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THE SPACE OF LOOPS ON THE EXCEPTIONAL LIE GROUP E_6

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1. Introduction

Let G be a compact 1-connected simple Lie group and ΩG the space of loops on G. As is well known ΩG is a homotopy commutative H-space and its integral homology $H_*(\Omega G)$ has no torsion and no odd dimensional part ([2]). Therefore $H_*(\Omega G)$ becomes a commutative Hopf algebra over the integers $\mathbb{Z}$. In [3] R. Bott introduced a "generating variety" and determined the Hopf algebra structure of $H_*(\Omega G)$ explicitly for $G = SU(n)$, $\mathrm{Spin}(n)$ and G_2. In [11] T. Watanabe determined $H_*(\Omega F_4)$ in a similar way. On the other hand A. Kono and K. Kozima determined $H_*(\Omega Sp(n))$ by different method using the Bott periodicity ([6]).

In this paper we carry out the Bott's program for $G = E_6$, where E_6 is the compact 1-connected exceptional Lie group of rank 6 and determine the Hopf algebra structure of $H_*(\Omega E_6)$ explicitly.

Let ψ be the coproduct of $H_*(\Omega G)$ induced by the diagonal map $\Omega G \longrightarrow \Omega G \times \Omega G$. To avoid the cumbersome notation, following [11] we introduce a map $\tilde{\psi}\colon H_*(\Omega G) \longrightarrow H_*(\Omega G) \otimes H_*(\Omega G)$ satisfying

$$\psi(\sigma) - (\sigma \otimes 1 + 1 \otimes \sigma) = \tilde{\psi}(\sigma) + T\tilde{\psi}(\sigma) \quad \text{for} \quad \sigma \in H_*(\Omega G)$$

where $T\colon H_*(\Omega G) \otimes H_*(\Omega G) \longrightarrow H_*(\Omega G) \otimes H_*(\Omega G)$ is defined by

$$T(\sigma \otimes \tau) = \begin{cases} \tau \otimes \sigma & \text{for} \quad \sigma \neq \tau, \\ 0 & \text{for} \quad \sigma = \tau. \end{cases}$$

Note that $\tilde{\psi}(\sigma) = 0$ if and only if $\sigma \in PH_*(\Omega G)$, where $PH_*(\Omega G)$ denotes the primitive module of the Hopf algebra $H_*(\Omega G)$.

Then our main results are stated as follows:

Theorem 1.1. *The Hopf algebra structure of $H_*(\Omega E_6)$ is given as follows*:
(i) *As an algebra*

$$H_*(\Omega E_6) = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8, \sigma_{11}]/({\sigma_1}^2 - 2\sigma_2, \sigma_1\sigma_2 - 3\sigma_3)$$

where $\deg(\sigma_i) = 2i$.

(ii) *With suitably chosen generators* σ_i, $i = 1, 2, 3, 4, 5, 7, 8, 11$, *the coproduct is given by*

$$\psi(\sigma_k) = \sum_{i+j=k} \sigma_i \otimes \sigma_j \quad \textit{for} \quad k = 1, 2, 3, 4 \quad (\sigma_0 = 1),$$
$$\tilde{\psi}(\sigma_5) = \tau_4 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_2,$$
$$\tilde{\psi}(\sigma_7) = (-\tau_6 + \sigma_1\sigma_5) \otimes \sigma_1 + \sigma_5 \otimes \sigma_2 + \tau_4 \otimes \sigma_3,$$
$$\begin{aligned}\tilde{\psi}(\sigma_8) = {} & (-\sigma_3\sigma_4 + \sigma_2\sigma_5) \otimes \sigma_1 + (\tau_6 - \sigma_2\sigma_4 + \sigma_1\sigma_5) \otimes \sigma_2 \\ & + (\sigma_1\tau_4 + \sigma_1\sigma_4 + \sigma_5) \otimes \sigma_3 + 2\tau_4 \otimes \tau_4 - \sigma_4 \otimes \sigma_4,\end{aligned}$$
$$\begin{aligned}\tilde{\psi}(\sigma_{11}) = {} & (-2\tau_4\tau_6 + \sigma_4\tau_6 + 2\sigma_1\tau_4\sigma_5 - \sigma_3\sigma_7 - \sigma_1\sigma_4\sigma_5 - \sigma_5{}^2) \otimes \sigma_1 \\ & + (-2\sigma_3\tau_6 + 4\tau_4\sigma_5 - \sigma_2\sigma_7 - \sigma_4\sigma_5) \otimes \sigma_2 \\ & + (-\sigma_2\tau_6 - \tau_4\sigma_4 + 4\sigma_3\sigma_5 - \sigma_1\sigma_7) \otimes \sigma_3 \\ & - \sigma_3\sigma_4 \otimes \tau_4 + \sigma_2\sigma_5 \otimes \tau_4 - \sigma_7 \otimes \sigma_4 - \tau_6 \otimes \sigma_1\tau_4 + 4\tau_6 \otimes \sigma_5 \\ & + \tau_6 \otimes \sigma_1\sigma_4 - \sigma_2\sigma_4 \otimes \sigma_5 + \sigma_1\sigma_5 \otimes \sigma_1\tau_4 - \sigma_1\sigma_5 \otimes \sigma_1\sigma_4 - 2\sigma_1\sigma_5 \otimes \sigma_5\end{aligned}$$

where $\tau_4 = \sigma_2{}^2 - \sigma_1\sigma_3$, $\tau_6 = \sigma_2{}^3 - 4\sigma_3{}^2$.

(iii) $$PH_*(\Omega E_6) = \langle \sigma_1, \tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_7, \tilde{\sigma}_8, \tilde{\sigma}_{11} \rangle$$

as a free module, where

$$\begin{aligned}\tilde{\sigma}_4 &= \tau_4 - 2\sigma_4,\\ \tilde{\sigma}_5 &= \sigma_1\tau_4 - 5\sigma_5,\\ \tilde{\sigma}_7 &= 4\sigma_1\tau_6 - 7\sigma_2\sigma_5 + 7\sigma_7,\\ \tilde{\sigma}_8 &= \sigma_2\tau_6 - 4\sigma_3\sigma_5 + 2\sigma_4{}^2 + 4\sigma_8,\\ \tilde{\sigma}_{11} &= 6\sigma_1\tau_4\tau_6 - 44\sigma_5\tau_6 + 11\sigma_1\sigma_5{}^2 + 11\sigma_4\sigma_7 + 11\sigma_{11}.\end{aligned}$$

The paper is organized as follows: In §2 we prepare various results which are needed in later sections. In §3 we carry out the Bott's program for E_6 and determine the primitive elements of $H^*(\Omega E_6)$. In §4 we determine the algebra structure of $H_*(\Omega E_6)$ depending on the results of mod p homology for all primes p. In the last section, §5 we determine the Hopf algebra structure of $H_*(\Omega E_6)$ explicitly using the "generating variety". Throughout this paper $\sigma_i(x_1, \ldots, x_n)$ denotes the i-th elementary symmetric function in the variables $x_1, \ldots, x_n$.

