} p_4 - \frac{1}{48} p_1^2 - \frac{1}{10} p_1 p_3 + \frac{13}{480} p_1^2 p_2 - \frac{11}{1280} p_1^4 + \frac{7}{8} ep_1 t_0 + \frac{7}{80} \left( -3 p_3 + \frac{7}{4} p_1 p_2 - \frac{11}{16} p_1^3 \right) t_0^2 + \frac{7}{32} e t_0^3 + \frac{7}{512} \left( 5 p_2 - \frac{11}{4} p_1^2 \right) t_0^4 \right\},\]

\[I_9 = 2^{11} \cdot 3^3 \cdot 7 \left\{ -\frac{1}{6} ep_2 - \frac{1}{8} e p_1^2 + \frac{1}{4} \left( p_4 - \frac{1}{12} p_2^2 + \frac{5}{24} p_1^2 p_2 - \frac{3}{64} p_1^4 \right) t_0 \right\},\]

\[I_{10} = 2^{15} \cdot 3^4 \cdot 5 \left\{ \frac{1}{180} p_5^2 + \frac{1}{24} p_2 p_4 - \frac{1}{864} p_3^2 - \frac{1}{540} p_1 p_2 p_3 + \frac{41}{864} p_1^2 p_4 + \frac{65}{108} e^2 p_1 + \frac{101}{17280} p_2^2 + \frac{13}{2160} p_1^2 p_3^2 + \frac{251}{69120} p_1^4 p_2 + \frac{19}{30720} p_1^6 \right\},\]

\[+ \frac{11}{12} \left( -\frac{1}{4} ep_3 - \frac{1}{4} ep_1 p_2 - \frac{1}{16} e p_2^3 \right) t_0 + \frac{11}{16} \left( \frac{11}{9} e^2 - \frac{1}{18} p_2 p_3 + \frac{7}{18} p_1 p_4 \right) \right\},\]

\[+ \frac{11}{144} p_1^2 p_2^2 + \frac{7}{72} p_2^3 p_3 - \frac{7}{96} p_1^3 p_2^2 + \frac{19}{1280} p_1^4 p_1^2 + \frac{77}{192} \left( -\frac{1}{3} ep_2 - \frac{1}{4} e p_1^2 \right) t_0^3 \right\},\]

\[+ \frac{11}{128} \left( -\frac{5}{12} p_1 + \frac{65}{144} p_2^2 + p_1 p_3 - \frac{253}{288} p_1^2 p_2 + \frac{57}{256} p_1^3 \right) t_0^4 \right\},\]

\[I_{12} = 2^{19} \cdot 3 \cdot 5 \left\{ \frac{1}{2} e^2 p_1 + \frac{101}{17280} p_2^2 + \frac{13}{2160} p_1^2 p_3^2 + \frac{251}{69120} p_1^4 p_2 + \frac{19}{30720} p_1^6 \right\},\]

\[+ \frac{11}{12} \left( -\frac{1}{4} ep_3 - \frac{1}{4} ep_1 p_2 - \frac{1}{16} e p_2^3 \right) t_0 + \frac{11}{16} \left( \frac{11}{9} e^2 - \frac{1}{18} p_2 p_3 + \frac{7}{18} p_1 p_4 \right) \right\},\]

\[+ \frac{11}{144} p_1^2 p_2^2 + \frac{7}{72} p_2^3 p_3 - \frac{7}{96} p_1^3 p_2^2 + \frac{19}{1280} p_1^4 p_1^2 + \frac{77}{192} \left( -\frac{1}{3} ep_2 - \frac{1}{4} e p_1^2 \right) t_0^3 \right\},\]

\[+ \frac{11}{128} \left( -\frac{5}{12} p_1 + \frac{65}{144} p_2^2 + p_1 p_3 - \frac{253}{288} p_1^2 p_2 + \frac{57}{256} p_1^3 \right) t_0^4 \right\},\]

\[+ \frac{11}{31072} \left( \frac{209}{5242880} p_1 t_0^4 + \frac{12977}{41943040} t_0^{12} \right) \right\} \cdot\]

