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# A $q$-CAUCHY IDENTITY FOR SCHUR FUNCTIONS AND IMPRIMITIVE COMPLEX REFLECTION GROUPS

Dedicated to Professor Shunichi Tanaka on his sixtieth birthday

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## 1. Introduction

Let

$$\rho\colon G \longrightarrow U_n(\mathbf{C})$$

be a finite group acting on $\mathbf{C}^n$ as a complex reflection group. For an irreducible character $\chi$ of $G$, we define a rational function

$$\Psi_G(\chi; q) = |G|^{-1} \sum_{w\in G} \chi(w^2) \frac{\det(1 + q\rho(w))}{\det(1 - q\rho(w))} \tag{1.1}$$

in an indeterminate $q$. Note that, at $q = 0$, this reduces to the Frobenius-Schur index of $\chi$. When $G$ is the symmetric group on $n$ letters, we have [6] an explicit formula for (1.1). In a recent work [4], [5] (this and the present work were done largely independently), A. Gyoja, K. Nishiyama and K. Taniguchi explicitly calculated (1.1) in the cases of real reflection groups of type $B_n, E_n, F_4, I_2(m)$ and $D_n$; in the case of type $D_n$, their proof depends upon one of the main result (Theorem 1.1 below) of the present paper. The authors of [4], [5] also observed a mysterious connection between $\Psi_G(\chi; q)$, Lusztig's cells and modular representations of Iwahori Hecke algebras.

The main purpose of this paper is to calculate $\Psi_G(\chi; q)$ explicitly when $G$ is an imprimitive complex reflection group $G(m, p, n)$ (in the notation of G.C. Shephard and J.A. Todd [12]). This includes, as special cases, the cases of real reflection groups of type $B_n, D_n,$ and $I_2(m)$.

**Theorem 1.1.** *Let $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ be two infinite sequences of independent variables. For a partition $\lambda$ let $s_\lambda(x) = s_\lambda(x_1, x_2, x_3, \ldots)$ and $s_\lambda(y) = s_\lambda(y_1, y_2, y_3, \ldots)$ be the corresponding Schur functions in $x$ and $y$ respectively.*

*Then we have the following identities*:

(1.2)
$$\prod_i \prod_{r=0}^{\infty} \frac{1+x_i q^{2r+1}}{1-x_i q^{2r+1}} \frac{1+y_i q^{2r+1}}{1-y_i q^{2r+1}} \prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda,\mu} q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2) s_\lambda(x) s_\mu(y),$$

*and*

$$\begin{aligned} \prod_i \prod_{r=0}^{\infty} \frac{1+x_i q^{2r+1}}{1-x_i q^{2r+1}} \frac{1+y_i q^{2r+1}}{1-y_i q^{2r+1}} \prod_{i,j} (1+x_i y_j) \\ = \sum_{\lambda,\mu} q^{|\lambda-\mu'|-|\mu'-\lambda|} J_{\lambda\mu'}(q^2) s_\lambda(x) s_\mu(y), \end{aligned} \tag{1.3}$$

*where $\mu'$ is the dual partition of $\mu$, $J_{\lambda\mu}(t)$ is a rational function defined in* Section 2.1, *and the sums are taken over all partitions $\lambda$ and $\mu$.*

Since, at $q = 0$, the identities (1.2) and (1.3) reduce to the classical Cauchy identities

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y), \tag{1.4}$$

and

$$\prod_{i,j} (1+x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda'}(y) \tag{1.5}$$

respectively, we shall refer to (1.2) and (1.3) as the *$q$-Cauchy identities*.

The paper is organized as follows. In Section 2, we give two different definitions of $J_{\lambda\mu}(t)$; one (2.3) is combinatorial and the other (2.6) is analytic, the equivalence of them being non-trivial (Lemma 2.3). We also recall some basic facts on the complex reflection groups $G(m, p, n)$. In Section 3, assuming the validity of the $q$-Cauchy identity (1.2), we show that $\Psi_G(\chi; q)$ for $G = G(m, p, n)$ can be written explicitly using $J_{\lambda\mu}(t)$ (Theorems 3.1 and 3.2). In Section 4, we derive some consequences of the $q$-Cauchy identity; in particular, we briefly discuss new inner products and new basis in the space of symmetric functions. Finally in Section 5, we prove the $q$-Cauchy identities (1.2), (1.3). The main technical tools for this are symmetrizing operators of A. Lascoux and P. Pragacz [7] and divided differences of higher order in the calculus of finite differences [10], [11].

The author thanks A. Gyoja for explaining the results in [5], by which the author came to realize that the calculation of $\Psi_G(\chi; q)$ for $G = G(m, p, n)$ can be reduced almost immeadiately to the one for $G = G(m, 1, n)$.

## 2. Preliminaries

**2.1. Combinatorics on partitions and diagrams.** A *partition*

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \tag{2.1}$$

is a finite sequence of non-negative integers $\lambda_i$ in non-increasing order; we consider $(\lambda_1, \lambda_2, \dots, \lambda_n) = (\lambda_1, \lambda_2, \dots, \lambda_n, 0)$. The set of partitions are denoted by $\mathcal{P}$. The sum of the parts of $\lambda \in \mathcal{P}$ is denoted by $|\lambda|$, the number of non-zero parts by $l(\lambda)$, and the multiplicity of $k(\neq 0)$ as a part by $m_k(\lambda)$. A partition (2.1) is often identified with the corresponding *Young diagram*, which is the set of points $(i, j) \in \mathbf{Z}^2$ such that $1 \leq j \leq \lambda_i$. In particular, the Young diagram corresponding to the partition $\phi = (0, 0, 0, \dots, 0)$ is empty. The set theoretical difference $\lambda - \mu = \lambda - \lambda \cap \mu$ of two Young diagrams $\lambda$ and $\mu$ is called a *skew diagram*. Note that we are *not* assuming $\lambda \supset \mu$. An element of a Young (or skew) diagram $\lambda$ is called a *node* of $\lambda$. For $\lambda \in \mathcal{P}$, we define the *dual* partition $\lambda'$ of $\lambda$ by

$$\lambda' = \{(i, j) \in \mathbf{Z}^2 \mid (j, i) \in \lambda\}.$$

For $\lambda \in \mathcal{P}$, the *hook-length* $h_\lambda$ of $\lambda$ is a positive-integer-valued function on $\lambda$ defined by

$$h_\lambda(v) = \lambda_i + \lambda'_j - i - j + 1, \qquad v = (i, j) \in \lambda.$$

As a natural generalization of $h_\lambda$, we define, for any $\lambda, \mu \in \mathcal{P}$, an integer-valued function $h_{\lambda\mu}$ on the set $\{v = (i, j) \mid i, j = 1, 2, 3, \dots\}$ by

$$h_{\lambda\mu}(v) = \lambda_i + \mu'_j - i - j + 1, \qquad v = (i, j).$$

We also define:

$$n(\lambda, \mu) = \sum_{(i,j)\in\lambda-\mu} (\lambda'_j - i) = \sum_{(i,j)\in\lambda-\mu} (i - \mu'_j - 1).$$

We put $n(\lambda) = n(\lambda, \phi)$, which coincides with the one appearing in [9], I, 1.

We have:

**Lemma 2.1.** (i) $h_{\lambda\lambda}(v) = h_\lambda(v), \quad v \in \lambda.$
(ii) $h_{\lambda'\mu'}(v') = h_{\mu\lambda}(v), \quad v \in \lambda \cup \mu,$
*where we put* $v' = (j, i) \in \lambda' \cup \mu'$ *for* $v = (i, j) \in \lambda \cup \mu$.
(iii) $h_{\lambda\phi}(v) = c_\lambda(\alpha(v)), \quad v \in \lambda,$
*where* $c_\lambda : \lambda \longrightarrow \mathbf{Z}$ *is the content function* (*see, e.g,* [9], I, 1, Ex. 3) *defined by*

$$c_\lambda((i, j)) = j - i, \quad (i, j) \in \lambda,$$

*and $\alpha$ is a permutation of $\lambda$ defined by*

$$\alpha((i, j)) = (i, \lambda_i - j + 1), \quad (i, j) \in \lambda.$$

(iv) *There exists a permutation $\beta$ of $\lambda - \mu$ such that*

$$h_{\lambda\mu}(\beta(v)) = -h_{\mu\lambda}(v), \quad v \in \lambda - \mu.$$

(v)
$$n(\lambda, \mu) + n(\mu, \lambda) + \sum_{v\in\mu} h_{\mu\lambda}(v) = n(\mu') + n(\lambda) + |\lambda \cap \mu|.$$

Proof. Parts (i)(ii) and (iii) are obvious. Part (iv) follows from a stronger assertion proved in Lemma 2.2 below. For a proof of part (v), we note

$$\sum_{v\in\mu} h_{\mu\lambda}(v) = \sum_{(i,j)\in\mu} \{(\mu_i - j) + 1 + (\lambda'_j - i)\} = n(\mu') + |\mu| + \sum_{(i,j)\in\mu} (\lambda'_j - i).$$

Hence

$$\begin{aligned} n(\lambda, \mu) + \sum_{v\in\mu} h_{\mu\lambda}(v) &= n(\mu') + |\mu| + \sum_{(i,j)\in\lambda\cup\mu} (\lambda'_j - i) \\ &= n(\mu') + |\mu| + \sum_{(i,j)\in\lambda} (\lambda'_j - i) + \sum_{(i,j)\in\mu-\lambda} \{(\lambda'_j - i + 1) - 1\} \\ &= n(\mu') + |\mu| + n(\lambda) - n(\mu, \lambda) - |\mu - \lambda| \\ &= n(\mu') + n(\lambda) + |\mu \cap \lambda| - n(\mu, \lambda), \end{aligned}$$

which proves part (v). □

Let $v = (i, j) \in \lambda - \mu$. We define

$$\begin{aligned} a_{\lambda\mu}(v) &= \lambda_i - j, & a'_{\lambda\mu}(v) &= j - \mu_i - 1, \\ l_{\lambda\mu}(v) &= \lambda'_j - i, & l'_{\lambda\mu}(v) &= i - \mu'_j - 1, \end{aligned}$$

so that

$$h_{\lambda\mu}(v) = a_{\lambda\mu}(v) - l'_{\lambda\mu}(v), \quad v \in \lambda - \mu,$$

and

$$h_{\mu\lambda}(v) = l_{\lambda\mu}(v) - a'_{\lambda\mu}(v), \quad v \in \lambda - \mu.$$

For a proof of Lemma 2.1 (iv), it is enough to show the following:

**Lemma 2.2.** *Let $\lambda, \mu \in \mathcal{P}$. Then there exists a permutation $\beta$ of $\lambda - \mu$ such that*

$$a_{\lambda\mu}(\beta(v)) = a'_{\lambda\mu}(v), \quad \textit{and} \quad l'_{\lambda\mu}(\beta(v)) = l_{\lambda\mu}(v),$$

*for $v \in \lambda - \mu$.*

Proof. Let us call a non-empty skew diagram of the form

$$\{(k,l) \mid b \le k \le B, \ c \le l \le C \} \tag{2.2}$$

a *rectangle*. The 'vertex' $(b,c)$ (resp. $(b,C),(B,c),(B,C)$) is called the NW (resp. NE, SW, SE)-vertex of the rectangle (2.2). We denote by $\mathcal{R}(\lambda-\mu)$ the set of all rectangles contained in $\lambda-\mu$. Note that a rectangle is in $\mathcal{R}(\lambda-\mu)$ if and only if its NW- and SE-vertices are both contained in $\lambda-\mu$. Let $R, R' \in \mathcal{R}(\lambda-\mu)$. We write $R \sim R'$, if $R' = R \pm (1,0)$ or $R \pm (0,1)$. Let $\approx$ be the equivalence relation in $\mathcal{R}(\lambda-\mu)$ generated by $\sim$. Let $v \in \lambda-\mu$. Then there exists a unique largest rectangle $R^v$ in $\mathcal{R}(\lambda-\mu)$ which has $v$ as its NE-vertex, and also there exists a unique largest rectangle ${}_vR$ which has $v$ as its SW-vertex. Moreover, there exists a unique rectangle $S \in \mathcal{R}(\lambda-\mu)$ such that $S \approx R^v$ and that $S = {}_wR$ for some $w \in \lambda-\mu$. (The existence of $S$ is trivial. The uniqueness follows from the fact that the condition '$R, R+(1,0), R+(0,1) \in \mathcal{R}(\lambda-\mu)$' implies $R+(1,1) \in \mathcal{R}(\lambda-\mu)$. ) Thus we can define a permutation $\beta$ of $\lambda-\mu$ by putting $\beta(v) = w$. Since, e.g., $a_{\lambda\mu}(\beta(v)) + 1$ and $a'_{\lambda\mu}(v)+1$ are the 'width' of $S = {}_wR$ and $R^v$ respectively, we get the lemma. $\square$

