



Title	The space of loops on the exceptional Lie group E_6
Author(s)	Nakagawa, Masaki
Citation	Osaka Journal of Mathematics. 2003, 40(2), p. 429-448
Version Type	VoR
URL	https://doi.org/10.18910/4329
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

THE SPACE OF LOOPS ON THE EXCEPTIONAL LIE GROUP E_6

MASAKI NAKAGAWA

(Received November 15, 2001)

1. Introduction

Let G be a compact 1-connected simple Lie group and ΩG the space of loops on G . As is well known ΩG is a homotopy commutative H-space and its integral homology $H_*(\Omega G)$ has no torsion and no odd dimensional part ([2]). Therefore $H_*(\Omega G)$ becomes a commutative Hopf algebra over the integers $\mathbb{Z}$. In [3] R. Bott introduced a “generating variety” and determined the Hopf algebra structure of $H_*(\Omega G)$ explicitly for $G = SU(n)$, $Spin(n)$ and G_2 . In [11] T. Watanabe determined $H_*(\Omega F_4)$ in a similar way. On the other hand A. Kono and K. Kozima determined $H_*(\Omega Sp(n))$ by different method using the Bott periodicity ([6]).

In this paper we carry out the Bott’s program for $G = E_6$, where E_6 is the compact 1-connected exceptional Lie group of rank 6 and determine the Hopf algebra structure of $H_*(\Omega E_6)$ explicitly.

Let ψ be the coproduct of $H_*(\Omega G)$ induced by the diagonal map $\Omega G \longrightarrow \Omega G \times \Omega G$. To avoid the cumbersome notation, following [11] we introduce a map $\tilde{\psi}: H_*(\Omega G) \longrightarrow H_*(\Omega G) \otimes H_*(\Omega G)$ satisfying

$$\psi(\sigma) - (\sigma \otimes 1 + 1 \otimes \sigma) = \tilde{\psi}(\sigma) + T\tilde{\psi}(\sigma) \quad \text{for } \sigma \in H_*(\Omega G)$$

where $T: H_*(\Omega G) \otimes H_*(\Omega G) \longrightarrow H_*(\Omega G) \otimes H_*(\Omega G)$ is defined by

$$T(\sigma \otimes \tau) = \begin{cases} \tau \otimes \sigma & \text{for } \sigma \neq \tau, \\ 0 & \text{for } \sigma = \tau. \end{cases}$$

Note that $\tilde{\psi}(\sigma) = 0$ if and only if $\sigma \in PH_*(\Omega G)$, where $PH_*(\Omega G)$ denotes the primitive module of the Hopf algebra $H_*(\Omega G)$.

Then our main results are stated as follows:

Theorem 1.1. *The Hopf algebra structure of $H_*(\Omega E_6)$ is given as follows:*

(i) *As an algebra*

$$H_*(\Omega E_6) = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8, \sigma_{11}] / (\sigma_1^2 - 2\sigma_2, \sigma_1\sigma_2 - 3\sigma_3)$$

where $\deg(\sigma_i) = 2i$.

(ii) With suitably chosen generators σ_i , $i = 1, 2, 3, 4, 5, 7, 8, 11$, the coproduct is given by

$$\begin{aligned}
 \psi(\sigma_k) &= \sum_{i+j=k} \sigma_i \otimes \sigma_j \quad \text{for } k = 1, 2, 3, 4 \quad (\sigma_0 = 1), \\
 \tilde{\psi}(\sigma_5) &= \tau_4 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_2, \\
 \tilde{\psi}(\sigma_7) &= (-\tau_6 + \sigma_1\sigma_5) \otimes \sigma_1 + \sigma_5 \otimes \sigma_2 + \tau_4 \otimes \sigma_3, \\
 \tilde{\psi}(\sigma_8) &= (-\sigma_3\sigma_4 + \sigma_2\sigma_5) \otimes \sigma_1 + (\tau_6 - \sigma_2\sigma_4 + \sigma_1\sigma_5) \otimes \sigma_2 \\
 &\quad + (\sigma_1\tau_4 + \sigma_1\sigma_4 + \sigma_5) \otimes \sigma_3 + 2\tau_4 \otimes \tau_4 - \sigma_4 \otimes \sigma_4, \\
 \tilde{\psi}(\sigma_{11}) &= (-2\tau_4\tau_6 + \sigma_4\tau_6 + 2\sigma_1\tau_4\sigma_5 - \sigma_3\sigma_7 - \sigma_1\sigma_4\sigma_5 - \sigma_5^2) \otimes \sigma_1 \\
 &\quad + (-2\sigma_3\tau_6 + 4\tau_4\sigma_5 - \sigma_2\sigma_7 - \sigma_4\sigma_5) \otimes \sigma_2 \\
 &\quad + (-\sigma_2\tau_6 - \tau_4\sigma_4 + 4\sigma_3\sigma_5 - \sigma_1\sigma_7) \otimes \sigma_3 \\
 &\quad - \sigma_3\sigma_4 \otimes \tau_4 + \sigma_2\sigma_5 \otimes \tau_4 - \sigma_7 \otimes \sigma_4 - \tau_6 \otimes \sigma_1\tau_4 + 4\tau_6 \otimes \sigma_5 \\
 &\quad + \tau_6 \otimes \sigma_1\sigma_4 - \sigma_2\sigma_4 \otimes \sigma_5 + \sigma_1\sigma_5 \otimes \sigma_1\tau_4 - \sigma_1\sigma_5 \otimes \sigma_1\sigma_4 - 2\sigma_1\sigma_5 \otimes \sigma_5
 \end{aligned}$$

where $\tau_4 = \sigma_2^2 - \sigma_1\sigma_3$, $\tau_6 = \sigma_2^3 - 4\sigma_3^2$.

$$(iii) \quad PH_*(\Omega E_6) = \langle \sigma_1, \tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_7, \tilde{\sigma}_8, \tilde{\sigma}_{11} \rangle$$

as a free module, where

$$\begin{aligned}
 \tilde{\sigma}_4 &= \tau_4 - 2\sigma_4, \\
 \tilde{\sigma}_5 &= \sigma_1\tau_4 - 5\sigma_5, \\
 \tilde{\sigma}_7 &= 4\sigma_1\tau_6 - 7\sigma_2\sigma_5 + 7\sigma_7, \\
 \tilde{\sigma}_8 &= \sigma_2\tau_6 - 4\sigma_3\sigma_5 + 2\sigma_4^2 + 4\sigma_8, \\
 \tilde{\sigma}_{11} &= 6\sigma_1\tau_4\tau_6 - 44\sigma_5\tau_6 + 11\sigma_1\sigma_5^2 + 11\sigma_4\sigma_7 + 11\sigma_{11}.
 \end{aligned}$$

The paper is organized as follows: In §2 we prepare various results which are needed in later sections. In §3 we carry out the Bott's program for E_6 and determine the primitive elements of $H^*(\Omega E_6)$. In §4 we determine the algebra structure of $H_*(\Omega E_6)$ depending on the results of mod p homology for all primes p . In the last section, §5 we determine the Hopf algebra structure of $H_*(\Omega E_6)$ explicitly using the "generating variety". Throughout this paper $\sigma_i(x_1, \dots, x_n)$ denotes the i -th elementary symmetric function in the variables $x_1, \dots, x_n$.