Let \( b_j \) denote the ideal in \( H^*(BT\; ; \mathbb{Q}^{w(G)}) \) generated by \( I_j \)'s for \( j < i, \ j \in 2, 5, 6, 8, 9, 12 \). By Lemma 2.4 we have immediately the following (here we put \( w = (1/6)p_2 + (9/16)t_0^4 \)
Lemma 2.5. In $H^*(BT; \mathbb{Q})^{W(C)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ we have

$$p_1 \equiv -\frac{3}{4} t_0^2 \mod b_5,$$

$$e \equiv \frac{3}{2} t_0 w - \frac{27}{32} t_0^5 \mod b_6,$$

$$p_3 \equiv \frac{21}{2} t_0^2 w - \frac{195}{32} t_0^6 \mod b_8,$$

$$p_4 \equiv -3 w^2 - \frac{63}{8} t_0^4 w + \frac{1413}{256} t_0^8 \mod b_9.$$

Using Lemmas 2.4, 2.5 we obtain

Lemma 2.6. In $H^*(BT; \mathbb{Q})^{W(C)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ we have

$$I_0 \equiv 2^{11} \cdot 3^3 \cdot 7 (t_0^9 - 3t_0 w^2) \mod b_9,$$

$$I_{12} \equiv 2^{15} \cdot 3^4 \cdot 5 (-3t_0^{12} + 9t_0^8 w - 6t_0^4 w^2 - w^3) \mod b_9.$$

Therefore by the classical results of A. Borel [1] we obtain the following rational cohomology ring of $EIII$:

$$H^*(EIII; \mathbb{Q}) \cong H^*(BT; \mathbb{Q})^{W(C)}/b_{13}$$

$$= \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]/(I_2, I_5, I_6, I_8, I_9, I_{12})$$

$$\cong \mathbb{Q}[t_0, w]/(t_0^9 - 3t_0 w^2, w^3 + 15t_0^4 w^2 - 9t_0^8 w).$$

Furthermore the integral cohomology ring of $EIII$ is determined in [10], Corollary C:

Theorem 2.7. The integral cohomology ring of $EIII$ is given as follows:

$$H^*(EIII) = \mathbb{Z}[t_0, w]/(t_0^9 - 3t_0 w^2, w^3 + 15t_0^4 w^2 - 9t_0^8 w)$$

where $t_0, w$ are as above.

Corollary 2.8. An additive basis of $H^*(EIII)$ as a free module for degree $\leq 22$ is given by Table 3.

3. The primitive elements of $H^*(\Omega E_6)$

In this section we determine the primitive elements of $H^*(\Omega E_6)$ which are needed for determination of the Hopf algebra structure of $H_*(\Omega E_6)$.

As is well known, over the rationals $E_6$ looks like the product of odd dimensional
spheres:

\[ S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23} . \]

Hence the rational cohomology ring of \( \Omega E_6 \) is given by

\[ H^*(\Omega E_6; \mathbb{Q}) = \mathbb{Q}[u_1, u_4, u_5, u_7, u_8, u_{11}] \]

where \( \deg(u_i) = 2i \). Furthermore we can choose the generators \( u_i \)'s as integral classes such that they are not divisible and primitive in \( H^*(\Omega E_6) \). Thus

\[ PH^*(\Omega E_6) = \langle u_1, u_4, u_5, u_7, u_8, u_{11} \rangle \]

as a free module.

Now we briefly review the results of R. Bott (for details see [3]): Let \( G \) be a compact connected semisimple Lie group with trivial center and \( \Omega G \) the space of loops on \( G \). Suppose that a homomorphism \( s: S^1 \to G \) is given. Let \( T \) be a maximal torus of \( G \) containing the image of \( s \) and \( C_s \) the centralizer of the image of \( s \) in \( G \). Define the map

\[ g_s: G/C_s \to \Omega_0 G \]

by \( g_s(q)(t) = x s(t) x^{-1} s(t)^{-1} \) for \( q = x C_s \in G/C_s \), \( t \in S^1 \) where \( \Omega_0 G \) denotes the identity component of \( \Omega G \).

Let \( \tilde{G} \) be the universal covering group of \( G \) and \( d \) the covering dimension. Note that \( \Omega_0 G \cong \Omega \tilde{G} \). Let \( \tilde{s}: S^1 \to \tilde{G} \) be a lift of \( s \) and \( C_{\tilde{s}} \) the centralizer of the image of \( \tilde{s} \) in \( \tilde{G} \). Note that \( G/C_{\tilde{s}} \cong \tilde{G}/C_{\tilde{s}} \). Define the map

\[ f_{\tilde{s}}: \tilde{G}/C_{\tilde{s}} \to \Omega \tilde{G} \]

by \( f_{\tilde{s}}(q)(t) = x \tilde{s}(t) x^{-1} \) for \( q = x C_{\tilde{s}} \in \tilde{G}/C_{\tilde{s}} \).