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2. Preliminaries

Let T be a maximal torus of E_6 and $\{\alpha_i\}_{1\le i\le 6}$ the root system given in [4], where the corresponding fundamental weights are given as follows:

$$
\begin{aligned}
w_1 &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6),\\
w_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,\\
w_3 &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6),\\
w_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6,\\
w_5 &= \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6),\\
w_6 &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6).
\end{aligned} \tag{2.1}
$$

As usual we regard roots and weights as elements of $H^1(T) \xrightarrow{\sim} H^2(BT)$. Then $\{w_i\}_{1\le i\le 6}$ forms a basis of $H^2(BT)$ and $H^*(BT) = \mathbb{Z}[w_1, w_2, \ldots, w_6]$. Let R_i denote the reflection to the hyperplane $\alpha_i = 0$, then the Weyl group of E_6, $W(E_6)$, is a finite group generated by R_i $(1 \le i \le 6)$ and the action on $\{w_i\}_{1\le i\le 6}$ is given as follows:

$$
R_i(w_i) = w_i - \sum_{j=1}^{6} \frac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)} w_j, \quad R_j(w_i) = w_i \quad \text{for} \quad j \ne i. \tag{2.2}
$$

Following [10], we introduce the elements

$$
\begin{aligned}
t_6 &= w_6,\\
t_5 &= R_6(t_6) = w_5 - w_6,\\
t_4 &= R_5(t_5) = w_4 - w_5,\\
t_3 &= R_4(t_4) = w_2 + w_3 - w_4,\\
t_2 &= R_3(t_3) = w_1 + w_2 - w_3,\\
t_1 &= R_1(t_2) = -w_1 + w_2,\\
c_i &= \sigma_i(t_1, \ldots, t_6),\ t = \frac{1}{3}c_1 = w_2.
\end{aligned}
$$

Then we have the following isomorphism:

$$
H^*(BT) = \mathbb{Z}[t_1, t_2, \ldots, t_6, t]/(c_1 - 3t).
$$

From (2.2) the action of $W(E_6)$ on these elememts is given by Table 1, where blanks indicate the trivial action.

Table 1.

	R_1	R_2	R_3	R_4	R_5	R_6
t_1	t_2	$t - t_2 - t_3$				
t_2	t_1	$t - t_1 - t_3$	t_3			
t_3		$t - t_1 - t_2$	t_2	t_4		
t_4				t_3	t_5	
t_5					t_4	t_6
t_6						t_5
t		$-t + t_4 + t_5 + t_6$				

Next as in [10] we put

$$x_i = 2t_i - t \quad (1 \leq i \leq 6).$$

From Table 1 we see easily that

$$S = \{x_i + x_j \ (1 \leq i < j \leq 6),\ t - x_i, -t - x_i \ (1 \leq i \leq 6)\} \subset H^2(BT;\mathbb{Q})$$

is invariant under the action of $W(E_6)$. (In fact S is an orbit of $2w_1$ under the action of $W(E_6)$) Thus we have $W(E_6)$-invariant forms

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT;\mathbb{Q})^{W(E_6)}.$$

As in [10] I_n is computed by the formula:

$$I_n = \frac{1}{2}\sum_{i=2}^{n-2}\binom{n}{i}s_i s_{n-i} + (6 - 2^{n-1})s_n + 2(-1)^n \sum_{j=0}^{[n/2]}\binom{n}{2j}s_{n-2j}t^{2j} \tag{2.3}$$

where $s_n = x_1^n + \cdots + x_6^n$ and s_n is written as a polynomial in d_i's, $d_i = \sigma_i(x_1, \ldots, x_6)$, by use of the Newton formula:

$$s_n = \sum_{i=1}^{n-1}(-1)^{i-1}s_{n-i}d_i + (-1)^{n-1}nd_n \quad (d_n = 0 \ \text{ for } \ n > 6). \tag{2.4}$$

Moreover since $x_i = 2t_i - t$ for $1 \leq i \leq 6$

$$\sum_{n=0}^{6} d_n = \prod_{i=1}^{6}(1 + x_i) = \prod_{i=1}^{6}(1 - t + 2t_i) = \sum_{i=0}^{6}(1 - t)^{6-i}2^i c_i$$

and we have

$$d_n = \sum_{i=0}^{n}\binom{6-i}{n-i}(-t)^{n-i}2^i c_i. \tag{2.5}$$

Table 2.

	R_2	R_3	R_4	R_5	R_6
ϵ_1	$-\epsilon_2$	ϵ_2			
ϵ_2	$-\epsilon_1$	ϵ_1	ϵ_3		
ϵ_3			ϵ_2	ϵ_4	
ϵ_4				ϵ_3	ϵ_5
ϵ_5					ϵ_4
t_0					

By using (2.3), (2.4), (2.5) I_n can be written as a polynomial in t and c_i's (for details see [10], §5).

Then the next lemma is proved in [10], Lemma 5.2:

Lemma 2.1. *The rational invariant subalgebra of the Weyl group $W(E_6)$ is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(E_6)} = \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}].$$

Let C_1 be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ $(i \neq 1)$. Then as is well known

$$C_1 = T^1 \cdot \mathrm{Spin}(10), \quad T^1 \cap \mathrm{Spin}(10) \cong \mathbb{Z}/4\mathbb{Z}$$

and the Weyl group of C_1, $W(C_1)$, is generated by R_i $(i \neq 1)$.

REMARK 2.2. The homogeneous space E_6/C_1 is the irreducible Hermitian symmetric space and denoted by $EIII$ in E. Cartan's notation.

We put

$$t_0 = t - t_1 = w_1, \quad \epsilon_i = t_{i+1} - \frac{1}{2}t_0 \quad (1 \leq i \leq 5). \tag{2.6}$$

Then we have

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_0, \epsilon_1, \epsilon_2, \dots, \epsilon_5]$$

and the action of $W(C_1)$ is given by Table 2.

