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Osaka University
MAXIMAL IDEAL CYCLES
OVER NORMAL SURFACE SINGULARITIES
OF BRIESKORN TYPE

KAZUHIRO KONNO and Daisuke NAGASHIMA

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Abstract
For normal two dimensional hypersurface singularities of Brieskorn type, concrete descriptions are given to both the fundamental cycle and the maximal ideal cycle on a star-shaped good resolution space. It is determined when these two cycles coincide.

Introduction
Let \((V, o)\) be a germ of a normal surface singularity and \(\phi : (X, E) \to (V, o)\) a resolution, where \(E = \phi^{-1}(o)\) denotes the exceptional set. Let \(E = \bigcup_{i=1}^{r} E_i\) be the irreducible decomposition of \(E\). A formal sum \(Y = \sum_{i=1}^{r} \lambda_i E_i (\lambda_i \in \mathbb{Z})\) is called a cycle on \(E\). For a cycle \(Y\), \(-Y\) is said to be nef on \(E\) if \(YE_i \leq 0\) for all \(i\). Since the intersection form is negative definite on \(E\), the set \(\{Y \succ 0 \mid -Y\text{ is nef on } E\}\) is non-empty and has the smallest element \(Z\), the fundamental cycle on \(E\). The arithmetic genus of \(Z\) is called the fundamental genus of \((V, o)\) and we denote it by \(p_f(V, o)\).

Let \(m\) be the maximal ideal of \(O_{V,a}\). For any non-zero \(f \in m\), the zero divisor of \(f \circ \phi\) can be written as \((f \circ \phi)_X + D\), where \((f \circ \phi)_X\) is a cycle on \(E\) and \(D\) is an effective divisor which does not involve any of \(E_i\)'s. We call \((f \circ \phi)_X\) the cycle on \(E\) led by \(f \in m\). The divisorial part \(M_E\) of the scheme theoretic fiber \(\phi^*o\) is said to be the maximal ideal cycle on \(E\). If \(f_1, \ldots, f_n \in m\) generate \(m\), then \(M_E = \inf_{1 \leq i \leq n} (f_i \circ \phi)_X\) by [14, Proposition 2.12]. Since \(-M_E\) is nef, we always have \(0 \prec Z_E \preceq M_E\).

It sometimes happens that \(M_E = Z_E\), as one can observe for rational singular points, Kodaira singular points and singularities of type \(\{z^n = f(x, y)\} (n \geq 2, f \in \mathbb{C}\{x, y\})\) when \(n \gg 0\). As for the last type, Tomaru proved in [10, Theorem 4.1] that two cycles coincide on any resolution when \(n\) divides \(\text{ord}(f)\), extending the well-known result for \(n = 2\) due to Dixon [2, Theorem 1]. However, even for a particular class of singularities, a more systematic study will be required in order to clarify when such a coincidence of important cycles occurs.

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In this paper, we consider the normal 2-dimensional hypersurface singularities of Brieskorn type,

\[(V_{a_0, a_1, a_2}, o) = \{(x_0, x_1, x_2) \in U \mid x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\},\]

where \(U \subset \mathbb{C}^3\) is a small neighborhood of the origin \(o = (0, 0, 0)\) and the \(a_i\)'s are integers with \(2 \leq a_0 \leq a_1 \leq a_2\), and give a necessary and sufficient condition for the coincidence of the maximal ideal cycle and the fundamental cycle.

Before stating the results, let us introduce some notation which will be used throughout the paper. We put \([x] := \min\{n \in \mathbb{Z} \mid n \geq x\}\) for \(x \in \mathbb{R}\). For integers \(d_1, d_2, \ldots, d_r\) \((d_i \geq 2\) for all \(i\)), we denote by \([[d_1, d_2, \ldots, d_r]]\) the continued fraction:

\[[d_1, d_2, \ldots, d_r]] := d_1 - \cfrac{1}{d_2 - \cfrac{1}{d_3 - \cfrac{1}{\ddots - \cfrac{1}{d_r}}}}.

Let \(n\) and \(\mu\) be positive integers that are relatively prime and \(0 < \mu < n\). The singularity

\[C_{n, \mu} := \mathbb{C}^2 \bigg/ \left\langle\begin{pmatrix} \epsilon_n \hfill 0 \\ 0 \hfill \epsilon_n^\mu \end{pmatrix}\right\rangle,\]

where \(\epsilon_n := \exp(2\pi \sqrt{-1}/n)\) denotes the primitive \(n\)-th root of unity, is called a cyclic quotient singularity. It is well-known (e.g., [4]) that, if \(E = \bigcup_{i=1}^{r} E_i\) is the exceptional set for the minimal resolution of \(C_{n, \mu}\), then \(E_i \simeq \mathbb{P}^1\) and the weighted dual graph of \(E\) is chain-shaped as in Fig. 1, where \(n/\mu = [[d_1, d_2, \ldots, d_r]]\).

To the singularity \((V_{a_0, a_1, a_2}, o) = \{(x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\}, 2 \leq a_0 \leq a_1 \leq a_2\), we associate the integers

\[l := \gcd(a_0, a_1, a_2), \quad l_i := \frac{\gcd(a_j, a_k)}{l}, \quad \alpha_i := \frac{a_i}{l_{jkl}}, \quad ([i, j, k] = \{0, 1, 2\}).\]

Furthermore, we let \(p_0, p_1, p_2\) be the integers determined by

\[p_i \alpha_j \alpha_k l_i + 1 \equiv 0 \pmod{\alpha_i}, \quad 0 \leq p_i < \alpha_i,\]
where \( \{i, j, k\} = \{0, 1, 2\} \). When \( \alpha_w > 1 \), we also put

\[
\alpha_w / p_w = [(d_{w,1}, d_{w,2}, \ldots, d_{w,r_w})]
\]

and \( e_{w,v} := [(d_{w,v}, d_{w,v+1}, \ldots, d_{w,r_w})] \) (1 \( \leq v \leq r_w ; w = 0, 1, 2 \)). With this notation, by [7], there exists a resolution \( \pi : (X, E_\pi) \to (V_{a_0,a_1,a_2}, o) \) such that the weighted dual graph of \( E_\pi \) is as in Fig. 2. It is star-shaped with \( l_w \) branches of type \( C_{a_{w,p_w}} \) (\( w = 0, 1, 2 \)) starting from a unique vertex. The non-singular curve \( E_0 \) corresponding to that vertex will be referred to as the central curve. We shall mainly work on the resolution \( \pi \).

Note that, when \( \alpha_w = 1 \), the corresponding branches do not appear and we understand \( C_{1,0} \) as a non-singular point on \( E_0 \).

Now, we are going to state our results in this paper. First, we concretely describe the fundamental cycle over \((V_{a_0,a_1,a_2}, o)\). We remark here that an algorithm computing the fundamental cycle from the exponents \( a_0, a_1, a_2 \) was established by Tomari (cf. [8, (3.3)]).
Theorem 1.4. Let
\[ Z = \theta_0 E_0 + \sum_{w=0}^{2} \sum_{v=1}^{l_w} \sum_{\xi=1}^{l_{w,v}} \theta_{w,v,\xi} E_{w,v,\xi} \]
be the fundamental cycle for resolution \( \pi \). Then the sequence \( \{\theta_{w,v,\xi}\}_{w=0}^{2} \) is defined by the following recurrence formula:

1. \( \theta_{w,0,\xi} := \theta_0 = \begin{cases} \alpha_0 \alpha_1 \alpha_2 & \text{if } \alpha_2 \leq l_2, \\ \alpha_0 \alpha_1 l_2 & \text{if } \alpha_2 \geq l_2, \end{cases} \)
2. \( \theta_{w,v,\xi} = \lfloor \theta_{w,v-1,\xi} / e_{w,v} \rfloor, \ 1 \leq v \leq r_w. \)

To show it, we first study the chain-shaped configuration obtained by plugging one extra vertex to the configuration of type \( C_{n,\mu} \). We consider the condition which should be satisfied by the smallest cycle \( Y_0 \) such that \( -Y \) is nef on the \( C_{n,\mu} \) part, when the multiplicity at the extra vertex is given. Then we apply it to branches of the star-shaped configuration as in Fig. 2. To determine the multiplicity of the central curve is our final task. Our method is so simple that it may apply also to the other singularities with \( \mathbb{C}^* \)-action in determining their fundamental cycles. As a by-product, we can reprove in Theorem 1.7 the formula computing the fundamental genus which was originally obtained in [8] and [9] by an entirely different method.

Next, we turn our attention to the maximal ideal cycle. With the help of Tomaru’s result in [12], we can determine the multiplicity of each component and give a formula similar to Theorem 1.4 also for \( (x_w \circ \pi)_Y \) \( (w = 0,1,2) \). See, Theorem 2.1 for the precise statement. This enables us to show in Theorem 3.1 that \( (x_2 \circ \pi)_Y \) is the maximal ideal cycle for resolution \( \pi \) (here, the assumption \( a_0 \leq a_1 \leq a_2 \) is essential).

