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REPRESENTATION FORMULAS OF THE SOLUTIONS TO THE CAUCHY PROBLEMS FOR FIRST ORDER SYSTEMS

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Abstract

Representation formulas of the solutions to the Cauchy problems for first order systems of the forms $\partial u/\partial t - \sum_{j=1}^{d} A_j(t)\partial u/\partial x_j - A_0(t)u = f$ are established. The coefficients A_j's are assumed to be matrix-valued functions of the forms $A_j(t) = \alpha_j(t)I + \beta_j(t)M_j$, where $\alpha_j(t), \beta_j(t)$, $j = 1, \dots, d$, are real-valued continuous functions, the eigenvalues of the matrices M_j, $j = 1, \dots, d$, are real, and the commutators $[M_j, M_l] = 0$ for all $j, l = 0, 1, \dots, d$. No restrictions on the multiplicities of the characteristic roots are imposed.

1. Introduction

In this note we establish a representation formula for the Cauchy problem for a first order system of the form

$$\frac{\partial u}{\partial t} - \sum_{j=1}^{d} A_j(t)\frac{\partial u}{\partial x_j} - A_0(t)u = f(t, x) \quad \text{in} \quad (0, T) \times \mathbb{R}_x^d \tag{1.1}$$

$$u(0, x) = \eta(x) \tag{1.2}$$

where $A_j(t)$, $j = 0, 1, \dots, d$, is a matrix-valued function of the form

$$A_j(t) = \alpha_j(t)I + \beta_j(t)M_j \tag{1.3}$$

with scalar valued functions $\alpha_j(t)$, $\beta_j(t)$ on the interval $[0, T]$, and a k by k complex matrix M_j. The k by k identity matrix is denoted by I. Both $f(t, x) = {}^t[f_1(t, x), \dots, f_k(t, x)]$ and $\eta(x) = {}^t[\eta_1(x), \dots, \eta_k(x)]$ are given functions.

We introduce notation in order to state the main theorem. For k by k matrices A and B, $[A, B]$ denotes the commutator:

$$[A, B] = AB - BA.$$

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The Fourier transform with respect the variable x is denoted by

$$\hat{\varphi}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x)\, dx.$$

ASSUMPTION (A). (i) $[M_j, M_l] = 0$ for all $j, l = 0, 1, \dots, d$.
(ii) For each $j = 1, 2, \dots, d$, the eigenvalues of M_j are real.
(iii) For each $j = 1, 2, \dots, d$, the functions $\alpha_j(t)$ and $\beta_j(t)$ are real-valued continuous functions, and $\alpha_0(t)$ and $\beta_0(t)$ are possibly complex-valued continuous functions.

Theorem 1.1. *Let* Assumption (A) *be verified and let f be a function such that $\hat{f}(t,\xi)$ is continuous with respect to t in the interval $[0, T]$ for each $\xi \in \mathbb{R}^d$. Suppose that there exist a constant C and a function $\psi \in L^1(\mathbb{R}^d)$ such that*

$$\langle\xi\rangle^{(m-1)d+1}\left(\left|\hat{f}(t,\xi)\right| + |\hat{\eta}(\xi)|\right) \le C\psi(\xi) \tag{1.4}$$

for all $(t,\xi) \in [0, T] \times \mathbb{R}^d$, where $m = 1$ if all M_j, $j = 1, \dots, d$ are semisimple, and otherwise

$$\begin{aligned} m := \max\{n \mid n \textit{ equals the algebraic multiplicity of} \\ \textit{an eigenvalue of some } M_j,\ 1 \le j \le d\}. \end{aligned} \tag{1.5}$$

Then the solution of the Cauchy problem (1.1)–(1.2) *is given by*

$$u(t,x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{B(t,\xi)} \left(\hat{\eta}(\xi) + \int_0^t e^{-B(s,\xi)} \hat{f}(s,\xi)\, ds\right) d\xi, \tag{1.6}$$

which is a C^1-function in $[0, T] \times \mathbb{R}^d$. Here

$$B(t,\xi) = i \sum_{j=1}^{n} \xi_j \int_0^t A_j(s)\, ds + \int_0^t A_0(s)\, ds. \tag{1.7}$$

It might be worthwhile to note that we do not need any restriction on the multiplicities of the characteristic roots for the equation (1.1), i.e. the roots of the characteristic polynomial

$$\det\left(\lambda I - \sum_{j=1}^{d} \xi_j A_j(t)\right).$$

If all the α_j's and β_j's are C^∞, and $f(t,x)$ and η satisfy some suitable conditions, then the solution $u(t,x)$ becomes C^∞. However, we shall not go into the discussions about this.