I would like to thank Professors Takashi Watanabe and Akira Kono for much helpful advice and encouragement.

2. Preliminaries

Let T be a maximal torus of E_6 and $\{\alpha_i\}_{1 \leq i \leq 6}$ the root system given in [4], where the corresponding fundamental weights are given as follows:

$$\begin{aligned}
 (2.1) \quad w_1 &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6), \\
 w_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\
 w_3 &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6), \\
 w_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6, \\
 w_5 &= \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6), \\
 w_6 &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6).
 \end{aligned}$$

As usual we regard roots and weights as elements of $H^1(T) \xrightarrow{\sim} H^2(BT)$. Then $\{w_i\}_{1 \leq i \leq 6}$ forms a basis of $H^2(BT)$ and $H^*(BT) = \mathbb{Z}[w_1, w_2, \dots, w_6]$. Let R_i denote the reflection to the hyperplane $\alpha_i = 0$, then the Weyl group of E_6 , $W(E_6)$, is a finite group generated by R_i ($1 \leq i \leq 6$) and the action on $\{w_i\}_{1 \leq i \leq 6}$ is given as follows:

$$(2.2) \quad R_i(w_i) = w_i - \sum_{j=1}^6 \frac{2(\alpha_i | \alpha_j)}{(\alpha_j | \alpha_j)} w_j, \quad R_j(w_i) = w_i \quad \text{for } j \neq i.$$

Following [10], we introduce the elements

$$\begin{aligned}
 t_6 &= w_6, \\
 t_5 &= R_6(t_6) = w_5 - w_6, \\
 t_4 &= R_5(t_5) = w_4 - w_5, \\
 t_3 &= R_4(t_4) = w_2 + w_3 - w_4, \\
 t_2 &= R_3(t_3) = w_1 + w_2 - w_3, \\
 t_1 &= R_1(t_2) = -w_1 + w_2, \\
 c_i &= \sigma_i(t_1, \dots, t_6), \quad t = \frac{1}{3}c_1 = w_2.
 \end{aligned}$$

Then we have the following isomorphism:

$$H^*(BT) = \mathbb{Z}[t_1, t_2, \dots, t_6, t]/(c_1 - 3t).$$

From (2.2) the action of $W(E_6)$ on these elements is given by Table 1, where blanks indicate the trivial action.

Table 1.

	R_1	R_2	R_3	R_4	R_5	R_6
t_1	t_2	$t - t_2 - t_3$				
t_2	t_1	$t - t_1 - t_3$	t_3			
t_3		$t - t_1 - t_2$	t_2	t_4		
t_4				t_3	t_5	
t_5					t_4	t_6
t_6						t_5
t		$-t + t_4 + t_5 + t_6$				

Next as in [10] we put

$$x_i = 2t_i - t \quad (1 \leq i \leq 6).$$

From Table 1 we see easily that

$$S = \{x_i + x_j \ (1 \leq i < j \leq 6), \ t - x_i, -t - x_i \ (1 \leq i \leq 6)\} \subset H^2(BT; \mathbb{Q})$$

is invariant under the action of $W(E_6)$. (In fact S is an orbit of $2w_1$ under the action of $W(E_6)$) Thus we have $W(E_6)$ -invariant forms

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_6)}.$$

As in [10] I_n is computed by the formula:

$$(2.3) \quad I_n = \frac{1}{2} \sum_{i=2}^{n-2} \binom{n}{i} s_i s_{n-i} + (6 - 2^{n-1}) s_n + 2(-1)^n \sum_{j=0}^{[n/2]} \binom{n}{2j} s_{n-2j} t^{2j}$$

where $s_n = x_1^n + \cdots + x_6^n$ and s_n is written as a polynomial in d_i 's, $d_i = \sigma_i(x_1, \dots, x_6)$, by use of the Newton formula:

$$(2.4) \quad s_n = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-i} d_i + (-1)^{n-1} n d_n \quad (d_n = 0 \text{ for } n > 6).$$

Moreover since $x_i = 2t_i - t$ for $1 \leq i \leq 6$

$$\sum_{n=0}^6 d_n = \prod_{i=1}^6 (1 + x_i) = \prod_{i=1}^6 (1 - t + 2t_i) = \sum_{i=0}^6 (1 - t)^{6-i} 2^i c_i$$

and we have

$$(2.5) \quad d_n = \sum_{i=0}^n \binom{6-i}{n-i} (-t)^{n-i} 2^i c_i.$$

Table 2.

	R_2	R_3	R_4	R_5	R_6
ϵ_1	$-\epsilon_2$	ϵ_2			
ϵ_2	$-\epsilon_1$	ϵ_1	ϵ_3		
ϵ_3			ϵ_2	ϵ_4	
ϵ_4				ϵ_3	ϵ_5
ϵ_5					ϵ_4
t_0					

By using (2.3), (2.4), (2.5) I_n can be written as a polynomial in t and c_i 's (for details see [10], §5).

Then the next lemma is proved in [10], Lemma 5.2:

Lemma 2.1. *The rational invariant subalgebra of the Weyl group $W(E_6)$ is given as follows:*

$$H^*(BT; \mathbb{Q})^{W(E_6)} = \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}].$$

Let C_1 be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ ($i \neq 1$). Then as is well known

$$C_1 = T^1 \cdot \text{Spin}(10), \quad T^1 \cap \text{Spin}(10) \cong \mathbb{Z}/4\mathbb{Z}$$

and the Weyl group of C_1 , $W(C_1)$, is generated by R_i ($i \neq 1$).

REMARK 2.2. The homogeneous space E_6/C_1 is the irreducible Hermitian symmetric space and denoted by $EIII$ in E . Cartan's notation.

We put

$$(2.6) \quad t_0 = t - t_1 = w_1, \quad \epsilon_i = t_{i+1} - \frac{1}{2}t_0 \quad (1 \leq i \leq 5).$$

Then we have

$$H^*(BT; \mathbb{Q}) = \mathbb{Q}[t_0, \epsilon_1, \epsilon_2, \dots, \epsilon_5]$$

and the action of $W(C_1)$ is given by Table 2.