Using the concrete descriptions thus obtained, we compare the fundamental cycle and the maximal ideal cycle on \( E_{\pi} \) and get the following:

**Theorem 3.2.** The maximal ideal cycle coincides with the fundamental cycle for resolution \( \pi \) if and only if \( \alpha_2 \geq l_2 \).

In particular, this implies that both cycles coincide on the minimal resolution, when \( a_0 \) is a prime number. Similarly, we obtain the following:

**Theorem 3.11.** The maximal ideal cycle coincides with the fundamental cycle for any resolution of \( (V_{a_0,a_1,a_2}, o) \), \( 2 \leq a_0 \leq a_1 \leq a_2 \), if and only if \( \alpha_2 \geq l_2 \) and \( 1 < a_1/a_2 + \gcd(a_0, a_1)/a_0 \). Furthermore, if this is the case, then the fundamental cycle is led by the holomorphic function \( x_2 \).

As an application, we give in Proposition 4.4 the necessary and sufficient condition for \( (V_{a_0,a_1,a_2}, o) \) to be a Kodaira singularity ([5], [6]), in order to supplement a result in [10]. We also describe the canonical cycle for \( \pi \) in Proposition 4.6.
The paper grew out of the second named author’s master thesis at Osaka University. The authors would like to express their deep gratitude to Professor Tadashi Tomaru for many helpful suggestions in the course of the study. Thanks are also due to Professors Masataka Tomari, Tomohiro Okuma and Tadashi Ashikaga for offering the second named author an opportunity to give a talk at their exciting seminar and for their precious comments.

1. The fundamental cycle

1.1. Minimal anti-nef cycle on a chain. Let \( n \) and \( \mu \) be integers that are relatively prime and \( 0 < \mu < n \), and put \( n/\mu = [[d_1, d_2, \ldots, d_r]] \). We consider a connected bunch \( \bigcup_{i=0}^r E_i \) of irreducible curves \( E_i \) on a smooth surface whose weighted dual graph is chain-shaped and \( E_1 + \cdots + E_r \) forms the configuration of type \( C_{n,\mu} \) as in Fig. 3. We put \( e_i := [[d_1, d_{i+1}, \ldots, d_r]] \) for \( 1 \leq i \leq r \). Then \( d_i = e_i + 1/e_{i+1} \) for \( 1 \leq i < r \), and \( d_r = e_r \). For a positive integer \( \lambda_0 \), consider the set

\[
D(\lambda_0) := \left\{ Y: \text{cycle on } \bigcup_{i=0}^r E_i \mid -Y \text{ is nef on } \bigcup_{i=1}^r E_i, \mult_{E_0}(Y) = \lambda_0 \right\}.
\]

**Lemma 1.1.** Take a positive integer \( \lambda_0 \) and define the sequence \( \{\lambda_i\}_{i=0}^r \) by the recurrence formula \( \lambda_i = \lceil \lambda_{i-1}/e_i \rceil \) for \( 1 \leq i \leq r \). Then the cycle \( Y_0 := \sum_{i=0}^r \lambda_i E_i \) is the smallest element of \( D(\lambda_0) \).

**Proof.** Let \( \rho_i \) \((0 \leq i \leq r)\) be positive integers and put \( Y = \sum_{i=0}^r \rho_i E_i \). We first claim that \( [\rho_{i-1}/e_i] \leq \rho_i \) holds for \( 1 \leq i \leq r \), if \( -Y \) is nef on \( \bigcup_{i=1}^r E_i \). This can be seen by induction as follows. For \( i = r \), we have \( 0 \geq YE_r = \rho_{r-1} - \rho_r d_r \). Since \( d_r = e_r \), we get \( \rho_{i-1}/e_i \leq \rho_r \) which implies \( [\rho_{i-1}/e_i] \leq \rho_r \). Take an index \( i \) with \( 1 \leq i < r \) and assume that \( [\rho_{i-1}/e_i] \leq \rho_{i-1} \) holds. We have \( 0 \geq YE_i = \rho_{i-1} - \rho_i d_i + \rho_{i+1} \). Then \( \rho_{i-1}/\rho_i \leq d_i - \rho_{i+1}/\rho_i \leq d_i - 1/e_{i+1} - e_i \), since \( \rho_i/e_{i+1} \leq \rho_{i+1} \) by the hypothesis. It follows \( \rho_{i-1}/e_i \leq \rho_i \) and, hence, \( [\rho_{i-1}/e_i] \leq \rho_i \).

We next consider the cycle \( Y_0 \). We have \( Y_0 E_i = \lambda_{i-1} - d_i \lambda_r = \lambda_{i-1} - e_i \lambda_r \leq 0 \), because \( \lambda_r = \lceil \lambda_{r-1}/e_r \rceil \geq \lambda_{r-1}/e_r \). Take any index \( i \) with \( 1 \leq i < r \). Since \( \lambda_i = [\lambda_{i-1}/e_i] \geq \lambda_{i-1}/e_i \), we get \( \lambda_{i-1}/\lambda_i \leq e_i = d_i - 1/e_{i+1} \), i.e., \( \lambda_{i-1} \leq d_i \lambda_i - \lambda_i/e_{i+1} \). It follows \( \lambda_{i-1} \leq d_i \lambda_i - [\lambda_i/e_{i+1}] = d_i \lambda_i - \lambda_i/e_{i+1} \), which shows \( Y_0 E_i \leq 0 \). Hence \( Y_0 \in \)
shows, if \( Y_D \) of otherwise stated explicitly.

If either (3) we have \( Y \). Then, from the first half of the proof, it is clear that \( Y_0 \) is the smallest element of \( D(\lambda_0) \).

Note that we did not specify the self-intersection number of \( E_0 \). As the proof shows, if \( Y_0 \) and \( Y_0' \) are the smallest elements in \( D(\lambda_0) \) and \( D(\lambda_0') \), respectively, then we have \( Y_0 \not\leq Y_0' \) if and only if \( \lambda_0 \neq \lambda_0' \).

Since the proof of the following lemma is elementary, we leave it to the reader.

**Lemma 1.2.** Let the sequence \( \{\lambda_i\}_{i=0}^{r} \) be as in the previous lemma and, for \( 1 \leq i \leq r \), take relatively prime positive integers \( n_i \) and \( \mu_i \) satisfying \( n_i/\mu_i = e_i \). Put \( \lambda_{i+1} := \lambda_i d_i - \lambda_{i-1} \).

1. If \( \lambda_{i-1} = \lambda_i d_i - \lambda_{i+1} \) holds for \( 1 \leq i \leq r \), then \( \lambda_1 = (\mu \lambda_0 + \lambda_{r+1})/n \).
2. If \( \lambda_0 \equiv 0 \pmod{n} \), then \( \lambda_i = \mu_i \lambda_i/\mu_i n_i \) for \( 1 \leq i \leq r \). If \( \mu \lambda_0 + 1 \equiv 0 \pmod{n} \), then \( \lambda_i = (\mu_i \lambda_{i-1} + 1)/n_i \) for \( 1 \leq i \leq r \).
3. If either \( \lambda_0 \equiv 0 \pmod{n} \) or \( \mu \lambda_0 + 1 \equiv 0 \pmod{n} \), then \( \lambda_{i-1} = \lambda_i d_i - \lambda_{i+1} \) holds for \( 1 \leq i \leq r \). Furthermore, \( \lambda_{r+1} = 0 \) when \( \lambda_0 \equiv 0 \pmod{n} \), and \( \lambda_{r+1} = 1 \) when \( \mu \lambda_0 + 1 \equiv 0 \pmod{n} \).
4. If \( \lambda_0 \equiv 0 \pmod{n} \), then \( \lambda_r = \lambda_0/n \). If \( \mu \lambda_0 + 1 \equiv 0 \pmod{n} \), then \( \lambda_r = [\lambda_0/n] \).

1.2. The fundamental cycle. We keep the notation in Introduction.

**Proposition 1.3.** There exists a resolution \( \pi : (X, E_\pi) \to (V_0, a_1, a_2, o) \) such that the weighted dual graph of \( E_\pi \) is as in Fig. 2. Furthermore, the genus \( g \) and the self-intersection number \( -d_0 \) of the central curve \( E_0 \) are given respectively by

\[
2g - 2 = l(l_0 l_1 l_2 l - l_0 l_1 - l_2),
\]

\[
d_0 = l \left( \frac{\sum_{w=0}^{2} p_w l_w}{\alpha_0} + \frac{1}{\alpha_0 \alpha_1 \alpha_2} \right).
\]

Proof. See [7, Proposition (3.5.1) and Theorem (3.6.1)]. We shall prove it in the course of the proof of Theorem 2.1 below.

In the sequel, we will work on the resolution space \( X \) in Proposition 1.3, unless otherwise stated explicitly.