From Table 2 we have

Lemma 2.3. *The rational invariant subalgebra of the Weyl group $W(C_1)$ is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$$

where

$$p_i = \sigma_i(\epsilon_1^2, \dots, \epsilon_5^2), \quad e = \prod_{i=1}^{5} \epsilon_i.$$

We shall find the relations between p_i's and I_n's. Put

$$b_i = \sigma_i(\epsilon_1, \dots, \epsilon_5)$$

so that

$$\begin{aligned}\sum_{i\geq 0}(-1)^i p_i &= \prod_{j=1}^{5}(1-\epsilon_j^2) = \prod_{j=1}^{5}(1+\epsilon_j)\prod_{j=1}^{5}(1-\epsilon_j)\\ &= \left(\sum_{k\geq 0} b_k\right)\left(\sum_{l\geq 0}(-1)^l b_l\right) = \sum_{k,l\geq 0}(-1)^l b_k b_l.\end{aligned}$$

Therefore

$$p_i = \sum_{k+l=2i}(-1)^{l+i} b_k b_l. \tag{2.7}$$

On the other hand since $t_i = \epsilon_{i-1} + (1/2)t_0$ for $2 \leq i \leq 6$

$$\begin{aligned}\sum_{n=0}^{6} c_n &= \prod_{i=1}^{6}(1+t_i) = (1+t_1)\prod_{i=2}^{6}(1+t_i)\\ &= (1+t_1)\prod_{i=1}^{5}\left(1+\frac{1}{2}t_0+\epsilon_i\right) = (1+t_1)\sum_{i=0}^{5}\left(1+\frac{1}{2}t_0\right)^{5-i} b_i\end{aligned}$$

and we have

$$c_n = \sum_{i=0}^{n}\binom{5-i}{n-i}\left(\frac{1}{2}t_0\right)^{n-i} b_i + \sum_{i=0}^{n-1}\binom{5-i}{n-i-1}\left(\frac{1}{2}t_0\right)^{n-i-1} t_1 b_i. \tag{2.8}$$

Since $I_n \in H^*(BT;\mathbb{Q})^{W(E_6)} \subset H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$, I_n can be written as a polynomial in t_0, p_1, p_2, e, p_3, p_4. Direct computation using (2.3), (2.4), (2.5), (2.7) and (2.8) then yields the following results:

Lemma 2.4. *In* $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ *we have*

$$I_2 = 2^4 \cdot 3\left(\frac{1}{2}p_1 + \frac{3}{8}t_0^2\right),$$

$$
\begin{aligned}
I_5 =& -2^7 \cdot 3 \cdot 5 \left\{ e + \frac{1}{4}\left(-p_2 + \frac{1}{4}p_1^2\right) t_0 + \frac{1}{32}p_1 t_0^3 - \frac{3}{256}t_0^5 \right\}, \\
I_6 =& \, 2^9 \cdot 3^2 \left\{ \frac{1}{4}p_3 - \frac{1}{24}p_1 p_2 + \frac{1}{32}p_1^3 - \frac{5}{4}et_0 + \frac{5}{32}\left(-p_2 + \frac{3}{4}p_1^2\right) t_0^2 + \frac{15}{512}p_1 t_0^4 \right. \\
& \left. + \frac{33}{2048}t_0^6 \right\}, \\
I_8 =& -2^{12} \cdot 3 \cdot 5 \left\{ -\frac{1}{4}p_4 - \frac{1}{48}p_2^2 - \frac{1}{10}p_1 p_3 + \frac{13}{480}p_1^2 p_2 - \frac{11}{1280}p_1^4 + \frac{7}{8}ep_1 t_0 \right. \\
& + \frac{7}{80}\left(-3p_3 + \frac{7}{4}p_1 p_2 - \frac{11}{16}p_1^3\right) t_0^2 + \frac{7}{32}et_0^3 + \frac{7}{512}\left(5p_2 - \frac{11}{4}p_1^2\right) t_0^4 \\
& \left. - \frac{77}{20480}p_1 t_0^6 - \frac{1419}{327680}t_0^8 \right\}, \\
I_9 =& \, 2^{11} \cdot 3^3 \cdot 7 \left\{ -\frac{1}{6}ep_2 - \frac{1}{8}ep_1^2 + \frac{1}{4}\left(p_4 - \frac{1}{12}p_2^2 + \frac{5}{24}p_1^2 p_2 - \frac{3}{64}p_1^4\right) t_0 \right. \\
& - \frac{5}{16}ep_1 t_0^2 + \frac{1}{16}\left(-p_3 + \frac{17}{12}p_1 p_2 - \frac{7}{16}p_1^3\right) t_0^3 - \frac{5}{128}et_0^4 \\
& \left. + \frac{1}{512}\left(11p_2 - \frac{21}{4}p_1^2\right) t_0^5 - \frac{3}{4096}p_1 t_0^7 + \frac{85}{65536}t_0^9 \right\}, \\
I_{12} =& \, 2^{15} \cdot 3^4 \cdot 5 \left\{ \frac{1}{180}p_3^2 + \frac{1}{24}p_2 p_4 - \frac{1}{864}p_2^3 - \frac{1}{540}p_1 p_2 p_3 + \frac{41}{864}p_1^2 p_4 \right. \\
& + \frac{65}{108}e^2 p_1 + \frac{101}{17280}p_1^2 p_2^2 + \frac{13}{2160}p_1^3 p_3 - \frac{251}{69120}p_1^4 p_2 + \frac{19}{30720}p_1^6 \\
& + \frac{11}{12}\left(-\frac{1}{3}ep_3 - \frac{1}{4}ep_1 p_2 - \frac{1}{16}ep_1^3\right) t_0 + \frac{11}{16}\left(\frac{11}{9}e^2 - \frac{1}{18}p_2 p_3 + \frac{7}{36}p_1 p_4 \right. \\
& \left. + \frac{11}{144}p_1 p_2^2 + \frac{7}{72}p_1^2 p_3 - \frac{7}{96}p_1^3 p_2 + \frac{19}{1280}p_1^5\right) t_0^2 + \frac{77}{192}\left(-\frac{1}{3}ep_2 - \frac{1}{4}ep_1^2\right) t_0^3 \\
& + \frac{11}{128}\left(-\frac{5}{12}p_4 + \frac{65}{144}p_2^2 + p_1 p_3 - \frac{253}{288}p_1^2 p_2 + \frac{57}{256}p_1^4\right) t_0^4 \\
& - \frac{77}{3072}ep_1 t_0^5 + \frac{77}{10240}\left(\frac{11}{3}p_3 - \frac{127}{36}p_1 p_2 + \frac{19}{16}p_1^3\right) t_0^6 - \frac{11}{12288}et_0^7 \\
& \left. + \frac{11}{131072}\left(-\frac{85}{3}p_2 + \frac{57}{4}p_1^2\right) t_0^8 + \frac{209}{5242880}p_1 t_0^{10} + \frac{12977}{41943040}t_0^{12} \right\}.
\end{aligned}
$$

Let $\mathfrak{b}_i$ denote the ideal in $H^*(BT;\mathbb{Q})^{W(C_1)}$ generated by I_j's for $j < i$, $j \in 2, 5, 6, 8, 9, 12$. By Lemma 2.4 we have immediately the following (here we put $w = (1/6)p_2 + (9/16)t_0^4$):

