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LAMINATION OF THE MODULI SPACE OF CIRCLES AND THEIR LENGTH SPECTRUM FOR A NON-FLAT COMPLEX SPACE FORM

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1. Introduction

A smooth curve $\gamma\colon \mathbb{R} \to M$ parametrized by its arclength on a complete Riemannian manifold M is called a *circle* of *geodesic curvature* κ if it satisfies the differential equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t) = -\kappa^2\dot\gamma(t)$. Here κ is a non-negative constant and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection on M. When $\kappa = 0$, as γ is parametrized by its arclength, this equation is equivalent to the equation of geodesics. In this paper we study the set of congruence classes of circles on a non-flat complex space form, which is either a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$. We call two circles γ_1 and γ_2 on M are *congruent* if there exist an isometry φ of M and a constant t_0 satisfying $\gamma_1(t) = \varphi\circ\gamma_2(t+t_0)$ for all t. We denote by $\mathrm{Cir}(M)$ the set of all congruence classes of circles on M.

In the preceding papers [5] and [3], we studied length spectum of circles on non-flat complex space forms. We call a circle γ *closed* if it satisfies $\gamma(t) = \gamma(t+t_c)$ for every t with some positive constant t_c. The minimum positive t_c with this property is called the *length* of γ and is denoted by $\mathrm{length}(\gamma)$. For an open circle γ, a circle which is not closed, we set $\mathrm{length}(\gamma) = \infty$. The *length spectrum* $\mathcal{L}\colon \mathrm{Cir}(M) \to \mathbb{R}\cup\{\infty\}$ of circles is defined by $\mathcal{L}([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ denotes the congruence class containing γ. In these papers [5], [3], we find that the moduli spaces $\mathrm{Cir}(\mathbb{C}P^n)$ and $\mathrm{Cir}(\mathbb{C}H^n)$ of circles on non-flat complex space forms have a natural lamination structure: If we restrict the length spectrum $\mathcal{L}$ on each leaf, it is continuous. In the first half of this paper we study the phenomenon of circles at the boundary of each leaf. For a sequence $\{\sigma_\iota\}$ of closed curves $\sigma_\iota\colon S^1 = [0,1]/\!\sim\; \to M$ on M we shall call $\lim\sigma_\iota$ its limit curve if it exists. We study this lamination from the viewpoint of limit curves of circles.

The second half of this paper is devoted to add some resluts on length functions of circles on non-flat complex space forms. As two circles have the same geodesic

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curvature when they are congruent, we denote by $\mathrm{Cir}_\kappa(M)$ the set of all congruence classes of circles of geodesic curvature κ on M and by $\mathcal{L}_\kappa$ the restriction of $\mathcal{L}$ onto $\mathrm{Cir}_\kappa(M)$. We also call the image on the real line $\mathrm{LSpec}_\kappa(M) = \mathcal{L}_\kappa(\mathrm{Cir}_\kappa(M)) \cap \mathbb{R}$ the length spectrum of circles of geodesic curvature κ. For non-flat complex space forms it is know that the length spectrum of circles of geodesic curvature κ is a discrete subset of $\mathbb{R}$ for each κ, and hence we can define the j-th length function $\lambda_j(\kappa)$. We investigate its asymptotic behaviour and continuity with respect to geodesic curvature and clarify the relationship between such properties and the lamination structure. Our study on length functions shows the difference between Kähler and totally real circles and other circles. The author is grateful to the referee for his valuable advice.

2. Moduli space of circles on a non-flat complex space form

On a real space form, which is either a standard sphere $S^n(c)$ of curvature c, a Euclidean space $\mathbb{R}^n$ or a real hyperbolic space $H^n(-c)$ of curvature $-c$, it is clear that two circles are congruent each other if and only if they have the same geodesic curvature. Therefore their moduli spaces are very simple: $\mathrm{Cir}(S^n(c)) = (0, 2\pi/\sqrt{c}\,)$, $\mathrm{Cir}(\mathbb{R}^n) = \mathrm{Cir}(H^n(-c)) = (0, \infty)$. The length spectrum of a circle of geodesic curvature κ on $S^n(c)$ is $\{2\pi/\sqrt{\kappa^2 + c}\,\}$, that on $\mathbb{R}^n$ is $\{2\pi/\kappa\}$, and that on $H^n(-c)$ is $\{2\pi/\sqrt{\kappa^2 - c}\,\}$ if $\kappa > \sqrt{c}$. Thus the length spectrum $\mathcal{L}$ is continuous with respect to the canonical topology induced from the topology on the real line $\mathbb{R}$.

For circles on a Kähler manifold M we have another invariant. For a circle γ of positive geodesic curvature on M with complex structure J, we set $\tau_\gamma = \langle\dot\gamma, J\nabla_{\dot\gamma}\dot\gamma\rangle/\|\nabla_{\dot\gamma}\dot\gamma\|$. It is constant along γ and is called *complex torsion* of γ. We call a circle γ *Kähler* if $\tau_\gamma = \pm 1$ and *totally real* if $\tau_\gamma = 0$. Kähler circles and geodesics are interpreted as motions of charged particles under some magnetic fields (see [1], [2]). On a non-flat complex space form of complex dimension greater than 1, all geodesics are congruent each other and two circles of positive geodesic curvature are congruent if and only if they have the same geodesic curvature and the same absolute value of complex torsion (see [8]). So set theoretically the moduli spaces $\mathrm{Cir}(\mathbb{C}P^n)$ and $\mathrm{Cir}(\mathbb{C}H^n)$ are bijective to the set $[0, \infty) \times [0, 1]/\sim$, where $(0, \tau)$ and $(0, \mu)$ are identified. From the viewpoint of the orthonormal frame $\{\dot\gamma, \nabla_{\dot\gamma}\dot\gamma/\|\nabla_{\dot\gamma}\dot\gamma\|\}$ of a circle γ of positive geodsesic curvature, on each moduli space we have a topology induced from the topology on $\mathbb{R}^2$, and trivial foliations by complex torsion and by geodesic curvature. In the following sections we treat another natural lamination on these moduli spaces. (For the definition of a lamination see for example [7].)

3. Lamination of circles on a complex projective space

In the preceding paper [6], we studied circles on a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c and of complex dimension $n \geq 2$, and showed the following:

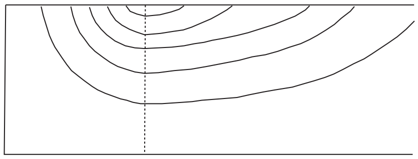

Fig. 1. Lamination on $\mathrm{Cir}\big(\mathbb{C}P^n(c)\big)$

(i) Every Kähler circle of geodesic curvature κ is a closed curve with length $2\pi/\sqrt{\kappa^2+c}$.
(ii) Every totally real circle of geodesic curvature κ is a closed curve with length $4\pi/\sqrt{4\kappa^2+c}$.
(iii) For $\kappa\ (>0)$ and $\tau\ (0<\tau<1)$ we denote by $a_{\kappa,\tau},\ b_{\kappa,\tau},\ d_{\kappa,\tau}\ (a_{\kappa,\tau}<b_{\kappa,\tau}<d_{\kappa,\tau})$ the solutions for the cubic equation

$$c\theta^3-(4\kappa^2+c)\theta+2\sqrt{c}\,\kappa\tau=0. \tag{3.1}$$

A circle of geodesic curvature κ and of complex torsion τ is closed if and only if one of (hence all of) the ratios $a_{\kappa,\tau}/b_{\kappa,\tau},\ b_{\kappa,\tau}/d_{\kappa,\tau},\ d_{\kappa,\tau}/a_{\kappa,\tau}$ is rational. When it is closed, its length is $(4\pi/\sqrt{c}\,)\times \mathrm{L.C.M}\{(b_{\kappa,\tau}-a_{\kappa,\tau})^{-1},(d_{\kappa,\tau}-a_{\kappa,\tau})^{-1}\}$.

For $\kappa>0$, we denote by $[\gamma_{\kappa,\tau}]$ the congruence classes of circles on $\mathbb{C}P^n(c)$ with geodesic curvature κ and complex torsion τ. Since the cubic equation (3.1) for (κ,τ) and that for $(\sqrt{2c}/4,\,3\sqrt{3}\,c\kappa\tau(4\kappa^2+c)^{-3/2})$ are homothetic, we have a map of normalization

$$\Phi_\kappa:\ \mathrm{Cir}_\kappa\big(\mathbb{C}P^n(c)\big)\setminus\{[\gamma_{\kappa,1}]\}\longrightarrow \mathrm{Cir}_{\sqrt{2c}/4}\big(\mathbb{C}P^n(c)\big)\setminus\{[\gamma_{\sqrt{2c}/4,1}]\}$$

between the sets of congruence classes of circles of prescribed geodesic curvature defined by

$$\Phi_\kappa\big([\gamma_{\kappa,\tau}]\big)=\Big[\gamma_{\sqrt{2c}/4,\,3\sqrt{3}\,c\kappa\tau(4\kappa^2+c)^{-3/2}}\Big].$$

This map gives a lamination on the moduli space $\mathrm{Cir}(\mathbb{C}P^n(c))$ of circles on $\mathbb{C}P^n(c)$ whose leaves are

$$\mathcal{F}_\mu=\begin{cases}\{[\gamma_{\kappa,0}]\mid \kappa>0\}, & \text{if } \mu=0,\\ \{[\gamma_{\kappa,\tau}]\mid 3\sqrt{3}\,c\kappa\tau(4\kappa^2+c)^{-3/2}=\mu,\ 0<\tau<1\}, & \text{if } 0<\mu<1,\\ \{[\gamma_{\kappa,1}]\mid \kappa\ge 0\}, & \text{if } \mu=1,\end{cases}$$

where $[\gamma_{0,1}]$ denotes the congruence class of geodesics (see Fig. 1).