**Theorem 1.4.** Let

\[
Z = \theta_0 E_0 + \sum_{w=0}^{2} \sum_{v=1}^{r_w} \sum_{\xi=1}^{l_{w,v}} \theta_w, v, \xi \ E_{w,v, \xi}
\]

be the fundamental cycle for resolution \( \pi \). Then the sequence \( \{\theta_w, v, \xi\}_{v=0}^{r_w} (w = 0, 1, 2) \) is defined by the following recurrence formula:
(1) \( \theta_{w,0,\xi} := \theta_0 = \begin{cases} \alpha_0 \alpha_1 \alpha_2 & \text{if } \alpha_2 \leq l_2, \\ \alpha_0 \alpha_1 l_2 & \text{if } \alpha_2 \geq l_2, \end{cases} \)

(2) \( \theta_{w,v,\xi} = [\theta_{w,v-1,\xi}/e_{w,v}] \) (1 \( \leq v \leq r_w \)).

Proof. By Lemma 1.1 applied to each branch of type \( C_{\alpha_w, p_w} \) plugged to \( E_0 \), we obtain (2) once \( \theta_0 \) is given. So, it suffices to show (1). Let \( u_w (w = 0, 1, 2) \) be the integer determined by \( p_w \theta_0 + u_w \equiv 0 \) (mod \( \alpha_w \)), \( 0 \leq u_w < \alpha_w \). Then \( \theta_{w,1,\xi} = [\theta_0/e_{w,1}] = (p_w \theta_0 + u_w)/\alpha_w \) by (2). By substituting the formula for \( d_0 \) in Proposition 1.3, the inequality \( \theta_0 d_0 \geq \sum_{w=0}^{2} \sum_{i=1}^{l_w} \theta_{w,1,\xi} \) coming from \(-ZE_0 \geq 0\) becomes

\[
\theta_0 \left( \sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0 \alpha_1 \alpha_2} \right) \geq \sum_{w=0}^{2} l_w \frac{p_w \theta_0 + u_w}{\alpha_w}.
\]

It follows \( \theta_0 \geq \alpha_1 \alpha_2 l_0 u_0 + \alpha_0 \alpha_2 l_1 u_1 + \alpha_0 \alpha_1 l_2 u_2 \). Put

\[
\Lambda := \left\{ \lambda \geq 1 \left| \begin{array}{ccc} \lambda \leq \alpha_1 \alpha_2 l_0 u_0 + \alpha_0 \alpha_2 l_1 u_1 + \alpha_0 \alpha_1 l_2 u_2, \\ p_w \lambda + u_w \equiv 0 \text{ (mod } \alpha_w \text{), } 0 \leq u_w < \alpha_w \text{ (} w = 0, 1, 2 \text{)} \end{array} \right. \right\}.
\]

Then, since \( Z \) is the fundamental cycle, \( \theta_0 = \min \Lambda \). Since \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are mutually coprime, we have \( \min \{ \lambda \in \Lambda \mid u_0 = u_1 = u_2 = 0 \} = \alpha_0 \alpha_1 \alpha_2 \). Also, since \( \alpha_0 \alpha_1 l_2 \leq \alpha_0 \alpha_2 l_1 \leq \alpha_1 \alpha_2 l_0 \) by \( a_0 \leq a_1 \leq a_2 \), we get \( \min \{ \lambda \in \Lambda \mid (u_0, u_1, u_2) \neq (0, 0, 0) \} = \alpha_0 \alpha_1 l_2 \). Therefore,

\[
\theta_0 = \begin{cases} \alpha_0 \alpha_1 \alpha_2 & \text{if } \alpha_2 \leq l_2, \\ \alpha_0 \alpha_1 l_2 & \text{if } \alpha_2 \geq l_2. \end{cases}
\]

This shows (1). \( \square \)

We put \( \theta_{w,v} := \theta_{w,v,\xi} \), because it does not depend on \( \xi \), and \( \theta_{w,r_w+1} := \theta_{w,r_w} d_{w,r_w} - \theta_{w,r_w-1} \). Furthermore, we sometimes write \( E_{w,v} \) for \( E_{w,v,\xi} \), when the index \( \xi \) is not important. By Lemma 1.2, we get the following:

**Lemma 1.5.** Let the situation be as above. Then

\[
\theta_{w,v-1} = \theta_{w,v} d_{w,v} - \theta_{w,v+1} \quad (w = 0, 1, 2; \ 1 \leq v \leq r_w).
\]

Furthermore, the following hold:

1. If \( \alpha_2 \leq l_2 \), then \( \theta_{i,j,k} = \alpha_i \alpha_k \) \((i, j, k) = (0, 1, 2)\) and \( \theta_{w,r_w+1} = 0 \) \((w = 0, 1, 2)\).
2. If \( \alpha_2 \geq l_2 \), then \( \theta_{0,r_0} = \alpha_1 l_2 \), \( \theta_{1,r_1} = \alpha_0 l_2 \), \( \theta_{2,r_2} = [\alpha_0 \alpha_1 l_2/\alpha_2] \), \( \theta_{0,r_0+1} = \theta_{1,r_1+1} = 0 \) and \( \theta_{2,r_2+1} = 1 \).
Proposition 1.6. The self-intersection number of the fundamental cycle is given by

\[-Z^2 = \begin{cases} 
  l\alpha_0\alpha_1\alpha_2 & \text{if } \alpha_2 \leq l_2, \\
  l_2[l_0\alpha_0\alpha_1l_2/\alpha_2] & \text{if } \alpha_2 \geq l_2.
\end{cases}\]

Proof. By Lemma 1.5, we have \(ZE_{w,v} = 0\) for \(w = 0, 1, 2, 1 \leq v \leq r_w - 1\). We first consider the case where \(\alpha_2 \leq l_2\). We already know that \(ZE_{w,r_w} = \theta_{w,r_w+1} = 0\) \((w = 0, 1, 2)\). Since

\[-Z E_0 = \theta_0d_0 - \sum_{w=0}^2 l_wl_0\theta_{w,1}\]

\[= l(p_0\alpha_0\alpha_2l_0 + p_1\alpha_0\alpha_2l_1 + p_2\alpha_0\alpha_1l_2 + 1) - l_0l_0\alpha_0\alpha_1l_2 - l_1p_1\alpha_0\alpha_2 - l_2p_2\alpha_0\alpha_1\]

we obtain \(-Z^2 = l\theta_0 = l\alpha_0\alpha_1\alpha_2\).

Next, we consider the case where \(\alpha_2 \geq l_2\). We have \(ZE_{0,r_0} = \theta_{0,r_0+1} = 0, ZE_{1,r_1} = \theta_{1,r_1+1} = 0\) and \(ZE_{2,r_2} = \theta_{2,r_2+1} = 1\). Furthermore,

\[-Z E_0 = \theta_0d_0 - \sum_{w=0}^2 l_wl_0\theta_{w,1}\]

\[= \alpha_0\alpha_1l_2\left(\sum_{w=0}^2 \frac{p_wl_w}{\alpha_0\alpha_1} + \frac{1}{\alpha_0\alpha_1\alpha_2}\right) - l_0l_0\alpha_0\alpha_1l_2 - l_1p_1\alpha_0\alpha_1l_2 - l_2p_2\alpha_0\alpha_1l_2 + 1\]

\[= 0.
\]

Therefore, \(-Z^2 = l_2\theta_{2,r_2} = l_2l_0[\alpha_0\alpha_1l_2/\alpha_2]\). \(\Box\)

Theorem 1.7 ((9, Theorem 2)). The fundamental genus \(p_f\) of \((V_{a_0,a_1,a_2}, o)\), \(2 \leq a_0 \leq a_1 \leq a_2\), is given as follows.

(1) If \(\alpha_2 \leq l_2\), then

\[p_f = \frac{1}{2}l(\text{lcm}(a_0, a_1, a_2) - \alpha_1\alpha_2l_0 - \alpha_0\alpha_2l_1 - \alpha_0\alpha_1l_2 - \alpha_0\alpha_1\alpha_2 + 1) + 1.
\]

(2) If \(\alpha_2 \geq l_2\), then

\[p_f = \frac{1}{2}\left\{(a_0 - 1)(a_1 - 1) - \left(2\left\lceil\frac{\alpha_0\alpha_1l_2}{\alpha_2}\right\rceil - 1\right)\text{gcd}(a_0, a_1) + 1\right\}.
\]

Proof. We consider the case where \(\alpha_0, \alpha_1, \alpha_2 \geq 2\). The other cases can be treated similarly. Let \(K\) be the canonical line bundle on \(X\). Since \(E_{w,v} \cong \mathbb{P}^1\) for \(v \neq 0\), we
have \( KE_{w,v} = -(E_{w,v})^2 + 2 \times 0 - 2 = d_{w,v} - 2 \). Similarly, since \( E_0 \) is of genus \( g \), \( KE_0 = -E_0^2 + 2g - 2 = d_0 + 2g - 2 \). It follows

\[
KZ = \theta_0(d_0 + 2g - 2) + \sum_{w=0}^{2} l_w l \sum_{v=1}^{r_w} \theta_{w,v}(d_{w,v} - 2)
\]

\[
= \theta_0[d_0 + l(lo_1l_2l - l_1 - l_2)] + \sum_{w=0}^{2} l_w l \left( \sum_{v=1}^{r_w} (\theta_{w,v-1} + \theta_{w,v+1}) - \sum_{v=1}^{r_w} 2 \theta_{w,v} \right)
\]

\[
= \theta_0[d_0 + l(lo_1l_2l - l_1 - l_2)] + \sum_{w=0}^{2} l_w l(\theta_0 - \theta_{w,1} - \theta_{w,r_w} + \theta_{w,r_w+1})
\]

\[
= \theta_0[l_0l_1l_2^2 + \left( \theta_0 \theta_0 - \sum_{w=0}^{2} l_w l \theta_{w,1} \right) - \sum_{w=0}^{2} l_w l \theta_{w,r_w} + l_2 l \theta_{2,r_2}
\]

\[
= \theta_0[l_0l_1l_2^2 - ZE_0 - \sum_{w=0}^{2} l_w l \theta_{w,r_w} + l_2 l \theta_{2,r_2}.
\]