According to Bott, if \( s \) is the “generating circle” the image of the homomorphism

\[ g_{s*}: H_*(G/C_s) \to H_*(\Omega_0 G) \]

generates \( H_*(\Omega_0 G) \) as an algebra. (In this case \( g_s \) is called the “generating map” and \( G/C_s \) the “generating variety”.) Dual statement is as follows:

\[ g_{s*}: H^*(\Omega_0 G) \to H^*(G/C_s) \]
is a split monomorphism when restricted to \( PH^*(\Omega_0 G) \). Moreover since \( \Omega_0 G \cong \Omega \tilde{G}, G/C_8 \cong \tilde{G}/C_{\tilde{8}} \) we can compare the images of \( g_S^* \) and \( f_S^* \). Then the following holds ([3], Proposition 7.1):

\[
f_S^*|_{PH^*(\Omega G)} = d \cdot g_S^*|_{PH^*(\Omega_0 G)}.
\]

Bott also gave the method of computing the image of \( f_S^* \) by means of the Borel description of the rational cohomology ring of \( \tilde{G}/C_{\tilde{8}} \) (see [3], Theorem 4). We carry out his program in the case of \( E_6 \).

As is well known \( E_6 \) has non-trivial center and it is of order 3. Hence the generating circle is defined in the adjoint group \( AdE_6 = E_6/center \). Since \( AdE_6 \) is simply laced, all minimal circles are generating circles ([3], page 43). So we take

\[
s = \text{the dual of } \alpha_1.
\]

Then the centralizer of the one dimensional torus \( \text{Im } s \) is \( C_1 = T^1 \cdot \text{Spin}(10) \) and the generating variety corresponding to \( s \) is

\[
E_6/C_1 = E_6/T^1 \cdot \text{Spin}(10) = EIII.
\]

In view of (2.1), (2.6) the derivation corresponding to \( s \) is

\[
\theta_s = \frac{4}{3} \frac{\partial}{\partial t_0} : \mathbb{Q}[t_0, \epsilon_1, \ldots, \epsilon_5] \longrightarrow \mathbb{Q}[t_0, \epsilon_1, \ldots, \epsilon_5].
\] (3.1)

We have to compute the image of the composition

\[
H^*(BT; \mathbb{Q})^{W(E_6)} \hookrightarrow H^*(BT; \mathbb{Q})^{W(C_1)} \xrightarrow{\theta_s} H^*(BT; \mathbb{Q})^{W(C_1)} \hookrightarrow H^*(EIII; \mathbb{Q})
\] (3.2)

where \( \iota \) is natural surjection.

Because \( \theta_s \) is a derivation and \( H^*(BT; \mathbb{Q})^{W(C_1)}/b_{13} \xrightarrow{\iota} H^*(EIII; \mathbb{Q}) \) we have only to compute the image of \( I_n, n = 2, 5, 6, 8, 9, 12 \). The computation proceeds as follows:

1. First we apply (3.1) to Lemma 2.4 and obtain \( \theta_s(I_n) \) for \( n = 2, 5, 6, 8, 9, 12 \).
2. Next using Lemma 2.5, 2.6 we rewrite \( \iota(\theta_s(I_n)) \) in terms of \( t_0, w \).

Then we obtain the results given by Table 4.

Therefore by a characterization of primitive elements given in [11], Proposition 5 (ii) we obtain

**Proposition 3.1.** There exist unique primitive elements \( a_i (\text{resp. } b_i, c_i, d_i, e_i, f_{11}) \) of \( H^*(\Omega E_6) \) such that \( g^*_S(a_1) = a \) (resp. \( g^*_S(b_4) = b \), \( g^*_S(c_5) = c \), \( g^*_S(d_7) = d \), \( g^*_S(e_8) = e \), \( g^*_S(f_{11}) = f \)) and

\[
PH^*(\Omega E_6) = \langle a_1, b_4, c_5, d_7, e_8, f_{11} \rangle
\]
as a free module.

4. The algebra structure of \( H_*(\Omega E_6) \)

The purpose of this section is to determine the algebra structure of \( H_*(\Omega E_6) \).

First note that the rational homology ring \( H_*(\Omega E_6; \mathbb{Q}) \) is of the form

\[
H_*(\Omega E_6; \mathbb{Q}) = \mathbb{Q}[x_2, x_8, x_{10}, x_{14}, x_{16}, x_{22}]
\]

where \( \deg(x_i) = i \quad ([9]) \).