2. Proof of the main theorem

We start with ordinary differential equations for $M_k(\mathbb{C})$-valued functions, where $M_k(\mathbb{C})$ denotes the set of all k by k complex matrices.

Let an $M_k(\mathbb{C})$-valued continuous function $L(t)$ on the interval $[0, T]$ be given, and consider the ordinary differential equation

$$\frac{dU}{dt} = L(t)U \tag{2.1}$$

with the initial condition

$$U(0) = I. \tag{2.2}$$

Here the unknown function $U(t)$ is an $M_k(\mathbb{C})$-valued function.

The solution $U(t)$ to the equation (2.1) subject to (2.2), which is called the fundamental solution, can be expressed in the form

$$U(t) = e^{\Omega(t)}, \tag{2.3}$$

where $\Omega(t)$ is generally given as an infinite series (cf. [2, Theorem III]).

As a preliminary to the proof of Theorem 1.1, we need the following

Lemma 2.1. *Suppose that* $[L(t), L(t')] = 0$ *for all* $t, t' \in [0, T]$. *Then the solution to the initial value problem* (2.1) *and* (2.2) *is written as*

$$U(t) = e^{\int_0^t L(s)\,ds} \quad (0 \le t \le T). \tag{2.4}$$

Proof. It is easy to see that

$$\left[\int_0^t L(s)\,ds, L(t')\right] = 0 \tag{2.5}$$

for all $t, t' \in [0, T]$. If we appeal to the definition

$$e^{\int_0^t L(s)\,ds} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\int_0^t L(s)\,ds\right)^j,$$

then (2.5) enables us to show that

$$\frac{d}{dt} e^{\int_0^t L(s)\,ds} = L(t) e^{\int_0^t L(s)\,ds} = e^{\int_0^t L(s)\,ds} L(t), \tag{2.6}$$

which implies (2.4). □

We should like to remark that the assertion of Lemma 2.1 is implicitly mentioned in [2]. See also [1].

Proof of Theorem 1.1. We shall give the proof only in the case $m \geq 2$. The proof in the case $m = 1$ is easier.

Taking the Fourier transform of (1.1) and (1.2), we obtain the ordinary differential equation

$$\frac{d}{dt}\hat{u}(t,\xi) = \left(i\sum_{j=1}^{d}\xi_j A_j(t) + A_0(t)\right)\hat{u}(t,\xi) + \hat{f}(t,\xi) \tag{2.7}$$

subject to the initial condition

$$\hat{u}(0,\xi) = \hat{\eta}(\xi), \tag{2.8}$$

where ξ should be regarded as a parameter. In view of (1.3), it follows from the assumptions of Theorem 1.1 that

$$i\sum_{j=1}^{d}\xi_j A_j(t) + A_0(t) \tag{2.9}$$

satisfies the assumption of Lemma 2.1. From Lemma 2.1, we see that the solution to (2.7), (2.8) is given by

$$e^{B(t,\xi)}\left(\hat{\eta}(\xi) + \int_0^t e^{-B(s,\xi)}\hat{f}(s,\xi)\,ds\right). \tag{2.10}$$

We now compute $e^{B(s,\xi)}$. For this purpose, we put

$$\widetilde{\alpha}_j(t) = \int_0^t \alpha_j(s)\,ds, \quad \widetilde{\beta}_j(t) = \int_0^t \beta_j(s)\,ds \tag{2.11}$$

for $j = 0, 1, \dots, d$. Then

$$B(t,\xi) = i\sum_{j=1}^{d}\xi_j\left(\widetilde{\alpha}_j(t)I + \widetilde{\beta}_j(t)M_j\right) + \widetilde{\alpha}_0(t)I + \widetilde{\beta}_0(t)M_0. \tag{2.12}$$

By Assumption (A) (i), we have

$$e^{B(t,\xi)} = \left(\prod_{j=1}^{d} e^{i\xi_j\widetilde{\alpha}_j(t)}e^{i\xi_j\widetilde{\beta}_j(t)M_j}\right)e^{\widetilde{\alpha}_0(t)}e^{\widetilde{\beta}_0(t)M_0}. \tag{2.13}$$

