<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Representation formulas of the solutions to the Cauchy problems for first order systems</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Tajiri, Masaki; Umeda, Tomio</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 44(1) P.197-P.205</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2007-03</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/4372">https://doi.org/10.18910/4372</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/4372</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
Tajiri, M. and Umeda, T.  
 Osaka J. Math.  
 44 (2007), 197–205

REPRESENTATION FORMULAS OF THE SOLUTIONS  
 TO THE CAUCHY PROBLEMS FOR FIRST ORDER SYSTEMS

MASAKI TAJIRI and TOMIO UMEDA

(Received September 30, 2005, revised May 10, 2006)

Abstract

Representation formulas of the solutions to the Cauchy problems for first order systems of the forms \( \frac{\partial u}{\partial t} - \sum_{j=1}^{d} A_j(t) \frac{\partial u}{\partial x_j} - A_0(t)u = f \) are established. The coefficients \( A_j \)'s are assumed to be matrix-valued functions of the forms \( A_j(t) = \alpha_j(t)I + \beta_j(t)M_j \), where \( \alpha_j(t), \beta_j(t), j = 1, \ldots, d \), are real-valued continuous functions, the eigenvalues of the matrices \( M_j, j = 1, \ldots, d \), are real, and the commutators \([M_j, M_l] = 0\) for all \( j, l = 0, 1, \ldots, d \). No restrictions on the multiplicities of the characteristic roots are imposed.

1. Introduction

In this note we establish a representation formula for the Cauchy problem for a first order system of the form

\[
\frac{\partial u}{\partial t} - \sum_{j=1}^{d} A_j(t) \frac{\partial u}{\partial x_j} - A_0(t)u = f(t, x) \quad \text{in} \quad (0, T) \times \mathbb{R}^d
\]

\[
u(0, x) = \eta(x)
\]

where \( A_j(t), j = 0, 1, \ldots, d \), is a matrix-valued function of the form

\[
A_j(t) = \alpha_j(t)I + \beta_j(t)M_j
\]

with scalar valued functions \( \alpha_j(t), \beta_j(t) \) on the interval \([0, T]\), and a \( k \) by \( k \) complex matrix \( M_j \). The \( k \) by \( k \) identity matrix is denoted by \( I \). Both \( f(t, x) = [f_1(t, x), \ldots, f_k(t, x)] \) and \( \eta(x) = [\eta_1(x), \ldots, \eta_k(x)] \) are given functions.

We introduce notation in order to state the main theorem. For \( k \) by \( k \) matrices \( A \) and \( B \), \([A, B]\) denotes the commutator:

\[
[A, B] = AB - BA.
\]

2000 Mathematics Subject Classification. 35C99, 35F10.
Research supported by Grant-in-Aid for Scientific Research (C) No. 15540178, Japan Society for the Promotion of Science.
The Fourier transform with respect to the variable $x$ is denoted by

$$
\hat{\varphi}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x) \, dx.
$$

**Assumption (A).** (i) $[M_j, M_l] = 0$ for all $j, l = 0, 1, \ldots, d$.

(ii) For each $j = 1, 2, \ldots, d$, the eigenvalues of $M_j$ are real.

(iii) For each $j = 1, 2, \ldots, d$, the functions $\alpha_j(t)$ and $\beta_j(t)$ are real-valued continuous functions, and $\alpha_0(t)$ and $\beta_0(t)$ are possibly complex-valued continuous functions.

**Theorem 1.1.** Let Assumption (A) be verified and let $f$ be a function such that $\hat{f}(t, \xi)$ is continuous with respect to $t$ in the interval $[0, T]$ for each $\xi \in \mathbb{R}^d$. Suppose that there exist a constant $C$ and a function $\psi \in L^1(\mathbb{R}^d)$ such that

$$
(\xi)^{(m-1)d+1} \left( |\hat{f}(t, \xi)| + |\hat{\eta}(\xi)| \right) \leq C \psi(|\xi|)
$$

for all $(t, \xi) \in [0, T] \times \mathbb{R}^d$, where $m = 1$ if all $M_j$, $j = 1, \ldots, d$ are semisimple, and otherwise

$$
m := \max\{n \mid n \text{ equals the algebraic multiplicity of an eigenvalue of some } M_j, 1 \leq j \leq d \}.\tag{1.5}
$$