By Corollary 2.8 we can choose an additive basis of \( H^*(EIII) \) for degree \( \leq 22 \) as in Table 5, where \( a, b, c, d, e \) and \( f \) are as in Proposition 3.1 and \( b', c', d', e', f' \) and \( e'' \) are determined by the following equations:

\[
B \cdot \begin{pmatrix} a^4 \\ w \end{pmatrix} = \begin{pmatrix} b \\ b' \end{pmatrix}, \quad C \cdot \begin{pmatrix} t_0^4 \\ t_0 w \end{pmatrix} = \begin{pmatrix} c \\ c' \end{pmatrix}, \quad D \cdot \begin{pmatrix} t_0^4 \\ t_0^2 w \end{pmatrix} = \begin{pmatrix} d \\ d' \end{pmatrix},
\]

\[
E \cdot \begin{pmatrix} t_0^4 \\ t_0^2 w \\ w^2 \end{pmatrix} = \begin{pmatrix} e \\ e' \\ e'' \end{pmatrix}, \quad F \cdot \begin{pmatrix} t_0^4 w \\ t_0^3 w^2 \end{pmatrix} = \begin{pmatrix} f \\ f' \end{pmatrix}
\]

where \( B, C, D, F \) (resp. \( E \)) are \( 2 \times 2 \) (resp. \( 3 \times 3 \)) matrices over \( \mathbb{Z} \) whose determinant is 1; for example

\[
B = \begin{pmatrix} 1 & -2 \\ k & l \end{pmatrix}
\]

with \( k, l \in \mathbb{Z} \) such that \( 2k + l = 1 \).
We take the corresponding dual basis of $H_* (EIII)$ and denote the dual of $x \in H^*$ by $x^* \in H_*$. Furthermore we define the elements of $H_* (\Omega E_6)$ by

\[
\sigma_1 = g_{s_+}(a_s), \quad \sigma_4 = g_{s_+}(b_s), \quad \sigma_5 = g_{s_+}(c_s), \quad \sigma_7 = g_{s_+}(d_s),
\]

\[
\sigma_8 = g_{s_+}(e_s), \quad \sigma_{11} = g_{s_+}(f_s),
\]

\[
\sigma_i = g_{s_+}(\bar{\theta}_i) \quad \text{for} \quad i = 2, 3.
\]

Then in view of Proposition 3.1 we obtain

**Lemma 4.1.**

(i) $\sigma_i$, $i = 1, 4, 5, 7, 8, 11$ are indecomposable and not divisible.

(ii) The coproduct of $\sigma_2$, $\sigma_3$ is given by

\[
\bar{\psi}(\sigma_2) = \sigma_1 \otimes \sigma_1,
\]

\[
\bar{\psi}(\sigma_3) = \sigma_2 \otimes \sigma_1.
\]

(iii) The following relations hold:

\[
\sigma_1^2 = 2\sigma_2, \quad \sigma_1\sigma_2 = 3\sigma_3.
\]

Proof of Theorem 1.1 (i). In view of (4.1) and Lemma 4.1, in order to determine the algebra structure of $H_* (\Omega E_6)$ we have to consider the “divisibility” among $\sigma_i$’s. Fortunately the mod $p$ homology rings $H_* (\Omega E_6; \mathbb{Z}/p\mathbb{Z})$ for each prime $p$ are studied by several authors and the algebra structure is determined for all cases ([8], [7], [5]). We exhibit their results:

\[
H_* (\Omega E_6; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_8, x_{10}, x_{14}, x_{16}, x_{22}] / (x_2^4),
\]

\[
H_* (\Omega E_6; \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}[x_2, x_6, x_8, x_{10}, x_{14}, x_{16}, x_{22}] / (x_2^3),
\]

\[
H_* (\Omega E_6; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}[x_2, x_8, x_{10}, x_{14}, x_{16}, x_{22}] \quad \text{for} \quad p \geq 5
\]

where $\deg(x_i) = i$ and moreover we can take $x_{2i}$ as the mod $p$ reduction of $\sigma_i$ for each case. Therefore if “divisibility” occurs in degree $k \geq 8$ there exists an integral class $\sigma$, $\deg(\sigma) = k$ and an integer $n \geq 2$ such that

\[
n \cdot \sigma = f(\sigma_1, \ldots, \sigma_{11})
\]

where $f$ is a homogeneous polynomial of degree $k$ in $\sigma_1, \ldots, \sigma_{11}$ with integer coefficients. Let $p$ be a prime number which divides $n$ and apply the mod $p$ reduction on both sides of (4.6). Then a relation of degree $k \geq 8$ arises in mod $p$ homology and it contradicts the previous results. Hence we obtain the required result. □
Table 6.

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Table 7.

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5. The Hopf algebra structure of $H_4(\Omega E_6)$

In this section we determine the Hopf algebra structure of $H_4(\Omega E_6)$ using the “generating variety” $E11$.

At first by Theorem 1.1 (i) we can choose an additive basis of $H_4(\Omega E_6)$ for degree $\leq 22$ as in Table 6, where $\tau_4 = \sigma_2^2 - \sigma_1 \sigma_3$, $\tau_6 = \sigma_2^3 - 4 \sigma_3^2$.