Lemma 2.5. *In* $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ *we have*

$$\begin{aligned} p_1 &\equiv -\frac{3}{4}t_0^2 \mod \mathfrak{b}_5, \\ e &\equiv \frac{3}{2}t_0 w - \frac{27}{32}t_0^5 \mod \mathfrak{b}_6, \\ p_3 &\equiv \frac{21}{2}t_0^2 w - \frac{195}{32}t_0^6 \mod \mathfrak{b}_8, \\ p_4 &\equiv -3w^2 - \frac{63}{8}t_0^4 w + \frac{1413}{256}t_0^8 \mod \mathfrak{b}_9. \end{aligned}$$

Using Lemmas 2.4, 2.5 we obtain

Lemma 2.6. *In* $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]$ *we have*

$$\begin{aligned} I_9 &\equiv 2^{11} \cdot 3^3 \cdot 7(t_0^9 - 3t_0 w^2) && \mod \mathfrak{b}_9, \\ I_{12} &\equiv 2^{15} \cdot 3^4 \cdot 5(-3t_0^{12} + 9t_0^8 w - 6t_0^4 w^2 - w^3) && \mod \mathfrak{b}_9. \end{aligned}$$

Therefore by the classical results of A. Borel [1] we obtain the following rational cohomology ring of $EIII$:

$$\begin{aligned} H^*(EIII;\mathbb{Q}) &\cong H^*(BT;\mathbb{Q})^{W(C_1)}/\mathfrak{b}_{13} \\ &= \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4]/(I_2, I_5, I_6, I_8, I_9, I_{12}) \\ &\cong \mathbb{Q}[t_0, w]/(t_0^9 - 3t_0 w^2, w^3 + 15t_0^4 w^2 - 9t_0^8 w). \end{aligned}$$

Furthermore the integral cohomology ring of $EIII$ is determined in [10], Corollary C:

Theorem 2.7. *The integral cohomology ring of* $EIII$ *is given as follows*:

$$H^*(EIII) = \mathbb{Z}[t_0, w]/(t_0^9 - 3t_0 w^2, w^3 + 15t_0^4 w^2 - 9t_0^8 w)$$

where t_0, w *are as above.*

Corollary 2.8. *An additive basis of* $H^*(EIII)$ *as a free module for degree* ≤ 22 *is given by* Table 3.

3. The primitive elements of $H^*(\Omega E_6)$

In this section we determine the primitive elements of $H^*(\Omega E_6)$ which are needed for determination of the Hopf algebra structure of $H_*(\Omega E_6)$.

As is well known, over the rationals E_6 looks like the product of odd dimensional

Table 3.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	t_0	t_0^2	t_0^3	t_0^4	t_0^5	t_0^6	t_0^7	t_0^8			
					w	t_0w	t_0^2w	t_0^3w	t_0^4w	t_0^5w	t_0^6w	t_0^7w
									w^2	t_0w^2	$t_0^2w^2$	$t_0^3w^2$

spheres:

$$S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}.$$

Hence the rational cohomology ring of ΩE_6 is given by

$$H^*(\Omega E_6; \mathbb{Q}) = \mathbb{Q}[u_1, u_4, u_5, u_7, u_8, u_{11}]$$

where $\deg(u_i) = 2i$. Furthermore we can choose the generators u_i's as integral classes such that they are not divisible and primitive in $H^*(\Omega E_6)$. Thus

$$PH^*(\Omega E_6) = \langle u_1, u_4, u_5, u_7, u_8, u_{11} \rangle$$

as a free module.

Now we briefly review the results of R. Bott (for details see [3]): Let G be a compact connected semisimple Lie group with trivial center and ΩG the space of loops on G. Suppose that a homomorphism $s\colon S^1 \longrightarrow G$ is given. Let T be a maximal torus of G containing the image of s and C_s the centralizer of the image of s in G. Define the map

$$g_s\colon G/C_s \longrightarrow \Omega_0 G$$

by $g_s(q)(t) = xs(t)x^{-1}s(t)^{-1}$ for $q = xC_s \in G/C_s$, $t \in S^1$ where $\Omega_0 G$ denotes the identity component of ΩG.

Let $\tilde{G}$ be the universal covering group of G and d the covering dimension. Note that $\Omega_0 G \cong \Omega\tilde{G}$. Let $\tilde{s}\colon S^1 \longrightarrow \tilde{G}$ be a lift of s and $C_{\tilde{s}}$ the centralizer of the image of $\tilde{s}$ in $\tilde{G}$. Note that $G/C_s \cong \tilde{G}/C_{\tilde{s}}$. Define the map

$$f_s\colon \tilde{G}/C_{\tilde{s}} \longrightarrow \Omega\tilde{G}$$

by $f_s(q)(t) = xs(t)x^{-1}$ for $q = xC_{\tilde{s}} \in \tilde{G}/C_{\tilde{s}}$.

Accoding to Bott, if s is the "generating circle" the image of the homomorphism

$$g_{s*}\colon H_*(G/C_s) \longrightarrow H_*(\Omega_0 G)$$

generates $H_*(\Omega_0 G)$ as an algebra. (In this case g_s is called the "generating map" and G/C_s the "generating variety") Dual statement is as follows:

$$g_s^*\colon H^*(\Omega_0 G) \longrightarrow H^*(G/C_s)$$

is a split monomorphism when restricted to $PH^*(\Omega_0 G)$. Moreover since $\Omega_0 G \cong \Omega\tilde{G}, G/C_s \cong \tilde{G}/C_{\tilde{s}}$ we can compare the images of g_s^* and f_s^*. Then the following holds ([3], Propositon 7.1):

$$f_s^*|_{PH^*(\Omega\tilde{G})} = d \cdot g_s^*|_{PH^*(\Omega_0 G)}.$$

Bott also gave the method of computing the image of f_s^* by means of the Borel description of the rational cohomology ring of $\tilde{G}/C_{\tilde{s}}$ (see [3], Theorem 4). We carry out his program in the case of E_6.

As is well known E_6 has non-trivial center and it is of order 3. Hence the generating circle is defined in the adjoint group $AdE_6 = E_6/\text{center}$. Since AdE_6 is simply laced, all minimal circles are generating circles ([3], page 43). So we take

$$s = \text{ the dual of } \alpha_1.$$

Then the centralizer of the one dimensional torus $\operatorname{Im} s$ is $C_1 = T^1 \cdot \mathrm{Spin}(10)$ and the generating variety corresponding to s is

$$E_6/C_1 = E_6/T^1 \cdot \mathrm{Spin}(10) = EIII.$$

In view of (2.1), (2.6) the derivation corresponding to s is

$$\theta_s = \frac{4}{3}\frac{\partial}{\partial t_0} : \mathbb{Q}[t_0, \epsilon_1, \ldots, \epsilon_5] \longrightarrow \mathbb{Q}[t_0, \epsilon_1, \ldots, \epsilon_5]. \tag{3.1}$$

We have to compute the image of the composition

$$H^*(BT;\mathbb{Q})^{W(E_6)} \hookrightarrow H^*(BT;\mathbb{Q})^{W(C_1)} \xrightarrow{\theta_s} H^*(BT;\mathbb{Q})^{W(C_1)} \xrightarrow{\iota} H^*(EIII;\mathbb{Q}) \tag{3.2}$$

where ι is natural surjection.