From Table 2 we have

Lemma 2.3. *The rational invariant subalgebra of the Weyl group $W(C_1)$ is given as follows:*

$$H^*(BT; \mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$$

where

$$p_i = \sigma_i(\epsilon_1^2, \dots, \epsilon_5^2), \quad e = \prod_{i=1}^5 \epsilon_i.$$

We shall find the relations between p_i 's and I_n 's. Put

$$b_i = \sigma_i(\epsilon_1, \dots, \epsilon_5)$$

so that

$$\begin{aligned} \sum_{i \geq 0} (-1)^i p_i &= \prod_{j=1}^5 (1 - \epsilon_j^2) = \prod_{j=1}^5 (1 + \epsilon_j) \prod_{j=1}^5 (1 - \epsilon_j) \\ &= \left(\sum_{k \geq 0} b_k \right) \left(\sum_{l \geq 0} (-1)^l b_l \right) = \sum_{k, l \geq 0} (-1)^l b_k b_l. \end{aligned}$$

Therefore

$$(2.7) \quad p_i = \sum_{k+l=2i} (-1)^{l+i} b_k b_l.$$

On the other hand since $t_i = \epsilon_{i-1} + (1/2)t_0$ for $2 \leq i \leq 6$

$$\begin{aligned} \sum_{n=0}^6 c_n &= \prod_{i=1}^6 (1 + t_i) = (1 + t_1) \prod_{i=2}^6 (1 + t_i) \\ &= (1 + t_1) \prod_{i=1}^5 \left(1 + \frac{1}{2}t_0 + \epsilon_i \right) = (1 + t_1) \sum_{i=0}^5 \left(1 + \frac{1}{2}t_0 \right)^{5-i} b_i \end{aligned}$$

and we have

$$(2.8) \quad c_n = \sum_{i=0}^n \binom{5-i}{n-i} \left(\frac{1}{2}t_0 \right)^{n-i} b_i + \sum_{i=0}^{n-1} \binom{5-i}{n-i-1} \left(\frac{1}{2}t_0 \right)^{n-i-1} t_1 b_i.$$

Since $I_n \in H^*(BT; \mathbb{Q})^{W(E_6)} \subset H^*(BT; \mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$, I_n can be written as a polynomial in $t_0, p_1, p_2, e, p_3, p_4$. Direct computation using (2.3), (2.4), (2.5), (2.7) and (2.8) then yields the following results:

Lemma 2.4. In $H^*(BT; \mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ we have

$$I_2 = 2^4 \cdot 3 \left(\frac{1}{2}p_1 + \frac{3}{8}t_0^2 \right),$$

$$\begin{aligned}
I_5 &= -2^7 \cdot 3 \cdot 5 \left\{ e + \frac{1}{4} \left(-p_2 + \frac{1}{4} p_1^2 \right) t_0 + \frac{1}{32} p_1 t_0^3 - \frac{3}{256} t_0^5 \right\}, \\
I_6 &= 2^9 \cdot 3^2 \left\{ \frac{1}{4} p_3 - \frac{1}{24} p_1 p_2 + \frac{1}{32} p_1^3 - \frac{5}{4} e t_0 + \frac{5}{32} \left(-p_2 + \frac{3}{4} p_1^2 \right) t_0^2 + \frac{15}{512} p_1 t_0^4 \right. \\
&\quad \left. + \frac{33}{2048} t_0^6 \right\}, \\
I_8 &= -2^{12} \cdot 3 \cdot 5 \left\{ -\frac{1}{4} p_4 - \frac{1}{48} p_2^2 - \frac{1}{10} p_1 p_3 + \frac{13}{480} p_1^2 p_2 - \frac{11}{1280} p_1^4 + \frac{7}{8} e p_1 t_0 \right. \\
&\quad + \frac{7}{80} \left(-3 p_3 + \frac{7}{4} p_1 p_2 - \frac{11}{16} p_1^3 \right) t_0^2 + \frac{7}{32} e t_0^3 + \frac{7}{512} \left(5 p_2 - \frac{11}{4} p_1^2 \right) t_0^4 \\
&\quad \left. - \frac{77}{20480} p_1 t_0^6 - \frac{1419}{327680} t_0^8 \right\}, \\
I_9 &= 2^{11} \cdot 3^3 \cdot 7 \left\{ -\frac{1}{6} e p_2 - \frac{1}{8} e p_1^2 + \frac{1}{4} \left(p_4 - \frac{1}{12} p_2^2 + \frac{5}{24} p_1^2 p_2 - \frac{3}{64} p_1^4 \right) t_0 \right. \\
&\quad - \frac{5}{16} e p_1 t_0^2 + \frac{1}{16} \left(-p_3 + \frac{17}{12} p_1 p_2 - \frac{7}{16} p_1^3 \right) t_0^3 - \frac{5}{128} e t_0^4 \\
&\quad \left. + \frac{1}{512} \left(11 p_2 - \frac{21}{4} p_1^2 \right) t_0^5 - \frac{3}{4096} p_1 t_0^7 + \frac{85}{65536} t_0^9 \right\}, \\
I_{12} &= 2^{15} \cdot 3^4 \cdot 5 \left\{ \frac{1}{180} p_3^2 + \frac{1}{24} p_2 p_4 - \frac{1}{864} p_2^3 - \frac{1}{540} p_1 p_2 p_3 + \frac{41}{864} p_1^2 p_4 \right. \\
&\quad + \frac{65}{108} e^2 p_1 + \frac{101}{17280} p_1^2 p_2^2 + \frac{13}{2160} p_1^3 p_3 - \frac{251}{69120} p_1^4 p_2 + \frac{19}{30720} p_1^6 \\
&\quad + \frac{11}{12} \left(-\frac{1}{3} e p_3 - \frac{1}{4} e p_1 p_2 - \frac{1}{16} e p_1^3 \right) t_0 + \frac{11}{16} \left(\frac{11}{9} e^2 - \frac{1}{18} p_2 p_3 + \frac{7}{36} p_1 p_4 \right. \\
&\quad + \frac{11}{144} p_1 p_2^2 + \frac{7}{72} p_1^2 p_3 - \frac{7}{96} p_1^3 p_2 + \frac{19}{1280} p_1^5 \left. \right) t_0^2 + \frac{77}{192} \left(-\frac{1}{3} e p_2 - \frac{1}{4} e p_1^2 \right) t_0^3 \\
&\quad + \frac{11}{128} \left(-\frac{5}{12} p_4 + \frac{65}{144} p_2^2 + p_1 p_3 - \frac{253}{288} p_1^2 p_2 + \frac{57}{256} p_1^4 \right) t_0^4 \\
&\quad - \frac{77}{3072} e p_1 t_0^5 + \frac{77}{10240} \left(\frac{11}{3} p_3 - \frac{127}{36} p_1 p_2 + \frac{19}{16} p_1^3 \right) t_0^6 - \frac{11}{12288} e t_0^7 \\
&\quad \left. + \frac{11}{131072} \left(-\frac{85}{3} p_2 + \frac{57}{4} p_1^2 \right) t_0^8 + \frac{209}{5242880} p_1 t_0^{10} + \frac{12977}{41943040} t_0^{12} \right\}.
\end{aligned}$$

Let $\mathfrak{b}_i$ denote the ideal in $H^*(BT; \mathbb{Q})^{W(C_1)}$ generated by I_j 's for $j < i$, $j \in 2, 5, 6, 8, 9, 12$. By Lemma 2.4 we have immediately the following (here we put $w = (1/6)p_2 + (9/16)t_0^4$):