Let $t$ be an indeterminate. For $\lambda, \mu \in \mathcal{P}$, we define a rational function $J_{\lambda\mu}(t)$ in $t$ by

$$J_{\lambda\mu}(t) = t^{n(\lambda,\mu)} \prod_{v\in\lambda} \frac{1+t^{h_{\lambda\mu}(v)}}{1-t^{h_\lambda(v)}} \ t^{n(\mu,\lambda)} \prod_{v\in\mu} \frac{1+t^{h_{\mu\lambda}(v)}}{1-t^{h_\mu(v)}}. \tag{2.3}$$

In particular, we have

$$\begin{aligned} J_{\lambda\mu}(t) &= J_{\mu\lambda}(t), \\ J_{\lambda\lambda}(t) &= \left( \prod_{v\in\lambda} \frac{1+t^{h_\lambda(v)}}{1-t^{h_\lambda(v)}} \right)^2, \end{aligned}$$

and

$$J_{\lambda\phi}(t) = t^{n(\lambda)} \prod_{v\in\lambda} \frac{1+t^{c_\lambda(v)}}{1-t^{h_\lambda(v)}}.$$

For an integer $a$, we put

$$[a] = \begin{cases} \prod_{h=0}^{a-1}(1+t^h), & \text{if } a \geq 1, \\ 1, & \text{if } a = 0, \\ \prod_{h=1}^{-a}(1+t^{-h})^{-1}, & \text{if } a \leq -1. \end{cases} \tag{2.4}$$

In other words, $[a]$ is defined by:

$$[0] = 1, \quad [a+1] = (1+t^a)[a]. \tag{2.5}$$

Then we have the following

**Lemma 2.3.** *Let $t = q^2$. Let $\lambda, \mu \in \mathcal{P}$, and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ $(n \geq l(\lambda))$ and $\mu' = (\mu'_1, \mu'_2, \mu'_3, \ldots, \mu'_N)$ $(N \geq l(\mu'))$. We put*

$$\sigma_i = \lambda_i + n - i, \ (1 \leq i \leq n), \quad \tau'_k = \mu'_k + N - k, \ (1 \leq k \leq N).$$

*Then*

$$\begin{aligned} q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(t) = q^{\mathcal{A}} & \prod_{i=1}^{n} [\sigma_i - n - N + 1] \prod_{k=1}^{N} [\tau'_k - n - N + 1] \\ & \times \prod_{i=1}^{n} \prod_{k=1}^{N} (1 + t^{\sigma_i + \tau'_k - n - N + 1}) \prod_{h=1}^{n+N-1} (1+t^h)^{n+N-h} \\ & \times \prod_{i<j} (1 - t^{\sigma_i - \sigma_j}) \prod_{k<l} (1 - t^{\tau'_k - \tau'_l}) \\ & \times \prod_{i=1}^{n} \prod_{h=1}^{\sigma_i} (1-t^h)^{-1} \prod_{k=1}^{N} \prod_{h=1}^{\tau'_k} (1-t^h)^{-1} \end{aligned} \tag{2.6}$$

*where*

$$\begin{aligned} \mathcal{A} = \mathcal{A}(n, N, \lambda, \mu) &= \sum_{i=1}^{n} (2i-1)\sigma_i + \sum_{k=1}^{N} (2k-1)\tau'_k - \sum_{s=1}^{n+N-1} s(s+1) \\ &\quad + \sum_{i=1}^{n} (2i-1)(-n+i) + \sum_{k=1}^{N} (2k-1)(-N+k) \\ &= 2n(\lambda) + 2n(\mu') + |\lambda| + |\mu| - \sum_{s=1}^{n+N-1} s(s+1). \end{aligned}$$

Proof. Since

$$\prod_{v \in \lambda} \frac{1}{1 - t^{h_\lambda(v)}} = \frac{\prod_{i<j} (1 - t^{\sigma_i - \sigma_j})}{\prod_i \prod_{h=1}^{\sigma_i} (1-t^h)},$$

([9], I.1, Ex. 1), it is enough to prove that

$$q^{\mathcal{B}} \prod_{v\in\lambda}(1+t^{h_{\lambda\mu}(v)}) \prod_{v\in\mu}(1+t^{h_{\mu\lambda}(v)}), \tag{2.7}$$

$$\text{with}\quad \mathcal{B} = 2n(\lambda,\mu) + 2n(\mu,\lambda) + |\lambda-\mu| + |\mu-\lambda| - \mathcal{A}(n,N,\lambda,\mu),$$

is equal to

$$\prod_i [\sigma_i - n - N + 1] \prod_k [\tau'_k - n - N + 1] \prod_{i,k}(1+t^{\sigma_i+\tau'_k-n-N+1}) \prod_{h=1}^{n+N-1}(1+t^h)^{n+N-h}$$

or to

$$\prod_i [\lambda_i - i - N + 1] \prod_k [\mu'_k - k - n + 1] \prod_{i,k}(1+t^{h_{\lambda\mu}(i,k)}) \prod_{h=1}^{n+N-1}(1+t^h)^{n+N-h}. \tag{2.8}$$

Denote the product (2.8) by $p(n,N,\lambda,\mu)(t)$. Then we have

$$p(n+1,N,\lambda,\mu)(t) = p(n,N+1,\lambda,\mu)(t) = t^{(n+N)(n+N+1)/2} p(n,N,\lambda,\mu) \tag{2.9}$$

for any $n \geq l(\lambda)$ and any $N \geq l(\mu')$. In fact, if we replace, e.g., $n$ with $n+1$ in (2.8), we should multiply (2.8) by

$$[-n-N] \prod_k \frac{[\mu'_k - k - n]}{[\mu'_k - k - n + 1]} \prod_k (1+t^{\mu'_k-k-n}) \prod_{h=1}^{n+N}(1+t^h),$$

which is equal to $t^{(n+N)(n+N+1)/2}$, as required. We assume, for the moment, that $n$ and $N$ to be large enough; in particular

$$\lambda_i - i - N + 1 < 0, \quad \mu_k - k - n + 1 < 0$$

for any $i$ and $k$. Then we have

$$\prod_{i=1}^{n} [\lambda_i - i - N + 1] = DEF,$$

where

$$D = \prod_{i=1}^{n} \prod_{h=1}^{i+A-1} (1+t^{-h})^{-1}, \quad E = \prod_{i=1}^{a} \prod_{h=i+A}^{-\lambda_i+i+N-1} (1+t^{-h})^{-1},$$

and

$$F = \prod_{i=a+1}^{n} \prod_{h=i+A}^{i+N-1} (1+t^{-h})^{-1},$$

where $a = l(\lambda)$ and $A = l(\mu')$. We also have

$$\prod_{k=1}^{N}[\mu'_k - k - n + 1] = HIK,$$

where

$$H = \prod_{k=1}^{A}\prod_{h=1}^{k+a-1}(1+t^{-h})^{-1}, \quad I = \prod_{k=1}^{A}\prod_{h=k+a}^{-\mu'_k+k+n-1}(1+t^{-h})^{-1},$$

and

$$K = \prod_{k=A+1}^{N}\prod_{h=1}^{k+n-1}(1+t^{-h})^{-1}.$$

We further write

$$\prod_{i=1}^{n}\prod_{k=1}^{N}(1+t^{h_{\lambda\mu}(i,k)}) = PQRS,$$

where

$$P = \prod_{i=1}^{a}\prod_{k=1}^{A}(1+t^{h_{\lambda\mu}(i,k)}), \quad Q = \prod_{i=1}^{a}\prod_{k=A+1}^{N}(1+t^{\lambda_i - i - k + 1}),$$

$$R = \prod_{i=a+1}^{n}\prod_{k=1}^{A}(1+t^{\mu'_k - i - k + 1}), \quad \text{and} \quad S = \prod_{i=a+1}^{n}\prod_{k=A+1}^{N}(1+t^{-i-k+1}),$$

and also

$$\prod_{h=1}^{n+N-1}(1+t^h)^{n+N-h} = \prod_{i=1}^{n+N-1}\prod_{h=1}^{i}(1+t^h) = TUV,$$

where

$$T = \prod_{i=1}^{A-1}\prod_{h=1}^{i}(1+t^h), \quad U = \prod_{i=A}^{n+A-1}\prod_{h=1}^{i}(1+t^h),$$

and

$$V = \prod_{i=n+A}^{n+N-1}\prod_{h=1}^{i}(1+t^h).$$

Then we have

$$D = t^{\delta}U^{-1}\left(\delta = \sum_{s=A}^{n+A-1}\frac{s(s+1)}{2}\right), \quad F = S^{-1},$$

and

$$K = t^{\kappa} V^{-1} \left( \kappa = \sum_{s=n+A}^{n+N-1} \frac{s(s+1)}{2} \right).$$

We also have

$$\begin{aligned} Q &= \prod_{i=1}^{a} \left\{ \prod_{h=-\lambda_i+i+A}^{i+A-1} (1+t^{-h}) \prod_{h=i+A}^{-\lambda_i+i+N-1} (1+t^{-h}) \right\} \\ &= \left\{ \prod_{i=1}^{a} \prod_{h=-\lambda_i+i+A}^{i+A-1} (1+t^{-h}) \right\} E^{-1}, \end{aligned}$$

and

$$\begin{aligned} R &= \prod_{i=1}^{A} \left\{ \prod_{h=-\mu'_k+k+a}^{k+a-1} (1+t^{-h}) \prod_{h=k+a}^{-\mu'_k+n+k-1} (1+t^{-h}) \right\} \\ &= \left\{ \prod_{i=1}^{A} \prod_{h=-\mu'_k+k+a}^{k+a-1} (1+t^{-h}) \right\} I^{-1}. \end{aligned}$$

Thus we have

$$\begin{aligned} p(n, N, \lambda, \mu) &= (DEF)(HIK)(PQRS)(TUV) \\ &= t^{\delta+\kappa} HPT \\ &\quad \times \prod_{i=1}^{a} \prod_{h=-\lambda_i+i+A}^{i+A-1} (1+t^{-h}) \prod_{k=1}^{A} \prod_{h=-\mu'_k+k+a}^{k+a-1} (1+t^{-h}) \\ &= t^{\mathcal{C}} \prod_{i=1}^{A-1} \prod_{h=1}^{i} (1+t^{-h}) \prod_{i=1}^{a} \prod_{h=-\lambda_i+i+A}^{i+A-1} (1+t^{-h}) \\ &\quad \times \prod_{k=1}^{A} \prod_{h=-\mu'_k+k+a}^{k+a-1} (1+t^{-h}) \prod_{i=1}^{a} \prod_{k=1}^{A} (1+t^{h_{\lambda\mu}(i,k)}) \\ &\quad \times \prod_{k=1}^{A} \prod_{h=1}^{k+a-1} (1+t^{-h})^{-1}, \end{aligned} \tag{2.10}$$

where

$$\mathcal{C} = \sum_{s=1}^{n+N-1} \frac{s(s+1)}{2}.$$

Although, in deriving (2.10), we have assumed $n$ and $N$ to be large, the formula (2.10) itself is true for any $n \geq l(\lambda)$ and any $N \geq l(\mu')$ by (2.9). Let

(2.11) $$a_0 = \min\{i \mid -\lambda_i + i + A \geq 1\}$$

and

(2.12) $$A_0 = \min\{k \mid -\mu'_k + k + a \geq 1\}.$$

Then, we can further rewrite (2.10) as follows.

$$\begin{aligned}
p(n, N, \lambda, \mu) = t^{\mathcal{C}} & \prod_{i=1}^{A-1} \prod_{h=1}^{i} (1 + t^{-h}) \prod_{i=a_0}^{a} \prod_{h=-\lambda_i+i+A}^{i+A-1} (1 + t^{-h}) \\
& \times \prod_{i=1}^{a_0-1} \left\{ \prod_{h=-\lambda_i+i+A}^{0} (1 + t^{-h}) \prod_{h=1}^{i+A-1} (1 + t^{-h}) \right\} \\
& \times \prod_{k=1}^{A_0-1} \prod_{h=-\mu'_k+k+a}^{0} (1 + t^{-h}) \\
& \times \prod_{i=1}^{a} \prod_{k=1}^{A} (1 + t^{h_{\lambda\mu}(i,k)}) \prod_{k=A_0}^{A} \prod_{h=1}^{-\mu'_k+k+a-1} (1 + t^{-h})^{-1} \\
= t^{\mathcal{C}} & \prod_{i=1}^{a_0+A-2} \prod_{h=1}^{i} (1 + t^{-h}) \prod_{i=a_0+A-1}^{a+A-1} \prod_{h=-\lambda_{i-A+1}+i+1}^{i} (1 + t^{-h}) \\
& \times \prod_{i=1}^{a_0-1} \prod_{h=-\lambda_i+i+A}^{0} (1 + t^{-h}) \prod_{k=1}^{A_0-1} \prod_{h=-\mu'_k+k+a}^{0} (1 + t^{-h}) \\
& \times \prod_{i=1}^{a} \prod_{k=1}^{A} (1 + t^{h_{\lambda\mu}(i,k)}) \prod_{k=A_0}^{A} \prod_{h=1}^{-\mu'_k+k+a-1} (1 + t^{-h})^{-1}.
\end{aligned}$$