Because θ_s is a derivation and $H^*(BT;\mathbb{Q})^{W(C_1)}/\mathfrak{b}_{13} \overset{\iota}{\underset{\sim}{\longrightarrow}} H^*(EIII;\mathbb{Q})$ we have only to compute the image of I_n, $n = 2, 5, 6, 8, 9, 12$. The computation proceeds as follows:

1. First we apply (3.1) to Lemma 2.4 and obtain $\theta_s(I_n)$ for $n = 2, 5, 6, 8, 9, 12$.
2. Next using Lemma 2.5, 2.6 we rewrite $\iota(\theta_s(I_n))$ in terms of t_0, w.

Then we obtain the results given by Table 4.

Therefore by a characterization of primitive elements given in [11], Proposition 5 (ii) we obtain

Proposition 3.1. *There exist unique primitive elements* a_1 (resp. b_4, c_5, d_7, e_8, f_{11}) *of* $H^*(\Omega E_6)$ *such that* $g_s^*(a_1) = a$ (resp. $g_s^*(b_4) = b$, $g_s^*(c_5) = c$, $g_s^*(d_7) = d$, $g_s^*(e_8) = e$, $g_s^*(f_{11}) = f$) *and*

$$PH^*(\Omega E_6) = \langle a_1, b_4, c_5, d_7, e_8, f_{11}\rangle$$

Table 4.

n	$\iota(\theta_s(I_n))$	deg	
2	$2^4 \cdot 3a$	2	$a = t_0$
5	$-2^7 \cdot 3 \cdot 5b$	8	$b = t_0^4 - 2w$
6	$2^9 \cdot 3^2 c$	10	$c = 3t_0^5 - 5t_0 w$
8	$-2^{12} \cdot 3 \cdot 5d$	14	$d = 4t_0^7 - 7t_0^3 w$
9	$2^{11} \cdot 3^3 \cdot 7e$	16	$e = 3t_0^8 - 4t_0^4 w - 2w^2$
12	$2^{15} \cdot 3^4 \cdot 5f$	22	$f = 11t_0^7 w - 19t_0^3 w^2$

Table 5.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	a	t_0^2	t_0^3	b	c	t_0^6	d	e			
					b'	c'	$t_0^2 w$	d'	e'	$t_0^5 w$	$t_0^6 w$	f
									e''	$t_0 w^2$	$t_0^2 w^2$	f'

as a free module.

4. The algebra strucutre of $H_*(\Omega E_6)$

The purpose of this section is to determine the algebra structure of $H_*(\Omega E_6)$. First note that the rational homology ring $H_*(\Omega E_6; \mathbb{Q})$ is of the form

$$H_*(\Omega E_6; \mathbb{Q}) = \mathbb{Q}[x_2, x_8, x_{10}, x_{14}, x_{16}, x_{22}] \tag{4.1}$$

where $\deg(x_i) = i$ ([9]).

By Corollary 2.8 we can choose an additive basis of $H^*(EIII)$ for degree ≤ 22 as in Table 5, where a, b, c, d, e and f are as in Proposition 3.1 and b', c', d', e', f' and e'' are determined by the following equations:

$$B \cdot \begin{pmatrix} t_0^4 \\ w \end{pmatrix} = \begin{pmatrix} b \\ b' \end{pmatrix}, \ C \cdot \begin{pmatrix} t_0^5 \\ t_0 w \end{pmatrix} = \begin{pmatrix} c \\ c' \end{pmatrix}, \ D \cdot \begin{pmatrix} t_0^7 \\ t_0^3 w \end{pmatrix} = \begin{pmatrix} d \\ d' \end{pmatrix},$$
$$E \cdot \begin{pmatrix} t_0^8 \\ t_0^4 w \\ w^2 \end{pmatrix} = \begin{pmatrix} e \\ e' \\ e'' \end{pmatrix}, \ F \cdot \begin{pmatrix} t_0^7 w \\ t_0^3 w^2 \end{pmatrix} = \begin{pmatrix} f \\ f' \end{pmatrix}$$

where B, C, D, F (resp. E) are 2×2 (resp. 3×3) matrices over $\mathbb{Z}$ whose determinant is 1; for example

$$B = \begin{pmatrix} 1 & -2 \\ k & l \end{pmatrix} \tag{4.2}$$

with k, $l \in \mathbb{Z}$ such that $2k + l = 1$.

We take the corresponding dual basis of $H_*(EIII)$ and denote the dual of $x \in H^*$ by $x_* \in H_*$. Furthermore we define the elements of $H_*(\Omega E_6)$ by

$$\begin{aligned}&\sigma_1 = g_{s*}(a_*),\ \sigma_4 = g_{s*}(b_*),\ \sigma_5 = g_{s*}(c_*),\ \sigma_7 = g_{s*}(d_*),\\&\sigma_8 = g_{s*}(e_*),\ \sigma_{11} = g_{s*}(f_*),\\&\sigma_i = g_{s*}((t_0^i)_*) \quad \text{for} \quad i = 2,\ 3.\end{aligned}$$

Then in view of Proposition 3.1 we obtain

Lemma 4.1. (i) σ_i, $i = 1, 4, 5, 7, 8, 11$ *are indecomposable and not divisible.*
(ii) *The coproduct of* σ_2, σ_3 *is given by*

$$\tilde{\psi}(\sigma_2) = \sigma_1 \otimes \sigma_1,$$
$$\tilde{\psi}(\sigma_3) = \sigma_2 \otimes \sigma_1.$$

(iii) *The following relations hold*:

$${\sigma_1}^2 = 2\sigma_2, \quad \sigma_1\sigma_2 = 3\sigma_3.$$

Proof of Theorem 1.1 (i). In view of (4.1) and Lemma 4.1, in order to determine the algebra structure of $H_*(\Omega E_6)$ we have to consider the "divisibility" among σ_i's. Fortunately the mod p homology rings $H_*(\Omega E_6; \mathbb{Z}/p\mathbb{Z})$ for each prime p are studied by several authors and the algebra structure is determined for all cases ([8], [7], [5]). We exhibit their results:

$$H_*(\Omega E_6; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_8, x_{10}, x_{14}, x_{16}, x_{22}]/(x_2^2), \tag{4.3}$$
$$H_*(\Omega E_6; \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}[x_2, x_6, x_8, x_{10}, x_{14}, x_{16}, x_{22}]/(x_2^3), \tag{4.4}$$
$$H_*(\Omega E_6; \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}[x_2, x_8, x_{10}, x_{14}, x_{16}, x_{22}] \quad \text{for} \quad p \geq 5 \tag{4.5}$$

where $\deg(x_i) = i$ and moreover we can take x_{2i} as the mod p reduction of σ_i for each case. Therefore if "divisibility" occurs in degree $k \geq 8$ there exists an integral class σ, $\deg(\sigma) = k$ and an intreger $n \geq 2$ such that

$$n \cdot \sigma = f(\sigma_1, \ldots, \sigma_{11}) \tag{4.6}$$

where f is a homogeneous polynomial of degree k in $\sigma_1, \ldots, \sigma_{11}$ with integer coeff-ficients. Let p be a prime number which divides n and apply the mod p reduction on both sides of (4.6). Then a relation of degree $k \geq 8$ arises in mod p homology and it contradicts the previous results. Hence we obtain the required result. □