Then the solution of the Cauchy problem (1.1)–(1.2) is given by

$$
u(t, x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{B(t, \xi)} \left( \hat{\eta}(\xi) + \int_0^t e^{-B(s, \xi)} \hat{f}(s, \xi) \, ds \right) d\xi,
$$

which is a $C^1$-function in $[0, T] \times \mathbb{R}^d$. Here

$$
B(t, \xi) = \sum_{j=1}^n \xi_j \int_0^t A_j(s) \, ds + \int_0^t A_0(s) \, ds.\tag{1.7}
$$

It might be worthwhile to note that we do not need any restriction on the multiplicities of the characteristic roots for the equation (1.1), i.e. the roots of the characteristic polynomial

$$
\det \left( \lambda I - \sum_{j=1}^d \xi_j A_j(t) \right).
$$

If all the $\alpha_j$’s and $\beta_j$’s are $C^\infty$, and $f(t, x)$ and $\eta$ satisfy some suitable conditions, then the solution $u(t, x)$ becomes $C^\infty$. However, we shall not go into the discussions about this.
2. Proof of the main theorem

We start with ordinary differential equations for $M_k(\mathbb{C})$-valued functions, where $M_k(\mathbb{C})$ denotes the set of all $k$ by $k$ complex matrices.

Let an $M_k(\mathbb{C})$-valued continuous function $L(t)$ on the interval $[0, T]$ be given, and consider the ordinary differential equation

\begin{equation}
\frac{dU}{dt} = L(t)U
\end{equation}

with the initial condition

\begin{equation}
U(0) = I.
\end{equation}

Here the unknown function $U(t)$ is an $M_k(\mathbb{C})$-valued function.

The solution $U(t)$ to the equation (2.1) subject to (2.2), which is called the fundamental solution, can be expressed in the form

\begin{equation}
U(t) = e^{\Omega(t)},
\end{equation}

where $\Omega(t)$ is generally given as an infinite series (cf. [2, Theorem III]).

As a preliminary to the proof of Theorem 1.1, we need the following

**Lemma 2.1.** Suppose that $[L(t), L(t')] = 0$ for all $t, t' \in [0, T]$. Then the solution to the initial value problem (2.1) and (2.2) is written as

\begin{equation}
U(t) = e^{\int_0^t L(s) \, ds} \quad (0 \leq t \leq T).
\end{equation}

Proof. It is easy to see that

\begin{equation}
\left[ \int_0^{t'} L(s) \, ds, L(t') \right] = 0
\end{equation}

for all $t, t' \in [0, T]$. If we appeal to the definition

\[ e^{\int_0^t L(s) \, ds} = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \int_0^t L(s) \, ds \right)^j, \]

then (2.5) enables us to show that

\begin{equation}
\frac{d}{dt} e^{\int_0^t L(s) \, ds} = L(t) e^{\int_0^t L(s) \, ds} = e^{\int_0^t L(s) \, ds} L(t),
\end{equation}

which implies (2.4).
We should like to remark that the assertion of Lemma 2.1 is implicitly mentioned in [2]. See also [1].

Proof of Theorem 1.1. We shall give the proof only in the case \( m \geq 2 \). The proof in the case \( m = 1 \) is easier.