Hence, by Lemma 2.4 given below, we have

$$\begin{aligned}
p(n, N, \lambda, \mu) = t^{\mathcal{C}} & \prod_{\substack{v \in \lambda - \mu \\ h_{\lambda\mu}(v) \geq 0}} (1 + t^{h_{\lambda\mu}(v)}) \prod_{\substack{v \in \mu - \lambda \\ h_{\lambda\mu}(v) \geq 0}} (1 + t^{h_{\lambda\mu}(v)}) \\
& \times \prod_{v \in \lambda \cap \mu} (1 + t^{h_{\lambda\mu}(v)}) \prod_{\substack{v \in \lambda \cup \mu \\ h_{\mu\lambda}(v) > 0}} (1 + t^{-h_{\mu\lambda}(v)}).
\end{aligned}$$

Hence, by Lemma 2.1 (iv), we get

$$\begin{aligned}
p(n,N,\lambda,\mu) &= t^{\mathcal{C}} \prod_{\substack{v\in\lambda-\mu\\ h_{\lambda\mu}(v)\geq 0}} (1+t^{h_{\lambda\mu}(v)}) \prod_{\substack{v\in\mu-\lambda\\ h_{\mu\lambda}(v)\leq 0}} (1+t^{-h_{\mu\lambda}(v)})\\
&\quad\times \prod_{v\in\lambda\cap\mu} (1+t^{h_{\lambda\mu}(v)}) \prod_{\substack{v\in\mu\\ h_{\mu\lambda}(v)>0}} (1+t^{-h_{\mu\lambda}(v)}) \prod_{\substack{v\in\lambda-\mu\\ h_{\lambda\mu}(v)<0}} (1+t^{h_{\lambda\mu}(v)})\\
&= t^{\mathcal{C}} \prod_{v\in\lambda}(1+t^{h_{\lambda\mu}(v)}) \prod_{v\in\mu}(1+t^{-h_{\mu\lambda}(v)})\\
&= t^{\mathcal{D}} \prod_{v\in\lambda}(1+t^{h_{\lambda\mu}(v)}) \prod_{v\in\mu}(1+t^{h_{\mu\lambda}(v)}),
\end{aligned}$$

where

$$\mathcal{D} = \mathcal{C} - \sum_{v\in\mu} h_{\mu\lambda}(v).$$

Thus, it is enough to prove

$$\mathcal{B} = 2\mathcal{D}.$$

But this is equivalent to the formula in Lemma 2.1 (v). □

**Lemma 2.4.** *Under the notation in the proof of* Lemma 2.3, *we have*:

$$\prod_{\substack{v\in\lambda-\mu\\ h_{\lambda\mu}(v)\geq 0}} (1+t^{h_{\lambda\mu}(v)}) = \prod_{i=1}^{a_0-1} \prod_{h=-\lambda_i+i+A}^{0} (1+t^{-h}) \prod_{\substack{v\in\Omega\\ v\in\lambda-\mu\\ h_{\lambda\mu}(v)\geq 0}} (1+t^{h_{\lambda\mu}(v)}), \tag{2.13}$$

$$\prod_{\substack{v\in\mu-\lambda\\ h_{\lambda\mu}(v)\geq 0}} (1+t^{h_{\lambda\mu}(v)}) = \prod_{k=1}^{A_0-1} \prod_{h=-\mu_k'+k+a}^{0} (1+t^{-h}) \prod_{\substack{v\in\Omega\\ v\in\mu-\lambda\\ h_{\lambda\mu}(v)\geq 0}} (1+t^{h_{\lambda\mu}(v)}), \tag{2.14}$$

*and*

$$\begin{aligned}
\prod_{\substack{v\in\lambda\cup\mu\\ h_{\mu\lambda}(v)>0}} (1+t^{-h_{\mu\lambda}(v)}) &= \prod_{i=1}^{a_0+A-2} \prod_{h=1}^{i}(1+t^{-h}) \prod_{i=a_0+A-1}^{a+A-1} \prod_{h=-\lambda_{i-A+1}+i+1}^{i} (1+t^{-h})\\
&\quad\times \prod_{\substack{v\in\Omega\\ h_{\lambda\mu}(v)<0}} (1+t^{h_{\lambda\mu}(v)}) \prod_{k=A_0}^{A} \prod_{h=1}^{-\mu_k'+k+a-1} (1+t^{-h})^{-1},
\end{aligned} \tag{2.15}$$

*where*

$$\Omega = \{(i,j) \mid 1\leq i\leq a, 1\leq j\leq A\}.$$

Proof. If $v = (i, j) \notin \Omega$, and $v \in \lambda - \mu$, then

$$h_{\lambda\mu}(v) = \lambda_i - i - j + 1, \qquad 1 \le i \le a, \ \ A + 1 \le j \le \lambda_i.$$

Hence (2.13) follows from (2.11). Similarly, we get (2.14). For a proof of (2.15), it is convenient to introduce the notion of $\lambda$-*sequences*; an infinite integer sequence $\{c_s\}$ ($s = 1, 2, 3, \ldots$) is a $\lambda$-sequence, if

$$c_s - \lambda'_s + s \quad \text{is independent of } s.$$

For a $\lambda$-sequence $\{c_s\}$, let

$$s_0 = \max\{s \mid c_s > 0\ \}.$$

Then

$$\{c_s\}_+ = \{c_s\}_{1 \le s \le s_0}$$

is called the *positive part* of the $\lambda$-sequence $\{c_s\}$. (If $c_1 \le 0$, then $\{c_s\}_+$ is empty.) Since we are only interested in the positive part of a $\lambda$-sequence, we call a finite sequence with $N$ terms a $\lambda$-sequence, if it is the first $N$ terms of an (infinite) $\lambda$-sequence $\{c_s\}$ containing $\{c_s\}_+$.

In connection with (2.15), we now state three constructions (i)(ii)(iii) of $\lambda$-sequences.

(i) For a fixed $i$, the sequence

$$h_{\mu\lambda}(i, j), \quad j = 1, 2, 3, \ldots$$

is a $\lambda$-sequence.

(ii) For a fixed $k$ such that $A_0 \le k \le A$ ($A = l(\mu') = \mu_1$, $A_0$ is as in (2.12)), we consider the sequence

$$-\mu'_k + a + k - 1, -\mu'_k + a + k - 2, -\mu'_k + a + k - 3, \ldots,$$

with $a = l(\lambda) = \lambda'_1$, and delete from it the terms contained in

$$-h_{\lambda\mu}(i, k), \quad i = a, a - 1, a - 2, \ldots, 1,$$

then we get a $\lambda$-sequence. (See [9], I, (1.7).)

(iii) Let $\tilde\lambda = (r, r, \ldots r, \lambda_1, \lambda_2, \ldots, \lambda_a) \in \mathcal{P}$ with $r$ sufficiently large, and $l(\tilde\lambda) = a + A - 1$. To each node $(i, j)$ of $\tilde\lambda$, we attach the number:

$$t_{ij} = i - j + 1.$$

We put

$$t_p^{(0)} = t_{\tilde\lambda'_p, p}, \quad p = 1, 2, 3, \ldots, r.$$

Then the (finite) sequence $\{t_p^{(0)}\}$ is a $\lambda$-sequence. More generally, if we put, for each $0 \le q \le a + A - 2$,

$$t_p^{(q)} = t_{\tilde{\lambda}'_p - q, p}, \quad p = 1, 2, 3, \ldots, s_q \qquad (s_q = \max\{p \mid \tilde{\lambda}'_p > q\})$$

then $\{t_p^{(q)}\}$ is a $\lambda$-sequence. (We get the sequence $t^{(0)} = \{t_p^{(0)}\}$ by reading, from west to east, the numbers attached to the 'south coast' $\{(\tilde{\lambda}'_p, p) \mid 1 \le p \le r\}$ of $\tilde{\lambda}$. Likewise we get the sequence $t^{(1)}$ by reading the numbers attached to the south coast of the diagram $\tilde{\lambda}$ – (the south coast of $\tilde{\lambda}$), and so on. The last term of each sequence $t^{(q)}$ lies either in the first row or in the $r$-th column; hence it must be less than or equal to 1. This shows the squence $t^{(q)}$ surely contains the positive part of the infinite $\lambda$-sequence with the first term $t_1^{(q)}$.)

Let $C^{(i)}, i = 1, 2, 3, \ldots,$ be the $\lambda$-sequences constructed in (i), and $C_+^{(i)} = \{c_j^{(i)}\}, i = 1, 2, 3, \ldots,$ their positive parts; we can similarly define, from the constructions (ii) and (iii), $D_+^{(k)} = \{d_l^{(k)}\}, A_0 \le k \le A$, and $E_+^{(q)} = \{e_p^{(q)}\}, 0 \le q \le a + A - 2$, respectively. Then we have:

$$\prod_{i,j}(1 + t^{-c_j^{(i)}}) = \prod_{\substack{v \in \lambda \cup \mu \\ h_{\mu\lambda}(v) > 0}} (1 + t^{-h_{\mu\lambda}(v)}),$$

$$\prod_{k,l}(1 + t^{-d_l^{(k)}})^{-1} = \prod_{\substack{v \in \Omega \\ h_{\lambda\mu}(v) < 0}} (1 + t^{h_{\lambda\mu}(v)}) \prod_{k=A_0}^{A} \prod_{h=1}^{-\mu'_k + a + k - 1} (1 + t^{-h})^{-1},$$

and

$$\prod_{p,q}(1 + t^{-e_p^{(q)}}) = \prod_{i=1}^{a_0 + A - 2} \prod_{h=1}^{i} (1 + t^{-h}) \prod_{i=a_0 + A - 1}^{a + A - 1} \prod_{h=-\lambda_{i-A+1}+i+1}^{i} (1 + t^{-h}).$$

Thus, for a proof of (2.15), it is enough to show that any $C_+^{(i)}, i = 1, 2, 3, \ldots,$ and any $D_+^{(k)}, A_0 \le k \le A$, appears among $E_+^{(q)}, 0 \le q \le a + A - 2$, and, conversely, any $E_+^{(q)}, 0 \le q \le a + A - 2$, appears either in $C_+^{(i)}, i = 1, 2, 3, \ldots,$ or in $D_+^{(k)}, A_0 \le k \le A$, exactly once. Since, a $\lambda$-sequence is determined by its first term, we concentrate on the first terms of $C_+^{(i)}, D_+^{(k)}$, and $E_+^{(q)}$. The first terms of $C_+^{(i)}, D_+^{(k)}$, and $E_+^{(q)}$ are

$$C_1 = \{\mu_1 + a - 1, \mu_2 + a - 2, \mu_3 + a - 3, \ldots, \mu_{j_0} + a - j_0\}$$
$$(j_0 = \max\{j \mid \mu_j + a - j \ge 1\}),$$

$$D_1 = \{-\mu'_A + a + A - 1, -\mu'_{A-1} + a + (A - 1) - 1,$$
$$- \mu'_{A-2} + a + (A - 2) - 1, \ldots, -\mu'_{k_0} + a + k_0 - 1\}$$
$$(k_0 = \min\{k \mid -\mu'_k + a + k - 1 \ge 1\}),$$

and

$$E_1 = \{a + A - 1, a + A - 2, a + A - 3, \dots, 2, 1\},$$

respectively. What we need to show is:

$$E_1 = C_1 \cup D_1, \quad \text{(disjoint)}. \tag{2.16}$$

Let $\alpha$ be a positive integer such that $a + \alpha > \mu'_1$. Then, by [9], I, (1.7),

$$\{a + \alpha + A - 1, a + \alpha + A - 2, a + \alpha + A - 3, \dots, 1, 0\} \tag{2.17}$$

is the disjoint union of

$$\{-\mu'_A + a + \alpha + A - 1, -\mu'_{A-1} + a + \alpha + (A - 1) - 1, \dots, -\mu'_1 + a + \alpha + 1 - 1\} \tag{2.18}$$

and

$$\{\mu_1 + a + \alpha - 1, \mu_2 + a + \alpha - 2, \dots, \mu_{a+\alpha-1} + 1, \mu_{a+\alpha} = 0\}. \tag{2.19}$$

Subtracting $\alpha$ from each member of (2.17)–(2.19), and then taking intersections with the set of positive numbers, we get (2.16). The proof of (2.15), and hence, that of Lemma 2.3 are now complete. □