Table 6.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	σ_1	σ_2	σ_3	τ_4	$\sigma_1\tau_4$	τ_6	$\sigma_1\tau_6$	$\sigma_2\tau_6$	$\sigma_3\tau_6$	$\tau_4\tau_6$	$\sigma_1\tau_4\tau_6$
					σ_4	$\sigma_1\sigma_4$	$\sigma_2\sigma_4$	$\sigma_3\sigma_4$	$\tau_4\sigma_4$	$\sigma_1\tau_4\sigma_4$	$\sigma_4\tau_6$	$\sigma_1\sigma_4\tau_6$
						σ_5	$\sigma_1\sigma_5$	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	$\tau_4\sigma_5$	$\sigma_1\tau_4\sigma_5$	$\sigma_5\tau_6$
								σ_7	$\sigma_1\sigma_7$	$\sigma_2\sigma_7$	$\sigma_3\sigma_7$	$\tau_4\sigma_7$
									${\sigma_4}^2$	$\sigma_1{\sigma_4}^2$	$\sigma_2{\sigma_4}^2$	$\sigma_3{\sigma_4}^2$
									σ_8	$\sigma_1\sigma_8$	$\sigma_2\sigma_8$	$\sigma_3\sigma_8$
										$\sigma_4\sigma_5$	$\sigma_1\sigma_4\sigma_5$	$\sigma_2\sigma_4\sigma_5$
											${\sigma_5}^2$	$\sigma_1{\sigma_5}^2$
												$\sigma_4\sigma_7$
												σ_{11}

Table 7.

deg	0	2	4	6	8	10	12	14	16	18	20	22
	1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
					b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
						c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
								d_7	d_8	d_9	d_{10}	d_{11}
									d_8'	${d_9}'$	${d_{10}}'$	${d_{11}}'$
									e_8	e_9	e_{10}	e_{11}
										${e_9}'$	${e_{10}}'$	${e_{11}}'$
											${e_{10}}''$	${e_{11}}''$
												${e_{11}}'''$
												f_{11}

5. The Hopf algebra structure of $H_*(\Omega E_6)$

In this section we determine the Hopf algebra structure of $H_*(\Omega E_6)$ using the "generating variety" $EIII$.

At first by Theorem 1.1 (i) we can choose an additive basis of $H_*(\Omega E_6)$ for degree ≤ 22 as in Table 6, where $\tau_4 = {\sigma_2}^2 - \sigma_1\sigma_3,\ \tau_6 = {\sigma_2}^3 - 4{\sigma_3}^2$.

We take the corresponding dual basis of $H^*(\Omega E_6)$ as in Table 7.

In view of Lemma 4.1 we obtain the results given by Table 8.

Next consider the case of degree 8: Since $\tau_4 = {\sigma_2}^2 - \sigma_1\sigma_3$, the coproduct of τ_4 is given by

(5.1) $$\tilde{\psi}(\tau_4) = 2\sigma_3 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_2.$$

Table 8.

deg	basis	$g_s{}^*$-image	relation
2	a_1	$a = t_0$	
4	a_2	t_0^2	$a_1^2 = a_2$
6	a_3	t_0^3	$a_2a_1 = a_1^3 = a_3$

We shall determine the coproduct of σ_4: We may put

$$\tilde{\psi}(\sigma_4) = m\sigma_3 \otimes \sigma_1 + \cdots$$

for some $m \in \mathbb{Z}$. On the other hand since $H^8(\Omega E_6) = \langle a_4, b_4 \rangle$, we can put

$$a_3a_1 = \mu a_4 + \nu b_4 \tag{5.2}$$

for some μ, $\nu \in \mathbb{Z}$. Then

$$\mu = \langle a_3a_1, \tau_4 \rangle = \langle a_3 \otimes a_1, \psi(\tau_4) \rangle = 2,$$
$$\nu = \langle a_3a_1, \sigma_4 \rangle = \langle a_3 \otimes a_1, \psi(\sigma_4) \rangle = m.$$

Therefore applying g_s^* on both sides of (5.2)

$$\begin{aligned} 2g_s^*(a_4) &= g_s^*(a_3a_1) - mg_s^*(b_4) \\ &= t_0^3 \cdot t_0 - m \cdot b \\ &= t_0^4 - m(t_0^4 - 2w) \\ &= (1 - m)t_0^4 + 2mw. \end{aligned}$$

On the other hand since $H^8(EIII) = \langle b, b' \rangle$, we can put

$$g_s^*(a_4) = \mu b + \nu b'$$

for some μ, $\nu \in \mathbb{Z}$. Then

$$\mu = \langle g_s^*(a_4), b_* \rangle = \langle a_4, g_{s*}(b_*) \rangle = \langle a_4, \sigma_4 \rangle = 0.$$

Hence

$$g_s^*(a_4) = \nu b' = \nu(kt_0^4 + lw)$$

with $2k + l = 1$. Combining these gives

$$(1 - m)t_0^4 + 2mw = 2\nu kt_0^4 + 2\nu lw.$$

Table 9.

deg	basis	$g_s{}^*$-image	relation
8	a_4	$b' = w$	$a_3 a_1 = 2a_4 + b_4$
8	b_4	$b = t_0^4 - 2w$	

Since $H^8(EIII) = \langle t_0^4, w \rangle$ we have

$$\begin{cases} 1 - m = 2\nu k, \\ \quad m \quad = \nu l. \end{cases} \tag{5.3}$$

Hence $\nu = 1$ and we may take $k = 0$, $l = 1$. Then $m = 1$ and $b' = w$.

Similarly we may put

$$\tilde{\psi}(\sigma_4) = \cdots + n\sigma_2 \otimes \sigma_2 + \cdots$$

for some $n \in \mathbb{Z}$. On the other hand we can put

$$a_2^2 = \mu a_4 + \nu b_4$$

for some μ, $\nu \in \mathbb{Z}$. Then

$$\mu = \langle a_2^2, \tau_4 \rangle = \langle a_2 \otimes a_2, \psi(\tau_4) \rangle = 2,$$
$$\nu = \langle a_2^2, \sigma_4 \rangle = \langle a_2 \otimes a_2, \psi(\sigma_4) \rangle = n.$$

But since $a_1^2 = a_2$, $a_1 a_2 = a_3$

$$a_2^2 = a_3 a_1 = 2a_4 + b_4.$$

Thus $n = \nu = 1$.