Taking the Fourier transform of (1.1) and (1.2), we obtain the ordinary differential equation

\[
\frac{d}{dt} \hat{u}(t, \xi) = \left( i \sum_{j=1}^{d} \hat{\xi}_j A_j(t) + A_0(t) \right) \hat{u}(t, \xi) + \hat{f}(t, \xi)
\]

subject to the initial condition

\[
\hat{u}(0, \xi) = \hat{\eta}(\xi),
\]

where \( \xi \) should be regarded as a parameter. In view of (1.3), it follows from the assumptions of Theorem 1.1 that

\[
i \sum_{j=1}^{d} \hat{\xi}_j A_j(t) + A_0(t)
\]

satisfies the assumption of Lemma 2.1. From Lemma 2.1, we see that the solution to (2.7), (2.8) is given by

\[
e^{B(t, \xi)} \left( \hat{\eta}(\xi) + \int_0^t e^{-B(s, \xi)} \hat{f}(s, \xi) \, ds \right).
\]

We now compute \( e^{B(s, \xi)} \). For this purpose, we put

\[
\tilde{\alpha}_j(t) = \int_0^t \alpha_j(s) \, ds, \quad \tilde{\beta}_j(t) = \int_0^t \beta_j(s) \, ds
\]

for \( j = 0, 1, \ldots, d \). Then

\[
B(t, \xi) = i \sum_{j=1}^{d} \hat{\xi}_j (\tilde{\alpha}_j(t)I + \tilde{\beta}_j(t)M_j) + \tilde{\alpha}_0(t)I + \tilde{\beta}_0(t)M_0.
\]

By Assumption (A) (i), we have

\[
e^{B(t, \xi)} = \left( \prod_{j=1}^{d} e^{i\hat{\xi}_j \tilde{\alpha}_j(t) + i\hat{\xi}_j \tilde{\beta}_j(t)M_j} \right) e^{\tilde{\alpha}_0(t)} e^{\tilde{\beta}_0(t)M_0}.
\]
For each $j$, the matrix $M_j$ can be expressed as the sum of a semisimple matrix $S_j$ and a nilpotent matrix $N_j$ that commutes with $S_j$:

\begin{equation}
M_j = S_j + N_j,
\end{equation}

which implies that

\begin{equation}
\exp\left[i \xi_j \beta_j(t)M_j\right] = \exp\left[i \xi_j \beta_j(t)S_j\right] \sum_{q=0}^{m-1} \frac{1}{q!} \left(i \xi_j \beta_j(t)N_j\right)^q
\end{equation}

for $j = 1, \ldots, d$, and that

\begin{equation}
\exp\left[i \beta_0(t)M_0\right] = \exp\left[i \beta_0(t)S_0\right] \sum_{q=0}^{m-1} \frac{1}{q!} \left(i \beta_0(t)N_0\right)^q,
\end{equation}

where $m$ is the integer defined in (1.5). It is now straightforward to show that the function defined by (1.6) is $C^1$ and gives the solution to the Cauchy problem (1.1)–(1.2).

\section{Examples}

The following proposition is useful in constructing examples to which Theorem 1.1 are applicable.

\textbf{Proposition 3.1.} \ Let $M$ be a $k$ by $k$ matrix of which eigenvalues are real, and let $M_j := p_j(M)$, $j = 0, 1, \ldots, d$, be real polynomials of $M$. Then Assumption (A) (i) is verified.

\textbf{Example 3.1.} \ We deal with the Cauchy problem for the partial differential equation of the form

\begin{equation}
\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & b(t) \\ 0 & a(t) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{bmatrix} u,
\end{equation}

where the coefficients are $M_2(\mathbb{C})$-valued continuous functions, and $a(t)$ and $b(t)$ are real-valued. (Note that (3.1) is the case where $k = 2$ and $d = 1$ in (1.1).) The aim here is to obtain the representation of the solution.

We would like to mention that the equation (3.1) is a special case of the equation that Matsumoto [3] investigated in order to study the hyperbolicity of systems with
double characteristic roots. Indeed, the equation he dealt with is reduced to the equation of the form

\[
\frac{\partial u}{\partial t} = \begin{bmatrix} a(t, x) & b(t, x) \\ 0 & a(t, x) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t, x) & \beta(t, x) \\ \gamma(t, x) & \delta(t, x) \end{bmatrix} u,
\]

where all the coefficients are \(C^\infty\)-functions of \((t, x)\) and \(b(t, x)\gamma(t, x) = 0\).