**2.2. Complex reflection groups $G(m, 1, n)$.** We fix a positive integer $m$. Let $C = C_m$ be the cyclic group of order $m$ generated by $\zeta = e^{2\pi i/m}$. The symmetric group $S_n$ acts on $C^n = C \times C \times \cdots \times C$ ($n$ times) by permuting the factors. Hence we can form the semidirect product

$$G_n = G(m, 1, n) = C^n S_n.$$

The group $G_n$ acts on the complex vector space $\mathbf{C}^n$ by

$$\rho = \rho_n : G_n \longrightarrow U_n(\mathbf{C}),$$

where

$$\rho(\zeta, 1, 1, \dots, 1) = \text{the diagonal matrix } (\zeta, 1, 1, \dots, 1)$$

and, for $s \in S_n$,

$$\rho(s) = \text{the permutation matrix corresponding to } s.$$

Hence, $(G_n, \rho)$ is a complex reflection group; in particular, if $m = 2$ , $(G_n, \rho)$ is a real reflection group of type $B_n$. Below, we summarize the character theory of $G_n$, due to

W. Specht, following [9].
Let $\mathcal{P}$ be the set of partitions, and $\mathcal{P}(C)$ the set of $\mathcal{P}$-valued functions on $C$. For $\mu \in \mathcal{P}(C)$, we put

$$|\mu| = \sum_{c \in C} |\mu(c)|.$$

The conjugacy classes of $G_n$ are parametrized by

$$\mathcal{P}_n(C) = \{\mu \in \mathcal{P}(C) \mid |\mu| = n\}.$$

If $w \in G_n$ is of type $\mu \in \mathcal{P}_n(C)$, we have

$$\det(1 - q\rho_n(w)) = \prod_{c \in C} \prod_k (1 - cq^k)^{m_k(\mu(c))}. \tag{2.20}$$

The number of elements of $G_n$ of type $\mu$ is equal to

$$|G_n| \Big/ \left\{ \prod_c \prod_k (km)^{m_k(\mu(c))} m_k(\mu(c))! \right\}. \tag{2.21}$$

For $c \in C$, and a positive integer $k$, let

$$p_k(x_c) = \sum_i x_{ic}^k \tag{2.22}$$

be the $k$-th power sum symmetric function in a sequence of variables $x_c = (x_{1c}, x_{2c}, x_{3c}, \ldots)$. For $\alpha \in \hat{C}$ =Hom$(C, \mathbf{C}^\times)$, we also put

$$p_k(x_\alpha) = m^{-1} \sum_{c \in C} \alpha(c) p_k(x_c), \tag{2.23}$$

and regard it as the $k$-th power sum of a new sequence of variables $x_\alpha = (x_{1\alpha}, x_{2\alpha}, x_{3\alpha}, \ldots)$, i.e.,

$$p_k(x_\alpha) = \sum_i x_{i\alpha}^k. \tag{2.24}$$

As usual, we also put

$$p_0(x_c) = p_0(x_\alpha) = 1.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

$$p_\lambda(x_c) = p_{\lambda_1}(x_c) p_{\lambda_2}(x_c) \ldots,$$

and similarly for $p_\lambda(x_\alpha)$. For $w \in G_n$ of type $\mu \in \mathcal{P}_n(C)$, we put

$$P_w = P_\mu = \prod_{c\in C} p_{\mu(c)}(x_c). \tag{2.25}$$

Let $\mathcal{P}(\hat{C})$ be the set of $\mathcal{P}$-valued functions on $\hat{C}$. For $\lambda \in \mathcal{P}(\hat{C})$, we put

$$|\lambda| = \sum_{\alpha\in\hat{C}} |\lambda(\alpha)|.$$

The irreducible characters of $G_n$ are parametrized by

$$\mathcal{P}_n(\hat{C}) = \{\lambda \in \mathcal{P}(\hat{C}) \mid |\lambda| = n \};$$

we denote by $X^\lambda$ the irreducible character corresponding to $\lambda \in \mathcal{P}_n(\hat{C})$. For $\lambda \in \mathcal{P}(\hat{C})$, let

$$S_\lambda = \prod_{\alpha\in\hat{C}} s_{\lambda(\alpha)}(x_\alpha), \tag{2.26}$$

where $s_{\lambda(\alpha)}(x_\alpha)$ is the Schur function

$$s_{\lambda(\alpha)}(x_{1\alpha}, x_{2\alpha}, \ldots).$$

Then we have

$$S_\lambda = |G_n|^{-1} \sum_{w\in G_n} X^\lambda(w) P_w, \tag{2.27}$$

where $n = |\lambda|$.

**2.3. Complex reflection groups $G(m, p, n)$.** Let $\pi : G_n \longrightarrow S_n$ be the natural homomorphism. Then $\theta : G_n \longrightarrow \mathbf{C}^\times$ defined by

$$\theta(w) = \det \rho(w) \ \det \rho(\pi(w)), \qquad w \in G_n$$

is a linear character of $G_n$. More explicitly, we have

$$\theta(w) = \prod_{c\in C} c^{l(\mu(c))} \tag{2.28}$$

if $w \in G_n$ is of type $\mu \in \mathcal{P}_n(C)$. Let $p$ be a natural number dividing $m$, and let

$$G_{n,p} = G(m, p, n) = \ker \theta^{m/p},$$

which is of index $p$ in $G_n$. We have (see [12], [1]):

(i) The group $G_{n,p}$ acts on $\mathbf{C}^n$ by $\rho$ (in 2.2) as a complex reflection group.
(ii) The group $(G(2,2,n),\rho)$ (resp. $(G(m,m,2),\rho)$) is a real reflection group of type $D_n$ (resp. $I_2(m)$).
(iii) The groups $(G(m,p,n),\rho)$ $(m \geq 2,\ \ p|m,\ \ n \geq 2, (m,p,n) \neq (2,2,2))$ can be characterized as the irreducible imprimitive finite complex reflection groups.
(iv) Besides $(G(m,p,n),\rho)$ and $S_n = G(1,1,n)$ acting on $\mathbf{C}^{n-1}$, there exist only finite (exactly speaking, 34) isomorphism classes of irreducible finite complex reflection groups.
Since, for each irreducible character $X^\lambda$ of $G_n$, $X^\lambda \otimes \theta$ is again irreducible, the cyclic group $\langle\theta\rangle$ (of order $m$) acts on the set of irreducible characters of $G_n$, and, hence, on the parameter space $\mathcal{P}_n(\hat{C})$; more explicitly, we have

$$X^\lambda \otimes \theta = X^\mu, \qquad \lambda, \mu \in \mathcal{P}_n(\hat{C})$$

if

$$\mu(\epsilon^k) = \lambda(\epsilon^{k-1}), \qquad 0 \leq k \leq m-1$$

with $\epsilon^k \in \hat{C}$ defined by

$$\epsilon^k(c) = c^k, \qquad c \in C. \tag{2.29}$$

By Clifford's theorem, we have the following description for the irreducible characters of $G_{n,p}$.

**Proposition 2.5.** *Let $p$ be a natural number dividing $m$. Let $s(\lambda,p)$ be the order of the stabilizer of $\lambda \in \mathcal{P}_n(\hat{C})$ in $\langle\theta^{m/p}\rangle$.*
(i) *The restriction $X^\lambda|G_{n,p}$ is a sum $\sum_i X_i^\lambda$ of $s(\lambda,p)$ distinct irreducible characters $X_i^\lambda$ $(1 \leq i \leq s(\lambda,p))$ of $G_{n,p}$, which are mutually conjugate under $G_n$.*
(ii) *We have*

$$X^\lambda|G_{n,p} = X^\mu|G_{n,p}$$

*if and only if $\lambda$ and $\mu$ are in the same orbit under $\langle\theta^{m/p}\rangle$.*
(iii) *Any irreducible character of $G_{n,p}$ can uniquely be obtained as $X_i^\lambda$ $(1 \leq i \leq s(\lambda,p))$, where $\lambda$ is taken over a complete set of representatives of the $\langle\theta^{m/p}\rangle$-orbits in $\mathcal{P}_n(\hat{C})$.*

## 3. *q*-Frobenius-Schur indices

**3.1. Statement of the results.** The main purpose of this section is to prove:

**Theorem 3.1.** *Under the notation in* 2.1–2.3, *let*

$$F(\lambda, d) = |G_n|^{-1} \sum_{w \in G_n} \overline{X}^{\lambda}(w^2)\theta^d(w) \, \frac{\det(1 + q\rho_n(w))}{\det(1 - q\rho_n(w))}, \tag{3.1}$$

*where $d$ is an integer, $\lambda \in \mathcal{P}(\hat{C})$, $n = |\lambda|$, and $\overline{X}^{\lambda}$ is the complex conjugate of $X^{\lambda}$. Then we have the following.*
(i)

$$F(\lambda, d) = \prod_{\substack{\{\alpha,\beta\} \subset \hat{C} \\ \alpha\beta = \epsilon^d}} F(\lambda \mid \{\alpha, \beta\}, d), \tag{3.2}$$

*where $\epsilon^d \in \hat{C}$ is defined by* (2.29), *and $\lambda \mid \{\alpha, \beta\}$ is an element of $\mathcal{P}(\hat{C})$ given by*

$$(\lambda \mid \{\alpha, \beta\})(\omega) = \begin{cases} \lambda(\omega), & \text{if } \omega = \alpha \text{ or } \beta, \\ \phi, & \text{otherwise}, \end{cases}$$

*for $\omega \in \hat{C}$.*
(ii) *Let $\{\alpha, \beta\} \subset \hat{C}$ be such that $\alpha\beta = \epsilon^d$. We put $\lambda_1 = \lambda(\alpha)$ and $\lambda_2 = \lambda(\beta)$. Then*

$$F(\lambda|\{\alpha, \beta\}, d) = \begin{cases} q^{[\alpha^2\overline{\epsilon}^d]|\lambda_1 - \lambda_2| + [\beta^2\overline{\epsilon}^d]|\lambda_2 - \lambda_1|} J_{\lambda_1\lambda_2}(q^m), & \text{if } \alpha \neq \beta, \\ I_{\lambda_1}(q^m), & \text{if } \alpha = \beta, \end{cases}$$

*where $J_{\lambda_1\lambda_2}(t)$ is as defined in* 2.1,

$$I_{\lambda_1}(t) = \prod_{v \in \lambda_1} \frac{1 + t^{h_{\lambda_1}(v)}}{1 - t^{h_{\lambda_1}(v)}},$$

*and, for $\alpha \in \hat{C}$, $[\alpha] \in \{0, 1, \ldots, m-1\}$ is defined by*

$$\alpha = \epsilon^{[\alpha]}. \tag{3.3}$$

Using Theorem 3.1, we can explicitly calculate the $q$-Frobenius-Schur index (1.1) of an irreducible character $X_i^{\lambda}$ of the group $G(m, p, n)$:

**Theorem 3.2.** *Let $p$ be a natural number dividing $m$, and let $G_{n,p} = G(m, p, n)$. For an irreducible character $X_i^\lambda$ $(\lambda \in \mathcal{P}_n(\hat{C}),\ 1 \le i \le s(\lambda, p))$ of $G_{n,p}$, we have*

$$\Psi_{G_{n,p}}(X_i^\lambda; q) = s(\lambda, p)^{-1} \sum_{k=0}^{p-1} F\left(\lambda, \frac{m}{p}k\right).$$

Remark. For $m = 1$ and $m = 2$, this was proved in [6] and [5] respectively. For $m = r = 2$, this was conjectured in [4] in a somewhat different form (and proved in [5] using a consequence (Theorem 4.2 below) of our $q$-Cauchy identities (1.2), (1.3)).