Lemma 2.5. In $H^*(BT; \mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ we have

$$\begin{aligned} p_1 &\equiv -\frac{3}{4}t_0^2 \pmod{\mathfrak{b}_5}, \\ e &\equiv \frac{3}{2}t_0w - \frac{27}{32}t_0^5 \pmod{\mathfrak{b}_6}, \\ p_3 &\equiv \frac{21}{2}t_0^2w - \frac{195}{32}t_0^6 \pmod{\mathfrak{b}_8}, \\ p_4 &\equiv -3w^2 - \frac{63}{8}t_0^4w + \frac{1413}{256}t_0^8 \pmod{\mathfrak{b}_9}. \end{aligned}$$

Using Lemmas 2.4, 2.5 we obtain

Lemma 2.6. In $H^*(BT; \mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ we have

$$\begin{aligned} I_9 &\equiv 2^{11} \cdot 3^3 \cdot 7(t_0^9 - 3t_0w^2) \pmod{\mathfrak{b}_9}, \\ I_{12} &\equiv 2^{15} \cdot 3^4 \cdot 5(-3t_0^{12} + 9t_0^8w - 6t_0^4w^2 - w^3) \pmod{\mathfrak{b}_9}. \end{aligned}$$

Therefore by the classical results of A. Borel [1] we obtain the following rational cohomology ring of $EIII$:

$$\begin{aligned} H^*(EIII; \mathbb{Q}) &\cong H^*(BT; \mathbb{Q})^{W(C_1)} / \mathfrak{b}_{13} \\ &= \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4] / (I_2, I_5, I_6, I_8, I_9, I_{12}) \\ &\cong \mathbb{Q}[t_0, w] / (t_0^9 - 3t_0w^2, w^3 + 15t_0^4w^2 - 9t_0^8w). \end{aligned}$$

Furthermore the integral cohomology ring of $EIII$ is determined in [10], Corollary C:

Theorem 2.7. The integral cohomology ring of $EIII$ is given as follows:

$$H^*(EIII) = \mathbb{Z}[t_0, w] / (t_0^9 - 3t_0w^2, w^3 + 15t_0^4w^2 - 9t_0^8w)$$

where t_0, w are as above.

Corollary 2.8. An additive basis of $H^*(EIII)$ as a free module for degree ≤ 22 is given by Table 3.

3. The primitive elements of $H^*(\Omega E_6)$

In this section we determine the primitive elements of $H^*(\Omega E_6)$ which are needed for determination of the Hopf algebra structure of $H_*(\Omega E_6)$.

As is well known, over the rationals E_6 looks like the product of odd dimensional

Table 3.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	t_0	t_0^2	t_0^3	t_0^4	t_0^5	t_0^6	t_0^7	t_0^8			
					w	$t_0 w$	$t_0^2 w$	$t_0^3 w$	$t_0^4 w$	$t_0^5 w$	$t_0^6 w$	$t_0^7 w$
									w^2	$t_0 w^2$	$t_0^2 w^2$	$t_0^3 w^2$

spheres:

$$S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}.$$

Hence the rational cohomology ring of ΩE_6 is given by

$$H^*(\Omega E_6; \mathbb{Q}) = \mathbb{Q}[u_1, u_4, u_5, u_7, u_8, u_{11}]$$

where $\deg(u_i) = 2i$. Furthermore we can choose the generators u_i 's as integral classes such that they are not divisible and primitive in $H^*(\Omega E_6)$. Thus

$$PH^*(\Omega E_6) = \langle u_1, u_4, u_5, u_7, u_8, u_{11} \rangle$$

as a free module.

Now we briefly review the results of R. Bott (for details see [3]): Let G be a compact connected semisimple Lie group with trivial center and ΩG the space of loops on G . Suppose that a homomorphism $s: S^1 \rightarrow G$ is given. Let T be a maximal torus of G containing the image of s and C_s the centralizer of the image of s in G . Define the map

$$g_s: G/C_s \rightarrow \Omega_0 G$$

by $g_s(q)(t) = xs(t)x^{-1}s(t)^{-1}$ for $q = xC_s \in G/C_s$, $t \in S^1$ where $\Omega_0 G$ denotes the identity component of ΩG .

Let $\tilde{G}$ be the universal covering group of G and d the covering dimension. Note that $\Omega_0 G \cong \Omega \tilde{G}$. Let $\tilde{s}: S^1 \rightarrow \tilde{G}$ be a lift of s and $C_{\tilde{s}}$ the centralizer of the image of $\tilde{s}$ in $\tilde{G}$. Note that $G/C_s \cong \tilde{G}/C_{\tilde{s}}$. Define the map

$$f_s: \tilde{G}/C_{\tilde{s}} \rightarrow \Omega \tilde{G}$$

by $f_s(q)(t) = xs(t)x^{-1}$ for $q = xC_{\tilde{s}} \in \tilde{G}/C_{\tilde{s}}$.

Accoding to Bott, if s is the “generating circle” the image of the homomorphism

$$g_{s*}: H_*(G/C_s) \rightarrow H_*(\Omega_0 G)$$

generates $H_*(\Omega_0 G)$ as an algebra. (In this case g_s is called the “generating map” and G/C_s the “generating variety”) Dual statement is as follows:

$$g_s^*: H^*(\Omega_0 G) \rightarrow H^*(G/C_s)$$

is a split monomorphism when restricted to $PH^*(\Omega_0 G)$. Moreover since $\Omega_0 G \cong \Omega \tilde{G}$, $G/C_s \cong \tilde{G}/C_{\tilde{s}}$ we can compare the images of g_s^* and f_s^* . Then the following holds ([3], Proposition 7.1):

$$f_s^*|_{PH^*(\Omega \tilde{G})} = d \cdot g_s^*|_{PH^*(\Omega_0 G)}.$$

Bott also gave the method of computing the image of f_s^* by means of the Borel description of the rational cohomology ring of $\tilde{G}/C_{\tilde{s}}$ (see [3], Theorem 4). We carry out his program in the case of E_6 .

As is well known E_6 has non-trivial center and it is of order 3. Hence the generating circle is defined in the adjoint group $AdE_6 = E_6/\text{center}$. Since AdE_6 is simply laced, all minimal circles are generating circles ([3], page 43). So we take

$$s = \text{the dual of } \alpha_1.$$

Then the centralizer of the one dimensional torus $\text{Im } s$ is $C_1 = T^1 \cdot \text{Spin}(10)$ and the generating variety corresponding to s is

$$E_6/C_1 = E_6/T^1 \cdot \text{Spin}(10) = EIII.$$

In view of (2.1), (2.6) the derivation corresponding to s is

$$(3.1) \quad \theta_s = \frac{4}{3} \frac{\partial}{\partial t_0} : \mathbb{Q}[t_0, \epsilon_1, \dots, \epsilon_5] \longrightarrow \mathbb{Q}[t_0, \epsilon_1, \dots, \epsilon_5].$$

We have to compute the image of the composition

$$(3.2) \quad H^*(BT; \mathbb{Q})^{W(E_6)} \hookrightarrow H^*(BT; \mathbb{Q})^{W(C_1)} \xrightarrow{\theta_s} H^*(BT; \mathbb{Q})^{W(C_1)} \xrightarrow{\iota} H^*(EIII; \mathbb{Q})$$

where ι is natural surjection.