In this section we study circles on each leaf of this lamination. Needless to say, the length $\lambda_1(\kappa)=2\pi/\sqrt{\kappa^2+c}$ of Kähler circles of geodesic curvature κ and the length $\lambda_2(\kappa) = 4\pi/\sqrt{4\kappa^2+c}$ of totally real circles satisfy

$$\lim_{\kappa\downarrow 0}\lambda_1(\kappa) = \frac{2\pi}{\sqrt{c}},\ \lim_{\kappa\downarrow 0}\lambda_2(\kappa) = \frac{4\pi}{\sqrt{c}},\ \lim_{\kappa\to\infty}\lambda_1(\kappa) = \lim_{\kappa\to\infty}\lambda_2(\kappa) = 0.$$

It is well-known that the length of an arbitrary geodesic on $\mathbb{C}P^n(c)$ is $2\pi/\sqrt{c}$. This assures that the point $[(0,\tau)] \in \mathrm{Cir}(\mathbb{C}P^n(c)) = [0,\infty)\times[0,1]/\sim$ corresponding to geodesics is not a limit point of the leaf $\mathcal{F}_0$ with respect to the canonical topology on $\mathrm{Cir}(\mathbb{C}P^n(c))$. To see this phenomenon more precisely we investigate their limit curves. Let $\varpi\colon S^{2n+1}(c/4) \to \mathbb{C}P^n(c)$ denote the Hopf fibration of a standard sphere onto a complex projective space. For the sake of simplicity, our discussion goes through only with the case $c = 4$ but the results hold for general c.

First of all, we study the leaf $\mathcal{F}_1$ of Kähler circles. Choose a unit tangent vector $u \in T_x\mathbb{C}P^n(c)$ at a point $x \in \mathbb{C}P^n(c)$ and denote by $\gamma_{\kappa,1}$ the Kähler circle of geodesic curvature κ on $\mathbb{C}P^n(c)$ with initial condition $\gamma_{\kappa,1}(0) = x$, $\dot\gamma_{\kappa,1}(0) = u$, $\nabla_{\dot\gamma_{\kappa,1}}\dot\gamma_{\kappa,1}(0) = -\kappa Ju$. Since it is a closed curve with length $2\pi/\sqrt{\kappa^2+c}$, we can define a curve $\sigma_{\kappa,1}\colon S^1 \to \mathbb{C}P^n(c)$ by $\sigma_{\kappa,1}(s) = \gamma_{\kappa,1}(2\pi s/\sqrt{\kappa^2+c})$. We are interested in the feature of the family $\mathcal{S}_1 = \{\sigma_{\kappa,1} \mid \kappa > 0\}$ of closed curves. When $c = 4$ and $x = \varpi(z)$, by identifying the tangent space $T_{\varpi(z)}\mathbb{C}P^n(4)$ with the horizontal subspace of T_zS^{2n+1} with respect to the Hopf fibration, we see a horizontal lift $\tilde\gamma_{\kappa,1}$ on $S^{2n+1}(1)$ of $\gamma_{\kappa,1}$ is of the form

$$\tilde\gamma_{\kappa,1}(t) = e^{-\kappa it/2}\Big\{\cos\frac{1}{2}\sqrt{\kappa^2+4}\,t\cdot z + (\kappa^2+4)^{-1/2}\sin\frac{1}{2}\sqrt{\kappa^2+4}\,t\cdot(\kappa iz+2u)\Big\}$$

(see [1] or [6]). This shows that

$$\sigma_{\kappa,1}(s) = \varpi\Big(\cos\pi s\cdot z + (\kappa^2+4)^{-1/2}\sin\pi s\cdot(\kappa iz+2u)\Big). \tag{3.2}$$

Hence we have

$$\lim_{\kappa\downarrow 0}\sigma_{\kappa,1}(s) = \varpi\big(\cos\pi s\cdot z + \sin\pi s\cdot u\big),$$

which is a closed geodesic of initial vector u, and $\lim_{\kappa\to\infty}\sigma_{\kappa,1}(s) = x$. Thus it is natural to think that geodesics are contained in the leaf of Kähler circles.

Next we study the leaf $\mathcal{F}_0$ of totally real circles. For a given pair $\{u,v\}$ of orthnormal tangent vectors at $x \in \mathbb{C}P^n(c)$ with $\langle u, Jv\rangle = 0$, we denote by $\gamma_{\kappa,0}$ the totally real circle on $\mathbb{C}P^n(c)$ with initial condition $\gamma_{\kappa,0}(0) = x$, $\dot\gamma_{\kappa,0}(0) = u$, $\nabla_{\dot\gamma_{\kappa,0}}\dot\gamma_{\kappa,0}(0) = \kappa v$. Since it is a closed curve with length $4\pi/\sqrt{4\kappa^2+c}$, we can define a curve $\sigma_{\kappa,0}\colon S^1 \to \mathbb{C}P^n(c)$ by $\sigma_{\kappa,0}(s) = \gamma_{\kappa,0}(4\pi s/\sqrt{4\kappa^2+c})$. The family $\mathcal{S}_0 = \{\sigma_{\kappa,0} \mid \kappa > 0\}$ of closed curves also has a limit curve. When $c = 4$ and $x = \varpi(z)$, since a horizontal lift $\tilde\gamma_{\kappa,0}$ on $S^{2n+1}(1)$ of $\gamma_{\kappa,0}$ is of the form

$$\tilde\gamma_{\kappa,0}(t) = \frac{\kappa}{\kappa^2+1}(\kappa z+v) + \frac{\cos\sqrt{\kappa^2+1}\,t}{\kappa^2+1}(z-\kappa v) + \frac{\sin\sqrt{\kappa^2+1}\,t}{\sqrt{\kappa^2+1}}u$$

(see [6]), we find

$$\lim_{\kappa\downarrow 0}\sigma_{\kappa,0}(s) = \varpi\big(\cos 2\pi s\cdot z + \sin 2\pi s\cdot u\big),$$

which is a double covering of the closed geodesic with initial vector u, and trivially find $\lim_{\kappa\to\infty}\sigma_{\kappa,0}(s) = x$.

Proposition 1. *On a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$ the following properties hold.*
(1) *The family $\mathcal{S}_1 = \{\sigma_{\kappa,1} \mid \kappa > 0\}$ of closed curves derived from Kähler circles has a limit curve. The curve $\lim_{\kappa\downarrow 0}\sigma_{\kappa,1}$ is a geodesic.*
(2) *The family $\mathcal{S}_0 = \{\sigma_{\kappa,0} \mid \kappa > 0\}$ of closed curves derived from totally real circles also has a limit curve. The curve $\lim_{\kappa\downarrow 0}\sigma_{\kappa,0}$ is a 2-fold covering of a geodesic.*

We now concern ourselves with general leaves. By the property (iii), we find a leaf $\mathcal{F}_\mu$ consists of congruence classes of closed circles if and only if $\mu = \mu(p,q) = q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ by use of a pair (p,q) of mutually prime positive integers with $p > q$ (see [5]). In order to study limit curves we only treat such leaves. By putting

$$\tau_\kappa(p,q;c) = \frac{q(9p^2-q^2)(4\kappa^2+c)^{3/2}}{3\sqrt{3}\,c\kappa(3p^2+q^2)^{3/2}},$$

we see the leaf $\mathcal{F}_{\mu(p,q)}$ is of the form $\{[\gamma_{\kappa,\tau_\kappa(p,q;c)}] \mid \zeta(p,q;c) < \kappa < \eta(p,q;c)\}$, where

$$\zeta(p,q;c) = q\sqrt{\frac{c}{9p^2-q^2}}, \quad \eta(p,q;c) = \frac{3p-q}{2}\sqrt{\frac{c}{2q(3p+q)}}.$$

We study the phenomena at $\zeta(p,q;c)$ and $\eta(p,q;c)$. Choose a unit tangent vector u at a point $x \in \mathbb{C}P^n(c)$ and a continuous map $\big(\zeta(p,q;c),\eta(p,q;c)\big) \ni \kappa \mapsto v_\kappa \in T_x\mathbb{C}P^n(c)$ such that $\langle u, v_\kappa\rangle = 0$ and $\langle u, Jv_\kappa\rangle = \tau_\kappa(p,q;c)$. We denote by $\gamma_{\kappa,\tau_\kappa(p,q;c)}$ a circle on $\mathbb{C}P^n(c)$ with initial condition

$$\gamma_{\kappa,\tau_\kappa(p,q;c)}(0) = x,\ \dot\gamma_{\kappa,\tau_\kappa(p,q;c)}(0) = u,\ \nabla_{\dot\gamma_{\kappa,\tau_\kappa(p,q;c)}}\dot\gamma_{\kappa,\tau_\kappa(p,q;c)}(0) = \kappa v_\kappa.$$

Since it is a closed curve with length

$$\lambda_\kappa(p,q;c) = 2\delta(p,q)\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2+c)}},$$

where $\delta(p,q) = 1$ when the product pq is odd and $\delta(p,q) = 2$ when pq is even, we can define a closed curve $\sigma_{\kappa,\tau_\kappa(p,q;c)} \colon S^1 \to \mathbb{C}P^n(c)$ by

$$\sigma_{\kappa,\tau_\kappa(p,q;c)}(s) = \gamma_{\kappa,\tau_\kappa(p,q;c)}\big(\lambda_\kappa(p,q;c)s\big).$$

We now show the following.