(1) If \( \alpha_2 \leq l_2 \), then \( \theta_0 = \alpha_0 \alpha_1 \alpha_2, -ZE_0 = l, \theta_{0,r_0} = \alpha_0 \alpha_2, \theta_{1,r_1} = \alpha_0 \alpha_2, \theta_{2,r_2} = \alpha_0 \alpha_1, \theta_{2,r_2+1} = 0 \) and \( Z^2 = -l \alpha_0 \alpha_1 \alpha_2 \) by Lemma 1.5 and Proposition 1.6. The assertion follows from the formula \( 2p_f - 2 = KZ + Z^2 \).

(2) If \( \alpha_2 \geq l_2 \), then \( \theta_0 = \alpha_0 \alpha_1 l_2, ZE_0 = 0, \theta_{0,r_0} = \alpha_1 l_2, \theta_{1,r_1} = \alpha_0 l_2, \theta_{2,r_2} = [\alpha_0 \alpha_1 l_2/\alpha_2], \theta_{2,r_2+1} = 1 \) and \( Z^2 = -l_2 [\alpha_0 \alpha_1 l_2/\alpha_2] \) by Lemma 1.5 and Proposition 1.6. Therefore, we obtain the assertion.

**Remark 1.8.** An algorithm computing \( Z \) from the exponents \( a_0, a_1, a_2 \) was first obtained by Masataka Tomari (cf. [8, (3.3)]). Based on it, the formula for \( p_f \) was shown by Tomaru in [8, Theorem 4.3] in the special case: \( \text{lcm}(a_0, a_1) \leq a_2 \), and later completed in [9, Theorem 2]. However, the hypothesis of [9, Theorem 2 (1)] needs a small correction from “\( m = l_2 \) to “\( m \leq l_2 \). Indeed, when \( \alpha_2 = l_2 = 1, [9, \text{Theorem 2 (1)}] \) yields \( 2p_f - 2 = (a_0 - 1)(a_1 - 1) - (2a_0a_1 + 1)l - 1 \), while it should be \( 2p_f - 2 = (a_0 - 1)(a_1 - 1) - (2a_0a_1 - 1)l - 1 (a_0 - 1)(a_1 - 1) - (2a_0a_1 - 1)l - 1 \) according to Theorem 1.7.

2. Cycles led by coordinate functions

The purpose of the section is to show the following:

**Theorem 2.1.** Let \( Z^{(k)} := (x_k \circ \pi)_X \) be the cycle on \( E_{\pi} \) led by \( x_k \) (\( k = 0, 1, 2 \)), and put

\[
Z^{(k)} = \lambda_0^{(k)} E_0 + \sum_{w=0}^{2} \sum_{v=1}^{l_w} \sum_{\xi=1}^{r_w} \lambda^{(k)}_{w,v,\xi} E_{w,v,\xi}.
\]
Then the sequence $\{\lambda_{w,v}^{(k)}\}$ $(k = 0, 1, 2)$ is determined by the following recurrence formula.

\[
\begin{align*}
\lambda_{w,v-1,1}^{(k)} &= \lambda_{w,v,1}^{(k)} - 1, \\
\lambda_{w,0,0}^{(k)} &= \lambda_0^{(k)} = \alpha_i \alpha_j l_k \quad (\{i, j, k\} = \{0, 1, 2\}), \\
\lambda_{w,r+1,1}^{(k)} &= \begin{cases} 1 & \text{if } w = k, \\
0 & \text{if } w \neq k.
\end{cases}
\end{align*}
\]

In particular, for $\{i, j, k\} = \{0, 1, 2\}$,

\[
\begin{align*}
\lambda_{i,1,1}^{(k)} &= p_i \alpha_j l_k, \\
\lambda_{k,1,1}^{(k)} &= \frac{p_k \alpha_i \alpha_j l_k + 1}{\alpha_k}, \\
\lambda_{j,r_i,1}^{(k)} &= \alpha_j l_k, \\
\lambda_{i,j,r_i,1}^{(k)} &= \frac{\alpha_i \alpha_j l_k}{\alpha_k}.
\end{align*}
\]

We divide the proof into three steps. During the proof, we will construct the resolution $\pi$ and show Proposition 1.3. Put $\{i, j, k\} = \{0, 1, 2\}$ and denote the primitive $n$-th root of unity by $\epsilon_n$.

**Step 1. The resolution of the branch locus.** We put $C := \{x_i^a + x_j^a = 0\} \subset \mathbb{C}^2$.

First, we compute the minimal embedded resolution of $C$. Though there are several methods for computing such resolutions (see [1]), we use a result in [11] here. Put $d := \text{lcm}(a_i, a_j)$, $n_1 := a_i / \gcd(a_i, a_j)$, $n_2 := a_j / \gcd(a_i, a_j)$. Furthermore, we put $\tilde{C} := \{\tilde{x}_i^d + \tilde{x}_j^d = 0\} \subset \mathbb{C}^2$, and let $\Psi: \mathbb{C}^2_{(\tilde{x}_i, \tilde{x}_j)} \to \mathbb{C}^2_{(x_i, x_j)}$ be the holomorphic map defined by $x_i = \tilde{x}_i^{n_1}, x_j = \tilde{x}_j^{n_2}$. Since $d = a_i a_j / \gcd(a_i, a_j) = a_i n_2 = a_j n_1$, we have $\Psi(\tilde{C}) = C$. The map $\Psi$ can be regarded as the quotient map by the natural action to $\mathbb{C}^2$ of the group

\[
G = \left\{ \left( \begin{array}{cc} \epsilon_{n_2} & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & \epsilon_{n_1} \end{array} \right) \right\}.
\]

Let $\Phi: \tilde{N} \to \mathbb{C}^2$ be the blowing-up at the origin of the $(\tilde{x}_i, \tilde{x}_j)$-plane. We denote by $\tilde{E}$ the exceptional $(-1)$-curve for $\Phi$. Then $\tilde{N}$ is covered by two open sets $U_0$ and $U_1$ each of which is isomorphic to $\mathbb{C}^2$. The action of $\tilde{G}$ is lifted onto $\tilde{N}$ through $\Phi$. Let $\mu_1, \mu_2$ be non-negative integers defined by

\[
\begin{align*}
n_2 \mu_1 + 1 &\equiv 0 \pmod{n_1}, \quad 0 \leq \mu_1 < n_1, \\
n_1 \mu_2 + 1 &\equiv 0 \pmod{n_2}, \quad 0 \leq \mu_2 < n_2.
\end{align*}
\]

Then, from [11, Theorem 2.3], we can easily see that the quotient space $\tilde{N}/\tilde{G}$ is covered by two cyclic quotient singularity spaces $U_0/G$ and $U_1/G$ whose respective types are $C_{n_1 \mu_1}$ and $C_{n_2 \mu_2}$; also those singular points are located on $\psi(\tilde{E}) \simeq \mathbb{P}^1$, where $\psi: \tilde{N} \to \tilde{N}/G$ is the quotient map. Let $\eta: N \to \tilde{N}/G$ be the minimal resolution.
of those two cyclic quotient singularities and \( \Phi : \tilde{N}/G \to \mathbb{C}^2 \) the natural map to the \((x_i, x_j)\)-plane. Then the composite \( \phi = \Phi \circ \eta : N \to \mathbb{C}^2 \) gives us the minimal embedded resolution of \( C \).