We take the corresponding dual basis of $H^*(\Omega E_6)$ as in Table 7.

In view of Lemma 4.1 we obtain the results given by Table 8.

Next consider the case of degree 8: Since $\tau_4 = \sigma_2^2 - \sigma_1 \sigma_3$, the coproduct of $\tau_4$ is given by

\begin{equation}
\tilde{\psi}(\tau_4) = 2 \sigma_3 \otimes \sigma_1 + 2 \sigma_2 \otimes \sigma_2.
\end{equation}
We shall determine the coproduct of $\sigma_4$: We may put

$$\bar{\psi}(\sigma_4) = m\sigma_3 \otimes \sigma_1 + \cdots$$

for some $m \in \mathbb{Z}$. On the other hand since $H^8(\Omega E_6) = \langle a_1, b_4 \rangle$, we can put

$$a_3a_1 = \mu a_4 + \nu b_4 \tag{5.2}$$

for some $\mu, \nu \in \mathbb{Z}$. Then

$$\mu = \langle a_3a_1, \tau_4 \rangle = \langle a_3 \otimes a_1, \psi(\tau_4) \rangle = 2,$$
$$\nu = \langle a_3a_1, \sigma_4 \rangle = \langle a_3 \otimes a_1, \psi(\sigma_4) \rangle = m.$$

Therefore applying $g_s^*$ on both sides of (5.2)

$$2g_s^*(a_4) = g_s^*(a_3a_1) - mg_s^*(b_4)$$
$$= t_0^3 \cdot t_0 - m \cdot b$$
$$= t_0^4 - m(t_0^4 - 2w)$$
$$= (1 - m)t_0^4 + 2mw.$$

On the other hand since $H^8(III) = \langle b, b' \rangle$, we can put

$$g_s^*(a_4) = \mu b + \nu b'$$

for some $\mu, \nu \in \mathbb{Z}$. Then

$$\mu = \langle g_s^*(a_4), b_4 \rangle = \langle a_4, g_s^*(b_4) \rangle = \langle a_4, \sigma_4 \rangle = 0.$$ 

Hence

$$g_s^*(a_4) = \nu b' = \nu(kt_0^4 + lw)$$

with $2k + l = 1$. Combining these gives

$$(1 - m)t_0^4 + 2mw = 2\nu kt_0^4 + 2\nu lw.$$
Since $H^8(EIII) = \langle t_0^4, w \rangle$ we have

\begin{equation}
\begin{cases}
1 - m = 2\nu k, \\
m = \nu l.
\end{cases}
\end{equation}

Hence $\nu = 1$ and we may take $k = 0$, $l = 1$. Then $m = 1$ and $b' = w$.

Similarly we may put

$$\tilde{\psi}(\sigma_4) = \cdots + n\sigma_2 \otimes \sigma_2 + \cdots$$

for some $n \in \mathbb{Z}$. On the other hand we can put

$$a_2^2 = \mu a_4 + \nu b_4$$

for some $\mu, \nu \in \mathbb{Z}$. Then

$$\mu = \langle a_2^2, \tau_4 \rangle = \langle a_2 \otimes a_2, \psi(\tau_4) \rangle = 2,$$

$$\nu = \langle a_2^2, \sigma_4 \rangle = \langle a_2 \otimes a_2, \psi(\sigma_4) \rangle = n.$$

But since $a_1^2 = a_2$, $a_1 a_2 = a_3$

$$a_2^2 = a_3 a_1 + 2a_4 + b_4.$$

Thus $n = \nu = 1$.

Hence we obtain the results given by Table 9 and

$$\tilde{\psi}(\tau_4) = 2\sigma_3 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_2,$$

$$\tilde{\psi}(\sigma_4) = \sigma_3 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2.$$

Note that for the element $\bar{\sigma}_4 = \tau_4 - 2\sigma_4$,

$$\tilde{\psi}(\bar{\sigma}_4) = 0.$$

We have to continue this argument up to degree $\leq 22$. But as well as the case of $F_4 ([111])$, the remainder is no more than a tedious computation. So we only exhibit the data and the results:

\begin{equation}
\tilde{\psi}(\sigma_3) = \tau_4 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_2,
\end{equation}
Table 10.