Because θ_s is a derivation and $H^*(BT; \mathbb{Q})^{W(C_1)}/\mathfrak{b}_{13} \xrightarrow{\iota} H^*(EIII; \mathbb{Q})$ we have only to compute the image of I_n , $n = 2, 5, 6, 8, 9, 12$. The computation proceeds as follows:

1. First we apply (3.1) to Lemma 2.4 and obtain $\theta_s(I_n)$ for $n = 2, 5, 6, 8, 9, 12$.
2. Next using Lemma 2.5, 2.6 we rewrite $\iota(\theta_s(I_n))$ in terms of t_0, w .

Then we obtain the results given by Table 4.

Therefore by a characterization of primitive elements given in [11], Proposition 5 (ii) we obtain

Proposition 3.1. *There exist unique primitive elements a_1 (resp. $b_4, c_5, d_7, e_8, f_{11}$) of $H^*(\Omega E_6)$ such that $g_s^*(a_1) = a$ (resp. $g_s^*(b_4) = b, g_s^*(c_5) = c, g_s^*(d_7) = d, g_s^*(e_8) = e, g_s^*(f_{11}) = f$) and*

$$PH^*(\Omega E_6) = \langle a_1, b_4, c_5, d_7, e_8, f_{11} \rangle$$

Table 4.

n	$\iota(\theta_s(I_n))$	deg	
2	$2^4 \cdot 3a$	2	$a = t_0$
5	$-2^7 \cdot 3 \cdot 5b$	8	$b = t_0^4 - 2w$
6	$2^9 \cdot 3^2c$	10	$c = 3t_0^5 - 5t_0w$
8	$-2^{12} \cdot 3 \cdot 5d$	14	$d = 4t_0^7 - 7t_0^3w$
9	$2^{11} \cdot 3^3 \cdot 7e$	16	$e = 3t_0^8 - 4t_0^4w - 2w^2$
12	$2^{15} \cdot 3^4 \cdot 5f$	22	$f = 11t_0^7w - 19t_0^3w^2$

Table 5.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	a	t_0^2	t_0^3	b	c	t_0^6	d	e			
					b'	c'	t_0^2w	d'	e'	t_0^5w	t_0^6w	f
									e''	t_0w^2	$t_0^2w^2$	f'

as a free module.

4. The algebra structure of $H_*(\Omega E_6)$

The purpose of this section is to determine the algebra structure of $H_*(\Omega E_6)$.

First note that the rational homology ring $H_*(\Omega E_6; \mathbb{Q})$ is of the form

$$(4.1) \quad H_*(\Omega E_6; \mathbb{Q}) = \mathbb{Q}[x_2, x_8, x_{10}, x_{14}, x_{16}, x_{22}]$$

where $\deg(x_i) = i$ ([9]).

By Corollary 2.8 we can choose an additive basis of $H^*(EIII)$ for degree ≤ 22 as in Table 5, where a, b, c, d, e and f are as in Proposition 3.1 and b', c', d', e', f' and e'' are determined by the following equations:

$$\begin{aligned} B \cdot \begin{pmatrix} t_0^4 \\ w \end{pmatrix} &= \begin{pmatrix} b \\ b' \end{pmatrix}, \quad C \cdot \begin{pmatrix} t_0^5 \\ t_0w \end{pmatrix} = \begin{pmatrix} c \\ c' \end{pmatrix}, \quad D \cdot \begin{pmatrix} t_0^7 \\ t_0^3w \end{pmatrix} = \begin{pmatrix} d \\ d' \end{pmatrix}, \\ E \cdot \begin{pmatrix} t_0^8 \\ t_0^4w \\ w^2 \end{pmatrix} &= \begin{pmatrix} e \\ e' \\ e'' \end{pmatrix}, \quad F \cdot \begin{pmatrix} t_0^7w \\ t_0^3w^2 \end{pmatrix} = \begin{pmatrix} f \\ f' \end{pmatrix} \end{aligned}$$

where B, C, D, F (resp. E) are 2×2 (resp. 3×3) matrices over $\mathbb{Z}$ whose determinant is 1; for example

$$(4.2) \quad B = \begin{pmatrix} 1 & -2 \\ k & l \end{pmatrix}$$

with $k, l \in \mathbb{Z}$ such that $2k + l = 1$.

We take the corresponding dual basis of $H_*(EIII)$ and denote the dual of $x \in H^*$ by $x_* \in H_*$. Furthermore we define the elements of $H_*(\Omega E_6)$ by

$$\begin{aligned}\sigma_1 &= g_{s*}(a_*), \sigma_4 = g_{s*}(b_*), \sigma_5 = g_{s*}(c_*), \sigma_7 = g_{s*}(d_*), \\ \sigma_8 &= g_{s*}(e_*), \sigma_{11} = g_{s*}(f_*), \\ \sigma_i &= g_{s*}((t_0^i)_*) \quad \text{for } i = 2, 3.\end{aligned}$$

Then in view of Proposition 3.1 we obtain

- Lemma 4.1.** (i) $\sigma_i, i = 1, 4, 5, 7, 8, 11$ are indecomposable and not divisible.
(ii) The coproduct of σ_2, σ_3 is given by

$$\begin{aligned}\tilde{\psi}(\sigma_2) &= \sigma_1 \otimes \sigma_1, \\ \tilde{\psi}(\sigma_3) &= \sigma_2 \otimes \sigma_1.\end{aligned}$$

- (iii) The following relations hold:

$$\sigma_1^2 = 2\sigma_2, \quad \sigma_1\sigma_2 = 3\sigma_3.$$

Proof of Theorem 1.1 (i). In view of (4.1) and Lemma 4.1, in order to determine the algebra structure of $H_*(\Omega E_6)$ we have to consider the “divisibility” among σ_i ’s. Fortunately the mod p homology rings $H_*(\Omega E_6; \mathbb{Z}/p\mathbb{Z})$ for each prime p are studied by several authors and the algebra structure is determined for all cases ([8], [7], [5]). We exhibit their results:

$$(4.3) \quad H_*(\Omega E_6; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_8, x_{10}, x_{14}, x_{16}, x_{22}]/(x_2^2),$$

$$(4.4) \quad H_*(\Omega E_6; \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}[x_2, x_6, x_8, x_{10}, x_{14}, x_{16}, x_{22}]/(x_2^3),$$

$$(4.5) \quad H_*(\Omega E_6; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}[x_2, x_8, x_{10}, x_{14}, x_{16}, x_{22}] \quad \text{for } p \geq 5$$

where $\deg(x_i) = i$ and moreover we can take x_{2i} as the mod p reduction of σ_i for each case. Therefore if “divisibility” occurs in degree $k \geq 8$ there exists an integral class σ , $\deg(\sigma) = k$ and an integer $n \geq 2$ such that

$$(4.6) \quad n \cdot \sigma = f(\sigma_1, \dots, \sigma_{11})$$

where f is a homogeneous polynomial of degree k in $\sigma_1, \dots, \sigma_{11}$ with integer coefficients. Let p be a prime number which divides n and apply the mod p reduction on both sides of (4.6). Then a relation of degree $k \geq 8$ arises in mod p homology and it contradicts the previous results. Hence we obtain the required result. $\square$