Theorem 1. *Let (p,q) be a pair of mutually prime positive integers with $p>q$.*
(1) *The family $\mathcal{S}_{\mu(p,q)}=\{\sigma_{\kappa,\tau_\kappa(p,q;c)}\mid \zeta(p,q;c)<\kappa<\eta(p,q;c)\}$ of closed curves on $\mathbb{C}P^n(c)$ $(n\geq 2)$ derived from closed circles whose congruency classes lie on the leaf $\mathcal{F}_{\mu(p,q)}$ has limit curves.*
(2) *The curve $\lim_{\kappa\downarrow\zeta(p,q;c)}\sigma_{\kappa,\tau_\kappa(p,q;c)}$ is a $\delta(p,q)p$-fold covering of a closed curve $\sigma_{\zeta(p,q;c),1}$ which is derived from a Kähler circle of geodesic curvature $\zeta(p,q;c)$.*
(3) *The curve $\lim_{\kappa\uparrow\eta(p,q;c)}\sigma_{\kappa,\tau_\kappa(p,q;c)}$ is a $\delta(p,q)(p+q)/2$-fold covering of a closed curve $\sigma_{\eta(p,q;c),1}$ which is derived from a Kähler circle of geodesic curvature $\eta(p,q;c)$.*

Proof. By direct computation, we have

$$\begin{aligned}
\lim_{\kappa\downarrow\zeta(p,q;c)}\lambda_\kappa(p,q;c)&=\frac{2\delta(p,q)\pi}{3}\sqrt{\frac{9p^2-q^2}{c}}=\frac{2\pi\delta(p,q)p}{\sqrt{\zeta(p,q;c)^2+c}},\\
\lim_{\kappa\uparrow\eta(p,q;c)}\lambda_\kappa(p,q;c)&=\frac{2\delta(p,q)\pi}{3}\sqrt{\frac{2q(3p+q)}{c}}=\frac{\pi\delta(p,q)(p+q)}{\sqrt{\eta(p,q;c)^2+c}},
\end{aligned}\tag{3.3}$$

which suggest our result. To see more precisely, we suppose $c=4$ and $x=\varpi(z)$. A horizontal lift $\tilde\gamma_{\kappa,\tau_\kappa(p,q;4)}$ on $S^{2n+1}(1)$ of $\gamma_{\kappa,\tau_\kappa(p,q;4)}$ is of the form

$$\tilde\gamma_{\kappa,\tau_\kappa(p,q;4)}(t)=A_\kappa e^{a_\kappa it}+B_\kappa e^{b_\kappa it}+D_\kappa e^{d_\kappa it},$$

where a_κ, b_κ, d_κ $(a_\kappa<b_\kappa<d_\kappa)$ are the solutions of the cubic equation $\theta^3-(\kappa^2+1)\times\theta+\tau_\kappa(p,q;4)\kappa=0$ and

$$\begin{cases}
A_\kappa=\dfrac{1}{(a_\kappa-b_\kappa)(d_\kappa-a_\kappa)}\{-(1+b_\kappa d_\kappa)z+a_\kappa Ju+\kappa v_\kappa\},\\
B_\kappa=\dfrac{1}{(b_\kappa-d_\kappa)(a_\kappa-b_\kappa)}\{-(1+d_\kappa a_\kappa)z+b_\kappa Ju+\kappa v_\kappa\},\\
D_\kappa=\dfrac{1}{(d_\kappa-a_\kappa)(b_\kappa-d_\kappa)}\{-(1+a_\kappa b_\kappa)z+d_\kappa Ju+\kappa v_\kappa\}.
\end{cases}$$

Since $\lim_{\kappa\downarrow\zeta(p,q;4)}\tau(p,q;4)=\lim_{\kappa\uparrow\eta(p,q;4)}\tau(p,q;4)=1$, one can easily see by the cubic equation that

$$\begin{aligned}
&\lim_{\kappa\downarrow\zeta(p,q;4)}a_\kappa=-\frac{3p+q}{\sqrt{9p^2-q^2}},\quad \lim_{\kappa\downarrow\zeta(p,q;4)}b_\kappa=\zeta(p,q;4),\quad \lim_{\kappa\downarrow\zeta(p,q;4)}d_\kappa=\frac{3p-q}{\sqrt{9p^2-q^2}},\\
&\lim_{\kappa\uparrow\eta(p,q;4)}a_\kappa=-\sqrt{\frac{3p+q}{2q}},\quad \lim_{\kappa\uparrow\eta(p,q;4)}b_\kappa=\sqrt{\frac{2q}{3p+q}},\quad \lim_{\kappa\uparrow\eta(p,q;4)}d_\kappa=\eta(p,q;4),
\end{aligned}$$

which are nothing but the solutions for the cubic equation corresponding to $(\kappa,\tau)=\big(\zeta(p,q;4),1\big)$ and $(\kappa,\tau)=\big(\eta(p,q;4),1\big)$. As $\lim_{\kappa\downarrow\zeta(p,q;4)}v_\kappa=\lim_{\kappa\uparrow\eta(p,q;4)}v_\kappa=-Ju$, these guarantee that

$$\begin{cases} \lim\limits_{\kappa\downarrow\zeta(p,q;4)} A_\kappa = \dfrac{1}{6p}\{(3p-q)z+\sqrt{9p^2-q^2}\,Ju\}, \\ \lim\limits_{\kappa\downarrow\zeta(p,q;4)} B_\kappa = 0, \\ \lim\limits_{\kappa\downarrow\zeta(p,q;4)} D_\kappa = \dfrac{1}{6p}\{(3p+q)z-\sqrt{9p^2-q^2}\,Ju\}, \end{cases}$$

$$\begin{cases} \lim\limits_{\kappa\uparrow\eta(p,q;4)} A_\kappa = \dfrac{1}{3(p+q)}\{2qz+\sqrt{2q(3p+q)}\,Ju\}, \\ \lim\limits_{\kappa\uparrow\eta(p,q;4)} B_\kappa = \dfrac{1}{3(p+q)}\{(3p+q)z-\sqrt{2q(3p+q)}\,Ju\}, \\ \lim\limits_{\kappa\uparrow\eta(p,q;4)} D_\kappa = 0. \end{cases}$$

We therefore obtain by use of (3.3) that the family $\mathcal{S}_{\mu(p,q)}$ has limit curves:

$$\begin{aligned} &\lim_{\kappa\downarrow\zeta(p,q;4)} \sigma_{\kappa,\tau_\kappa(p,q;4)}(s) \\ &\qquad = \varpi\left(\cos\delta(p,q)p\pi s\cdot z+\frac{\sqrt{9p^2-q^2}}{6p}\sin\delta(p,q)p\pi s\cdot\big(\zeta(p,q;4)iz+2u\big)\right), \\ &\lim_{\kappa\uparrow\eta(p,q;4)} \sigma_{\kappa,\tau_\kappa(p,q;4)}(s) \\ &\qquad = \varpi\bigg(\cos\frac{1}{2}(p+q)\delta(p,q)\pi s\cdot z \\ &\qquad\qquad +\frac{\sqrt{2q(3p+q)}}{3(p+q)}\sin\frac{1}{2}(p+q)\delta(p,q)\pi s\cdot\big(\eta(p,q;4)iz+2u\big)\bigg), \end{aligned}$$

Thus in comparing these with (3.2) we get our conclusion. □

Our theorem shows that the lamination $\{\mathcal{F}_\mu\}$ of the moduli space $\mathrm{Cir}(\mathbb{C}P^n(c))$ is not a foliation. But we can easily see that $\{\mathcal{F}_\mu\}_{\mu\in[0,1)}$ is a foliation of the moduli space $\mathrm{Cir}(\mathbb{C}P^n(c))\setminus\{[\gamma_{\kappa,1}]\mid\kappa\geq 0\}$ of non-Kähler circles.

4. Lamination of bounded circles on a complex hyperbolic space

In this section we study corresponding results on a lamination on the moduli space of circles on a complex hyperbolic space $\mathbb{C}H^n(-c)$ of constant holomorphic sectional curvature $-c$ whose complex dimension n is greater than 1. We define a non-negative function $\nu\colon [0,\infty)\to\mathbb{R}$ by

$$\nu(\kappa)=\begin{cases}0, & \text{if } 0\le\kappa<\dfrac{\sqrt{c}}{2},\\ \dfrac{(4\kappa^2-c)^{3/2}}{3\sqrt{3}\,c\kappa}, & \text{if } \dfrac{\sqrt{c}}{2}\le\kappa\le\sqrt{c},\\ 1, & \text{if } \kappa>1.\end{cases}$$

We studied in [4] circles on a complex hyperbolic space and showed the following:
(i) A circle of geodesic curvature κ is bounded as a image in $\mathbb{C}H^n(-c)$ if and only if either $\kappa>\sqrt{c}$ or its complex torsion τ is smaller than $\nu(\kappa)$.
(ii) Every Kähler circle of geodesic curvature $\kappa>\sqrt{c}$ is a closed curve with length $2\pi/\sqrt{\kappa^2-c}$.
(iii) Every totally real circle of geodesic curvature $\kappa>\sqrt{c}/2$ is a closed curve with length $4\pi/\sqrt{4\kappa^2-c}$.
(iv) For κ $(>\sqrt{c}/2)$ and τ $(0<\tau<\nu(\kappa))$ we denote by $a_{\kappa,\tau}$, $b_{\kappa,\tau}$, $d_{\kappa,\tau}$ $(a_{\kappa,\tau}<b_{\kappa,\tau}<d_{\kappa,\tau})$ the solutions for the cubic equation

$$c\theta^3-(4\kappa^2-c)\theta+2\sqrt{c}\,\kappa\tau=0. \tag{4.1}$$

A circle of geodesic curvature κ and of complex torsion τ is closed if and only if one of (hence all of) the ratios $a_{\kappa,\tau}/b_{\kappa,\tau}$, $b_{\kappa,\tau}/d_{\kappa,\tau}$, $d_{\kappa,\tau}/a_{\kappa,\tau}$ is rational. When it is closed, its length is $(4\pi/\sqrt{c})\times \mathrm{L.C.M}\{(b_{\kappa,\tau}-a_{\kappa,\tau})^{-1},(d_{\kappa,\tau}-a_{\kappa,\tau})^{-1}\}$.