Second, we describe \( \phi^*C \). The strict transform \( \tilde{\sigma}^{-1}C \) of \( C \) by \( \tilde{\sigma} \) consists of disjoint \( d \) branches each of which intersects \( \tilde{E} \) transversally at a point. Then \( \psi(\tilde{\sigma}^{-1}C) \) consists of \( \gcd(a_i, a_j) \) irreducible components each of which intersects \( \psi(\tilde{E}) \) transversally at a point. For \( \tilde{f} := \tilde{x}_i^{a_i} + \tilde{x}_j^{a_j} \), the multiplicity of \( \tilde{f} \circ \tilde{\sigma} \) along \( \tilde{E} \) is \( d \). If \( f \) denotes the holomorphic function on \( \tilde{N}/G \) induced by \( \tilde{f} \), then the multiplicity of \( f \) along \( \psi(\tilde{E}) \) is also \( d \). Furthermore, the multiplicity of \( f \) along each component of \( \psi(\tilde{\sigma}^{-1}C) \) is one. Since \( f = (x_i^{a_i} + x_j^{a_j}) \circ \Phi \), the dual graph of the divisor \( \phi^*C \) becomes as in Fig. 4. In that figure, \( F_0 \) is the strict transform of \( \psi(\tilde{E}) \) by \( \eta \) and \( F_{m,v_m} \) \((m = 1, 2; 1 \leq v_m \leq s_m)\) is the exceptional curve arising from \( C_{n_m, \mu_m} \) with self-intersection number \(-e_{m,v_m}\), where \( n_m/\mu_m = [(e_{m,1}, e_{m,2}, \ldots, e_{m,s_m})] \). For \( m = 1, 2 \), we denote by \( \rho_{m,v_m} \) the multiplicity of \( \phi^*C \) along \( F_{m,v_m} \). Since \( F_{m,v_m} \phi^*C = 0 \), we have

\[
(2.1) \quad \rho_{m,v_m-1} = \rho_{m,v_m} e_{m,v_m} - \rho_{m,v_m+1}, \quad 1 \leq v_m \leq s_m,
\]

with \( \rho_{m,0} = d \) and \( \rho_{m,s_m+1} = 0 \). Then, by Lemma 1.2 (1), we get \( \rho_{m,1} = \mu_m d/n_m \), that is,

\[
(2.2) \quad \rho_{1,1} = \mu_1 a_j, \quad \rho_{2,1} = \mu_2 a_i.
\]

We also have \( \rho_{1,1} + \rho_{2,1} + \gcd(a_i, a_j) = d \) by \( F_0 \phi^*C = 0 \), since \( -F_0^3 = F_0 F_{1,1} = F_0 F_{2,1} = 1 \) and \( F_0 \phi^{-1}C = \gcd(a_i, a_j) \).

**Step 2. The resolution of the cyclic covering.** We consider the resolution of \( \{x_i^{a_i} + x_j^{a_j} = x_k^{a_k}\} \) regarding \( V(a_i, a_j, a_k) \) as an \( a_k \)-fold cyclic covering of \( \mathbb{C}^2 \). Let \( \phi \) \( N \to \mathbb{C}^2 \) be the holomorphic map constructed in Step 1. We consider the normalization \( W \) of
the fiber product $V_{a_0, a_1, a_2} \times_{\mathbb{C}^2} N$. For this purpose, we use the following result due to Tomaru [12].

**Theorem 2.2** ([12]). Let $(U, o)$ be the cyclic quotient singularity of type $C_{n, \mu}$, and $m$ the maximal ideal of $O_{U, o}$. Assume that the zero divisor of the pull-back of $h \in m$ on the minimal resolution of $(U, o)$ has the weighted dual graph as in Fig. 5, where $n/\mu = [(c_1, \ldots, c_s)]$ and the $\rho_i$'s denote multiplicities. For a positive integer $a$, put

$$\tilde{a} = \frac{a}{\gcd(a, \rho_0)} \text{ and } \tilde{n} = \frac{\rho_{s+1}}{\gcd(a, \rho_0, \rho_{s+1})}$$

and $\alpha = \tilde{a} \tilde{n}$. Furthermore, let $p$ be the integer defined by

$$p \equiv \frac{a}{\gcd(a, \rho_0)} \mu \beta + \frac{\rho_{s+1}}{\gcd(a, \rho_{s+1})} \gamma \pmod{\alpha}, \quad 0 \leq p < \alpha,$$

where $\beta$ and $\gamma$ are integers determined by

$$\frac{a}{\gcd(a, \rho_0)} \beta \equiv 1 \pmod{\rho_0/\gcd(a, \rho_0)}, \quad 0 \leq \beta < \frac{\rho_0}{\gcd(a, \rho_0)},$$

$$\frac{\rho_{s+1}}{\gcd(a, \rho_0)} \gamma = \frac{a}{\gcd(a, \rho_0)} \beta - 1.$$  

Then the normalization of the $a$-fold covering of $U$ defined by $z^a = h$ has exactly $\gcd(a, \rho_0, \ldots, \rho_{s+1})$ cyclic quotient singularities of type $C_{a, p}$.

In our application, we always have $a = a_k$ and $\rho_0 = d = \lcm(a_i, a_j)$. So, $\beta$ and $\gamma$ are determined by

$$\alpha_k \beta \equiv 1 \pmod{\alpha_k \alpha_j l_k}, \quad 0 \leq \beta < \alpha_k \alpha_j l_k; \quad \alpha_k \alpha_j l_k \gamma = \alpha_k \beta - 1.$$

**Case 1.** We study $W$ over a neighborhood of $F_0 \cap \phi_a^{-1}C$ on $N$. Let $u = 0$ and $v = 0$ be local analytic equations of $F_0$, $\phi_a^{-1}C$ in a small neighborhood of each intersection point of $F_0$ and $\phi_a^{-1}C$. Recall that there are $\gcd(a_i, a_j) = l_k l$ such points in total. The $a_k$-fold cyclic covering is locally isomorphic to the singularity $\{u^d v = x_k^{a_k}\}$. Then, by Theorem 2.2 applied to $(n, \mu) = (1, 0), s = 0$ and $h = u^d v (\rho_0 = d, \rho_1 = 1)$, we see that $W$ has one cyclic quotient singularity of type $C_{\alpha_k, \rho_k}$ for each point of $F_0 \cap \phi_a^{-1}C$. 

![Fig. 5.](image-url)
where \( p_k \) is the integer defined by \( p_k \equiv \gamma \pmod{\alpha_k} \) (see also [10, Lemma 2.5]). Hence it is determined by the property \( p_k \alpha \alpha_j l_k + 1 \equiv 0 \pmod{\alpha_k} \), \( 0 \leq p_k < \alpha_k \) by (2.3).

By resolving these \( l_j l \) singular points according to [3], we easily see that the central curve \( E_0 \), which is nothing more than the proper inverse image of \( F_0 \), has a simple intersection with the curve \( E_{k,1,\xi} \) (\( 1 \leq \xi \leq l_j l \)) of self-intersection number \(-d_{k,1}\) as in Fig. 2, where we put \( \alpha_k/p_k = \left[[d_{k,1}, \ldots, d_{k,r_k}]\right]\) as before. Similarly, the proper inverse image of \( \phi^{-1}_*C \) has a simple intersection with each \( E_{k,r_k,\xi} \).

**Case 2.** We study \( W \) over a neighborhood of \( C_{n_1,\mu_1} \) and \( C_{n_2,\mu_2} \) on \( N \). We consider \( W \) over a neighborhood of \( C_{n_1,\mu_1} \) by applying Theorem 2.2 to \( F_0 + F_{1,1} + \cdots + F_{1,s_1} \), that is, the curve \( * \) on the left side in Fig. 5 is \( F_0 \) and \( \rho_{s_1+1} = 0 \). We take the pull-back to \( N \) of the equation \( x_i^n + x_j^a_j \) of \( C \) as the function \( h \). Then \( \rho_0 = d, \rho_i = \rho_{1,i} (1 \leq i \leq s_1) \). By (2.1) and (2.2), we have \( \gcd(\alpha_k, d, \rho_{1,1}, \ldots, \rho_{1,s_1}, \rho_{1,s_1+1}) = \gcd(\alpha_k, d, \rho_{1,1}) = l_j l \). Then, since \( \tilde{\alpha}_i = \alpha_i \) and \( \tilde{\alpha}_k = 1 \), Theorem 2.2 implies that \( W \) has \( l_j l \) cyclic quotient singularities of type \( C_{\alpha_i,\rho_i} \), where \( p_i \) is the integer defined by \( p_i \equiv \mu_i \beta \pmod{\alpha_i} \), \( 0 \leq p_i < \alpha_i \). Note that \( p_i \) satisfies \( p_i \alpha_j \alpha_k l_i + 1 \equiv 0 \pmod{\alpha_i} \) by the choice of \( \mu_1 \) and (2.3). Similarly, by considering \( C_{n_2,\mu_2} \), we see that \( W \) also has \( l_j l \) cyclic quotient singularities of type \( C_{\alpha_i,\rho_j} \), where \( p_j \) is the integer defined by \( p_j \equiv \mu_j \beta \pmod{\alpha_j} \), \( 0 \leq p_j < \alpha_j \). Then, \( p_j \) satisfies \( p_j \alpha_i \alpha_k l_j + 1 \equiv 0 \pmod{\alpha_j} \).

From Cases 1 and 2, we know that \( W \) has \( l(l_0 + l_1 + l_2) \) cyclic quotient singularities in total. Then we obtain the desired resolution \( \pi : X \to V_{a_1,a_2} \) by performing the minimal resolutions of all such cyclic quotient singularities of \( W \). Now, it is clear that the resolution dual graph is just as in Fig. 2.

**Step 3. The cycle \( Z^{(k)} \) and the central curve \( E_0 \).** In this final step, we determine \( Z^{(k)} \). We also calculate the genus \( g \) and the self-intersection number \(-d_0\) of the central curve \( E_0 \), and complete Proposition 1.3.