Hence we obtain the results given by Table 9 and

$$\tilde{\psi}(\tau_4) = 2\sigma_3 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_2,$$
$$\tilde{\psi}(\sigma_4) = \sigma_3 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2.$$

Note that for the element $\tilde{\sigma}_4 = \tau_4 - 2\sigma_4$,

$$\tilde{\psi}(\tilde{\sigma}_4) = 0.$$

We have to continue this argument up to degree ≤ 22. But as well as the case of F_4 ([11]), the remainder is no more than a tedious computation. So we only exhibit the data and the results:

$$\tilde{\psi}(\sigma_5) = \tau_4 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_2, \tag{5.4}$$

Table 10.

deg	basis		$g_s{}^*$-image	relation
10	a_5	c'	$= -t_0^5 + 2t_0 w$	$a_4 a_1 = 5a_5 + 2b_5 + c_5$
10	b_5	$-c'$	$= t_0^5 - 2t_0 w$	$b_4 a_1 = b_5$
10	c_5	c	$= 3t_0^5 - 5t_0 w$	
12	a_6		$-5t_0^6 + 9t_0^2 w$	$a_5 a_1 = a_6 + b_6 + c_6$
12	b_6		$t_0^6 - 2t_0^2 w$	$b_5 a_1 = b_6$
12	c_6		$3t_0^6 - 5t_0^2 w$	$c_5 a_1 = c_6$
14	a_7		0	$a_6 a_1 = 7a_7 + 2b_7 + 3c_7 - d_7$
14	b_7	$-d'$	$= t_0^7 - 2t_0^3 w$	$b_6 a_1 = b_7$
14	c_7	d'	$= -t_0^7 + 2t_0^3 w$	$c_6 a_1 = c_7 + d_7$
14	d_7	d	$= 4t_0^7 - 7t_0^3 w$	

(5.5) $$\tilde{\psi}(\sigma_7) = (-\tau_6 + \sigma_1\sigma_5) \otimes \sigma_1 + \sigma_5 \otimes \sigma_2 + \tau_4 \otimes \sigma_3.$$

Note that for the elements $\tilde{\sigma}_5 = \sigma_1\tau_4 - 5\sigma_5$, $\tilde{\sigma}_7 = 4\sigma_1\tau_6 - 7\sigma_2\sigma_5 + 7\sigma_7$,

$$\tilde{\psi}(\tilde{\sigma}_5) = 0,$$
$$\tilde{\psi}(\tilde{\sigma}_7) = 0.$$

Since rank $H^{16}(EIII) = 3$, the computation for degree = 16 is a little complicated. So first we take

$$E = \begin{pmatrix} 3 & -4 & -2 \\ 2 & -2 & -3 \\ -2 & 3 & 1 \end{pmatrix}$$

so that

$$\begin{pmatrix} e \\ e' \\ e'' \end{pmatrix} = \begin{pmatrix} 3 & -4 & -2 \\ 2 & -2 & -3 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} t_0^8 \\ t_0^4 w \\ w^2 \end{pmatrix} = \begin{pmatrix} 3t_0^8 - 4t_0^4 w - 2w^2 \\ 2t_0^8 - 2t_0^4 w - 3w^2 \\ -2t_0^8 + 3t_0^4 w + w^2 \end{pmatrix}.$$

Then we obtain the results given by Table 11 and

(5.6) $$\begin{aligned}\tilde{\psi}(\sigma_8) =& (\sigma_3\sigma_4 + \sigma_2\sigma_5) \otimes \sigma_1 + (\tau_6 - \sigma_2\sigma_4 + \sigma_1\sigma_5) \otimes \sigma_2 \\ &+ (\sigma_1\tau_4 + \sigma_1\sigma_4 + \sigma_5) \otimes \sigma_3 + 2\tau_4 \otimes \tau_4 - \sigma_4 \otimes \sigma_4.\end{aligned}$$

Note that for the element $\tilde{\sigma}_8 = \sigma_2\tau_6 - 4\sigma_3\sigma_5 + 2\sigma_4{}^2 + 4\sigma_8$,

$$\tilde{\psi}(\tilde{\sigma}_8) = 0.$$

Table 11.

deg	basis	$g_s{}^*$-image	relation		
16	a_8	$-e' - 3e''$	a_7a_1	$=$	$4a_8 + b_8 + c_8 - d_8$
16	b_8	$e' + e''$	b_7a_1	$=$	$2b_8 + 2d_8' - e_8$
16	c_8	$2e' + 8e''$	c_7a_1	$=$	$c_8 + 2d_8 + e_8$
16	d_8	$-e' - 3e''$	d_7a_1	$=$	d_8
16	d_8'	$-e' - 2e''$	$a_4{}^2$	$=$	$70a_8 + 34b_8 + 28c_8 + 4d_8 + 17d_8' + 2e_8$
16	e_8	e	a_4b_4	$=$	b_8
			$b_4{}^2$	$=$	$2d_8' - e_8$

$$
\begin{aligned}
\tilde{\psi}(\sigma_{11}) =& (-2\tau_4\tau_6 + \sigma_4\tau_6 + 2\sigma_1\tau_4\sigma_5 - \sigma_3\sigma_7 - \sigma_1\sigma_4\sigma_5 - \sigma_5{}^2) \otimes \sigma_1 \\
& + (-2\sigma_3\tau_6 + 4\tau_4\sigma_5 - \sigma_2\sigma_7 - \sigma_4\sigma_5) \otimes \sigma_2 \\
& + (-\sigma_2\tau_6 - \tau_4\sigma_4 + 4\sigma_3\sigma_5 - \sigma_1\sigma_7) \otimes \sigma_3 \\
& - \sigma_3\sigma_4 \otimes \tau_4 + \sigma_2\sigma_5 \otimes \tau_4 - \sigma_7 \otimes \sigma_4 - \tau_6 \otimes \sigma_1\tau_4 + 4\tau_6 \otimes \sigma_5 + \tau_6 \otimes \sigma_1\sigma_4 \\
& - \sigma_2\sigma_4 \otimes \sigma_5 + \sigma_1\sigma_5 \otimes \sigma_1\tau_4 - \sigma_1\sigma_5 \otimes \sigma_1\sigma_4 - 2\sigma_1\sigma_5 \otimes \sigma_5.
\end{aligned} \tag{5.7}
$$

Note that for the element $\tilde{\sigma}_{11} = 6\sigma_1\tau_4\tau_6 - 44\sigma_5\tau_6 + 11\sigma_1\sigma_5{}^2 + 11\sigma_4\sigma_7 + 11\sigma_{11}$,

$$\tilde{\psi}(\tilde{\sigma}_{11}) = 0.$$

Consequently we established Theorem 1.1.