We denote by $[\gamma_{\kappa,\tau}]$ the congruence class of circles on $\mathbb{C}H^n(-c)$ with geodesic curvature κ and complex torsion τ. We also denote by $\mathrm{BCir}_\kappa(\mathbb{C}H^n(-c))$ the set of all congruence classes of bounded circles of geodesic curvature κ on $\mathbb{C}H^n(-c)$. Since the cubic equation (4.1) for (κ,τ) and that for $(\sqrt{c},3\sqrt{3}\,c\kappa\tau(4\kappa^2-c)^{-3/2})$ are homothetic, we have a map of normalization

$$\Phi_\kappa\colon\ \mathrm{BCir}_\kappa\bigl(\mathbb{C}H^n(-c)\bigr)\setminus\{[\gamma_{\kappa,1}]\}\longrightarrow \mathrm{Cir}_{\sqrt{c}}\bigl(\mathbb{C}H^n(-c)\bigr)\setminus\{[\gamma_{\sqrt{c},1}]\}$$

between the sets of congruence classes of bounded circles of prescribed geodesic curvatures defined by

$$\Phi_\kappa([\gamma_{\kappa,\tau}])=\Bigl[\gamma_{\sqrt{c},3\sqrt{3}\,c\kappa\tau(4\kappa^2-c)^{-3/2}}\Bigr].$$

This map gives a lamination on the moduli space

$$\begin{aligned}\mathrm{BCir}\bigl(\mathbb{C}H^n(-c)\bigr)&=\bigcup_{\kappa>\sqrt{c}/2}\mathrm{BCir}_\kappa\bigl(\mathbb{C}H^n(-c)\bigr)\\ &\simeq\Bigl\{(\kappa,\tau)\Bigm|\frac{\sqrt{c}}{2}<\kappa\le\sqrt{c},\ 0\le\tau<\nu(\kappa)\Bigr\}\cup\bigl(\sqrt{c},\infty\bigr)\times[0,1]\end{aligned}$$

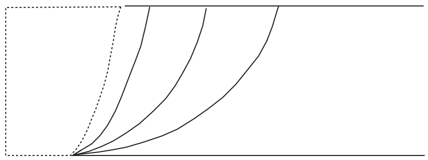

Fig. 2. Lamination on $\mathrm{BCir}\big(\mathbb{C}H^n(-c)\big)$

of bounded circles on $\mathbb{C}H^n(-c)$ whose leaves are

$$\mathcal{F}_\mu = \begin{cases} \left\{[\gamma_{\kappa,0}] \mid \kappa > \dfrac{\sqrt{c}}{2}\right\}, & \text{if } \mu = 0, \\ \{[\gamma_{\kappa,\tau}] \mid 3\sqrt{3}\,c\kappa\tau(4\kappa^2 - c)^{-3/2} = \mu,\ 0 < \tau < 1\}, & \text{if } 0 < \mu < 1, \\ \{[\gamma_{\kappa,1}] \mid \kappa > \sqrt{c}\,\}, & \text{if } \mu = 1. \end{cases}$$

It is known that a leaf $\mathcal{F}_\mu$ consists of congruence classes of closed circles if and only if $\mu = \mu(p,q) = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ by use of a pair (p,q) of mutually prime positive integers with $p > q$ (see [3]). By putting

$$\tau_\kappa(p,q;-c) = \frac{q(9p^2 - q^2)(4\kappa^2 - c)^{3/2}}{3\sqrt{3}\,c\kappa(3p^2 + q^2)^{3/2}},$$

we see the leaf $\mathcal{F}_{\mu(p,q)}$ is of the form $\{[\gamma_{\kappa,\tau_\kappa(p,q;-c)}] \mid \sqrt{c}/2 < \kappa < \eta(p,q;-c)\}$, where $\eta(p,q;-c) = (3p+q)\sqrt{c/8q(3p-q)}$. In order to see the lamination $\mathcal{F}$ on $\mathrm{BCir}(\mathbb{C}H^n(-c))$ is not a foliation, we study limit curves of a family of closed curves corresponding to $\mathcal{F}_{\mu(p,q)}$.

Choose a unit tangent vector u at a point $x \in \mathbb{C}H^n(-c)$ and denote by $\gamma_{\kappa,1}$ for $\kappa > \sqrt{c}$ the Kähler circle of geodesic curvature κ on $\mathbb{C}H^n(-c)$ with initial condition $\gamma_{\kappa,1}(0) = x$, $\dot\gamma_{\kappa,1}(0) = u$, $\nabla_{\dot\gamma_{\kappa,1}}\dot\gamma_{\kappa,1}(0) = -\kappa Ju$. Since it is a closed curve with length $2\pi/\sqrt{\kappa^2 - c}$, we can define a curve $\sigma_{\kappa,1}\colon S^1 \to \mathbb{C}H^n(-c)$ by $\sigma_{\kappa,1}(s) = \gamma_{\kappa,1}(2\pi s/\sqrt{\kappa^2 - c}\,)$.

On the other hand, for given a continuous map $\big(\sqrt{c}/2, \eta(p,q;-c)\big) \ni \kappa \mapsto v_\kappa \in T_x\mathbb{C}H^n(-c)$ with $\langle u, v_\kappa\rangle = 0$ and $\langle u, Jv_\kappa\rangle = \tau_\kappa(p,q;-c)$ we denote by $\gamma_{\kappa,\tau_\kappa(p,q;-c)}$ a circle on $\mathbb{C}H^n(-c)$ with initial condition

$$\gamma_{\kappa,\tau_\kappa(p,q;-c)}(0) = x,\ \dot\gamma_{\kappa,\tau_\kappa(p,q;-c)}(0) = u,\ \nabla_{\dot\gamma_{\kappa,\tau_\kappa(p,q;-c)}}\dot\gamma_{\kappa,\tau_\kappa(p,q;-c)}(0) = \kappa v_\kappa.$$

Since it is also a closed curve with length

$$\lambda_\kappa(p,q;-c) = 2\delta(p,q)\pi\sqrt{\frac{3p^2 + q^2}{3(4\kappa^2 - c)}},$$

where $\delta(p,q) = 1$ when the product pq is odd and $\delta(p,q) = 2$ when pq is even, we can define a closed curve $\sigma_{\kappa,\tau_\kappa(p,q;-c)}\colon S^1 \to \mathbb{C}H^n(-c)$ by

$$\sigma_{\kappa,\tau_\kappa(p,q;-c)}(s) = \gamma_{\kappa,\tau_\kappa(p,q;-c)}(\lambda_\kappa(p,q;-c)s).$$

We now show the following.

Theorem 2. *Let (p,q) be a pair of mutually prime positive integers with $p > q$.*
(1) *A family $\mathcal{S}_{\mu(p,q)} = \{\sigma_{\kappa,\tau_\kappa(p,q;-c)} \mid \sqrt{c}/2 < \kappa < \eta(p,q;c)\}$ of closed curves on $\mathbb{C}H^n(-c)$ $(n \geq 2)$ derived from closed circles whose congruence classes lie on the leaf $\mathcal{F}_{\mu(p,q)}$ has a limit curve.*
(2) *The curve $\lim_{\kappa\uparrow\eta(p,q;-c)} \sigma_{\kappa,\tau_\kappa(p,q;-c)}$ is a $\delta(p,q)(p-q)/2$-fold covering of a closed curve $\sigma_{\eta(p,q;-c),1}$ which is derived from a Kähler circle of geodesic curvature $\eta(p,q;-c)$.*

Proof. First we see

$$\lim_{\kappa\uparrow\eta(p,q;c)} \lambda_\kappa(p,q;c) = \frac{2\delta(p,q)\pi}{3}\sqrt{\frac{2q(3p-q)}{c}} = \frac{\pi\delta(p,q)(p-q)}{\sqrt{\eta(p,q;c)^2+c}}, \tag{4.2}$$

which suggests our result. To see more precisely, we make use of the standard fibration $\varpi\colon H_1^{2n+1} \to \mathbb{C}H^n(-4)$ of an anti de-Sitter space

$$H_1^{2n+1} = \{(z_0,\dots,z_n \in \mathbb{C}^{n+1} \mid -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = -1\}.$$

If $x = \varpi(z)$, horizontal lifts $\tilde\gamma_{\kappa,1}$ and $\tilde\gamma_{\kappa,\tau_\kappa(p,q;-4)}$ on H_1^{2n+1} of circles $\gamma_{\kappa,1}$ and $\gamma_{\kappa,\tau_\kappa(p,q;-4)}$ on $\mathbb{C}H^n(-4)$ are of the forms

$$\begin{aligned}\tilde\gamma_{\kappa,1}(t) &= e^{\kappa it/2}\Big\{\cos\frac{1}{2}\sqrt{\kappa^2-4}\,tz + (\kappa^2-4)^{-1/2}\sin\frac{1}{2}\sqrt{\kappa^2-4}\,t(\kappa iz+2u)\Big\},\\ \tilde\gamma_{\kappa,\tau_\kappa(p,q;-4)}(t) &= A_\kappa e^{a_\kappa it} + B_\kappa e^{b_\kappa it} + D_\kappa e^{d_\kappa it},\end{aligned}$$

where $a_\kappa,\ b_\kappa,\ d_\kappa$ $(a_\kappa < b_\kappa < d_\kappa)$ are the solutions of the cubic equation $\theta^3 - (\kappa^2-1)\theta - \tau_\kappa(p,q;-4)\kappa = 0$ and

$$\begin{cases} A_\kappa = \dfrac{1}{(a_\kappa-b_\kappa)(d_\kappa-a_\kappa)}\{(1-b_\kappa d_\kappa)z + a_\kappa Ju + \kappa v_\kappa\},\\ B_\kappa = \dfrac{1}{(b_\kappa-d_\kappa)(a_\kappa-b_\kappa)}\{(1-d_\kappa a_\kappa)z + b_\kappa Ju + \kappa v_\kappa\},\\ D_\kappa = \dfrac{1}{(d_\kappa-a_\kappa)(b_\kappa-d_\kappa)}\{(1-a_\kappa b_\kappa)z + d_\kappa Ju + \kappa v_\kappa\}.\end{cases}$$

Since we have

$$\lim_{\kappa\uparrow\eta(p,q;-4)} \tau_\kappa(p,q;-4) = 1, \qquad \lim_{\kappa\uparrow\eta(p,q;-4)} v_\kappa = -Ju,$$
$$\lim_{\kappa\uparrow\eta(p,q;-4)} a_\kappa = -\sqrt{\frac{3p-q}{2q}}, \qquad \lim_{\kappa\uparrow\eta(p,q;-4)} b_\kappa = -\sqrt{\frac{2q}{3p-q}},$$
$$\lim_{\kappa\uparrow\eta(p,q;-4)} d_\kappa = \eta(p,q;-4),$$

we find

$$\begin{cases} \displaystyle\lim_{\kappa\uparrow\eta(p,q;-4)} A_\kappa = \frac{1}{3(p-q)}\left\{-2qz + \sqrt{2q(3p-q)}\,Ju\right\}, \\ \displaystyle\lim_{\kappa\uparrow\eta(p,q;-4)} B_\kappa = \frac{1}{3(p-q)}\left\{(3p-q)z - \sqrt{2q(3p-q)}\,Ju\right\}, \\ \displaystyle\lim_{\kappa\uparrow\eta(p,q;-4)} D_\kappa = 0. \end{cases}$$

Together with (4.2) these guarantee that

$$\begin{aligned} &\lim_{\kappa\uparrow\eta(p,q;-4)} \sigma_{\kappa,\tau_\kappa(p,q;-c)} \\ &\quad = \varpi\Big(\cos\frac{1}{2}(p-q)\delta(p,q)\pi s\cdot z \\ &\qquad\qquad + \frac{\sqrt{2q(3p-q)}}{3(p-q)}\sin\frac{1}{2}(p-q)\delta(p,q)\pi s\cdot\big(\eta(p,q;4)iz+2u\big)\Big). \end{aligned}$$

Since we have

$$\sigma_{\kappa,1}(s) = \varpi\big(\cos\pi s\cdot z + (\kappa^2-4)^{-1/2}\sin\pi s\cdot(\kappa iz+2u)\big),$$

we get our conclusion. □

Our theorem shows that the lamination $\mathcal{F}$ is not a foliation on $\mathrm{BCir}(\mathbb{C}H^n(-c))$. But we can easily see that $\{\mathcal{F}_\mu\}_{\mu\in[0,1)}$ is a foliation on the moduli space of bounded non-Kähler circles.