<table>
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<th>deg</th>
<th>basis</th>
<th>$g_s^*$-image</th>
<th>relation</th>
</tr>
</thead>
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<td>$a_5$</td>
<td>$c' = -t_0^3 + 2t_0 w$</td>
<td>$a_4a_1 = 5a_5 + 2b_5 + c_5$</td>
</tr>
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</tbody>
</table>

(5.5) \[ \tilde{\psi}(\sigma_7) = (-\sigma_6 + \sigma_1\sigma_5) \otimes \sigma_1 + \sigma_5 \otimes \sigma_2 + \tau_4 \otimes \sigma_3. \]

Note that for the elements $\tilde{\sigma}_5 = \sigma_1\tau_4 - 5\sigma_5$, $\tilde{\sigma}_7 = 4\sigma_1\tau_6 - 7\sigma_2\sigma_5 + 7\sigma_7$,

\[ \tilde{\psi}(\tilde{\sigma}_5) = 0, \]
\[ \tilde{\psi}(\tilde{\sigma}_7) = 0. \]

Since rank $H^{16}(EIII) = 3$, the computation for degree $= 16$ is a little complicated. So first we take

\[ E = \begin{pmatrix} 3 & -4 & -2 \\ 2 & -2 & -3 \\ -2 & 3 & 1 \end{pmatrix} \]

so that

\[ \begin{pmatrix} e' \\ e'' \end{pmatrix} = \begin{pmatrix} 3 & -4 & -2 \\ 2 & -2 & -3 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} t_0^8 \\ t_0^3 w \\ w^2 \end{pmatrix} = \begin{pmatrix} 3t_0^8 - 4t_0^4 w - 2w^2 \\ 2t_0^8 - 2t_0^4 w - 3w^2 \\ -2t_0^8 + 3t_0^4 w + w^2 \end{pmatrix}. \]

Then we obtain the results given by Table 11 and

(5.6) \[ \tilde{\psi}(\sigma_8) = (\sigma_1\sigma_4 + \sigma_2\sigma_5) \otimes \sigma_1 + (\tau_6 - \sigma_2\sigma_4 + \sigma_1\sigma_5) \otimes \sigma_2 + \sigma_4 \otimes \sigma_3 + 2\tau_4 \otimes \tau_4 - \sigma_4 \otimes \sigma_4. \]

Note that for the element $\tilde{\sigma}_8 = \sigma_2\tau_6 - 4\sigma_3\sigma_5 + 2\sigma_4^2 + 4\sigma_8$,

\[ \tilde{\psi}(\tilde{\sigma}_8) = 0. \]
Table 11.

<table>
<thead>
<tr>
<th>deg</th>
<th>basis</th>
<th>$g^*_\text{image}$</th>
<th>relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$a_8$</td>
<td>$-e' - 3e''$</td>
<td>$\sigma a_1 = 4a_8 + b_8 + c_8 - d_8$</td>
</tr>
<tr>
<td>16</td>
<td>$b_8$</td>
<td>$e' + e''$</td>
<td>$\beta a_1 = 2b_8 + 2d_8 - e_8$</td>
</tr>
<tr>
<td>16</td>
<td>$c_8$</td>
<td>$2e' + 8e''$</td>
<td>$\gamma a_1 = c_8 + 2d_8 + e_8$</td>
</tr>
<tr>
<td>16</td>
<td>$d_8$</td>
<td>$-e' - 3e''$</td>
<td>$\phi a_1 = d_8$</td>
</tr>
<tr>
<td>16</td>
<td>$d_8'$</td>
<td>$-e' - 2e''$</td>
<td>$a_4 = 70a_8 + 34b_8 + 28c_8 + 4d_8 + 17d_8' + 2e_8$</td>
</tr>
<tr>
<td>16</td>
<td>$e_8$</td>
<td>$e$</td>
<td>$a_4 b_4 = b_8$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b_4^2$</td>
<td>$= 2d_8' - e_8$</td>
</tr>
</tbody>
</table>

$$
\tilde{\psi}(\sigma_{11}) = (-2\tau_4\tau_6 + \sigma_4\tau_6 + 2\sigma_1\tau_4\sigma_5 - \sigma_3\sigma_7 - \sigma_1\sigma_4\sigma_5 - \sigma_5^2) \otimes \sigma_1

+ (-2\sigma_3\tau_6 + 4\tau_3\sigma_5 - \sigma_2\sigma_7 - \sigma_4\sigma_5) \otimes \sigma_2

+ (-\sigma_2\tau_6 - \tau_4\sigma_4 + 4\sigma_3\sigma_5 - \sigma_1\sigma_7) \otimes \sigma_3

- \sigma_3\sigma_4 \otimes \tau_4 + \sigma_2\sigma_5 \otimes \tau_4 - \sigma_7 \otimes \sigma_4 - \tau_6 \otimes \sigma_1\tau_4 + 4\tau_6 \otimes \sigma_5 + \tau_6 \otimes \sigma_1\sigma_4

- \sigma_2\sigma_4 \otimes \sigma_5 + \sigma_1\sigma_5 \otimes \sigma_1\tau_4 - \sigma_1\sigma_5 \otimes \sigma_1\sigma_4 - 2\sigma_1\sigma_5 \otimes \sigma_5.