**3.2. Proof of Theorem 3.1.** Let $S_\lambda, X^\lambda$ $(\lambda \in \mathcal{P}(\hat{C}))$ and $P_w$ $(w \in G_n)$ be as in 2.2. We put

$$F(\lambda, d; q, t) = |G_n|^{-1} \sum_{w \in G_n} \bar{X}^\lambda(w^2)\theta^d(w) \frac{\det(1 + t\rho(w))}{\det(1 - q\rho(w))},$$

where $n = |\lambda|$. Then, by (2.27),

$$\sum_{\lambda \in \mathcal{P}(\hat{C})} F(\lambda, d; q, t) S_\lambda = \sum_{n=0}^{\infty} |G_n|^{-1} \sum_{w \in G_n} \theta^d(w) \frac{\det(1 + t\rho(w))}{\det(1 - q\rho(w))}\ P_{w^2}. \tag{3.4}$$

By (2.20), (2.21), (2.25), and (2.28), the right hand side of (3.4) is equal to

$$\begin{aligned}
&\sum_{\mu \in \mathcal{P}(C)} \prod_{\substack{k \text{ odd} \\ c \in C}} \frac{p_k(x_{c^2})^{m_k(\mu(c))}}{(km)^{m_k(\mu(c))} m_k(\mu(c))!} \left(c^d \frac{1 + ct^k}{1 - cq^k}\right)^{m_k(\mu(c))} \\
&\quad \times \prod_{\substack{l \\ c \in C}} \frac{p_l(x_c)^{2m_{2l}(\mu(c))}}{(2lm)^{m_{2l}(\mu(c))} m_{2l}(\mu(c))!} \left(c^d \frac{1 - ct^{2l}}{1 - cq^{2l}}\right)^{m_{2l}(\mu(c))} \\
&= \prod_{\substack{k \text{ odd} \\ l,c}} \sum_{(i_k)(j_l)} \frac{1}{i_k!} \left(c^d \frac{1 + ct^k}{1 - cq^k} \frac{p_k(x_{c^2})}{km}\right)^{i_k} \frac{1}{j_l!} \left(c^d \frac{1 - ct^{2l}}{1 - cq^{2l}} \frac{p_l(x_c)^2}{2lm}\right)^{j_l} \\
&= \prod_c \left\{ \prod_{k \text{ odd}} \exp\left(c^d \frac{1 + ct^k}{1 - cq^k} \frac{p_k(x_{c^2})}{km}\right) \prod_l \exp\left(c^d \frac{1 - ct^{2l}}{1 - cq^{2l}} \frac{p_l(x_c)^2}{2lm}\right) \right\}
\end{aligned}$$

Hence, by (2.23), we have

$$\begin{aligned}
\log\left(\sum_\lambda F(\lambda, d; q, t) S_\lambda\right) &= \sum_{\substack{k \text{ odd} \\ c}} c^d \frac{1 + ct^k}{1 - cq^k} \frac{1}{km} \left(\sum_{\sigma \in \hat{C}} p_k(x_\sigma)\bar{\sigma}(c^2)\right) \\
&\quad + \sum_{l,c} c^d \frac{1 - ct^{2l}}{1 - cq^{2l}} \frac{1}{2lm} \left(\sum_{\sigma \in \hat{C}} p_l(x_\sigma)\bar{\sigma}(c)\right)\left(\sum_{\tau \in \hat{C}} p_l(x_\tau)\bar{\tau}(c)\right).
\end{aligned}$$

We note

$$\frac{1}{m}\sum_{c} c^{d}\frac{1-c(-t)^{k}}{1-cq^{k}}\overline{\sigma}(c^{2}) = \sum_{\substack{u=0\\ u\equiv[\sigma^{2}\overline{\epsilon}^{d}]}}^{\infty} q^{ku} - \sum_{\substack{u=0\\ u\equiv[\sigma^{2}\overline{\epsilon}^{d}]-1}}^{\infty} (-tq^{u})^{k},$$

and

$$\frac{1}{m}\sum_{c} c^{d}\frac{1-ct^{2l}}{1-cq^{2l}}\overline{\sigma}(c)\overline{\tau}(c) = \sum_{\substack{u=0\\ u\equiv[\sigma\tau\overline{\epsilon}^{d}]}}^{\infty} q^{2lu} - \sum_{\substack{u=0\\ u\equiv[\sigma\tau\overline{\epsilon}^{d}]-1}}^{\infty} (tq^{u})^{2l},$$

where $\equiv$ means the congruence modulo $m$, and, for $\alpha \in \hat{C}$, $[\alpha]$ is defined by (3.3). Hence, we have

$$\begin{aligned}
&\log\left(\sum_{\lambda} F(\lambda, d; q, t)S_{\lambda}\right)\\
&= \sum_{k,\sigma}\left(\sum_{u\equiv[\sigma^{2}\overline{\epsilon}^{d}]}\sum_{i=1}^{\infty}\frac{(q^{u}x_{i\sigma})^{k}}{k} - \sum_{u\equiv[\sigma^{2}\overline{\epsilon}^{d}]-1}\sum_{i=1}^{\infty}\frac{(-tq^{u}x_{i\sigma})^{k}}{k}\right)\\
&\quad + \sum_{l,\sigma}\left(-\sum_{u\equiv[\sigma^{2}\overline{\epsilon}^{d}]}\sum_{i=1}^{\infty}\frac{(q^{u}x_{i\sigma})^{2l}}{2l} + \sum_{u\equiv[\sigma^{2}\overline{\epsilon}^{d}]-1}\sum_{i=1}^{\infty}\frac{(tq^{u}x_{i\sigma})^{2l}}{2l}\right)\\
&\quad + \sum_{l,\sigma,\tau}\left(\sum_{u\equiv[\sigma\tau\overline{\epsilon}^{d}]}\sum_{i,j=1}^{\infty}\frac{(q^{2u}x_{i\sigma}x_{j\tau})^{l}}{2l} - \sum_{u\equiv[\sigma\tau\overline{\epsilon}^{d}]-1}\sum_{i,j=1}^{\infty}\frac{(t^{2}q^{2u}x_{i\sigma}x_{j\tau})^{l}}{2l}\right)\\
&= \sum_{i,k,\sigma}\left(\sum_{u\equiv[\sigma^{2}\overline{\epsilon}^{d}]}\frac{(q^{u}x_{i\sigma})^{k}}{k} - \sum_{u\equiv[\sigma^{2}\overline{\epsilon}^{d}]-1}\frac{(-tq^{u}x_{i\sigma})^{k}}{k}\right)\\
&\quad + \sum_{\substack{l\\(\sigma,\tau,i,j)\in\mathcal{S}}}\left(\sum_{u\equiv[\sigma\tau\overline{\epsilon}^{d}]}\frac{(q^{2u}x_{i\sigma}x_{j\tau})^{l}}{l} - \sum_{u\equiv[\sigma\tau\overline{\epsilon}^{d}]-1}\frac{(t^{2}q^{2u}x_{i\sigma}x_{j\tau})^{l}}{l}\right),
\end{aligned}$$

where

$$\mathcal{S} = \{(\sigma,\tau,i,j) \mid [\sigma] < [\tau]\} \cup \{(\sigma,\tau,i,j) \mid \sigma=\tau,\ i<j\}.$$

Hence, we have

$$\begin{aligned}
\log\left(\sum_{\lambda} F(\lambda, d; q, t)S_{\lambda}\right) &= -\sum_{\sigma,i}\sum_{\substack{u\geq 0\\ u\equiv[\sigma^{2}\overline{\epsilon}^{d}]}}\log(1-q^{u}x_{i\sigma}) + \sum_{\sigma,i}\sum_{\substack{u\geq 0\\ u\equiv[\sigma^{2}\overline{\epsilon}^{d}]-1}}\log(1+tq^{u}x_{i\sigma})\\
&\quad - \sum_{(\sigma,\tau,i,j)\in\mathcal{S}}\sum_{\substack{u\geq 0\\ u\equiv[\sigma\tau\overline{\epsilon}^{d}]}}\log(1-q^{2u}x_{i\sigma}x_{j\tau}) + \sum_{(\sigma,\tau,i,j)\in\mathcal{S}}\sum_{\substack{u\geq 0\\ u\equiv[\sigma\tau\overline{\epsilon}^{d}]-1}}\log(1-t^{2}q^{2u}x_{i\sigma}x_{j\tau}).
\end{aligned}$$

Hence

$$\sum_{\lambda} F(\lambda, d; q, t) S_\lambda = \prod_{\sigma,i} \frac{\prod_{u+1\equiv[\sigma^2\bar{\epsilon}^d]}(1+tq^u x_{i\sigma})}{\prod_{u\equiv[\sigma^2\bar{\epsilon}^d]}(1-q^u x_{i\sigma})} \prod_{(\sigma,\tau,i,j)\in\mathcal{S}} \frac{\prod_{u+1\equiv[\sigma\tau\bar{\epsilon}^d]}(1-t^2q^{2u}x_{i\sigma}x_{j\tau})}{\prod_{u\equiv[\sigma\tau\bar{\epsilon}^d]}(1-q^{2u}x_{i\sigma}x_{j\tau})}. \tag{3.5}$$

Now we consider the case $t = q$. Noting that $u$ only takes non-negative integer values, we have

$$\begin{aligned}\sum_{\lambda} F(\lambda, d) S_\lambda &= \sum_{\lambda} F(\lambda, d; q, q) S_\lambda \\ &= \prod_{\substack{\sigma\in\hat{C} \\ \sigma^2=\epsilon^d}} \left( \prod_{i=1}^{\infty}\prod_{r=0}^{\infty} \frac{1+q^{m(r+1)}x_{i\sigma}}{1-q^{mr}x_{i\sigma}} \prod_{i<j}\frac{1}{1-x_{i\sigma}x_{j\sigma}} \right) \\ &\quad \times \prod_{\substack{\{\sigma,\tau\}\subset\hat{C} \\ \sigma\tau=\epsilon^d \\ \sigma\neq\tau}} \left( \prod_{i=1}^{\infty}\prod_{r=0}^{\infty} \frac{1+q^{[\sigma^2\bar{\epsilon}^d]+mr}x_{i\sigma}}{1-q^{[\sigma^2\bar{\epsilon}^d]+mr}x_{i\sigma}} \frac{1+q^{[\tau^2\bar{\epsilon}^d]+mr}x_{i\tau}}{1-q^{[\tau^2\bar{\epsilon}^d]+mr}x_{i\tau}} \prod_{i,j}\frac{1}{1-x_{i\sigma}x_{j\tau}} \right).\end{aligned} \tag{3.6}$$

We fix an $\alpha \in \hat{C}$ such that $\alpha^2 = \epsilon^d$ (if it exists). If we put, in (3.6), $x_{i\sigma} = 0$ ($i = 1, 2, 3, \ldots$) for any $\sigma \neq \alpha$, then we have

$$\sum_{\substack{\lambda \\ \lambda(\sigma)=\phi \\ \text{if } \sigma\neq\alpha}} F(\lambda, d) s_{\lambda(\alpha)}(x_\alpha) = \prod_{i}\prod_{r=0}^{\infty} \frac{1+q^{m(r+1)}x_{i\alpha}}{1-q^{mr}x_{i\alpha}} \prod_{i<j}\frac{1}{1-x_{i\alpha}x_{j\alpha}}. \tag{3.7}$$

Similarly, if we fix $\{\alpha, \beta\} \subset \hat{C}$ such that $\alpha\beta = \epsilon^d$, $\alpha \neq \beta$, and put, in (3.6), $x_{i\sigma} = 0$ for any $\sigma \neq \alpha, \beta$, then we have

$$\begin{aligned}&\sum_{\substack{\lambda \\ \lambda(\sigma)=\phi \\ \text{if } \sigma\neq\alpha,\beta}} F(\lambda, d) s_{\lambda(\alpha)}(x_\alpha) s_{\lambda(\beta)}(x_\beta) \\ &= \prod_{i=1}^{\infty}\prod_{r=0}^{\infty} \frac{1+q^{[\alpha^2\bar{\epsilon}^d]+mr}x_{i\alpha}}{1-q^{[\alpha^2\bar{\epsilon}^d]+mr}x_{i\alpha}} \frac{1+q^{[\beta^2\bar{\epsilon}^d]+mr}x_{i\beta}}{1-q^{[\beta^2\bar{\epsilon}^d]+mr}x_{i\beta}} \prod_{i,j}\frac{1}{1-x_{i\alpha}x_{j\beta}}.\end{aligned} \tag{3.8}$$

By (3.6), (3.7), and (3.8), we have

$$\sum_{\lambda} F(\lambda, d) S_\lambda = \prod_{\substack{\{\alpha,\beta\}\subset\hat{C} \\ \alpha\beta=\epsilon^d}} \left\{ \sum_{\substack{\lambda \\ \lambda(\sigma)=\phi \\ \text{if } \sigma\neq\alpha,\beta}} F(\lambda, d) \prod_{\sigma\in\{\alpha,\beta\}} s_{\lambda(\sigma)}(x_\sigma) \right\}.$$

Comparing the coefficients of $S_\lambda$ on both sides, we get Part (i). By [6], we know that the right hand side of (3.7) is equal to

$$\sum_{\lambda_1\in\mathcal{P}}\prod_{v\in\lambda_1}\frac{1+(q^m)^{h_{\lambda_1}(v)}}{1-(q^m)^{h_{\lambda_1}(v)}}s_{\lambda_1}(x_\alpha)=\sum_{\lambda_1\in\mathcal{P}}I_{\lambda_1}(q^m)s_{\lambda_1}(x_\alpha),$$

which implies Part (ii) in the case $\alpha=\beta$. Now we put

$$q^{[\alpha^2\bar{\epsilon}^d]-m/2}x_{i\alpha}=x_i,\quad q^{[\beta^2\bar{\epsilon}^d]-m/2}x_{i\beta}=y_i,\quad i=1,2,3,\dots$$

on the right hand side of (3.8), then the product is transformed to

$$\prod_i\prod_r\frac{1+q^{m(2r+1)/2}x_i}{1-q^{m(2r+1)/2}x_i}\frac{1+q^{m(2r+1)/2}y_i}{1-q^{m(2r+1)/2}y_i}\prod_{i,j}\frac{1}{1-x_iy_j}. \tag{3.9}$$

By the $q$-Cauchy identity (1.2), whose proof is given in Section 5, the product (3.9) is equal to

$$\sum_{\lambda_1,\lambda_2}q^{(|\lambda_1-\lambda_2|+|\lambda_2-\lambda_1|)m/2}J_{\lambda_1\lambda_2}(q^m)s_{\lambda_1}(x)s_{\lambda_2}(y),$$

which, after going back to the original variables $x_\alpha=(x_{i\alpha})$ and $x_\beta=(x_{j\beta})$, is equal to

$$\sum_{\lambda_1,\lambda_2}q^{[\alpha^2\bar{\epsilon}^d]|\lambda_1-\lambda_2|+[\beta^2\bar{\epsilon}^d]|\lambda_2-\lambda_1|}J_{\lambda_1\lambda_2}(q^m)s_{\lambda_1}(x_\alpha)s_{\lambda_2}(x_\beta).$$

Comparing this with the left hand side of (3.8), we get Part (ii) in the case $\alpha\neq\beta$. The proof of Part (ii) is now completely reduced to that of Theorem 1.1.