5. Length functions on a complex projective space

We devote the rest of this paper to study length functions for circles on a non-flat complex space form. For each length $\lambda \in \mathrm{LSpec}_\kappa(M)$ of circles of geodesic curvature κ we call the cardinality $m(\lambda)$ of the set $\mathcal{L}_\kappa^{-1}(\lambda)$ the *multiplicity* of λ. When $m(\lambda) = 1$ we call λ *simple*. In case the length spectrum $\mathrm{LSpec}_\kappa(M)$ is a discrete set and each length is of finite multiplicity, we denote by $\lambda_j(\kappa)$ the j-th length of circles of geodesic curvature κ. That is,

$$\mathrm{LSpec}_\kappa(M) = \{\lambda_1(\kappa) \le \lambda_2(\kappa) \le \lambda_3(\kappa) \le \cdots \le \lambda_j(\kappa) \le \lambda_{j+1}(\kappa) \le \cdots\},$$

where each length is repeated according to its multiplicity. The first length $\lambda_1(\kappa)$ shows the length of shortest closed circles of geodesic curvature κ. Trivially for a real space form the unique length function $\lambda(\kappa) = \lambda_1(\kappa)$ is continuous, monotone decreasing and satisfies $\lim_{\kappa\to\infty} \kappa\lambda(\kappa) = 2\pi$. Also it is trivial that

$$\begin{cases} \lim_{\kappa\downarrow 0} \lambda(\kappa) = \dfrac{2\pi}{\sqrt{c}}, & \text{for } S^n(c), \\ \lim_{\kappa\downarrow 0} \kappa\lambda(\kappa) = 2\pi, & \text{for } \mathbb{R}^n, \\ \lim_{\kappa\downarrow\sqrt{c}} (\kappa - \sqrt{c}\,)^{1/2}\lambda(\kappa) = \sqrt{2}\,\pi c^{-1/4}, & \text{for } H^n(-c). \end{cases}$$

We now study length functions of circles on a complex projective space. In view of Section 3 the length spectrum of circles on $\mathbb{C}P^n(c)$ $(n \geq 2)$ is of the following form (see [5]):

$$\begin{aligned} \mathrm{LSpec}_\kappa\big(\mathbb{C}P^n(c)\big) = &\left\{ \frac{2\pi}{\sqrt{\kappa^2 + c}},\ \frac{4\pi}{\sqrt{4\kappa^2 + c}} \right\} \\ &\bigcup \left\{ \lambda_\kappa(p, q; c) \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are mutually prime positive} \\ \text{integers which satisfy } p > \alpha_{\kappa;c}q \end{array} \right\}, \end{aligned}$$

where $\alpha_{\kappa;c}(\geq 1)$ denotes the unique positive number which satisfies

$$3\sqrt{3}\,c\kappa(4\kappa^2 + c)^{-3/2} = (9\alpha_{\kappa;c}^2 - 1)(3\alpha_{\kappa,;c}^2 + 1)^{-3/2}. \tag{5.1}$$

Hence $\mathrm{LSpec}_\kappa(\mathbb{C}P^n(c))$ is a discrete set and each length is of finite multiplicity. Also one can easily see that the first length is $\lambda_1(\kappa) = 2\pi/\sqrt{\kappa^2 + c}$, which is the length of Kähler circles, and the second length is $\lambda_2(\kappa) = 4\pi/\sqrt{4\kappa^2 + c}$, which is the length of totally real circles. They are simple and satisfy the following properties:

(1) They are smooth and monotone decreasing with respect to κ, and satisfy

$$\lim_{\kappa\downarrow 0} \lambda_1(\kappa) = \frac{2\pi}{\sqrt{c}},\ \lim_{\kappa\downarrow 0} \lambda_2(\kappa) = \frac{4\pi}{\sqrt{c}},$$
$$\lim_{\kappa\to\infty} \kappa\lambda_1(\kappa) = \lim_{\kappa\to\infty} \kappa\lambda_2(\kappa) = 2\pi,$$

in particular, $\lim_{\kappa\to\infty} \lambda_1(\kappa) = \lim_{\kappa\to\infty} \lambda_2(\kappa) = 0$.

(2) The gap between these lengths is monotone decreasing with respect to κ and satisfies

$$\lim_{\kappa\to\infty} \kappa^3\big(\lambda_2(\kappa) - \lambda_1(\kappa)\big) = \frac{3\pi c}{4}, \quad \lim_{\kappa\downarrow 0}\big(\lambda_2(\kappa) - \lambda_1(\kappa)\big) = \frac{2\pi}{\sqrt{c}}.$$

We here study corresponding properties for $\lambda_j(\kappa)$ with $j \geq 3$.

Theorem 3. *For $j \geq 3$ the j-th length function of circles on $\mathbb{C}P^n(c)$ $(n \geq 2)$ satisfies the following properties*:

(1) *This function is not continuous. The points where this function is not right continuous are contained in the set*

$$\{\zeta(p,q;c) \mid (p,q) \text{ are mutually prime positive integers with } p>q\},$$

and the points where this function is not left continuous are contained in the set

$$\{\eta(p,q;c) \mid (p,q) \text{ are mutually prime positive integers with } p>q\}.$$

(2) *On each interval where λ_j is continuous, it is monotone decreasing and satisfies* $\lim_{\kappa\to\infty}\lambda_j(\kappa)=\lim_{\kappa\downarrow 0}\lambda_j(\kappa)=\infty$. *More precisely, it satisfies*

$$\lim_{\kappa\to\infty}\frac{\lambda_j(\kappa)}{\kappa}=\frac{8\pi}{3c},\ \lim_{\kappa\downarrow 0}\kappa\lambda_j(\kappa)=\frac{2\pi}{3}.$$

(3) *The gap $\lambda_{j+1}(\kappa)-\lambda_j(\kappa)$ is monotone decreasing on each interval where this gap function is continuous, and is uniformly bounded*;

$$\lambda_{j+1}(\kappa)-\lambda_j(\kappa)<\frac{4\pi}{\sqrt{4\kappa^2+c}}<\frac{4\pi}{\sqrt{c}}$$

for every κ (>0). It satisfies

$$\lim_{\kappa\to\infty}\kappa\bigl(\lambda_{j+1}(\kappa)-\lambda_j(\kappa)\bigr)=2\pi,\ \lim_{\kappa\downarrow 0}\bigl(\lambda_{j+1}(\kappa)-\lambda_j(\kappa)\bigr)=\frac{4\pi}{\sqrt{c}}.$$

In order to understand the structure of $\mathrm{LSpec}_\kappa(\mathbb{C}P^n(c))$, the properties of $\alpha_{\kappa;c}$ are important.

Lemma 1. (1) $\alpha_{\sqrt{2c}/4;c}=1$ *and* $\alpha_{\kappa;c}>1$ *for* $\kappa\neq\sqrt{2c}/4$.
(2) *The function $\kappa\mapsto\alpha_{\kappa;c}$ is monotone decreasing on the interval* $(0,\sqrt{2c}/4]$ *and is monotone increasing on the interval* $[\sqrt{2c}/4,\infty)$.
(3) $\lim_{\kappa\to\infty}\kappa^{-2}\alpha_\kappa=8/(3c)$ *and* $\lim_{\kappa\downarrow 0}\alpha_\kappa\kappa=\sqrt{c}/3$.

Proof. (2) Differentiating both sides of (5.1) by κ, we have

$$\frac{d\alpha_{\kappa;c}}{d\kappa}=\frac{\sqrt{3}\,c(8\kappa^2-c)(3\alpha_{\kappa;c}^2+1)^{5/2}}{9\alpha_{\kappa;c}(4\kappa^2+c)^{5/2}(\alpha_{\kappa;c}^2-1)}.$$

Since $\alpha_{\kappa;c}>1$ when $\kappa\neq\sqrt{2c}/4$, we get the assertion.
(3) By the assertion of (2) and (5.1) we see $\lim_{\kappa\to\infty}\alpha_{\kappa;c}=\lim_{\kappa\downarrow 0}\alpha_{\kappa;c}=\infty$. The last assertion follows the following computation:

$$\lim_{\kappa\to\infty}\frac{\alpha_{\kappa;c}}{\kappa^2}=\lim_{\kappa\to\infty}\frac{\alpha_{\kappa;c}}{\kappa^2}\times\frac{(4\kappa^2+c)^{3/2}(9\alpha_{\kappa;c}-1)}{3\sqrt{3}\,c\kappa(3\alpha_{\kappa;c}^2+1)^{3/2}}$$

$$= \lim_{\kappa\to\infty} \frac{(4+c\kappa^{-2})^{3/2}(9-\alpha_{\kappa;c}^{-1})}{3\sqrt{3}\,c(3+\alpha_{\kappa;c}^{-2})^{3/2}} = \frac{8}{3c},$$

$$\lim_{\kappa\downarrow 0} \alpha_{\kappa;c}\kappa = \lim_{\kappa\downarrow 0} \frac{(4\kappa^2+c)^{3/2}(9-\alpha_{\kappa;c}^{-1})}{3\sqrt{3}\,c(3+\alpha_{\kappa;c}^{-2})^{3/2}} = \frac{\sqrt{c}}{3}. \qquad \square$$

In order to understand Theorem 3, the following result on the third length $\lambda_3(\kappa)$ will be a good guide to the readers.