(5.7)

Note that for the element $\tilde{\sigma}_{11} = 6\sigma_1\tau_4\tau_6 - 44\sigma_5\tau_6 + 11\sigma_1\sigma_5^2 + 11\sigma_4\sigma_7 + 11\sigma_{11}$,

$$
\tilde{\psi}(\tilde{\sigma}_{11}) = 0.
$$

Consequently we established Theorem 1.1.
Table 12.

<table>
<thead>
<tr>
<th>deg</th>
<th>basis</th>
<th>$g_3$-image</th>
<th>relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>$a_9$</td>
<td>$-18t_0^2w + 31t_0w^2$</td>
<td>$a_9a_1 = 3a_9 + 2b_9 + c_9 - d_9$</td>
</tr>
<tr>
<td>18</td>
<td>$b_9$</td>
<td>$4t_0^3w - 7t_0w^2$</td>
<td>$b_9a_1 = 5b_9 + 4d_9 - 2e_9 + e_9'$</td>
</tr>
<tr>
<td>18</td>
<td>$c_9$</td>
<td>$32t_0^3w - 55t_0w^2$</td>
<td>$c_9a_1 = 2c_9 + 3d_9 + 3e_9 + e_9'$</td>
</tr>
<tr>
<td>18</td>
<td>$d_9$</td>
<td>$-7t_0^5w + 12t_0w^2$</td>
<td>$d_9a_1 = d_9$</td>
</tr>
<tr>
<td>18</td>
<td>$d_9'$</td>
<td>$-4t_0^5w + 7t_0w^2$</td>
<td>$d_9'a_1 = d_9'$</td>
</tr>
<tr>
<td>18</td>
<td>$e_9$</td>
<td>$-4t_0^5w + 7t_0w^2$</td>
<td>$e_9a_1 = e_9$</td>
</tr>
<tr>
<td>18</td>
<td>$e_9'$</td>
<td>$-11t_0^5w + 19t_0w^2$</td>
<td>$e_9'a_1 = e_9'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_9a_4 = 42a_9 + 60b_9 + 25c_9 + 3d_9 + 29d_9'$ + $4e_9 + 12e_9'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9a_4 = 5b_9 + 4d_9' - 2e_9 + e_9'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_9a_4 = c_9 + 3d_9' + 2e_9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_9b_4 = b_9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9b_4 = 2d_9' - e_9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_9b_4 = e_9'$</td>
</tr>
<tr>
<td>20</td>
<td>$a_{10}$</td>
<td>$-22t_0^6w + 38t_0^2w^2$</td>
<td>$a_{10}a_1 = 5a_{10} + b_{10} + 3c_{10} - d_{10}$</td>
</tr>
<tr>
<td>20</td>
<td>$b_{10}$</td>
<td>$19t_0^6w - 33t_0^2w^2$</td>
<td>$b_{10}a_1 = b_{10} + 2d_{10}' - e_{10} + e_{10}'$</td>
</tr>
<tr>
<td>20</td>
<td>$c_{10}$</td>
<td>$22t_0^6w - 38t_0^2w^2$</td>
<td>$c_{10}a_1 = 5c_{10} + 2d_{10} + 3e_{10} + 2e_{10}' + 2e_{10}''$</td>
</tr>
<tr>
<td>20</td>
<td>$d_{10}$</td>
<td>$-7t_0^8w + 12t_0^2w^2$</td>
<td>$d_{10}a_1 = d_{10}$</td>
</tr>
<tr>
<td>20</td>
<td>$d_{10}'$</td>
<td>$-4t_0^8w + 7t_0^2w^2$</td>
<td>$d_{10}'a_1 = d_{10}'$</td>
</tr>
<tr>
<td>20</td>
<td>$e_{10}$</td>
<td>$-4t_0^8w + 7t_0^2w^2$</td>
<td>$e_{10}a_1 = e_{10}$</td>
</tr>
<tr>
<td>20</td>
<td>$e_{10}'$</td>
<td>$-11t_0^8w + 19t_0^2w^2$</td>
<td>$e_{10}'a_1 = e_{10}'$</td>
</tr>
<tr>
<td>20</td>
<td>$e_{10}''$</td>
<td>$-15t_0^8w + 26t_0^2w^2$</td>
<td>$e_{10}''a_1 = e_{10}''$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_9a_{10} = 210a_{10} + 97b_{10} + 246c_{10} + 6d_{10}$ + $135d_{10}' + 21e_{10} + 114e_{10}' + 48e_{10}''$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9a_{10} = 5b_{10} + 14d_{10}' - 7e_{10} + 6e_{10}'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_9a_{10} = 5c_{10} + 5d_{10} + 5e_{10} + 2e_{10}' + 2e_{10}''$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_9b_{10} = b_{10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9b_{10} = 2d_{10}' - e_{10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_9b_{10} = e_{10}'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_9s^2 = 42a_{10} + 20b_{10} + 50c_{10} + 2d_{10}$ + $29d_{10}' + 4e_{10} + 24e_{10}' + 10e_{10}''$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9s = b_{10} + 2d_{10}' - e_{10} + e_{10}'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_9c_5 = c_{10} + d_{10} + e_{10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9s = 2d_{10}' - e_{10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_9c_5 = e_{10}'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_5^2 = 2e_{10}''$</td>
</tr>
</tbody>
</table>
Table 13.