**3.3. Proof of Theorem 3.2.** We first note that $\Psi_{G_{n,p}}(X_i^\lambda;q)$ does not depend on $i$, since, by Propsition 2.1, $X_i^\lambda$ $(1\le i\le s(\lambda,p))$ are mutually conjugate under $G_n$. Hence

$$s(\lambda,p)\Psi_{G_{n,p}}(X_i^\lambda)=\sum_i\Psi_{G_{n,p}}(X_i^\lambda)=\Psi_{G_{n,p}}(X^\lambda|G_{n,p}).$$

On the other hand, since

$$p^{-1}\sum_{k=0}^{p-1}\theta^{mk/p}(w)=\begin{cases}1, & \text{if } w\in G_{n,p},\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}\sum_{k=0}^{p-1}F\left(\lambda,\frac{m}{p}k\right)&=p|G_n|^{-1}\sum_{w\in G_{n,p}}X^\lambda(w^2)\frac{\det(1+q\rho(w))}{\det(1-q\rho(w))}\\&=\Psi_{G_{n,p}}(X^\lambda|G_{n,p}).\end{aligned}$$

Hence Theorem 3.2 follows.

### 4. Skew Schur functions and Inner Products.

Here we discuss some results which follow easily from the $q$-Cauchy identity (1.2). Let $\Lambda = \Lambda_u$ be the ring of symmetric functions [9] in the variables $u = (u_1, u_2, u_3, \ldots)$. We consider the following specialization of $\Lambda_u$:

$$\phi_u : h_m(u) \longrightarrow q^m \prod_{i=1}^{m} \frac{1+q^{2i-2}}{1-q^{2i}}, \quad m = 0, 1, 2, \ldots,$$

where $h_m(u)$ is the $m$-th complete symmetric function, i.e., the Schur function $s_{(m)}(u)$. In general, we have [9], I, 3. Ex. 3:

$$\phi_u(s_\lambda(u)) = q^{2n(\lambda)+|\lambda|} \prod_{v\in\lambda} \frac{1+q^{2c_\lambda(v)}}{1-q^{2h_\lambda(v)}} = q^{|\lambda|} J_{\lambda\phi}(q^2) \tag{4.1}$$

for any $\lambda \in \mathcal{P}$. Since

$$\prod_{h,i} \frac{1}{1-u_h x_i} = \prod_i \left( \sum_{m=0}^{\infty} h_m(u) x_i^m \right),$$

we have

$$\phi_u \left( \prod_{h,i} \frac{1}{1-u_h x_i} \right) = \prod_i \prod_{r=0}^{\infty} \frac{1+q^{2r+1}x_i}{1-q^{2r+1}x_i} \tag{4.2}$$

by [9], I, 2, Ex. 5. For $\lambda, \beta \in \mathcal{P}$, the *skew Schur function* $s_{\lambda/\beta}(u)$ is defined by

$$s_{\lambda/\beta}(u) = \sum_{\alpha\in\mathcal{P}} c^\lambda_{\alpha,\beta} s_\alpha(u), \tag{4.3}$$

where $\{c^\lambda_{\alpha\beta}\}_{\lambda,\alpha,\beta\in\mathcal{P}}$ are the *Littlewood-Richardson coefficients* appearing in the expansion

$$s_\alpha(u) s_\beta(u) = \sum_{\lambda\in\mathcal{P}} c^\lambda_{\alpha\beta} s_\lambda(u). \tag{4.4}$$

Let us consider the infinite matrix $S(u) = (s_{\lambda/\mu}(u))_{\lambda,\mu\in\mathcal{P}}$. If we choose the total order on $\mathcal{P}$ compatible with the (set theoretical) inclusion, then the matrix $S(u)$ is lower unitriangular.

**Lemma 4.1.** *The matrix $S(u)$ has the following properties.*
(i) *Let $v = (v_1, v_2, v_3, \dots)$ be another series of variables. Then the matrix $S(u)$ and the transpose ${}^tS(v)$ of $S(v)$ almost commute with each other. More explicitly, we have*

$$\prod_{i,j} \frac{1}{1 - u_i v_j} S(u)^t S(v) = {}^t S(v) S(u).$$

(ii) *The inverse $S(u)^{-1}$ of $S(u)$ is given by*

$$S(u)^{-1} = \left((-1)^{|\lambda/\mu|} s_{\lambda'/\mu'}(u)\right)_{\lambda,\mu}.$$

Proof. (i) This is nothing but the identity

$$\sum_{\rho\in\mathcal{P}} s_{\rho/\mu}(u) s_{\rho/\lambda}(v) = \sum_{\sigma\in\mathcal{P}} s_{\lambda/\sigma}(u) s_{\mu/\sigma}(v) \prod_{i,j} \frac{1}{1 - u_i v_j}. \tag{4.5}$$

given in [9], I, 5, Ex. 26.
(ii) Let $S_{\nu/\mu}(u/v) (\nu, \mu \in \mathcal{P})$ be the skew Schur function associated with the series

$$f(t) = \prod_i \frac{1 + t v_i}{1 - t u_i}$$

in the sense of Littlewood (see [7] and [9], I, 3, Ex. 23). Then

$$S_{\nu/\mu}\left(\frac{u}{v}\right) = \sum_{\lambda} s_{\nu'/\lambda'}(v) s_{\lambda/\mu}(u).$$

If we put $v_i = -u_i, i = 1, 2, 3, \dots$, then, since $f(t) = 1$, we have

$$\sum_{\lambda} (-1)^{|\nu'/\lambda'|} s_{\nu'/\lambda'}(u) s_{\lambda/\mu}(u) = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu, \end{cases}$$

which proves Part (ii). □

**Theorem 4.2.** *Under the above notation, the following holds.*
(i) *For $\lambda, \mu \in \mathcal{P}$, we have*

$$\phi_u \left( \sum_{\sigma\in\mathcal{P}} s_{\lambda/\sigma}(u) s_{\mu/\sigma}(u) \right) = q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2).$$

(ii) *For a 'rectangular' Young diagram*

$$\Omega = \{(i, j) \mid 1 \le i \le l,\ 1 \le j \le m\}$$

*and a Young diagram $\omega$ contained in $\Omega$, we put*

$$\Omega(\omega) = \{(l+1-i, m+1-j) \mid (i,j) \in \Omega - \omega \},$$

*which is again a Young diagram contained in $\Omega$. For two rectangular Young diagrams $\Omega$ and $\Theta$, we have*

$$\phi_u\left(\sum_{\omega\subset\Omega\cap\Theta} s_{\Omega(\omega)}(u)s_{\Theta(\omega)}(u)\right) = q^{|\Omega-\Theta|+|\Theta-\Omega|}J_{\Omega\Theta}(q^2).$$

*In particular, when $\Omega = \Theta$, we have*

$$\phi_u\left(\sum_{\omega\subset\Omega} s_{\omega}(u)^2\right) = J_{\Omega\Omega}(q^2). \tag{4.6}$$

(iii) *For $\lambda, \mu \in \mathcal{P}$, we have*

$$\phi_u\left(\sum_{\rho} s_{\rho/\lambda}(u)s_{\rho/\mu}(u)\right) = q^{|\lambda-\mu|+|\mu-\lambda|}J_{\lambda\mu}(q^2)\prod_{i=1}^{\infty}\left(\frac{1+q^{2i}}{1-q^{2i}}\right)^{2i}. \tag{4.7}$$

Proof. (i) Let $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$, $u = (u_1, u_2, u_3, \ldots)$, $v = (v_1, v_2, v_3, \ldots)$. By the classical Cauchy identity (1.4), we have

$$\begin{aligned}
\prod_{h,i}\frac{1}{1-u_hx_i}\prod_{k,j}\frac{1}{1-v_ky_j}\prod_{i,j}\frac{1}{1-x_iy_j}
&= \left(\sum_{\pi} s_{\pi}(u)s_{\pi}(x)\right)\left(\sum_{\rho} s_{\rho}(v)s_{\rho}(y)\right)\left(\sum_{\sigma} s_{\sigma}(x)s_{\sigma}(y)\right)\\
&= \sum_{\pi,\rho,\sigma}(s_{\pi}(x)s_{\sigma}(x))(s_{\rho}(y)s_{\sigma}(y))s_{\pi}(u)s_{\rho}(v)\\
&= \sum_{\pi,\rho,\sigma}\left(\sum_{\lambda} c^{\lambda}_{\pi\sigma}s_{\lambda}(x)\right)\left(\sum_{\mu} c^{\mu}_{\rho\sigma}s_{\mu}(y)\right)s_{\pi}(u)s_{\rho}(v)\\
&= \sum_{\lambda,\mu}\left(\sum_{\sigma} s_{\lambda/\sigma}(u)s_{\mu/\sigma}(v)\right)s_{\lambda}(x)s_{\mu}(y).
\end{aligned}$$

Putting $v = u$ in the above equality and applying $\phi_u$, we get

$$\prod_{r,i,j}\frac{1+q^{2r+1}x_i}{1-q^{2r+1}x_i}\frac{1+q^{2r+1}y_j}{1-q^{2r+1}y_j}\prod_{i,j}\frac{1}{1-x_iy_j} = \sum_{\lambda,\mu}\phi_u\left(\sum_{\sigma} s_{\lambda/\sigma}(u)s_{\mu/\sigma}(u)\right)s_{\lambda}(x)s_{\mu}(y) \tag{4.8}$$

by (4.2). Comparing this with the $q$-Cauchy identity (1.2), we get Part (i).

(ii) Let $\omega$ be a Young diagram contained in a rectangular Young diagram $\Omega$. By the Littlewood-Richardson rule (see, e.g. [9], Ch.I, 9), we easily see that

$$c^{\Omega}_{\alpha\omega} = \begin{cases} 1, & \text{if } \alpha = \Omega(\omega), \\ 0, & \text{otherwise.} \end{cases}$$

This means

$$s_{\Omega/\omega}(u) = s_{\Omega(\omega)}(u).$$

Hence Part (ii) follows from Part (i) by putting $\lambda = \Omega$, $\mu = \Theta$.

(iii) We first prove this in the case $\lambda = \mu = \phi$. Then the left hand side of (4.7) is equal to

$$\phi_u\left(\sum_{\lambda} s_\lambda(u)^2\right) = \lim_{n\to\infty} \phi_u\left(\sum_{\omega\subset\Omega^n} s_\lambda(u)^2\right),$$

where $\Omega^n = \{(i,j) |\ 1 \le i, j \le n\ \}$. Hence, by Part (ii), we have

$$\phi_u\left(\sum_{\lambda\subset\Omega^n} s_\lambda(u)^2\right) = \lim_{n\to\infty} J_{\Omega^n\Omega^n}(q^2) = \prod_{i=1}^{\infty}\left(\frac{1+q^{2i}}{1-q^{2i}}\right)^{2i}.$$

This proves (4.7) in the case $\lambda = \mu = \phi$. For a proof in the general case, we use the identity (4.5). Putting $v = u$ in (4.5) and specializing under $\phi_u$, we get, by Part (i) and (4.6),

$$\begin{aligned} \phi_u\left(\sum_{\rho\in\mathcal{P}} s_{\rho/\mu}(u)s_{\rho/\lambda}(u)\right) &= \phi_u\left(\sum_{\sigma\in\mathcal{P}} s_{\lambda/\sigma}(u)s_{\mu/\sigma}(u)\right)\phi_u\left(\sum_{\lambda} s_\lambda(u)^2\right) \\ &= q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2)\prod_i\left(\frac{1+q^{2i}}{1-q^{2i}}\right)^{2i}, \end{aligned}$$

which proves Part (iii). □

The equality (4.8) can also be writtn as:

(4.9) $$\prod_{r,i,j}\frac{1+q^{2r+1}x_i}{1-q^{2r+1}x_i}\frac{1+q^{2r+1}y_j}{1-q^{2r+1}y_j}\prod_{i,j}\frac{1}{1-x_iy_j} = \sum_{\lambda}\tilde{s}_\lambda(x)\tilde{s}_\lambda(y),$$

where $\tilde{s}_\lambda(x)$ is the infinite formal sum defined by

(4.10) $$\tilde{s}_\lambda(x) = \sum_{\rho}\phi_u(s_{\rho/\lambda}(u))s_\rho(x).$$