Table 6.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	σ_1	σ_2	σ_3	τ_4	$\sigma_1\tau_4$	τ_6	$\sigma_1\tau_6$	$\sigma_2\tau_6$	$\sigma_3\tau_6$	$\tau_4\tau_6$	$\sigma_1\tau_4\tau_6$
					σ_4	$\sigma_1\sigma_4$	$\sigma_2\sigma_4$	$\sigma_3\sigma_4$	$\tau_4\sigma_4$	$\sigma_1\tau_4\sigma_4$	$\sigma_4\tau_6$	$\sigma_1\sigma_4\tau_6$
					σ_5	$\sigma_1\sigma_5$	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	$\tau_4\sigma_5$	$\sigma_1\tau_4\sigma_5$	$\sigma_5\tau_6$	
							σ_7	$\sigma_1\sigma_7$	$\sigma_2\sigma_7$	$\sigma_3\sigma_7$	$\tau_4\sigma_7$	
								σ_4^2	$\sigma_1\sigma_4^2$	$\sigma_2\sigma_4^2$	$\sigma_3\sigma_4^2$	
								σ_8	$\sigma_1\sigma_8$	$\sigma_2\sigma_8$	$\sigma_3\sigma_8$	
									$\sigma_4\sigma_5$	$\sigma_1\sigma_4\sigma_5$	$\sigma_2\sigma_4\sigma_5$	
										σ_5^2	$\sigma_1\sigma_5^2$	
											$\sigma_4\sigma_7$	
											σ_{11}	

Table 7.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
					b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
						c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
								d_7	d_8	d_9	d_{10}	d_{11}
									d'_8	d'_9	d'_{10}	d'_{11}
									e_8	e_9	e_{10}	e_{11}
										e'_9	e'_{10}	e'_{11}
											e''_{10}	e''_{11}
												e'''_{11}
												f_{11}

5. The Hopf algebra structure of $H_*(\Omega E_6)$

In this section we determine the Hopf algebra structure of $H_*(\Omega E_6)$ using the “generating variety” $EIII$.

At first by Theorem 1.1 (i) we can choose an additive basis of $H_*(\Omega E_6)$ for degree ≤ 22 as in Table 6, where $\tau_4 = \sigma_2^2 - \sigma_1\sigma_3$, $\tau_6 = \sigma_2^3 - 4\sigma_3^2$.

We take the corresponding dual basis of $H^*(\Omega E_6)$ as in Table 7.

In view of Lemma 4.1 we obtain the results given by Table 8.

Next consider the case of degree 8: Since $\tau_4 = \sigma_2^2 - \sigma_1\sigma_3$, the coproduct of τ_4 is given by

$$(5.1) \quad \tilde{\psi}(\tau_4) = 2\sigma_3 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_2.$$

Table 8.

deg	basis	g_s^* -image	relation
2	a_1	$a = t_0$	
4	a_2	t_0^2	$a_1^2 = a_2$
6	a_3	t_0^3	$a_2 a_1 = a_1^3 = a_3$

We shall determine the coproduct of σ_4 : We may put

$$\tilde{\psi}(\sigma_4) = m\sigma_3 \otimes \sigma_1 + \cdots$$

for some $m \in \mathbb{Z}$. On the other hand since $H^8(\Omega E_6) = \langle a_4, b_4 \rangle$, we can put

$$(5.2) \quad a_3 a_1 = \mu a_4 + \nu b_4$$

for some $\mu, \nu \in \mathbb{Z}$. Then

$$\begin{aligned} \mu &= \langle a_3 a_1, \tau_4 \rangle = \langle a_3 \otimes a_1, \psi(\tau_4) \rangle = 2, \\ \nu &= \langle a_3 a_1, \sigma_4 \rangle = \langle a_3 \otimes a_1, \psi(\sigma_4) \rangle = m. \end{aligned}$$

Therefore applying g_s^* on both sides of (5.2)

$$\begin{aligned} 2g_s^*(a_4) &= g_s^*(a_3 a_1) - m g_s^*(b_4) \\ &= t_0^3 \cdot t_0 - m \cdot b \\ &= t_0^4 - m(t_0^4 - 2w) \\ &= (1 - m)t_0^4 + 2mw. \end{aligned}$$

On the other hand since $H^8(EIII) = \langle b, b' \rangle$, we can put

$$g_s^*(a_4) = \mu b + \nu b'$$

for some $\mu, \nu \in \mathbb{Z}$. Then

$$\mu = \langle g_s^*(a_4), b_* \rangle = \langle a_4, g_{s*}(b_*) \rangle = \langle a_4, \sigma_4 \rangle = 0.$$

Hence

$$g_s^*(a_4) = \nu b' = \nu(kt_0^4 + lw)$$

with $2k + l = 1$. Combining these gives

$$(1 - m)t_0^4 + 2mw = 2\nu kt_0^4 + 2\nu lw.$$

Table 9.

deg	basis	g_s^* -image	relation
8	a_4	$b' = w$	$a_3a_1 = 2a_4 + b_4$
8	b_4	$b = t_0^4 - 2w$	

Since $H^8(EIII) = \langle t_0^4, w \rangle$ we have

$$(5.3) \quad \begin{cases} 1 - m = 2\nu k, \\ m = \nu l. \end{cases}$$

Hence $\nu = 1$ and we may take $k = 0, l = 1$. Then $m = 1$ and $b' = w$.

Similarly we may put

$$\tilde{\psi}(\sigma_4) = \cdots + n\sigma_2 \otimes \sigma_2 + \cdots$$

for some $n \in \mathbb{Z}$. On the other hand we can put

$$a_2^2 = \mu a_4 + \nu b_4$$

for some $\mu, \nu \in \mathbb{Z}$. Then

$$\begin{aligned} \mu &= \langle a_2^2, \tau_4 \rangle = \langle a_2 \otimes a_2, \psi(\tau_4) \rangle = 2, \\ \nu &= \langle a_2^2, \sigma_4 \rangle = \langle a_2 \otimes a_2, \psi(\sigma_4) \rangle = n. \end{aligned}$$

But since $a_1^2 = a_2, a_1a_2 = a_3$

$$a_2^2 = a_3a_1 = 2a_4 + b_4.$$

Thus $n = \nu = 1$.