As in Step 2, we regard \( V_{a_1,a_2} \) as an \( a_k \)-fold cyclic covering of the \((x_i, x_j)\)-plane. We saw in Step 1 that \( \phi^{-1}_*C \) meets the \((-1)\)-curve \( F_0 \) at \( \gcd(a_i, a_j) = l_j l \) distinct points, and the multiplicity of \( \phi^*C \) along \( F_0 \) is \( d \equiv \text{lcm}(a_i, a_j) \). Then we obtain \( E_0 \) as an \( l_j l \)-fold cyclic covering of \( F_0 \), because \( \gcd(a_i, \text{lcm}(a_i, a_j)) = \gcd(\alpha_i l_j l, \alpha_j l_j l, \alpha_i l_j l_j l_1 2_1 l_2 1_2 l_1) = l_j l \).

Moreover, the vanishing order of \( x_k \circ \pi \) along \( E_0 \), i.e., the multiplicity of \( Z^{(k)} \) along \( E_0 \), is given by \( d/l_j l = \alpha_i \alpha_j l_k = : \lambda_{0}^{(k)} \). With this, we can determine the sequence \( \{ \lambda_{w,v,\xi}^{(k)} \} \) (\( w = 0, 1, 2; v = 1, \ldots, r_w; \xi = 1, \ldots, l_w l \)). In fact, since the intersection number of \( E_{w,v,\xi} \) with \((x_k \circ \pi)\) is zero, we obtain

\[
\lambda_{w,v,\xi}^{(k)} = \lambda_{w,v,\xi}^{(k)} d_{w,v} - \lambda_{w,v,\xi+1}^{(k)},
\]

with \( \lambda_{0,0,\xi}^{(k)} : = \lambda_{0,0,\xi}^{(k)} = \alpha_i \alpha_j l_k \) and \( \lambda_{r_w,1,\xi}^{(k)} = \lambda_{r_w,1,\xi+1}^{(k)} = 0, \lambda_{k,1,\xi} = 1 \) (recall that \( E_{k,1,\xi} \) meets the proper inverse image of \( \phi^{-1}_*C \) at a point). Note that one can compute all \( \lambda_{w,v,\xi}^{(k)} \) from these data. In particular, \( \lambda_{w,1,\xi}^{(k)} \) and \( \lambda_{w,\xi}^{(k)} \) are determined by Lemma 1.2 (1) and (4).
Let us calculate $g$ and $d_0$. In order to compute $g$, we observe how ramifies the $l/l$-sheeted covering map $E_0 \to F_0 \simeq \mathbb{P}^1$. Since there are $l/l$ branches of type $C_{\alpha_i, \beta_i}$, there exist $l/l$ points on $E_0$ of ramification index $l_j$. Similarly, considering $C_{\alpha_j, \beta_j}$, we have $l/j$ points of ramification index $l_i$. Also, considering $C_{\alpha_k, \beta_k}$, there exist $l/k$ points of ramification index $l/l$ over $F_0 \cap \phi^{-1}_x C$. Hence, by the Riemann–Hurwitz formula, we get
\[
2g - 2 = l(l(l(2 \times 0 - 2) + l(l_j - 1) + l_j(k - 1) + l_l(k - 1))
= \lambda_0 d_0 - l_l - l_l - l_j.
\]
Since the intersection number of $E_0$ with $(x_k \circ \pi)$ is zero, we get
\[
\lambda_0 d_0 = 2 \sum \sum \lambda_{w, 1, \xi}.
\]
Hence,
\[
d_0 = \frac{1}{\alpha_l \alpha_j \alpha_k} \left( p_l \alpha_l \alpha_j \alpha_k + p_j \alpha_j \alpha_k \alpha_l + \frac{1}{\alpha_k} \alpha_l \right)
= l \left( \sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0 \alpha_1 \alpha_2} \right).
\]
In sum, we have shown Theorem 2.1 and Proposition 1.3.

3. The maximal ideal cycle

We keep the notation in the previous section, but put $\lambda_{(w, v)}^{(k)} := \lambda_{w, v, \xi}^{(k)}$ for simplicity, because it does not depend on $\xi$.

**Theorem 3.1.** $Z^{(2)} \subseteq Z^{(1)} \subseteq Z^{(0)}$. In particular, $Z^{(2)}$ is the maximal ideal cycle for resolution $\pi$.

**Proof.** \{$\lambda_{w, v}^{(k)} \}_{w=0}^2$ satisfies $\lambda_{w, v}^{(k)} = [\lambda_{w, v-1}/c_{w, v}]$. Since $\lambda_0^{(2)} \leq \lambda_0^{(1)} \leq \lambda_0^{(0)}$ by $a_0 \leq a_1 \leq a_2$, we obtain inductively $Z^{(2)} \subseteq Z^{(1)} \subseteq Z^{(0)}$. Needless to say, the maximal ideal $m$ of $\mathcal{O}_{\phi_0, \phi_1, \phi_2}$ is generated by $x_0, x_1, x_2$. It follows from [14, Proposition 2.12] that $Z^{(2)}$ is the maximal ideal cycle.

**Theorem 3.2.** The maximal ideal cycle coincides with the fundamental cycle for resolution $\pi$ if and only if $\alpha_2 \geq l_2$.

**Proof.** By Theorems 1.4, 2.1 and Lemma 1.5, we see that $Z^{(2)}$ is the fundamental cycle if and only if $\alpha_2 \geq l_2$. Since $Z^{(2)}$ is the maximal ideal cycle by Theorem 3.1, we obtain the assertion.
fundamental cycle. Nothing more than the fundamental cycle of a rational double point of type $E_6$.

**Example 3.3 ($\alpha_2 \geq l_2$).** If $(a_0, a_1, a_2) = (2, 3, 4)$, then $l = l_0 = l_2 = 1$, $l_1 = 2$, $\alpha_0 = 1$, $\alpha_1 = 3$, $\alpha_2 = 2$, $p_0 = 0$, $p_1 = 2$, $p_2 = 1$. The maximal ideal cycle $Z^{(2)}$ is nothing more than the fundamental cycle of a rational double point of type $E_6$.

**Example 3.4 ($\alpha_2 < l_2$).** If $(a_0, a_1, a_2) = (6, 15, 20)$, then $l = 1$, $l_0 = 5$, $l_1 = 2$, $l_2 = 3$, $\alpha_0 = \alpha_1 = 1$, $\alpha_2 = 2$, $p_0 = p_1 = 0$, $p_2 = 1$. Hence the weighted dual graph of the maximal ideal cycle $Z^{(2)}$ is as in Fig. 6. It is clear that $Z^{(2)}$ is not the fundamental cycle.

**Lemma 3.5.** If $\pi$ is not the minimal resolution, then $l = 1$, $\{l_0, l_1, l_2\} = \{1, 1, n\}$ for some $n \geq 1$.

Proof. Assume that $\pi$ is not the minimal resolution. Since, in $E_6$, the self-intersection number of any component except $E_0$ is less than or equal to $-2$, we see that $E_0$ must be a $(-1)$-curve: $g = 0$ and $d_0 = 1$. By the formula in Proposition 1.3, we have $g = 0$ if and only if $(l_0, l_1, l_2, l) = (1, 1, 1, 2)$ or $l = 1$, $\{l_0, l_1, l_2\} = \{1, 1, n\}$.

Assume that $(l_0, l_1, l_2, l) = (1, 1, 1, 2)$. We have $2(p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1) = \alpha_0\alpha_1\alpha_2$ by $d_0 = 1$. Note that $\alpha_0\alpha_1\alpha_2$ divides $p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1$ by the choice of the $p_w$’s. Hence one has $p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1 \geq \alpha_0\alpha_1\alpha_2 = 2(p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1)$, which is absurd. Therefore, $(l_0, l_1, l_2, l) \neq (1, 1, 1, 2)$ and we are left the case: $l = 1$, $\{l_0, l_1, l_2\} = \{1, 1, n\}$. □

**Theorem 3.6.** The maximal ideal cycle coincides with the fundamental cycle for the minimal resolution of $(V_{a_0, a_1, a_2}, o)$, $2 \leq a_0 \leq a_1 \leq a_2$, if and only if $\alpha_2 \geq l_2$.

Proof. Assume that $\alpha_2 \geq l_2$. It is obvious that the fundamental cycle coincides with the maximal ideal cycle also on the minimal resolution by the assumption and Theorem 3.2.

Assume that $\alpha_2 < l_2$. If $\pi$ is not the minimal resolution, then, by Lemma 3.5, $l_0 = l_1 = l = 1$, $l_2 \geq 2$, because $l_2 > \alpha_2 \geq 1$. Then we would have $a_2 = \alpha_2 < l_2 \leq a_0$, which is absurd. Hence $\pi$ is minimal. By Theorems 1.4, 2.1 and 3.2, the maximal ideal cycle $Z^{(2)}$ cannot be the fundamental cycle. □
Corollary 3.7. If \( a_0 \) is a prime number, then the maximal ideal cycle coincides with the fundamental cycle for the minimal resolution of \( (V_{a_0 a_1 a_2}, o) \), \( 2 \leq a_0 \leq a_1 \leq a_2 \).