Proposition 2. (1) *The third length is simple for arbitrary* $\kappa\,(> 0)$.
(2) *Denoting by* $p_\kappa(\geq 3)$ *the smallest odd integer which is greater than* $\alpha_{\kappa;c}$ *we have*

$$\lambda_3(\kappa) = \lambda_\kappa(p_\kappa, 1; c) = 2\pi\sqrt{\frac{3p_\kappa^2+1}{3(4\kappa^2+c)}}.$$

It satisfies

$$\lim_{\kappa\to\infty} \kappa^{-1}\lambda_3(\kappa) = \frac{8\pi}{3c}, \quad \lim_{\kappa\downarrow 0} \kappa\lambda_3(\kappa) = \frac{2\pi}{3}.$$

(3) *The function* $\lambda_3\colon (0,\infty) \to \mathbb{R}$ *is right continuous except at* $\zeta(2m+1, 1; c)$, $m = 1, 2, 3, \ldots$, *and is left continuous except at* $\eta(2m+1, 1; c)$, $m = 1, 2, 3, \ldots$.
(4) *The gap* $\lambda_3(\kappa) - \lambda_2(\kappa)$ *is monotone decreasing on each interval where* λ_3 *is continuous but is not uniformly bounded and satisfies*

$$\lim_{\kappa\to\infty} \kappa^{-1}\big(\lambda_3(\kappa) - \lambda_2(\kappa)\big) = \frac{8\pi}{3c}, \quad \lim_{\kappa\downarrow 0} \kappa\big(\lambda_3(\kappa) - \lambda_2(\kappa)\big) = \frac{2\pi}{3}.$$

Proof. First we note a property of $\lambda_\kappa(p, q; c)$. Let (p_1, q_1), (p_2, q_2) be two pairs of mutually prime positive integers with $p_i > q_i$. If either both of the products p_1q_1 and p_2q_2 are odd or both of them are even, and if they satisfy $p_1 \leq p_2$ and $q_1 \leq q_2$, then $\lambda_\kappa(p_1, q_1; c) \leq \lambda_\kappa(p_2, q_2; c)$. Under this hypothesis the equality holds if and only if $(p_1, q_1) = (p_2, q_2)$. On the other hand, if p_1q_1 is odd, p_2q_2 is even, $p_1 \leq p_2$ and $q_1 \leq q_2$, then $\lambda_\kappa(p_1, q_1; c) < \lambda_\kappa(p_2, q_2; c)$.

What we have to do is to find out the smallest $\lambda_\kappa(p, q; c)$. When $p_\kappa - 1 \leq \alpha_{\kappa;c}$, as every pair (p, q) of mutually prime positive integers with $p > \alpha_{\kappa;c}q$, $q \geq 2$ satisfies $p - p_\kappa > (q-1)\alpha_{\kappa;c} - 1 > 0$, it is clear that $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa, 1; c)$ if $(p, q) \neq (p_\kappa, 1)$. When $\alpha_{\kappa;c} < p_\kappa - 1$, as every pair (p, q) of mutually prime positive integers with $p > \alpha_{\kappa;c}q$, $q \geq 2$ satisfies $p - p_\kappa \geq (q-1)\alpha_{\kappa;c} - 2 \geq 0$, we have only to compare $\lambda_\kappa(p_\kappa, 1; c)$ and $\lambda_\kappa(p_\kappa - 1, 1; c)$. Since

$$4\{3(p_\kappa - 1)^2 + 1\} - \{3p_\kappa^2 + 1\} = 3(3p_\kappa - 5)(p_\kappa - 1) > 0,$$

we find also in this case that $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa, 1; c)$ if $(p, q) \neq (p_\kappa, 1)$. Thus we see $\lambda_3(\kappa) = \lambda_\kappa(p_\kappa, 1; c)$ and is simple for arbitrary κ. By this form we find $\lambda_3(\kappa)$ is not continuous at points where $\alpha_{\kappa;c}$ is a odd integer which is greater than 1. For given α (≥ 1) we can easily check that positive solutions of the equation

$$(9\alpha^2 - 1)^2(4\kappa^2 + c)^3 - 27c^2(3\alpha^2 + 1)^3\kappa^2 = 0$$

are $\kappa = \sqrt{c/(9\alpha^2 - 1)}$ and $\kappa = (3\alpha - 1)\sqrt{c/\{8(3\alpha + 1)\}}$. Hence these points are $\zeta(2m + 1, 1; c)$ and $\eta(2m + 1, 1; c)$ with $m = 1, 2, \ldots$. As $\alpha_{\kappa;c} < p_\kappa \leq \alpha_{\kappa;c} + 2$, our assertion follows from Lemma 1. $\square$

The congruence classes corresponding to the third length of circles on $\mathbb{C}P^n(c)$ lie on the leaf $\mathcal{F}_{\mu(2m+1,1)}$ when $\zeta(2m + 1, 1; c) < \kappa \leq \zeta(2m - 1, 1; c)$ or $\eta(2m - 1, 1; c) \leq \kappa < \eta(2m + 1, 1; c)$. By our study in Section 3 each family $\{\sigma_{\kappa,\tau_\kappa(2m+1,1;c)} \mid \zeta(2m + 1, 1; c) < \kappa \leq \zeta(2m - 1, 1; c)\}$ of closed curves derived from circles corresponding to the third length has a limit curve $\lim_{\kappa\downarrow\zeta(2m+1,1;c)} \sigma_{\kappa,\tau_\kappa(2m+1,1;c)}$, which is a $(2m+1)$-fold covering of a closed curve derived from a Kähler circle. When κ goes to 0, we find by Lemma 1 that m goes to infinity. One can easily guess that $\lim_{\kappa\downarrow 0} \lambda_3(\kappa) = \infty$.

REMARK 1. We should note that among the gap functions $\{\lambda_{j+1} - \lambda_j \mid j = 1, 2, \ldots\}$ only the gap function $\lambda_3 - \lambda_2$ is not bounded. One should compare the rates of convergence of $\lambda_{j+1} - \lambda_j$ for $j \geq 3$ and for $j = 1$ when κ goes to infinity and when it goes to 0. Also one should note the rates of convergence of λ_j, $j = 1, 2$ and the rates of divergence of λ_j, $j > 3$.

The following lemma gives us information on asymptotic behaviours of length functions.

Lemma 2. *Let $j(\geq 4)$ be a positive integer.*
(1) *If $\alpha_{\kappa;c} \geq 2(j - 3)$ then $\lambda_j(\kappa) = \lambda_\kappa(p_\kappa + 2(j - 3), 1; c)$.*
(2) *If $2j - 7 \leq \alpha_{\kappa;c} < 2(j - 3)$, then $\lambda_j(\kappa) = \lambda_\kappa(p_\kappa - 1, 1; c) = \lambda_\kappa(2(j - 3), 1; c)$.*
(3) *If $2(j - 4) \leq \alpha_{\kappa;c} < 2j - 7$ and $j \geq 5$, then*

$$\begin{aligned}\lambda_j(\kappa) &= \lambda_\kappa(p_\kappa + 2(j - 3), 1; c)\\ &= \lambda_\kappa(4j - 13, 1; c).\end{aligned}$$

(4) *The j-th length is simple on the intervals where $\alpha_{\kappa;c} \geq 2j - 8$, which is $(0, \zeta(2j - 8, 1; c)]$ and $[\eta(2j - 8, 1; c), \infty)$.*

Proof. We study $\lambda_\kappa(p, q; c)$ along the same lines as in the proof of Proposition 2.

In the first place we consider the case $p_\kappa - 1 \leq \alpha_{\kappa;c}$. Since $\alpha_{\kappa;c} \geq 2(j - 4)$, we see $p_\kappa \geq 2j - 7$. We shall show that for a pair (p, q) of mutually prime positive integers with $p > \alpha_{\kappa;c}q$ and $(p, q) \neq (p_\kappa + 2i, 1)$, $i = 0, 1, \dots, j - 3$ the length $\lambda_\kappa(p, q; c)$ is greater than $\lambda_\kappa(p_\kappa + 2(j-3), 1; c)$. We first compare $\lambda_\kappa(p_\kappa + 1, 1; c)$ and $\lambda_\kappa(p_\kappa + 2(j-3), 1; c)$. Since

$$\begin{aligned}&4\{3(p_\kappa + 1)^2 + 1\} - 3\{p_\kappa + 2(j-3)\}^2 - 1\\ &\quad = 3\{3p_\kappa^2 - 4(j-5)p_\kappa + 5 - 4(j-3)^2\} \geq 24(j-3) > 0,\end{aligned}$$

we find $\lambda_\kappa(p_\kappa + 1, 1; c) > \lambda_\kappa(p_\kappa + 2(j-3), 1; c)$. Next we consider $\lambda_\kappa(p, q; c)$ with $q \geq 2$. Note that $p > \alpha_\kappa q \geq q(p_\kappa - 1)$.

(i) When q is even, as we have

$$p - (p_\kappa + 1) \geq (q-1)(p_\kappa - 1) - 1 \geq p_\kappa - 2 > 0,$$

we obtain $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa + 1, 1; c) > \lambda_\kappa(p_\kappa + 2(j-3), 1; c)$.

(ii) When q is odd and $j \geq 5$, as we have

$$p - \{p_\kappa + 2(j-3)\} \geq (q-1)(p_\kappa - 1) - 2(j-3) \geq 2(j-5) \geq 0,$$

we obtain $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa + 2(j-3), 1; c)$.

(iii) When q is odd and $j = 4$, as we have

$$p - (p_\kappa + 2) \geq (q-1)p_\kappa - 2 \geq 2 \cdot 3 - 2 > 0,$$

we obtain $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa + 2, 1; c)$.

Thus we get the assertion of this case.

In the second place we consider the case $\alpha_{\kappa;c} < p_\kappa - 1$. Since $\alpha_{\kappa;c} \geq 2(j-4)$, we see $p_\kappa \geq 2j - 5$. First we compare $\lambda_\kappa(p_\kappa - 1, 1; c)$ and $\lambda_\kappa(p_\kappa + 2(j-3), 1; c)$. Since the quantity

$$\begin{aligned}&4\{3(p_\kappa - 1)^2 + 1\} - 3\{p_\kappa + 2(j-3)\}^2 - 1\\ &\qquad\qquad = 3\{3p_\kappa^2 - 4(j-1)p_\kappa + 5 - 4(j-3)^2\}\end{aligned}$$

is equal to $24(3-j) < 0$ when $p_\kappa = 2j - 5$, and is not smaller than $24(j-2) > 0$ when $p_\kappa \geq 2j - 3$, we have

$$\begin{cases}\lambda_\kappa(p_\kappa - 1, 1; c) < \lambda_\kappa\big(p_\kappa + 2(j-3), 1; c\big), & \text{if } p_\kappa = 2j - 5,\\ \lambda_\kappa(p_\kappa - 1, 1; c) > \lambda_\kappa\big(p_\kappa + 2(j-3), 1; c\big), & \text{if } p_\kappa \geq 2j - 3.\end{cases}$$

Next we consider $\lambda_\kappa(p, q; c)$ with $q \geq 2$. Note that $p > \alpha_\kappa q \geq q(p_\kappa - 2)$.