Hence we obtain the results given by Table 9 and

$$\begin{aligned} \tilde{\psi}(\tau_4) &= 2\sigma_3 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_2, \\ \tilde{\psi}(\sigma_4) &= \sigma_3 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2. \end{aligned}$$

Note that for the element $\tilde{\sigma}_4 = \tau_4 - 2\sigma_4$,

$$\tilde{\psi}(\tilde{\sigma}_4) = 0.$$

We have to continue this argument up to degree ≤ 22 . But as well as the case of F_4 ([11]), the remainder is no more than a tedious computation. So we only exhibit the data and the results:

$$(5.4) \quad \tilde{\psi}(\sigma_5) = \tau_4 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_2,$$

Table 10.

deg	basis	g_s^* -image	relation
10	a_5	$c' = -t_0^5 + 2t_0w$	$a_4a_1 = 5a_5 + 2b_5 + c_5$
10	b_5	$-c' = t_0^5 - 2t_0w$	$b_4a_1 = b_5$
10	c_5	$c = 3t_0^5 - 5t_0w$	
12	a_6	$-5t_0^6 + 9t_0^2w$	$a_5a_1 = a_6 + b_6 + c_6$
12	b_6	$t_0^6 - 2t_0^2w$	$b_5a_1 = b_6$
12	c_6	$3t_0^6 - 5t_0^2w$	$c_5a_1 = c_6$
14	a_7	0	$a_6a_1 = 7a_7 + 2b_7 + 3c_7 - d_7$
14	b_7	$-d' = t_0^7 - 2t_0^3w$	$b_6a_1 = b_7$
14	c_7	$d' = -t_0^7 + 2t_0^3w$	$c_6a_1 = c_7 + d_7$
14	d_7	$d = 4t_0^7 - 7t_0^3w$	

$$(5.5) \quad \tilde{\psi}(\sigma_7) = (-\tau_6 + \sigma_1\sigma_5) \otimes \sigma_1 + \sigma_5 \otimes \sigma_2 + \tau_4 \otimes \sigma_3.$$

Note that for the elements $\tilde{\sigma}_5 = \sigma_1\tau_4 - 5\sigma_5$, $\tilde{\sigma}_7 = 4\sigma_1\tau_6 - 7\sigma_2\sigma_5 + 7\sigma_7$,

$$\tilde{\psi}(\tilde{\sigma}_5) = 0,$$

$$\tilde{\psi}(\tilde{\sigma}_7) = 0.$$

Since $\text{rank } H^{16}(EIII) = 3$, the computation for degree = 16 is a little complicated. So first we take

$$E = \begin{pmatrix} 3 & -4 & -2 \\ 2 & -2 & -3 \\ -2 & 3 & 1 \end{pmatrix}$$

so that

$$\begin{pmatrix} e \\ e' \\ e'' \end{pmatrix} = \begin{pmatrix} 3 & -4 & -2 \\ 2 & -2 & -3 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} t_0^8 \\ t_0^4w \\ w^2 \end{pmatrix} = \begin{pmatrix} 3t_0^8 - 4t_0^4w - 2w^2 \\ 2t_0^8 - 2t_0^4w - 3w^2 \\ -2t_0^8 + 3t_0^4w + w^2 \end{pmatrix}.$$

Then we obtain the results given by Table 11 and

$$(5.6) \quad \begin{aligned} \tilde{\psi}(\sigma_8) = & (\sigma_3\sigma_4 + \sigma_2\sigma_5) \otimes \sigma_1 + (\tau_6 - \sigma_2\sigma_4 + \sigma_1\sigma_5) \otimes \sigma_2 \\ & + (\sigma_1\tau_4 + \sigma_1\sigma_4 + \sigma_5) \otimes \sigma_3 + 2\tau_4 \otimes \tau_4 - \sigma_4 \otimes \sigma_4. \end{aligned}$$

Note that for the element $\tilde{\sigma}_8 = \sigma_2\tau_6 - 4\sigma_3\sigma_5 + 2\sigma_4^2 + 4\sigma_8$,

$$\tilde{\psi}(\tilde{\sigma}_8) = 0.$$

Table 11.

deg	basis	g_s^* -image	relation
16	a_8	$-e' - 3e''$	$a_7a_1 = 4a_8 + b_8 + c_8 - d_8$
16	b_8	$e' + e''$	$b_7a_1 = 2b_8 + 2d'_8 - e_8$
16	c_8	$2e' + 8e''$	$c_7a_1 = c_8 + 2d_8 + e_8$
16	d_8	$-e' - 3e''$	$d_7a_1 = d_8$
16	d'_8	$-e' - 2e''$	$a_4^2 = 70a_8 + 34b_8 + 28c_8 + 4d_8 + 17d'_8 + 2e_8$
16	e_8	e	$a_4b_4 = b_8$ $b_4^2 = 2d'_8 - e_8$

$$\begin{aligned}
 \tilde{\psi}(\sigma_{11}) = & (-2\tau_4\tau_6 + \sigma_4\tau_6 + 2\sigma_1\tau_4\sigma_5 - \sigma_3\sigma_7 - \sigma_1\sigma_4\sigma_5 - \sigma_5^2) \otimes \sigma_1 \\
 & + (-2\sigma_3\tau_6 + 4\tau_4\sigma_5 - \sigma_2\sigma_7 - \sigma_4\sigma_5) \otimes \sigma_2 \\
 (5.7) \quad & + (-\sigma_2\tau_6 - \tau_4\sigma_4 + 4\sigma_3\sigma_5 - \sigma_1\sigma_7) \otimes \sigma_3 \\
 & - \sigma_3\sigma_4 \otimes \tau_4 + \sigma_2\sigma_5 \otimes \tau_4 - \sigma_7 \otimes \sigma_4 - \tau_6 \otimes \sigma_1\tau_4 + 4\tau_6 \otimes \sigma_5 + \tau_6 \otimes \sigma_1\sigma_4 \\
 & - \sigma_2\sigma_4 \otimes \sigma_5 + \sigma_1\sigma_5 \otimes \sigma_1\tau_4 - \sigma_1\sigma_5 \otimes \sigma_1\sigma_4 - 2\sigma_1\sigma_5 \otimes \sigma_5.
 \end{aligned}$$

Note that for the element $\tilde{\sigma}_{11} = 6\sigma_1\tau_4\tau_6 - 44\sigma_5\tau_6 + 11\sigma_1\sigma_5^2 + 11\sigma_4\sigma_7 + 11\sigma_{11}$,

$$\tilde{\psi}(\tilde{\sigma}_{11}) = 0.$$

Consequently we established Theorem 1.1.

Table 12.