Let $(\ ,\ )$ be the standard inner product on the space $\Lambda_x$ of finite linear combinations of Schur functions defined by

$$(s_\lambda(x), s_\mu(x)) = \delta_{\lambda\mu}, \qquad \lambda, \mu \in \mathcal{P}.$$

We extend it formally to infinite linear conbinations of Schur functions, although in some cases this does not make sense. Then Theorem 4.2 (iii) is equivalent to

$$(\tilde{s}_\lambda(x), \tilde{s}_\mu(x)) = q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2) \prod_{i=1}^{\infty} \left(\frac{1+q^{2i}}{1-q^{2i}}\right)^{2i}.$$

Similarly, Theorem 4.2 (i) is equivalent to

$$(\hat{s}_\lambda(x), \hat{s}_\mu(x)) = q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2),$$

where

$$\hat{s}_\lambda(x) = \sum_\sigma \phi_u(s_{\lambda/\sigma}(u)) s_\sigma(x) = \sum_\sigma \phi_u(s_\sigma(u)) s_{\lambda/\sigma}(x) = \phi_u(s_\lambda(u, x)).$$

Just as the classical Cauchy identity is related to the standard inner product $(\ ,\ )$ (see [9], Ch.I, 4), the $q$-Cauchy identity (4.9) suggests to introduce an inner product $\langle\ ,\ \rangle_1$ on finite linear combinations of $\tilde{s}_\lambda(x)$'s by putting

(4.11)
$$\langle \tilde{s}_\lambda(x), \tilde{s}_\mu(x)\rangle_1 = \delta_{\lambda\mu}.$$

Then, since

$$\begin{aligned} s_\mu(x) &= \sum_\lambda (-1)^{|\lambda'/\mu'|} \phi_u(s_{\lambda'/\mu'}(u)) \tilde{s}_\lambda(x) \\ &= \sum_\lambda (-1)^{|\lambda/\mu|} \phi_u(s_{\lambda/\mu}(u)) \tilde{s}_\lambda(x) \end{aligned}$$

by Lemma 4.1 (ii), after extending $\langle\ ,\ \rangle_1$ to infinite linear conminations of $\tilde{s}_\lambda(x)$'s, we have

$$\begin{aligned} \langle s_\mu(x), s_\nu(x)\rangle_1 &= \sum_\lambda (-1)^{|\lambda/\mu|+|\lambda/\nu|} \phi_u(s_{\lambda/\mu}(u) s_{\lambda/\nu}(u)) \\ &= (-1)^{|\mu|+|\nu|} q^{|\mu-\nu|+|\nu-\mu|} J_{\mu\nu}(q^2) \prod_{i=1}^{\infty} \left(\frac{1+q^{2i}}{1-q^{2i}}\right)^{2i}. \end{aligned}$$

One can also define an inner product $\langle\ ,\ \rangle_2$ on $\Lambda_x$ by putting

$$\langle \hat{s}_\lambda(x), \hat{s}_\mu(x)\rangle_2 = \delta_{\lambda\mu}.$$

Then, since

$$s_\rho(x) = \sum_\lambda (-1)^{|\rho/\lambda|}\phi_u(s_{\rho/\lambda}(u))\hat{s}_\lambda(x),$$

we have

$$\langle s_\rho(x), s_\sigma(x)\rangle_2 = (-1)^{|\rho|+|\sigma|}q^{|\rho-\sigma|+|\sigma-\rho|}J_{\rho\sigma}(q^2).$$

## 5. Proof of the $q$-Cauchy identities

**5.1. Symmetrizing operators.** Let $F_n$ be the ring of series in $n$ variables $x_1, x_2, \ldots, x_n$. For an element $f$ of $F_n$, and an element $s$ of the symmetric group $S_n$,we put

$$f^s(x_1, x_2, \ldots, x_n) = f(x_{s^{-1}(1)}, x_{s^{-1}(2)}, \ldots, x_{s^{-1}(n)}).$$

The *symmetrizing operator* [7]

$$\pi_n\colon F_n \longrightarrow F_n$$

is defined by

$$\pi_n(f) = \left(\prod_{i<j}(x_i - x_j)\right)^{-1} \sum_{s\in S_n} \mathrm{sgn}(s)(f x^{\delta(n)})^s, \quad f \in F_n, \tag{5.1}$$

where

$$x^{\delta(n)} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}.$$

We recall some of the fundamental properties of $\pi_n$:

$$\pi_n(f)^s = \pi_n(f), \quad f \in F_n, \quad s \in S_n. \tag{5.2}$$

$$\pi_n(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = 0, \tag{5.3}$$

unless $a_i + n - i, 1 \le i \le n$, are all distinct.

$$s_\lambda(x_1, x_2, \ldots, x_n) = \pi_n(x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}), \tag{5.4}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}$, and the symmetric polynomial on the left hand side is obtained from $s_\lambda(x) = s_\lambda(x_1, x_2, x_3, \ldots)$ by putting $x_{n+1} = x_{n+2} = \cdots = 0$.

**Lemma 5.1** ([7]). *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ be a partition with $l(\lambda) \le n-1$, and $m$ a non-negative integer. Then we have*

$$\pi_n(s_\lambda(x_1, x_2, \ldots, x_{n-1})\ x_n^m) = 0$$

*if* $m = \lambda_i + n - i$ *for some* $1 \le i \le n-1$, *and*

$$\pi_n(s_\lambda(x_1, x_2, \dots, x_{n-1})\ x_n^m) = \mathrm{sgn}(w) s_\mu(x_1, x_2, \dots, x_n)$$

*otherwise, where the element* $w = w(\lambda, m)$ *of* $S_n$ *and the partition* $\mu = \mu(\lambda, m) = (\mu_1, \mu_2, \dots, \mu_n)$ *are uniquely determined by*:

$$(\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_{n-1} + 1, m) = w(\mu_1 + n - 1, \mu_2 + n - 2, \dots, \mu_n),$$

*where, on the right hand side,* $w$ *acts as a permutation of* $n$ *numbers.*

**5.2. Divided differences.** Let $f(x)$ be a function in a variable $x$, and let $P_1, P_2, P_3, \dots, P_n$ be distinct numbers. We put

$$\delta^{n-1}[f(x)] = \delta^{n-1}[f(x)]\{P_1, P_2, \dots, P_n\} = \sum_{t=1}^{n} \frac{f(P_t)}{\prod_{i \neq t}(P_t - P_i)}, \tag{5.5}$$

which is called the $(n-1)$-*th divided difference* of $f$ at $\{P_1, P_2, \dots, P_n\}$. What we need is the following easy properties of $\delta^{n-1}$, which can be extracted from text books (e.g. [10], [11]) on the calculus of finite differences. See also [8].

**Lemma 5.2.** *Under the above notation, we have the following.*

(i) $\delta^{n-1}[f(cx+d)]\{P_1, P_2, \dots, P_n\} = c^{n-1}\delta^{n-1}[f(x)]\{cP_1+d, cP_2+d, \dots, cP_n+d\}$

(ii)

$$\delta^{n-1}[f(x)] = \prod_{i<j}(P_j - P_i)^{-1} \begin{vmatrix} 1 & P_1 & P_1^2 & \cdots & P_1^{n-2} & f(P_1) \\ 1 & P_2 & P_2^2 & \cdots & P_2^{n-2} & f(P_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & P_n & P_n^2 & \cdots & P_n^{n-2} & f(P_n) \end{vmatrix}. \tag{5.6}$$

(iii)
$$\delta^{n-1}[x^k] = \begin{cases} 1 & \text{if } k = n-1, \\ 0 & \text{if } 0 \le k \le n-2. \end{cases}$$

(iv)
$$\delta^{n-1}[x^{-1}] = (-1)^{n-1}\left(\prod_{i=1}^{n} P_i\right)^{-1}.$$

(v)
$$\delta^{n-1}[(cx+d)^{-1}] = (-c)^{n-1}\left\{\prod_i (cP_i + d)\right\}^{-1}.$$

Proof. Parts (i) and (ii) follow from (5.5). Part (iii) follows from Part (ii), Part (iv) from Parts (ii) and (iii), and Part (v) from Parts (i) and (iv). ☐

**5.3. Proof of Theorem 1.1.** Since (1.2) can be obtained from (1.3) by applying the involution $\omega$ in [9], Ch.I, to the symmetric functions of the $y$ variables, it is enough to prove (1.3). Moreover, for a proof of (1.3), it is enough to prove it in the case $x_{n+1} = x_{n+2} = \cdots = 0$. Thus we want to show:

$$\prod_r \left( \prod_{i=1}^n \frac{1+q^{2r+1}x_i}{1-q^{2r+1}x_i} \prod_{j=1}^\infty \frac{1+q^{2r+1}y_j}{1-q^{2r+1}y_j} \right) \prod_{i=1}^n \prod_{j=1}^\infty (1+x_i y_j) \tag{5.7}$$

$$= \sum_{\substack{\lambda,\mu \\ l(\lambda)\le n}} q^{|\lambda-\mu'|+|\mu'-\lambda|} J_{\lambda,\mu'}(q^2) s_\lambda(x_1, x_2, \ldots, x_n) s_\mu(y)$$

for any non-negative integer $n$. When $n = 0$, (5.7) amounts to say:

$$\prod_j \prod_r \frac{1+q^{2r+1}y_j}{1-q^{2r+1}y_j} = \sum_\mu q^{|\mu|} J_{\phi\mu}(q^2) s_\mu(y),$$

which follows from (1.4), (4.1) and (4.2). By induction assumption, the left hand side of (5.7) is equal to

$$\left( \sum_{\substack{\alpha,\beta \\ l(\alpha)\le n-1}} K_{\alpha\beta'}(q) s_\alpha(x_1, \ldots, x_{n-1}) s_\beta(y) \right) \prod_r \frac{1+q^{2r+1}x_n}{1-q^{2r+1}x_n} \prod_{j=1}^\infty (1+x_n y_j),$$

where $K_{\alpha\beta'}(q) = q^{|\alpha-\beta'|+|\beta'-\alpha|} J_{\alpha\beta'}(q^2)$. By the $q$-binomial identity (see [9], I, 2, Ex. 5, or [2])

$$\sum_{n=0}^\infty \left( \prod_{i=1}^n \frac{1+tq^{i-1}}{1-q^i} \right) z^n = \prod_{r=0}^\infty \frac{1+tzq^r}{1-zq^r},$$

this is equal to

$$\begin{aligned} &\left( \sum_{\substack{\alpha,\beta \\ l(\alpha)\le n-1}} K_{\alpha\beta'}(q) s_\alpha(x_1, \ldots, x_{n-1}) s_\beta(y) \right) \\ &\qquad \times \left( \sum_{l=0}^\infty \prod_{i=1}^l \frac{1+q^{2i-2}}{1-q^{2i}} (qx_n)^l \right) \left( \sum_{m=0}^\infty e_m(y) x_n^m \right), \end{aligned} \tag{5.8}$$

where $e_m(y)$ is the $m$-th elementary symmetric function in $y$. We recall the well-known formula (see [9], I, 5):

$$s_\beta(y)e_m(y) = \sum_{\substack{\mu\supset\beta \\ \mu-\beta\in V(m)}} s_\mu(y), \tag{5.9}$$

where $V(m)$ is the set of vertical strips with $m$ nodes; a skew diagram $\nu$ is called a vertical strip if the condition '$(i, j), (i, k) \in \nu$' implies $j = k$. By (5.9), the coefficient of $s_\mu(y)$ in (5.8) is equal to

$$\sum_{m=0}^{\infty}\left\{\sum_{\substack{l(\alpha)\leq n-1 \\ \nu\in V(\mu,m)}} K_{\alpha\nu'}(q)s_\alpha(x_1,\ldots,x_{n-1})\right\}\left(\sum_{l=0}^{\infty}\prod_{i=1}^{l}\frac{1+q^{2i-2}}{1-q^{2i}}q^l x_n^{l+m}\right),$$

where

$$V(\mu,m) = \{\nu\in\mathcal{P} \mid \mu\supset\nu,\ \ \mu-\nu\in V(m)\}.$$

Thus, for a proof of (5.7), it is enough to show

$$\begin{aligned}\sum_{l(\lambda)\leq n} & K_{\lambda\mu'}(q)s_\lambda(x_1,\ldots,x_n) \\ &= \sum_{l,m}\sum_{\substack{l(\alpha)\leq n-1 \\ \nu\in V(\mu,m)}} K_{\alpha\nu'}(q)q^l\prod_{i=1}^{l}\frac{1+q^{2i-2}}{1-q^{2i}}s_\alpha(x_1,\ldots,x_{n-1})x_n^{l+m}.\end{aligned} \tag{5.10}$$