Taking into account this, we shall establish a representation formula for the equation (3.1) in the case where

\[b(t) = 0, \quad \alpha(t) = \delta(t), \quad \beta(t) = \gamma(t).\]

In this case, the equation (3.1) becomes

\[
\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & 0 \\ 0 & a(t) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \alpha(t) \end{bmatrix} u.
\]

It is straightforward to check that the coefficients of (3.3) satisfy Assumption (A). Indeed, with

\[M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\]

we have \(A_1(t) = a(t)I + M_1, \ A_0(t) = \alpha(t)I + \beta(t)M_0\). Furthermore, we have

\[e^{B(t, \xi)} = e^\xi \bar{a}(t) \tilde{a}(t) \begin{bmatrix} e^{\tilde{\beta}(t)} + e^{-\tilde{\beta}(t)} \\ e^{\tilde{\beta}(t)} - e^{-\tilde{\beta}(t)} \end{bmatrix} I + \begin{bmatrix} \eta_1(x + \tilde{a}(t)) \\ \eta_2(x + \tilde{a}(t)) \end{bmatrix} M_0 \]

where we have used the fact that \(M_0^2 = I\). Since \(M_0\) is semisimple, the assumption (1.4) on \(\eta\) becomes that \((\xi) \tilde{\eta} \in L^1(\mathbb{R})\). Then (1.6) defines a \(C^1\)-function and leads to the representation of the solution of the Cauchy problem (3.1):

\[
u(t, x) = \frac{1}{2} \begin{bmatrix} e^{\tilde{\alpha}(t)+\tilde{\beta}(t)} + e^{\tilde{\alpha}(t)-\tilde{\beta}(t)} \\ e^{\tilde{\alpha}(t)-\tilde{\beta}(t)} - e^{\tilde{\alpha}(t)+\tilde{\beta}(t)} \end{bmatrix} \begin{bmatrix} \eta_1(x + \tilde{a}(t)) \\ \eta_2(x + \tilde{a}(t)) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} e^{\tilde{\alpha}(t)+\tilde{\beta}(t)} + e^{\tilde{\alpha}(t)-\tilde{\beta}(t)} \\ e^{\tilde{\alpha}(t)-\tilde{\beta}(t)} - e^{\tilde{\alpha}(t)+\tilde{\beta}(t)} \end{bmatrix} \begin{bmatrix} \eta_1(x + \tilde{a}(t)) \\ \eta_2(x + \tilde{a}(t)) \end{bmatrix}.
\]

Following the proof of Theorem 1.1, we can deduce a direct generalization of the theorem in the manner described in the next theorem.

**Theorem 3.1.** Let

\[
A_j(t) = \alpha_j(t)I + \sum_{P: \text{finite}} \beta_{jp}(t) M_{jp}
\]
for $j = 0, 1, \ldots, d$, with $[M_{jp}, M_{iq}] = 0$ for all pairs $(j, p)$ and $(l, q)$. Suppose that for each $j = 1, 2, \ldots, d$, the eigenvalues of $M_{jp}$'s are real. Suppose, in addition, that for each $j = 1, 2, \ldots, d$, $\alpha_j(t)$ and $\beta_{jp}(t)$'s are real-valued continuous functions, and $\alpha_0(t)$ and $\beta_{0p}(t)$'s are possibly complex-valued continuous functions. Furthermore, suppose that $f(t, x)$ and $\eta(x)$ satisfy the same assumptions as in Theorem 1.1, where $m = 1$ if all the $M_{jp}$'s are semisimple, and otherwise with $m$ in (1.5) replaced by

$$m := \max\{n \mid n \text{ equals the algebraic multiplicity of an eigenvalue of some } M_{jp}, \ 1 \leq j \leq d, p, \}. \quad (3.5)$$

Then the solution of the Cauchy problem (1.1)–(1.2) is given by (1.6), which is a $C^1$-function in $[0, T] \times \mathbb{R}^d$.