deg	basis	g_s^* -image	relation
18	a_9	$-18t_0^5w + 31t_0w^2$	$a_8a_1 = 3a_9 + 2b_9 + c_9 - d_9$
18	b_9	$4t_0^5w - 7t_0w^2$	$b_8a_1 = 5b_9 + 4d'_9 - 2e_9 + e'_9$
18	c_9	$32t_0^5w - 55t_0w^2$	$c_8a_1 = 2c_9 + 3d_9 + 3e_9 + e'_9$
18	d_9	$-7t_0^5w + 12t_0w^2$	$d_8a_1 = d_9$
18	d'_9	$-4t_0^5w + 7t_0w^2$	$d'_8a_1 = d'_9$
18	e_9	$-4t_0^5w + 7t_0w^2$	$e_8a_1 = e_9$
18	e'_9	$-11t_0^5w + 19t_0w^2$	$a_5a_4 = 42a_9 + 60b_9 + 25c_9 + 3d_9 + 29d'_9 + 4e_9 + 12e'_9$
			$b_5a_4 = 5b_9 + 4d'_9 - 2e_9 + e'_9$
			$c_5a_4 = c_9 + 3d_9 + 2e_9$
			$a_5b_4 = b_9$
			$b_5b_4 = 2d'_9 - e_9$
			$c_5b_4 = e'_9$
20	a_{10}	$-22t_0^6w + 38t_0^2w^2$	$a_9a_1 = 5a_{10} + b_{10} + 3c_{10} - d_{10}$
20	b_{10}	$19t_0^6w - 33t_0^2w^2$	$b_9a_1 = b_{10} + 2d'_{10} - e_{10} + e'_{10}$
20	c_{10}	$22t_0^6w - 38t_0^2w^2$	$c_9a_1 = 5c_{10} + 2d_{10} + 3e_{10} + 2e'_{10} + 2e''_{10}$
20	d_{10}	$-7t_0^6w + 12t_0^2w^2$	$d_9a_1 = d_{10}$
20	d'_{10}	$-4t_0^6w + 7t_0^2w^2$	$d'_9a_1 = d'_{10}$
20	e_{10}	$-4t_0^6w + 7t_0^2w^2$	$e_9a_1 = e_{10}$
20	e'_{10}	$-11t_0^6w + 19t_0^2w^2$	$e'_9a_1 = e'_{10}$
20	e''_{10}	$-15t_0^6w + 26t_0^2w^2$	$a_6a_4 = 210a_{10} + 97b_{10} + 246c_{10} + 6d_{10} + 135d'_{10} + 21e_{10} + 114e'_{10} + 48e''_{10}$
			$b_6a_4 = 5b_{10} + 14d'_{10} - 7e_{10} + 6e'_{10}$
			$c_6a_4 = 5c_{10} + 5d_{10} + 5e_{10} + 2e'_{10} + 2e''_{10}$
			$a_6b_4 = b_{10}$
			$b_6b_4 = 2d'_{10} - e_{10}$
			$c_6b_4 = e'_{10}$
			$a_5^2 = 42a_{10} + 20b_{10} + 50c_{10} + 2d_{10} + 29d'_{10} + 4e_{10} + 24e'_{10} + 10e''_{10}$
			$a_5b_5 = b_{10} + 2d'_{10} - e_{10} + e'_{10}$
			$a_5c_5 = c_{10} + d_{10} + e_{10}$
			$b_5^2 = 2d'_{10} - e_{10}$
			$b_5c_5 = e'_{10}$
			$c_5^2 = 2e''_{10}$

Table 13.

deg	basis	g_s^* -image	relation
22	a_{11}	0	$a_{10}a_1 = 11a_{11} + 2b_{11} + c_{11} - d_{11} - 2f_{11}$
22	b_{11}	0	$b_{10}a_1 = 7b_{11} + 4d'_{11} - 2e_{11} + 3e'_{11} - e'''_{11} + f_{11}$
22	c_{11}	$-f'$	$c_{10}a_1 = c_{11} + d_{11} + e_{11} + e'_{11} + 2e''_{11} + 2f_{11}$
22	d_{11}	$-f'$	$d_{10}a_1 = 2d_{11} + e'''_{11} - f_{11}$
22	d'_{11}	f'	$d'_{10}a_1 = d'_{11}$
22	e_{11}	f'	$e_{10}a_1 = e_{11}$
22	e'_{11}	$-f'$	$e'_{10}a_1 = e'_{11} + e'''_{11} - f_{11}$
22	e''_{11}	f'	$e''_{10}a_1 = e''_{11} - f_{11}$
22	e'''_{11}	f'	
22	f_{11}	f	
			$a_7a_4 = 330a_{11} + 147b_{11} + 636c_{11} + d_{11}$ $+65d'_{11} + 11e_{11} + 84e'_{11} + 72e''_{11}$
			$b_7a_4 = 35b_{11} + 34d'_{11} - 17e_{11} + 21e'_{11}$ $+e'''_{11} - f_{11}$
			$c_7a_4 = 5c_{11} + 14d_{11} + 10e_{11} + 7e'_{11} + 12e''_{11}$ $+7e'''_{11} + f_{11}$
			$d_7a_4 = d_{11}$
			$a_7b_4 = b_{11}$
			$b_7b_4 = 2d'_{11} - e_{11}$
			$c_7b_4 = e'_{11}$
			$d_7b_4 = e'''_{11} - f_{11}$
			$a_6a_5 = 462a_{11} + 217b_{11} + 91c_{11} + 9d_{11} + 103d'_{11}$ $+15e_{11} + 129e'_{11} + 108e''_{11} + 5e'''_{11} - f_{11}$
			$b_6a_5 = 7b_{11} + 6d'_{11} - 3e_{11} + 4e'_{11}$
			$c_6a_5 = c_{11} + 3d_{11} + 2e_{11} + e'_{11} + 2e''_{11} + e'''_{11} + f_{11}$
			$a_6b_5 = 7b_{11} + 4d'_{11} - 2e_{11} + 3e'_{11} - e'''_{11} + f_{11}$
			$b_6b_5 = 2d'_{11} - e_{11}$
			$c_6b_5 = e'_{11} + e'''_{11} - f_{11}$
			$a_6c_5 = c_{11} + 3d_{11} + 2e_{11} + 4f_{11}$
			$b_6c_5 = e'_{11} + e'''_{11} - f_{11}$
			$c_6c_5 = 2e''_{11} - 2f_{11}$

where $f = 11t_0^7w - 19t_0^3w^2$, $f' = -4t_0^7w + 7t_0^3w^2$.

References

- [1] A. Borel: *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.
- [2] R. Bott: *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–282.
- [3] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.
- [4] N. Bourbaki: *Groupes et Algèbre de Lie IV–VI*, Masson, Paris, 1968.
- [5] H. Hamanaka and S. Hara: *The mod 3 homology of the space of loops on the exceptional Lie groups and adjoint action*, J. Math. Kyoto Univ. **37** (1997), 441–453.
- [6] A. Kono and K. Kozima: *The space of loops on a symplectic group*, Japan J. Math. **4** (1978), 461–486.
- [7] A. Kono and K. Kozima: *The mod 2 homology of the space of loops on the exceptional Lie group*, Proc. Royal Society of Edinburgh **112** (1989), 187–202.
- [8] T. Petrie: *The weakly complex bordism of Lie groups*, Ann. of Math. **88** (1968), 371–402.
- [9] J.P. Serre: *Groupe d’homotopie et classes de groupes abéliens*, Ann. of Math. **58** (1953), 258–294.
- [10] H. Toda and T. Watanabe: *The integral cohomology ring of F_4/T and E_6/T* , J. Math. Kyoto Univ. **14** (1974), 257–286.
- [11] T. Watanabe: *The homology of the loop space of the exceptional group F_4* , Osaka J. Math. **15** (1978), 463–474.

Department of Mathematics
 Graduate School of Science
 Kyoto University
 Kyoto, 606-8502
 Japan

Current Address:
 Department of General Education
 Takamatsu National College of Technology
 Takamatsu, 761-8058
 Japan
 e-mail: nakagawa@takamatsu-nct.ac.jp