<table>
<thead>
<tr>
<th>deg</th>
<th>basis</th>
<th>$g_3^*$-image</th>
<th>relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>$a_{11}$</td>
<td>0</td>
<td>$a_{10}a_{11} = 11a_{11} + 2b_{11} + c_{11} - d_{11} - 2f_{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$b_{11}$</td>
<td>0</td>
<td>$b_{10}a_{11} = 7b_{11} + 4d'<em>{11} - 2e</em>{11} + 3e'<em>{11} - e''</em>{11} + f_{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$c_{11}$</td>
<td>$-f'$</td>
<td>$c_{10}a_{11} = c_{11} + d_{11} + e_{11} + e'<em>{11} + 2e''</em>{11} + 2f_{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$d_{11}$</td>
<td>$-f'$</td>
<td>$d_{10}a_{11} = 2d_{11} + e''<em>{11} - f</em>{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$d'_{11}$</td>
<td>$f'$</td>
<td>$d'<em>{10}a</em>{11} = d'_{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$e_{11}$</td>
<td>$f'$</td>
<td>$e_{10}a_{11} = e_{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$e'_{11}$</td>
<td>$-f'$</td>
<td>$e'<em>{10}a</em>{11} = e'<em>{11} + e''</em>{11} - f_{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$e''_{11}$</td>
<td>$f'$</td>
<td>$e''<em>{10}a</em>{11} = e''<em>{11} - f</em>{11}$</td>
</tr>
<tr>
<td>22</td>
<td>$f_{11}$</td>
<td>$f$</td>
<td>$f_{10}a_{11} = 330a_{11} + 147b_{11} + 636c_{11} + d_{11}$ $+65d'<em>{11} + 11e</em>{11} + 84e'<em>{11} + 72e''</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_{11}a_{11} = 35b_{11} + 34d'<em>{11} - 17e</em>{11} + 21e'<em>{11}$ $+e''</em>{11} - f_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_{11}a_{11} = 5c_{11} + 14d_{11} + 10e_{11} + 7e'<em>{11} + 12e''</em>{11}$ $+7e'''<em>{11} + f</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$d_{11}a_{11} = d_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_{11}b_{11} = b_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_{11}b_{11} = 2d'<em>{11} - e</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_{11}b_{11} = e''_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$d_{11}b_{11} = e'''<em>{11} - f</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_{11}c_{11} = 462a_{11} + 217b_{11} + 91c_{11} + 9d_{11} + 103d'<em>{11}$ $+15e</em>{11} + 129e'<em>{11} + 108e''</em>{11} + 5e'''<em>{11} - f</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_{11}c_{11} = 7b_{11} + 6d'<em>{11} - 3e</em>{11} + 4e'_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_{11}c_{11} = c_{11} + 3d_{11} + 2e_{11} + e'<em>{11} + 2e''</em>{11} + e'''<em>{11} + f</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_{11}d_{11} = 7b_{11} + 4d'<em>{11} - 2e</em>{11} + 3e'<em>{11} - e''</em>{11} + f_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_{11}d_{11} = 2d'<em>{11} - e</em>{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_{11}d_{11} = e'<em>{11} + e''</em>{11} - f_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a_{11}c_{11} = c_{11} + 3d_{11} + 2e_{11} + 4f_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$b_{11}c_{11} = e'<em>{11} + e''</em>{11} - f_{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$c_{11}c_{11} = 2e''<em>{11} - 2f</em>{11}$</td>
</tr>
</tbody>
</table>

where $f = 11t_0^7w - 19t_0^3w^2$, $f' = -4t_0^7w + 7t_0^3w^2$. 

References


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