Table 12.

deg	basis	$g_s{}^*$-image	relation
18	a_9	$-18t_0^5w + 31t_0w^2$	$a_8a_1 = 3a_9 + 2b_9 + c_9 - d_9$
18	b_9	$4t_0^5w - 7t_0w^2$	$b_8a_1 = 5b_9 + 4d'_9 - 2e_9 + e'_9$
18	c_9	$32t_0^5w - 55t_0w^2$	$c_8a_1 = 2c_9 + 3d_9 + 3e_9 + e'_9$
18	d_9	$-7t_0^5w + 12t_0w^2$	$d_8a_1 = d_9$
18	d'_9	$-4t_0^5w + 7t_0w^2$	$d'_8a_1 = d'_9$
18	e_9	$-4t_0^5w + 7t_0w^2$	$e_8a_1 = e_9$
18	e'_9	$-11t_0^{+}19t_0w^2$	
			$a_5a_4 = 42a_9 + 60b_9 + 25c_9 + 3d_9 + 29d'_9$
			$+4e_9 + 12e'_9$
			$b_5a_4 = 5b_9 + 4d'_9 - 2e_9 + e'_9$
			$c_5a_4 = c_9 + 3d_9 + 2e_9$
			$a_5b_4 = b_9$
			$b_5b_4 = 2d'_9 - e_9$
			$c_5b_4 = e'_9$
20	a_{10}	$-22t_0^6w + 38t_0^2w^2$	$a_9a_1 = 5a_{10} + b_{10} + 3c_{10} - d_{10}$
20	b_{10}	$19t_0^6w - 33t_0^2w^2$	$b_9a_1 = b_{10} + 2d'_{10} - e_{10} + e'_{10}$
20	c_{10}	$22t_0^6w - 38t_0^2w^2$	$c_9a_1 = 5c_{10} + 2d_{10} + 3e_{10} + 2e'_{10} + 2e''_{10}$
20	d_{10}	$-7t_0^6w + 12t_0^2w^2$	$d_9a_1 = d_{10}$
20	d'_{10}	$-4t_0^6w + 7t_0^2w^2$	$d'_9a_1 = d'_{10}$
20	e_{10}	$-4t_0^6w + 7t_0^2w^2$	$e_9a_1 = e_{10}$
20	e'_{10}	$-11t_0^6w + 19t_0^2w^2$	$e'_9a_1 = e'_{10}$
20	e''_{10}	$-15t_0^6w + 26t_0^2w^2$	
			$a_6a_4 = 210a_{10} + 97b_{10} + 246c_{10} + 6d_{10}$
			$+135d'_{10} + 21e_{10} + 114e'_{10} + 48e''_{10}$
			$b_6a_4 = 5b_{10} + 14d'_{10} - 7e_{10} + 6e'_{10}$
			$c_6a_4 = 5c_{10} + 5d_{10} + 5e_{10} + 2e'_{10} + 2e''_{10}$
			$a_6b_4 = b_{10}$
			$b_6b_4 = 2d'_{10} - e_{10}$
			$c_6b_4 = e'_{10}$
			${a_5}^2 = 42a_{10} + 20b_{10} + 50c_{10} + 2d_{10}$
			$+29d'_{10} + 4e_{10} + 24e'_{10} + 10e''_{10}$
			$a_5b_5 = b_{10} + 2d'_{10} - e_{10} + e'_{10}$
			$a_5c_5 = c_{10} + d_{10} + e_{10}$
			${b_5}^2 = 2d'_{10} - e_{10}$
			$b_5c_5 = e'_{10}$
			${c_5}^2 = 2e''_{10}$

Table 13.

deg	basis	$g_s{}^*$-image	relation
22	a_{11}	0	$a_{10}a_1 = 11a_{11} + 2b_{11} + c_{11} - d_{11} - 2f_{11}$
22	b_{11}	0	$b_{10}a_1 = 7b_{11} + 4d'_{11} - 2e_{11} + 3e'_{11} - e'''_{11} + f_{11}$
22	c_{11}	$-f'$	$c_{10}a_1 = c_{11} + d_{11} + e_{11} + e'_{11} + 2e''_{11} + 2f_{11}$
22	d_{11}	$-f'$	$d_{10}a_1 = 2d_{11} + e'''_{11} - f_{11}$
22	d'_{11}	f'	$d'_{10}a_1 = d'_{11}$
22	e_{11}	f'	$e_{10}a_1 = e_{11}$
22	e'_{11}	$-f'$	$e'_{10}a_1 = e'_{11} + e'''_{11} - f_{11}$
22	e''_{11}	f'	$e''_{10}a_1 = e''_{11} - f_{11}$
22	e'''_{11}	f'	
22	f_{11}	f	
			$a_7a_4 = 330a_{11} + 147b_{11} + 636c_{11} + d_{11} + 65d'_{11} + 11e_{11} + 84e'_{11} + 72e''_{11}$
			$b_7a_4 = 35b_{11} + 34d'_{11} - 17e_{11} + 21e'_{11} + e'''_{11} - f_{11}$
			$c_7a_4 = 5c_{11} + 14d_{11} + 10e_{11} + 7e'_{11} + 12e''_{11} + 7e'''_{11} + f_{11}$
			$d_7a_4 = d_{11}$
			$a_7b_4 = b_{11}$
			$b_7b_4 = 2d'_{11} - e_{11}$
			$c_7b_4 = e'_{11}$
			$d_7b_4 = e'''_{11} - f_{11}$
			$a_6a_5 = 462a_{11} + 217b_{11} + 91c_{11} + 9d_{11} + 103d'_{11} + 15e_{11} + 129e'_{11} + 108e''_{11} + 5e'''_{11} - f_{11}$
			$b_6a_5 = 7b_{11} + 6d'_{11} - 3e_{11} + 4e'_{11}$
			$c_6a_5 = c_{11} + 3d_{11} + 2e_{11} + e'_{11} + 2e''_{11} + e'''_{11} + f_{11}$
			$a_6b_5 = 7b_{11} + 4d'_{11} - 2e_{11} + 3e'_{11} - e'''_{11} + f_{11}$
			$b_6b_5 = 2d'_{11} - e_{11}$
			$c_6b_5 = e'_{11} + e'''_{11} - f_{11}$
			$a_6c_5 = c_{11} + 3d_{11} + 2e_{11} + 4f_{11}$
			$b_6c_5 = e'_{11} + e'''_{11} - f_{11}$
			$c_6c_5 = 2e''_{11} - 2f_{11}$

where $f = 11t_0^7w - 19t_0^3w^2$, $f' = -4t_0^7w + 7t_0^3w^2$.

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