**Example 3.2 $(d = 1)$.** We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = A(t) \frac{\partial u}{\partial x} \quad (3.6)$$

$$u(0, x) = \eta(x) \quad (3.7)$$

where

$$A(t) = \alpha(t) I + \beta_1(t) M_1 + \beta_2(t) M_2 \quad (3.8)$$

$$M_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & J_2 \end{bmatrix} \quad (3.9)$$

$$J_l = \begin{bmatrix} \lambda_l & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \lambda_l & 1 \end{bmatrix}, \quad (l = 1, 2) \quad (3.10)$$

$I$ being the $k$ by $k$ identity matrix, $J_1$ and $J_2$ being $k_1$ by $k_1$ and $k_2$ by $k_2$ matrices respectively, and $k_1 + k_2 = k$. We suppose that $\alpha(t)$, $\beta_1(t)$ and $\beta_2(t)$ are real-valued continuous functions. It is easy to see that all the assumptions of Theorem 3.1 are verified. Thus, if the initial data is assumed to satisfy that $\xi R \in L^1(\mathbb{R})$, where $m = \max\{k_1, k_2\}$, then Theorem 3.1 gives the $C^1$-solution $u(t, x)$ to the Cauchy problem (3.6), (3.7).

We shall compute $u(t, x)$. To this end, we write

$$J_1 = \lambda_1 I_1 + N_1, \quad J_2 = \lambda_2 I_2 + N_2 \quad (3.11)$$

where $I_l$ is the $k_l$ by $k_l$ identity matrix $(l = 1, 2)$. Noting that $N_l^{\xi t} = 0$, we have

$$e^{i\xi \tilde{\beta}_1(t) J_l} = e^{i\xi \tilde{\beta}_1(t) k_l} \sum_{q=0}^{k_l-1} \frac{1}{q!} (i\xi \tilde{\beta}_1(t) N_l)^q, \quad l = 1, 2.$$
Here $\tilde{\beta}_l(t)$ is defined similarly to (2.11). Since

$$e^{B(t,k)} = e^{i\xi\tilde{\alpha}(t)} \begin{bmatrix} e^{i\xi \tilde{\beta}_1(t)J_1} & 0 \\ 0 & e^{i\xi \tilde{\beta}_2(t)J_2} \end{bmatrix},$$

the formula (1.6), together with (3.11), leads to the representation

$$u(t, x) = \begin{bmatrix} \sum_{q=0}^{k_1-1} \frac{1}{q!} \tilde{\beta}_1(t)^q N_1^q (\partial_x^q \eta^s) \left( x + \tilde{\alpha}(t) + \lambda_1 \tilde{\beta}_1(t) \right) \\ \sum_{q=0}^{k_2-1} \frac{1}{q!} \tilde{\beta}_2(t)^q N_2^q (\partial_x^q \eta^s) \left( x + \tilde{\alpha}(t) + \lambda_2 \tilde{\beta}_2(t) \right) \end{bmatrix},$$

where $\eta^s(x) = [\eta_1(x), \ldots, \eta_{k_1}(x)]$, $\eta^{ss}(x) = [\eta_{k_1+1}(x), \ldots, \eta_k(x)]$ and

$$\eta(x) = \begin{bmatrix} \eta^s(x) \\ \eta^{ss}(x) \end{bmatrix}.$$

**Remark.** Neither of Theorems 1.1 and 3.1 is applicable to the system of the form

$$\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & b(t) \\ c(t) & -a(t) \end{bmatrix} \frac{\partial u}{\partial x},$$

which is a special case of the system that was studied in Nishitani [4].

**Acknowledgement.** The authors would like to thank Professor Waichiro Matsumoto of Ryukoku University and Professor Tatsuo Nishitani of Osaka University for their helpful advice. Also, they would like to thank Takashi Okaji of Kyoto University and Professor Masaru Taniguchi of Waseda University for their fruitful discussions with us.

**References**

Masaki Tajiri
Himeji Municipal Office
Yasuda, Himeji 670–8501
Japan

Tomio Umeda
Department of Mathematics
University of Hyogo
Shosha, Himeji 671–2201
Japan
e-mail: umeda@sci.u-hyogo.ac.jp