(i) When q is even, or when q is odd and p is even, as we have

$$p - (p_\kappa - 1) \geq (q-1)(p_\kappa - 2) > 0,$$

we obtain $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa - 1, 1; c)$.

(ii) When p and q are odd, as we have

$$p - \{p_\kappa + 2(j-3)\} \geq (q-1)(p_\kappa - 2) - 2(j-3) \geq 2(j-2) > 0,$$

we obtain $\lambda_\kappa(p, q; c) > \lambda_\kappa(p_\kappa + 2(j-3), 1; c)$.

Summarizing up these two cases under the condition $\alpha_{\kappa;c} \geq 2(j-4)$, we get the following. If either $\alpha_\kappa \geq p_\kappa - 1$, or $\alpha_\kappa < p_\kappa - 1$ and $p_\kappa \geq 2j-3$ holds, which is equivalent to the condition that $\alpha_{\kappa;c} \geq 2(j-3)$ or $2(j-4) \leq \alpha_{\kappa;c} < 2j-7$, then

$$\lambda_\kappa(p_\kappa, 1; c) < \lambda_\kappa(p_\kappa + 2, 1; c) < \cdots < \lambda_\kappa\big(p_\kappa + 2(j-3), 1; c\big)$$

are smaller than other lengths $\lambda_\kappa(p, q; c)$. This leads us to the assertion (1) and (3). If $\alpha_{\kappa;c} < p_\kappa - 1$ and $p_\kappa = 2j-5$, which is equivalent to the condition that $2j-7 \leq \alpha_\kappa < 2(j-3)$, then

$$\lambda_\kappa(p_\kappa, 1; c) < \lambda_\kappa(p_\kappa + 2, 1; c) < \cdots < \lambda_\kappa\big(p_\kappa + 2(j-4), 1; c\big) < \lambda_\kappa(p_\kappa - 1, 1; c)$$

are smaller than other lengths $\lambda_\kappa(p, q; c)$. Thus we get the rest of our conclusion. $\square$

We are now in the position to prove Theorem 3. By Lemma 2, we have

$$\lambda_j(\kappa) = \lambda_\kappa\big(p_\kappa + 2(j-3), 1; c\big) = 2\pi\sqrt{\frac{3(p_\kappa + 2j - 6)^2 + 1}{3(4\kappa^2 + c)}},$$

if $\alpha_{\kappa;c} \geq 2(j-3)$. Hence the second assertion is a direct consequence of Lemma 1.

The first assertion follows from the property that functions $\big(\zeta(p, q; c), \eta(p, q; c)\big) \ni \kappa \mapsto \lambda_\kappa(p, q; c) \in \mathbb{R}$ are continuous and the first paragraph of the proof of Proposition 2, which guarantees that the property $\lambda(p_1, q_1; c) < \lambda(p_2, q_2; c)$ depends only on (p_i, q_i) and does not depend on κ. The function λ_j is continuous at point κ if and only if $(\kappa, 1)$ is not a boundary point of leaves containing congruence classes of circles corresponding this length. By the form of $\lambda_j(\kappa)$, if we restrict ourselves on the intervals $(0, \zeta(2j-8, 1; c)]$ and $\eta(2j-8, 1; c), \infty)$, it is right continuous except at $\zeta(2j-6, 1; c)$ and $\zeta(2m+1, 1; c)$, $m = j-4, j-3, \ldots,$ and is left continuous except at $\eta(2j-6, 1; c)$ and $\eta(2m+1, 1; c)$, $m = j-4, j-3, \ldots.$

The assertion on asymptotic behaviours of gaps between two lengths follows directly from Lemma 2. We only need to give a unifom estimate of gaps. We first estimate $\lambda_\kappa(p, q; c)$ with $q \geq 2$ by $\lambda_\kappa(r, 1; c)$. When pq is odd, we choose $r = r(p, q)$ so that it is the maximum positive odd integer which satisfies $3r^2 + 1 \leq 3p^2 + q^2$. Clearly we have $r \geq p > \alpha_{\kappa;c} q > \alpha_{\kappa;c}$, and $3r^2 + 1 \leq 3p^2 + q^2 < 3(r+2)^2 + 1$ because $r+2$ is odd. We hence obtain $\lambda_\kappa(r, 1; c) \leq \lambda_\kappa(p, q; c) < \lambda_\kappa(r+2, 1; c)$. When pq is even and $q \geq 2$, we choose $r = r(p, q)$ so that it is the maximum positive odd integer which satisfies $3r^2+1 \le 4(3p^2+q^2)$. Then we have

$$\begin{cases} 3r^2+1 \le 4(3p^2+q^2) < 3(r+2)^2+1, \\ r \ge 2p-1 > 2\alpha_{\kappa;c}q-1 \ge 4\alpha_{\kappa;c}-1 > \alpha_{\kappa;c}. \end{cases}$$

Thus we get also in this case that $\lambda_\kappa(r,1;c) \le \lambda_\kappa(p,q;c) < \lambda_\kappa(r+2,1;c)$. Since the function $\lambda_j(\kappa)$ is of the form $\lambda_\kappa(p_{\kappa,j},q_{\kappa,j};c)$ with some pair $(p_{\kappa,j},q_{\kappa,j})$ of mutually prime positive integers, by choosing the corresponding $r_{\kappa,j}$ in the above argument we have

$$\begin{aligned} \lambda_{j+1}(\kappa)-\lambda_j(\kappa) &\le \lambda_\kappa(r_{\kappa,j}+2,1;c)-\lambda_\kappa(r_{\kappa,j},1;c) \\ &= \frac{2\pi}{\sqrt{3(4\kappa^2+c)}} \times \left(\sqrt{3(r_{\kappa,j}+2)^2+1}-\sqrt{3r_{\kappa,j}^2+1}\right) < \frac{4\pi}{\sqrt{4\kappa^2+c}}, \end{aligned}$$

as the function $f(\theta)=\sqrt{3(\theta+2)^2+1}-\sqrt{3\theta^2+1}$ is monotone increasing and satisfies $f(\theta) < 2\sqrt{3}$. This completes the proof of Theorem 3.

REMARK 2. The fourth length function is of the following form:

$$\lambda_4(\kappa) = \begin{cases} \lambda_\kappa(2,1;c) = 4\pi\sqrt{\dfrac{13}{3(4\kappa^2+c)}}, & \text{if } 1 \le \alpha_\kappa < 2, \\ \lambda_\kappa(p_\kappa+2,1;c) = 2\pi\sqrt{\dfrac{3(p_\kappa+2)^2+1}{3(4\kappa^2+c)}}, & \text{if } \alpha_\kappa \ge 2. \end{cases}$$

It is right continuous except at $\zeta(2,1;c)$ and $\zeta(2m+1,1;c)$, $m=1, 2, \dots$, and is left continuous except at $\eta(2,1;c)$ and $\eta(2m+1,1;c)$, $m=1, 2, \dots$.

Checking each length functions carefully we can conclude the following:

Proposition 3. (1) *The lengths* $\lambda_j(\kappa)$ $(j \le 15)$ *are simple everywhere.*
(2) *The length* $\lambda_{16}(\kappa)$ *is simple except on the intervals where* $9/7 \le \alpha_{\kappa;c} < 5/4$. *On these two intervals its multiplicity is* 2; $\lambda_{16}(\kappa)=\lambda_{17}(\kappa)$.
(3) *The length* $\lambda_{17}(\kappa)$ *is simple except on the interval where* $1 \le \alpha_{\kappa;c} < 5/4$. *On this interval its multiplicity is* 2; $\lambda_{17}(\kappa)=\lambda_{18}(\kappa)$ *when* $1 \le \alpha_{\kappa;c} < 9/7$, *and* $\lambda_{17}(\kappa)=\lambda_{16}(\kappa)$ *when* $9/7 \le \alpha_{\kappa;c} < 5/4$.

6. Length functions on a complex hyperbolic space

In this final section we mention briefly corresponding results on length functions for circles on a complex hyperbolic space. In view of Section 4 the length spectrum of circles of geodesic curvature κ on $\mathbb{C}H^n(-c)$ $(n \ge 2)$ is of the following form (see [3]): When $\kappa \le \sqrt{c}/2$, every circle is an unbounded curve and $\mathrm{LSpec}_\kappa(\mathbb{C}H^n(-c)) = \emptyset$, when $\sqrt{c}/2 < \kappa \le \sqrt{c}$,

$$\mathrm{LSpec}_\kappa\big(\mathbb{C}H^n(-c)\big) = \left\{\frac{4\pi}{\sqrt{4\kappa^2 - c}}\right\} \bigcup \left\{ \lambda_\kappa(p, q; -c) \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are mutually} \\ \text{prime positive integers} \\ \text{which satisfy } p > q \end{array} \right\},$$

and when $\kappa > \sqrt{c}$,

$$\begin{aligned} \mathrm{LSpec}_\kappa\big(\mathbb{C}H^n(-c)\big) = &\left\{\frac{4\pi}{\sqrt{4\kappa^2 - c}}, \frac{2\pi}{\sqrt{\kappa^2 - c}}\right\} \\ &\bigcup \left\{ \lambda_\kappa(p, q; -c) \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are mutually prime positive} \\ \text{integers which satisfy } p > \alpha_{\kappa;-c} q \end{array} \right\}. \end{aligned}$$

Here, for $\kappa \ge \sqrt{c}$, the constant $\alpha_{\kappa;-c}$ denotes the unique positive number which satisfies

$$3\sqrt{3}\, c\kappa(4\kappa^2 - c)^{-3/2} = (9\alpha_{\kappa,-c}^2 - 1)(3\alpha_{\kappa,-c}^2 + 1)^{-3/2}. \tag{6.1}$$

The structures of the length spectrum $\mathrm{LSpec}_\kappa(\mathbb{C}H^n(-c))$ of circles of prescribed geodesic curvatures are essentially same each other if $\sqrt{c}/2 < \kappa \le \sqrt{c}$. We are hence interested in the behaviour when κ goes to infinity. From now on we consider only for $\kappa > \sqrt{c}/2$. The first length is $\lambda_1(\kappa) = 4\pi/\sqrt{4\kappa^2 - c}$, which is the length of totally real circles. Hence it is continuous and monotone decreasing, and satisfies

$$\lim_{\kappa\to\infty} \kappa\lambda_1(\kappa) = 2\pi, \quad \lim_{\kappa\downarrow\sqrt{c}/2} \left(\kappa - \frac{\sqrt{c}}{2}\right)^{1/2} \lambda_1(\kappa) = 2\pi c^{-1/4},$$

in particular, $\lim_{\kappa\to\infty} \lambda_1(\kappa) = 0$.