Since both sides of (5.10) are symmetric functions in $x$, (5.10) is equivalent to

$$\begin{aligned}\sum_{l(\lambda)\leq n} & K_{\lambda\mu'}(q)s_\lambda(x_1,\ldots,x_n) \\ &= \sum_{l,m}\sum_{\substack{l(\alpha)\leq n-1 \\ \nu\in V(\mu,m)}} K_{\alpha\nu'}(q)q^l\prod_{i=1}^{l}\frac{1+q^{2i-2}}{1-q^{2i}}\pi_n(s_\alpha(x_1,\ldots,x_{n-1})x_n^{l+m}),\end{aligned} \tag{5.11}$$

where $\pi_n$ is the symmetrizing operator defined in 5.1. By Lemma 5.1, for partitions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $l(\alpha) \leq n-1, l(\lambda) \leq n$, we have

$$\pi_n(s_\alpha(x_1,x_2,\ldots,x_{n-1})x_n^{l+m}) = cs_\lambda(x_1,x_2,\ldots,x_n)$$

for some constant $c$, if and only if

$$l+m = \lambda_e + n - e,$$

and

$$(\alpha_1,\alpha_2,\ldots,\alpha_{n-1}) = \lambda(e)$$

for some $e$ with $1 \le e \le n$, where

$$\lambda(e) = (\lambda_1, \lambda_2, \dots, \lambda_{e-1}, \lambda_{e+1} - 1, \lambda_{e+2} - 1, \dots, \lambda_n - 1);$$

moreover, in that case we have $c = (-1)^{n-e}$. Hence, by comparing the coefficients of $s_\lambda(x_1, \dots, x_n)$ on both sides of (5.11), we see that (5.11) is equivalent to

$$K_{\lambda\mu'}(q) = \sum_m \sum_{\nu \in V(\mu,m)} \sum_{e=1}^{n} (-1)^{n-e} K_{\lambda(e)\nu'}(q)[\sigma_e - m]\{\sigma_e - m\} q^{\sigma_e - m}, \tag{5.12}$$

where $\sigma_e = \lambda_e + n - e$, and, for an integer $a$, $[a]$ is defined by (2.4), and $\{a\}$ by

$$\{a\} = \begin{cases} \prod_{1=1}^{a} (1-t^i)^{-1}, & \text{if } a \ge 1, \\ 1, & \text{if } a = 0, \\ 0, & \text{if } a \le -1, \end{cases} \tag{5.13}$$

or by

$$\{0\} = 1, \quad \{a\} = (1 - t^{a+1})\{a+1\} \tag{5.14}$$

with $t = q^2$. Let $N = l(\mu)$. For a subset $S$ of the set $\{1, 2, \dots, N\}$, we define a sequence

$$\mu_S = (\mu_{1S}, \mu_{2S}, \dots, \mu_{NS})$$

of non-negative integers $\mu_{kS}$ as follows:

$$\mu_{kS} = \begin{cases} \mu_k - 1, & \text{if } k \in S, \\ \mu_k, & \text{if } k \notin S. \end{cases}$$

Then $\mu_S \in V(\mu, m)$ if and only if $\mu_S \in \mathcal{P}$. Moreover every $\nu \in V(\mu, m)$ can be obtained uniquely in this way. Hence we can write (5.12) in the following form:

$$K_{\lambda\mu'}(q) = \sum_{m=0}^{N} \sum_{\substack{|S|=m \\ \mu_S \in \mathcal{P}}} \sum_{e=1}^{n} (-1)^{n-e} K_{\lambda(e)\mu_S'}(q)[\sigma_e - m]\{\sigma_e - m\} q^{\sigma_e - m}. \tag{5.15}$$

Now, by Lemma 2.3, we see that, after cancelling out some factors using (2.5) and (5.14), (5.15) is equivalent to

$$
\begin{aligned}
&\prod_{h=1}^{L}(1+t^h)\prod_{i,k}(1+t^{\sigma_i+\tau_k-L})\prod_{i<j}(1-t^{\sigma_i-\sigma_j})\prod_{k<l}(1-t^{\tau_k-\tau_l})\\
&\quad=\sum_{e=1}^{n}(-1)^{n-e}\sum_{m=0}^{N}\sum_{\substack{|S|=m\\ \mu_S\in\mathcal{P}}} q^{\mathcal{D}}\prod_{i\neq e}(1-t^{\sigma_i})\prod_{h=0}^{m-1}(1-t^{\sigma_e-h})\prod_{h=m+1}^{L}(1+t^{\sigma_e-h})\\
&\qquad\times\prod_{k\in S}(1-t^{\tau_k})\prod_{k\notin S}(1+t^{\tau_k-L})\prod_{\substack{i\neq e\\ k}}(1+t^{\sigma_i+\tau_{kS}-L})\\
&\qquad\times\prod_{\substack{i<j\\ i,j\neq e}}(1-t^{\sigma_i-\sigma_j})\prod_{k<l}(1-t^{\tau_{kS}-\tau_{lS}}),
\end{aligned}
\tag{5.16}
$$

where

$$
\begin{aligned}
\mathcal{D} &= \mathcal{D}(e,S)=\sigma_e-|S|+\mathcal{A}(n-1,N,\lambda(e),\mu'_S)-\mathcal{A}(n,N,\lambda,\mu'),\\
L &= n+N-1,
\end{aligned}
$$

and, for $1\le k\le N$, $\tau_k=\mu_k+N-k$ and $\tau_{kS}=\mu_{kS}+N-k$. Note that, on the right hand side of (5.16), the sum on $S$ can be taken over the set of all subsets $S$ of $\{1,2,\dots,N\}$ such that $|S|=m$, not necessarily satisfying $\mu_S\in\mathcal{P}$. In fact, if $\mu_S\notin\mathcal{P}$, then there exists a $k$ such that $\mu_{kS}<\mu_{k+1S}$; this means that $\mu_k=\mu_{k+1}$ and that $k\in S, k+1\notin S$. Hence, we get $\tau_{kS}=\tau_{k+1S}$, i.e., $(1-t^{\tau_{kS}-\tau_{k+1S}})=0$, which means the corresponding term vanishes.

Thus, if we put

$$P_i=t^{\sigma_i},\qquad 1\le i\le n,$$

and

$$A_k=t^{\tau_k},\qquad 1\le k\le N,$$

then (5.16) can be written as

$$
\begin{aligned}
&\prod_{h=1}^{L}(1+t^h)\prod_{i,k}(t^L+P_iA_k)\prod_{i<j}(P_j-P_i)\prod_{k<l}(A_l-A_k)\\
&\quad=\sum_{e=1}^{n}(-1)^{n-e}\sum_{m=0}^{N}\sum_{|S|=m} t^{mL}\prod_{i\neq e}(1-P_i)\prod_{h=0}^{m-1}(t^h-P_e)\prod_{h=m+1}^{L}(t^h+P_e)\\
&\qquad\times\prod_{k\in S}(1-A_k)\prod_{k\notin S}(t^L+A_k)\\
&\qquad\times\prod_{\substack{i\neq e\\ k}}(t^L+P_iA_{kS})\prod_{\substack{i<j\\ i,j\neq e}}(P_j-P_i)\prod_{k<l}(A_{lS}-A_{kS}),
\end{aligned}
$$

where

$$A_{kS} = \begin{cases} t^{-1}A_k, & \text{if } \ k \in S, \\ A_k, & \text{if } \ k \notin S, \end{cases}$$

or, equivalently, as

$$\prod_{h=1}^{L}(1+t^h)\prod_{i,k}(t^L + P_i A_k)\prod_{k<l}(A_l - A_k) = \sum_{m=0}^{N}\sum_{e=1}^{n}\frac{f_m(P_e)}{\prod_{i\neq e}(P_e - P_i)}, \tag{5.17}$$

where

$$\begin{aligned} f_m(x) =& t^{mL}\prod_{i=1}^{n}(1-P_i) \\ &\times \left\{ \sum_{|S|=m}\prod_{k<l}(A_{lS} - A_{kS})\prod_{i,k}(t^L + P_i A_{kS})\prod_{k\in S}(1-A_k)\prod_{k\notin S}(t^L + A_k)\right\} \\ &\times \frac{\prod_{h=0}^{m-1}(t^h - x)\prod_{h=m+1}^{L}(t^h + x)}{(1-x)\prod_{k=1}^{N}(t^L + A_{kS}x)}. \end{aligned}$$

We are going to prove (5.17) viewing it as an identity for rational functions in independent variables $t, P_i (1 \leq i \leq n), A_k (1 \leq k \leq N)$. We have the partial fraction expansion:

$$\frac{\prod_{h=0}^{m-1}(t^h - x)\prod_{h=m+1}^{L}(t^h + x)}{(1-x)\prod_{k=1}^{N}(t^L + A_{kS}x)} = A(x) + \frac{B}{1-x} + \sum_{d=1}^{N}\frac{C_d}{t^L + A_{dS}x}, \tag{5.18}$$

where $A(x)$ is a polynomial of degree $n-2$ in $x$,

$$B = \begin{cases} \prod_{h=1}^{L}(1+t^h)\prod_{k=1}^{N}(t^L + A_{kS})^{-1}, & \text{if } \ m = 0, \\ 0, & \text{if } \ m \neq 0, \end{cases}$$

and

$$C_d = \frac{\prod_{h=0}^{m-1}(t^h + A_{dS}^{-1}t^L)\prod_{h=m+1}^{L}(t^h - A_{dS}^{-1}t^L)}{(1 + A_{dS}^{-1}t^L)\prod_{l\neq d}(t^L - A_{dS}^{-1}A_{lS}t^L)}, \quad 1 \leq d \leq N.$$

If $m = 0$, then $S = \phi$ and $A_{kS} = A_{k\phi} = A_k$ for any $k$. Hence, by Lemma 5.2(iii)(v) and

(5.18), we have

$$\begin{aligned}\sum_{e=1}^{n}\frac{f_0(P_e)}{\prod_{i\neq e}(P_e-P_i)}
&=\prod_{h=1}^{L}(1+t^h)\prod_{i,k}(t^L+P_iA_k)\prod_{k<l}(A_l-A_k)\\
&\quad+t^{-(N-1)L}\prod_{i}(1-P_i)\left\{\sum_{d=1}^{N}(-1)^{d+L}\prod_{\substack{i\\k\neq d}}(t^L+P_iA_k)\right.\\
&\qquad\left.\times\prod_{k\neq d}(t^L+A_k)\prod_{h=1}^{L}(A_dt^h-t^L)\prod_{\substack{k<l\\k,l\neq d}}(A_l-A_k)\right\}.\end{aligned}\tag{5.19}$$

Similarly, for $m\geq 1$, we have

$$\begin{aligned}\sum_{e=1}^{n}\frac{f_m(P_e)}{\prod_{i\neq e}(P_e-P_i)}
&=t^{(m-N+1)L}\prod_{i}(1-P_i)\sum_{|S|=m}\prod_{k\in S}(1-A_k)\prod_{k\notin S}(t^L+A_k)\\
&\quad\times\left\{\sum_{d=1}^{N}(-1)^{d+L}\prod_{\substack{i\\k\neq d}}(t^L+P_iA_{kS})\prod_{h=1}^{m-1}(t^L+A_{dS}t^h)\right.\\
&\qquad\left.\times\prod_{h=m+1}^{L}(A_{dS}t^h-t^L)\prod_{\substack{k<l\\k,l\neq d}}(A_{lS}-A_{kS})\right\}.\end{aligned}\tag{5.20}$$

Note that the first term of the right hand side of (5.19) is equal to the left hand side of (5.17). Hence, by (5.19) and (5.20), for a proof of (5.17), it is enough to show:

$$\begin{aligned}&\sum_{d=1}^{N}(-1)^{d}\prod_{\substack{i\\k\neq d}}(t^L+P_iA_k)\prod_{k\neq d}(t^L+A_k)\prod_{h=1}^{L}(A_dt^h-t^L)\prod_{\substack{k<l\\k,l\neq d}}(A_l-A_k)\\
&+\sum_{m=1}^{N}t^{mL}\sum_{|S|=m}\prod_{k\in S}(1-A_k)\prod_{k\notin S}(t^L+A_k)\left\{\sum_{d=1}^{N}(-1)^{d}\prod_{\substack{i\\k\neq d}}(t^L+P_iA_{kS})\right.\\
&\qquad\left.\times\prod_{h=1}^{m-1}(t^L+A_{dS}t^h)\prod_{h=m+1}^{L}(A_{dS}t^h-t^L)\prod_{\substack{k<l\\k,l\neq d}}(A_{lS}-A_{kS})\right\}=0.\end{aligned}\tag{5.21}$$

The left hand side of (5.21) is the sum of the terms $T(m, S, d)$ corresponding to $m \in \{0, 1, 2, \ldots, N\}$, $S \subset \{1, 2, \ldots, N\}$ with $|S| = m$, and $d \in \{1, 2, \ldots, N\}$. For any $m$ and $d$, it is easy to see:

$$T(m, S, d) + T(m + 1, S \cup \{d\}, d) = 0, \qquad \text{if } d \notin S.$$

Hence, (5.21) holds. The proof of Theorem 1.1 is now complete.

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