Proof. It can be checked directly that \( \alpha_2 \geq l_2 \) holds, when \( a_0 \) is prime. \( \square \)

Lemma 3.8. \(-(Z^{(k)})^2 = l_k [\alpha_i \alpha_j l_k / \alpha_k] \), where \( \{ i, j, k \} = \{ 0, 1, 2 \} \). In particular, \(-(Z^{(2)})^2 = a_0 = \mult(\mathcal{O}_{V_{a_0 a_1 a_2}, o}) \) holds if and only if \( [\alpha_0 \alpha_1 l_2 / \alpha_2] = \alpha_0 l_1 \), i.e., \( 1 < a_1 / a_2 + \gcd(a_0, a_1) / a_0 \).

Proof. By Lemma 1.2 (4) and Theorem 2.1, we have \( \lambda^{(k)}_{k, r, t, k} = [\alpha_i \alpha_j l_k / \alpha_k] \). Then the self-intersection number of \( Z^{(k)} \) can be computed similarly as in the proof of Proposition 1.6. Hence we omit the detail. Note that
\[
-(Z^{(2)})^2 = l_2 l \left[ \frac{\alpha_0 \alpha_1 l_2}{\alpha_2} \right] = a_0 \iff \left[ \frac{\alpha_0 \alpha_1 l_2}{\alpha_2} \right] = \alpha_0 l_1
\]
\[
\iff \frac{\alpha_0 \alpha_1 l_2}{\alpha_2} \leq \alpha_0 l_1 < \frac{\alpha_0 \alpha_1 l_2}{\alpha_2} + 1.
\]
In the last inequalities, we need not care the left hand side one, because it always holds true by \( a_1 \leq a_2 \). As to the right hand side inequality, we have
\[
\alpha_0 l_1 < \frac{\alpha_0 \alpha_1 l_2}{\alpha_2} + 1 \iff 1 < \frac{\alpha_1 l_2 l_0}{\alpha_2 l_1 l_0} + \frac{l_2 l}{\alpha_2 l_1 l_2} = \frac{a_1}{a_2} + \frac{\gcd(a_0, a_1)}{a_0}
\]
from which the assertion follows. \( \square \)

Proposition 3.9. Put \( \delta := \alpha_0 l_1 - [\alpha_0 \alpha_1 l_2 / \alpha_2] \geq 0 \). The base points on \( E_\pi \) of the linear system \( |\mathcal{O}_X(Z^{(2)})| \) can be resolved by a succession of \( \delta l_2 l \) simple blowing-ups. In particular, the linear system \( |\mathcal{O}_X(Z^{(2)})| \) has no base points on \( E_\pi \) if and only if \( \delta = 0 \).

Proof. The second assertion is clear from Lemma 3.8, because \( Z^{(2)} \) is the maximal ideal cycle and one has \( \mathfrak{m} \mathcal{O}_X \simeq \mathcal{O}_X(-Z^{(2)}) \). See, e.g., [13, Theorem 2.7].

Now, we prove the first assertion. Note that we have \( \lambda^{(1)}_{2, r_2} = \alpha_0 l_1 \) and \( \lambda^{(2)}_{2, r_2} = [\alpha_0 \alpha_1 l_2 / \alpha_2] \) by Theorem 2.1. Hence, \( \delta \) is nothing but the difference of the multiplicities of \( Z^{(1)} \) and \( Z^{(2)} \) along \( E_{2, r_2, \xi} \). Assume that \( \delta > 0 \) and put \( D = (x_2 \circ \pi) - (x_2 \circ \pi)_X \).

It is clear that the base points of \( |\mathcal{O}_X(Z^{(2)})| \) on \( E_\pi \) are \( l_2 l \) intersection points \( P_\xi = E_{2, r_2, \xi} \cap D, \ 1 \leq \xi \leq l_2 l \). Let \( \phi : \tilde{X} \to X \) be the composite of blowing-ups (performed \( \delta \) times for each \( \xi \)) at \( P_\xi \) and the \( \delta - 1 \) points infinitely near to it on the proper transform of \( D \) (see, Fig. 7). Thus, \( \phi \) blows up \( \delta l_2 l \) points in total. Put \( A := K_{\tilde{X}} - \phi^* K_X \) and let \( A_\xi \) be the \((-1\))-curve over \( P_\xi \) lastly appeared in \( \phi \). Then the multiplicity of \( A \) along \( A_\xi \) is \( \delta \), and the cycle on \( \tilde{X} \) is \( \tilde{Z}^{(2)} : = \phi^* Z^{(2)} + A \). We have \( \mult(A_\xi(\tilde{Z}^{(2)})) = \lambda^{(2)}_{2, r_2} + \delta = [\alpha_0 \alpha_1 l_2 / \alpha_2] + \delta = \alpha_0 l_1 \) for each \( \xi \). Then, \( x_1 \circ \pi \circ \phi \) gives us
a section of $\mathcal{O}_X(-\tilde{Z}^{(2)})$ that is a non-zero constant on each $A_\xi$, because $\tilde{Z}^{(2)} \preceq \phi^*Z^{(1)}$, mult$_{A_\xi}(\phi^*Z^{(1)}) = \alpha_0 l_1$ and $(x_1 \circ \pi \circ \phi) = \phi^*Z^{(1)}$ in a neighborhood of the $l_2 l$ branches containing the $A_\xi$'s. Therefore, $|\mathcal{O}_X(-\tilde{Z}^{(2)})|$ has no base points on $E_{\pi = \phi}$. Needless to say, $\tilde{Z}^{(2)}$ is the maximal ideal cycle on $\tilde{X}$ and $-(\tilde{Z}^{(2)})^2 = a_0$.

**Example 3.10.** If $(a_0,a_1,a_2) = (6,10,15)$, then $l = 1$, $l_0 = 5$, $l_1 = 3$, $l_2 = 2$, $\alpha_0 = \alpha_1 = \alpha_2 = 1$, $p_0 = p_1 = p_2 = 0$. The exceptional set is a non-singular curve $E_0$ of genus 11, and $d_0 = 1$, $\lambda_0 = 2$. Then $Z = E_0$ and $Z^{(2)} = 2E_0$. We have $-(Z^{(2)})^2 = 4$, while mult$(\mathcal{O}_{\mathcal{X}_{6,10,15}}) = 6$. Two intersection points $E_0 \cap D$, where $D = (x_2 \circ \pi) - (x_2 \circ \pi)_X$, are base points of $|\mathcal{O}_X(-2E_0)|$. Indeed, since the vanishing order of $x_2 \circ \pi$ along $E_0$ is exactly 2, it induces a non-zero element of $H^0(X, -2E_0)/H^0(X, -3E_0) \subset H^0(E_0, -2E_0)$. On the other hand, $\dim H^0(E_0, -2E_0) = 1$, because $E_0$ is a non-hyperelliptic curve. Therefore, $H^0(E_0, -2E_0)$ is generated by the image of $x_2 \circ \pi$ which vanishes at two intersection points $P_1$, $P_2$ mentioned above. Let $\phi$ be the blowing-up at $P_1$, $P_2$, and put $A = A_1 + A_2$, where $A_i = \phi^{-1}(P_i)$ for $i = 1, 2$. Then $2\phi^*E_0 + A$ is the cycle led by $x_2$ and we obtain $(2\phi^*E_0 + A)^2 = -6$.

**Theorem 3.11.** The maximal ideal cycle coincides with the fundamental cycle for any resolution of $(V_{0,a_0,a_1,a_2},0)$, $2 \leq a_0 \leq a_1 \leq a_2$, if and only if $a_2 \geq l_2$ and $1 < a_1/a_2 + \gcd(a_0,a_1)/a_0$. If this is the case, then the fundamental cycle is led by the holomorphic function $x_2$.

Proof. The fundamental cycle on a resolution is obtained as the pull-back of that on the minimal resolution. The same holds for the maximal ideal cycle, if the minus of it defines a free linear system on the minimal resolution. Therefore, the first assertion follows from Theorem 3.6, Lemma 3.8 and Proposition 3.9. The second assertion is clear, because $Z^{(2)}$ is led by $x_2$.

**Example 3.12.** For $(V_{2,3,2n+1,0})$, the above implies that the maximal ideal cycle coincides with the fundamental cycle for any resolution when $n = 1, 2$, while it holds not for all but for the minimal resolution when $n \geq 3$. 

\[
\begin{array}{cccc}
\lambda_0^{(2)} & \lambda_2^{(2)} & \lambda_2^{(2)} + 1 & \lambda_2^{(2)} + \delta \\
-d_0 & -d_2 & -d_2 - 1 & -1 \\
E_0 & E_{2,1,\xi} & E_{2,2,\xi} & \phi^{-1}D \\
\end{array}
\]
4. Further remarks

4.1. Kodaira singularities. Let \( S \) be a non-singular complex surface and \( D \subset \mathbb{C} \) a small open disc around the origin. A surjective holomorphic map \( \Phi: S \to D \) is said to be a pencil of curves of genus \( g \), if it is proper and connected, and fibers \( S_t := \Phi^{-1}(t) \ (t \neq 0) \) are smooth curves of genus \( g \).