The structure of length spectrum of circles on $\mathbb{C}H^n(-c)$ is a bit more complicated than that of circles on a complex projective space because Kähler circles of geodesic curvature $\sqrt{c}$ are unbounded. It follows from [3] that the second length is

$$\lambda_2(\kappa) = \begin{cases} \lambda_\kappa(3, 1; -c) = 4\pi\sqrt{\dfrac{7}{3(4\kappa^2 - c)}}, & \text{if } \dfrac{\sqrt{c}}{2} < \kappa < \dfrac{5\sqrt{c}}{4}, \\[2ex] \dfrac{2\pi}{\sqrt{\kappa^2 - c}}, & \text{if } \kappa \ge \dfrac{5\sqrt{c}}{4}, \end{cases}$$

which is the length of Kähler circles when $\kappa \ge 5\sqrt{c}/4$, and is simple everywhere.

Proposition 4. (1) *The second length function* $\lambda_2 \colon (\sqrt{c}/2, \infty) \to \mathbb{R}$ *on* $\mathbb{C}H^n(-c)$ $(n \ge 2)$ *is also continuous and monotone decreasing. It also satisfies*

$$\lim_{\kappa\to\infty} \kappa\lambda_2(\kappa) = 2\pi, \quad \lim_{\kappa\downarrow\sqrt{c}/2} \left(\kappa - \frac{\sqrt{c}}{2}\right)^{1/2} \lambda_2(\kappa) = 2\pi c^{-1/4}\sqrt{\frac{7}{3}},$$

in particular, $\lim_{\kappa\to\infty}\lambda_2(\kappa)=0$.
(2) *The gap function* $\lambda_2-\lambda_1$ *is monotone decreasing and satisfies*

$$\lim_{\kappa\to\infty}\kappa^3\bigl(\lambda_2(\kappa)-\lambda_1(\kappa)\bigr)=\frac{3\pi c}{4},\quad \lim_{\kappa\downarrow\sqrt{c}/2}\bigl(\lambda_2(\kappa)-\lambda_1(\kappa)\bigr)=\infty.$$

When $\sqrt{c}/2<\kappa<5\sqrt{c}/4$ the congruence classes corresponding to the second length of circles lie on the leaf $\mathcal{F}_{\mu(2,1)}$. By the study in Section 4, the limit curve $\lim_{\kappa\uparrow 5\sqrt{c}/4}\sigma_{\kappa,\tau_\kappa(2,1;-c)}$ is the closed curve derived from a Kähler circle of geodesic curvature $5\sqrt{c}/4$. This suggests the second length function is continuous also at the point $\kappa=5\sqrt{c}/4$.

Let p_κ denote the smallest odd integer which is greater than $\alpha_{\kappa;-c}$. By a similar argument as in the proof of Proposition 2, we find the third length is of the following form:

$$\lambda_3(\kappa)=\begin{cases}\lambda_\kappa(2,1;-c)=4\pi\sqrt{\dfrac{13}{3(4\kappa^2-c)}}, & \text{if } \dfrac{\sqrt{c}}{2}<\kappa<\dfrac{7\sqrt{10c}}{20},\\[2ex] \dfrac{2\pi}{\sqrt{\kappa^2-c}}, & \text{if } \dfrac{7\sqrt{10c}}{20}\le\kappa<\dfrac{5\sqrt{c}}{4},\\[2ex] \lambda_\kappa(p_\kappa,1;-c)=2\pi\sqrt{\dfrac{3p_\kappa^2+1}{3(4\kappa^2-c)}} & \text{if } \kappa\ge\dfrac{5\sqrt{c}}{4}.\end{cases}$$

This length is also simple everywhere. Since we have
(i) $\alpha_{\sqrt{c};-c}=1$ and $\alpha_{\kappa;-c}>1$ for $\kappa>\sqrt{c}$,
(ii) the function $\kappa\mapsto\alpha_{\kappa;-c}$ is monotone increasing for $\kappa\ge\sqrt{c}$ and satisfies $\lim_{\kappa\to\infty}\kappa^{-2}\alpha_{\kappa;-c}=8/(3c)$,
(iii) for an arbitrary $\alpha\,(\ge 1)$ the unique positive solution for the cubic equation

$$3\sqrt{3}\,c\kappa(3\alpha^2+1)^{3/2}=(9\alpha^2-1)(4\kappa^2-c)^{3/2}$$

is $\kappa=(3\alpha+1)\sqrt{c/\{8(3\alpha-1)\}}$,
we obtain the following.

Proposition 5. (1) *The third length function* $\lambda_3\colon(\sqrt{c}/2,\infty)\to\mathbb{R}$ *of circles on* $\mathbb{C}H^n(-c)$ $(n\ge 2)$ *is right continuous at each point, and is left continuous except at* $\eta(2m+1,1;-c)$, $m=1,2,3,\dots$.
(2) *On each interval where* λ_3 *is continuous, it is monotone decreasing.*
(3) *The gap function* $\lambda_3-\lambda_2$ *satisfies*

$$\lim_{\kappa\to\infty}\kappa^{-1}\bigl(\lambda_3(\kappa)-\lambda_2(\kappa)\bigr)=\frac{8\pi}{3c},\quad \lim_{\kappa\downarrow\sqrt{c}/2}\bigl(\lambda_3(\kappa)-\lambda_2(\kappa)\bigr)=\infty.$$

Along the same lines as in Section 5, we obtain

Lemma 3. *Let $j\,(\geq 4)$ be a positive integer.*
(1) *If $\alpha_{\kappa;-c} \geq 2(j-3)$, then $\lambda_j(\kappa) = \lambda_\kappa(p_\kappa + 2(j-3), 1; -c)$.*
(2) *If $2j-7 \leq \alpha_{\kappa;-c} < 2(j-3)$, then*

$$\lambda_j(\kappa) = \lambda_\kappa(p_\kappa - 1, 1; -c) = \lambda_\kappa(2(j-3), 1; -c).$$

(3) *If $2(j-4) \leq \alpha_{\kappa;-c} < 2j-7$, then*

$$\lambda_j(\kappa) = \lambda_\kappa(p_\kappa + 2(j-3), 1; -c) = \lambda_\kappa(4j-13, 1; -c).$$

(4) *The j-th length is simple on the interval where $\alpha_{\kappa;-c} \geq 2j-8$, which is $[\eta(2j-8, 1; -c), \infty)$.*

For a pair (p,q) of positive integers we put

$$\xi(p,q;-c) = \sqrt{\frac{\{\delta(p,q)^2(3p^2+q^2)-3\}c}{\delta(p,q)^2(3p^2+q^2)-12}}.$$

When $\kappa = \xi(p,q;-c)$, the lengths of a Kähler circle of geodesic curvature κ and a circle of complex torsion $\tau_\kappa(p,q;-c)$ and of geodesic curvature κ are the same. By Lemma 3 we get the following result.

Theorem 4. *For $j \geq 3$ the j-th length function of circles on $\mathbb{C}H^n(-c)$ (≥ 2) satisfies the following properties*:
(1) *This function is right continuous.*
(2) *This function is not left continuous. Such points are contained in the set*

$$\left\{ \eta(p,q;-c), \xi(p,q;-c) \;\middle|\; \begin{array}{l} p \text{ and } q \text{ are mutually prime} \\ \text{positive integers with } p > q \end{array} \right\}.$$

(3) *On each interval where λ_j is continuous, it is monotone decreasing and satisfies $\lim_{\kappa\to\infty} \lambda_j(\kappa) = \infty$. More precisely it satisfies*

$$\lim_{\kappa\to\infty} \frac{\lambda_j(\kappa)}{\kappa} = \frac{8\pi}{3c}.$$

(4) *The gap $\lambda_{j+1}(\kappa) - \lambda_j(\kappa)$ is smaller than $4\pi/\sqrt{4\kappa^2 - c}$ for every $\kappa\,(> \sqrt{c}/2)$. This gap function is monotone decreasing on each interval where it is continuous and $\kappa \geq 5\sqrt{c}/4$. It satisfies*

$$\lim_{\kappa\to\infty} \kappa\big(\lambda_{j+1}(\kappa) - \lambda_j(\kappa)\big) = 2\pi, \quad \lim_{\kappa\downarrow\sqrt{c}/2} \big(\lambda_{j+1}(\kappa) - \lambda_j(\kappa)\big) = \infty.$$

REMARK 3. For every $j \geq 1$ the gap satisfies

$$0 \leq \lim_{\kappa\downarrow\sqrt{c}/2} (2\kappa - \sqrt{c}\,)^{1/2}\big(\lambda_{j+1}(\kappa) - \lambda_j(\kappa)\big) < 4\pi c^{-1/4},$$

where the limit exists and the equality holds if and only if $\lambda_{j+1} = \lambda_j$ on the interval $\sqrt{c}/2 < \kappa \leq \sqrt{c}$.

REMARK 4. When we restrict ourselves on the interval $\kappa \geq \sqrt{c}$, among the gap functions $\{\lambda_{j+1} - \lambda_j \mid j = 1, 2, \ldots\}$ only the gap function $\lambda_3 - \lambda_2$ is not bounded. One should compare the rate of convergence of $\lambda_{j+1} - \lambda_j$ for $j \geq 3$ and for $j = 1$ when κ goes to infinity.

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