**Definition 4.1** ([5]). A normal surface singularity \((V, o)\) is said to be a Kodaira singularity, if there exists a pencil of curves \( \Phi: S \to D \) such that, after a finite number of blowing-ups at non-singular points in non-multiple components of the central fiber \( S_0 \), \( \Psi: S' \to S \), there is a holomorphic map \( \varphi: M \to V \) from an open neighborhood \( M \) of the proper transform of \( \text{Supp}(S_0) \) in \( S' \) which defines a resolution of \((V, o)\).

**Proposition 4.2** ([5, p. 46], [6]). Let \( \phi: (X, E) \to (V, o) \) be the minimal good resolution of a normal surface singularity and \( \mathfrak{m} \) the maximal ideal of \( \mathcal{O}_{V, o} \). Then \((V, o)\) is a Kodaira singularity if and only if \( \text{mult}_E(Z_E) = 1 \) holds for every component \( E_j \) satisfying \( Z_E E_j < 0 \) and there exists an element \( f \in \mathfrak{m} \) such that the divisor \((f \circ \phi)\) is normal crossing and \((f \circ \phi)_X = Z_E\).

Now, we return to the situation we are interested in. Consider the singularity of Brieskorn type and let \( \pi: (X, E_\pi) \to (V_{(a_0, a_1, a_2)}, o) \) be the resolution as before.

**Lemma 4.3.** \( \pi \) is not the minimal good resolution if and only if \( a_0 = a_1 = 2 \), \( a_2 = 2m + 1 \) for a positive integer \( m \) (the rational double point of type \( A_{2m} \)).

**Proof.** Clearly, \( \pi \) is not the minimal good resolution if and only if \( E_0 \) is a \((-1)\)-curve and the number of branches plugged to it is at most two.

Assume that \( \pi \) is not the minimal good resolution. Then, since \( E_0 \) is a \((-1)\)-curve, we have \( l = 1 \), \((l_i, l_j, l_k) = (1, 1, n) \) \((n \geq 1, \{i, j, k\} = \{0, 1, 2\})\) by Lemma 3.5. First, assume that \( n = 1 \). Then \( \alpha_0, \alpha_1, \alpha_2 \geq 2 \), because \( 2 \leq a_0 \leq a_1 \leq a_2 \). But, this implies that there are three branches, a contradiction. Second, assume that \( n \geq 3 \). Then \( \alpha_k \geq 2 \) and, we obtain a contradiction, because the number of branches is at least \( l_k l = n \geq 3 \). Finally assume that \( n = 2 \). Then \( \alpha_k \geq 2 \). Furthermore, \( \alpha_k \) is odd, because it is coprime to \( l_k = 2 \). We have \( \alpha_i = \alpha_j = 1 \), because \( l_k l = 2 \) and the number of branches must be at most two. Therefore, by \( 2 \leq a_0 \leq a_1 \leq a_2 \), we see \( a_0 = a_1 = 2 \) and \( a_2 \) is an odd integer not less than three. Then it is a rational double point of type \( A_{2m} \) \((m \geq 1)\).

Conversely, assume that \( a_0 = a_1 = 2 \) and \( a_2 = 2m + 1 \) for some positive integer \( m \). Then \( l = l_0 = l_1 = 1, l_2 = 2, \alpha_0 = \alpha_1 = 1, \alpha_2 = 2m + 1, p_0 = p_1 = 0, p_2 = m \). Hence \( g = 0, d_0 = 1 \) by Proposition 1.3, and exactly two branches are plugged to \( E_0 \). So, we obtain the minimal good resolution by contracting \( E_0 \).

The following shows that the sufficient condition given in [10, Corollary 4.6] is also necessary.
Proposition 4.4. \((V_{a_0,a_1,a_2}, o), 2 \leq a_0 \leq a_1 \leq a_2,\) is a Kodaira singularity if and only if \(\alpha_0 \alpha_1 l_2 \leq \alpha_2,\) i.e., \(\text{lcm}(a_0, a_1) \leq a_2.\) If this is the case, it is associated to a pencil of curves of genus \((1/2)((a_0 - 1)(a_1 - 1) - \gcd(a_0, a_1) + 1).\)

Proof. (i) We first assume that \((V_{a_0,a_1,a_2}, o)\) is not a rational double point of type \(A_{2m} (m \geq 1).\) Then \(\pi\) is the minimal good resolution by Lemma 4.3. Let \(Z\) denote the fundamental cycle for \(\pi.\) Assume that \(\alpha_2 < l_2.\) Let \(m\) be the maximal ideal of \(\mathcal{O}_{V_{a_0,a_1,a_2}, o}.\) We have no \(f \in m\) such that \((f \circ \pi)_X = Z,\) because \(Z\) is not the maximal ideal cycle by Theorem 3.2. Hence \((V_{a_0,a_1,a_2}, o)\) is not a Kodaira singularity by Proposition 4.2. Assume that \(\alpha_2 \geq l_2.\) We already know that \((x_2 \circ \pi)\) is a normal crossing divisor, and \(Z^{(2)} = (x_2 \circ \pi)_X = Z\) by Theorem 3.2. Furthermore, we have

\[
Z E_{w,v,\xi} = \begin{cases} 
-1 & \text{if } w = 2, v = r_2, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence, by Proposition 4.2, \((V_{a_0,a_1,a_2}, o)\) is a Kodaira singularity if and only if \(\text{mult}_{E_{2,2,2}}(Z) = [\alpha_0 \alpha_1 l_2/\alpha_2] = 1,\) i.e., \(\alpha_0 \alpha_1 l_2 \leq \alpha_2.\)

(ii) Next, we consider \((V_{2,2,2m+1}, o).\) Then \(\alpha_0 \alpha_1 l_2 = 2 < \alpha_2 = 2m + 1.\) Let \(Z'\) be the fundamental cycle on the minimal good resolution \(\pi': X' \to V_{2,2,2m+1}.\) By Theorem 3.11, \(Z'\) is led by \(x_2\) and it is clear that \((x_2 \circ \pi')\) is normal crossing. Hence \((V_{2,2,2m+1}, o)\) is a Kodaira singularity, by Proposition 4.2.

The last assertion for the genus follows from [8, Theorem 4.3] or Theorem 1.7 (2).

4.2. Canonical cycle.

Definition 4.5. Let \(\phi: (X,E) \to (V, o)\) be a resolution of a normal surface singularity. A \(\mathbb{Q}\) divisor \(Z_K\) with support in \(E = \bigcup_{i=1}^{r} E_i\) is said to be the canonical cycle, if \(-Z_K E_i = KE_i\) holds for any irreducible component \(E_i.\)

For the singularity of Brieskorn type, we can express \(Z_K\) in terms of some previously known cycles.

Proposition 4.6. Let \(Z_K\) be the canonical cycle for the resolution \(\pi: (X, E_\pi) \to (V_{a_0,a_1,a_2}, o)\) in Proposition 1.3. Then

\[
Z_K = E + l_0 l_1 l_2 l Z_0 - Z^{(0)} - Z^{(1)} - Z^{(2)},
\]

where \(E\) is the reduced exceptional divisor,

\[
E = E_0 + \sum_{w=0}^{2} \sum_{v=1}^{r_w} \sum_{\xi=1}^{l_{w,v}} E_{w,v,\xi}.
\]
and $Z_0$ is the cycle with $\text{mult}_{K}(Z_0) = \alpha_0\alpha_1\alpha_2$ appeared in Theorem 1.4 as the fundamental cycle for the case $\alpha_2 \leq l_2$.

Proof. We only consider the case where $\alpha_0, \alpha_1, \alpha_2 \geq 2$, because the other cases can be carried out similarly. For short, we put $E_{w,v} := E_{w,v,\xi}$, and $E_{w,0} := E_0$. By Proposition 1.3 and the fact that $E_{w,v} \simeq \mathbb{P}^1$ when $v \neq 0$, we have

$$KE_{w,v} = \begin{cases} d_0 + l_0l_1l_2l - l_0 - l_1 - l_2 & \text{if } v = 0, \\ d_{w,v} - 2 & \text{otherwise.} \end{cases}$$

On the other hand, as in the proof of Theorem 1.6, we have

$$-EE_{w,v} = \begin{cases} d_0 - l_0l - l_1l - l_2l & \text{if } v = 0, \\ d_{w,v} - 1 & \text{if } v = r_w, \\ d_{w,v} - 2 & \text{otherwise,} \end{cases}$$

$$-Z_0E_{w,v} = \begin{cases} 1 & \text{if } w = k, v = r_w, \\ 0 & \text{otherwise,} \end{cases}$$

$$-Z^{(k)}E_{w,v} = \begin{cases} l & \text{if } v = 0, \\ 0 & \text{otherwise,} \end{cases}$$

in view of Theorems 1.4 and 2.1. Hence, for our $Z_K$, it can be checked directly that $-Z_K E_{w,v} = KE_{w,v}$ holds for all $w$ and $v$. \qed

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