For each j, the matrix M_j can be expressed as the sum of a semisimple matrix S_j and a nilpotent matrix N_j that commutes with S_j:

$$M_j = S_j + N_j, \tag{2.14}$$

which implies that

$$e^{i\xi_j\widetilde{\beta}_j(t)M_j} = e^{i\xi_j\widetilde{\beta}_j(t)S_j}\sum_{q=0}^{m-1}\frac{1}{q!}\big(i\xi_j\widetilde{\beta}_j(t)N_j\big)^q \tag{2.15}$$

for $j = 1, \ldots, d$, and that

$$e^{\widetilde{\beta}_0(t)M_0} = e^{\widetilde{\beta}_0(t)S_0}\sum_{q=0}^{m-1}\frac{1}{q!}\big(\widetilde{\beta}_0(t)N_0\big)^q, \tag{2.16}$$

where m is the integer defined in (1.5). It is now straightforward to show that the function defined by (1.6) is C^1 and gives the solution to the Cauchy problem (1.1)–(1.2). □

3. Examples

The following proposition is useful in constructing examples to which Theorem 1.1 are applicable.

Proposition 3.1. *Let M be a k by k matrix of which eigenvalues are real, and let $M_j := p_j(M)$, $j = 0, 1, \ldots, d$, be real polynomials of M. Then* Assumption (A) (i) *is verified.*

EXAMPLE 3.1. We deal with the Cauchy problem for the partial differential equation of the form

$$\begin{gathered}\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & b(t) \\ 0 & a(t)\end{bmatrix}\frac{\partial u}{\partial x} + \begin{bmatrix}\alpha(t) & \beta(t) \\ \gamma(t) & \delta(t)\end{bmatrix} u, \\ u(0,x) = \eta(x) = {}^t[\eta_1(x), \eta_2(x)]\end{gathered} \tag{3.1}$$

where the coefficients are $M_2(\mathbb{C})$-valued continuous functions, and $a(t)$ and $b(t)$ are real-valued. (Note that (3.1) is the case where $k = 2$ and $d = 1$ in (1.1).) The aim here is to obtain the representation of the solution.

We would like to mention that the equation (3.1) is a special case of the equation that Matsumoto [3] investigated in order to study the hyperbolicity of systems with double characteristic roots. Indeed, the equation he dealt with is reduced to the equation of the form

$$\frac{\partial u}{\partial t} = \begin{bmatrix} a(t,x) & b(t,x) \\ 0 & a(t,x) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t,x) & \beta(t,x) \\ \gamma(t,x) & \delta(t,x) \end{bmatrix} u, \tag{3.2}$$

where all the coefficients are C^∞-functions of (t,x) and $b(t,x)\gamma(t,x) \equiv 0$.

Taking into account this, we shall establish a representation formula for the equation (3.1) in the case where

$$b(t) \equiv 0, \quad \alpha(t) \equiv \delta(t), \quad \beta(t) \equiv \gamma(t).$$

In this case, the equation (3.1) becomes

$$\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & 0 \\ 0 & a(t) \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \alpha(t) \end{bmatrix} u. \tag{3.3}$$

It is straightforward to check that the coefficients of (3.3) satisfy Assumption (A). Indeed, with

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have $A_1(t) = a(t)I + M_1$, $A_0(t) = \alpha(t)I + \beta(t)M_0$. Furthermore, we have

$$e^{B(t,\xi)} = e^{i\xi\widetilde{a}(t)+\widetilde{\alpha}(t)} \left\{ \frac{e^{\widetilde{\beta}(t)} + e^{-\widetilde{\beta}(t)}}{2} I + \frac{e^{\widetilde{\beta}(t)} - e^{-\widetilde{\beta}(t)}}{2} M_0 \right\},$$

where we have used the fact that $M_0^2 = I$. Since M_0 is semisimple, the assumption (1.4) on η becomes that $\langle\xi\rangle\hat{\eta} \in L^1(\mathbb{R})$. Then (1.6) defines a C^1-function and leads to the representation of the solution of the Cauchy problem (3.1):

$$\begin{aligned} u(t,x) = {} & \frac{1}{2}\left(e^{\widetilde{\alpha}(t)+\widetilde{\beta}(t)} + e^{\widetilde{\alpha}(t)-\widetilde{\beta}(t)}\right) \begin{bmatrix} \eta_1(x+\widetilde{a}(t)) \\ \eta_2(x+\widetilde{a}(t)) \end{bmatrix} \\ & + \frac{1}{2}\left(e^{\widetilde{\alpha}(t)+\widetilde{\beta}(t)} - e^{\widetilde{\alpha}(t)-\widetilde{\beta}(t)}\right) \begin{bmatrix} \eta_2(x+\widetilde{a}(t)) \\ \eta_1(x+\widetilde{a}(t)) \end{bmatrix} \end{aligned}$$

Following the proof of Theorem 1.1, we can deduce a direct generalization of the theorem in the manner described in the next theorem.

Theorem 3.1. *Let*

$$A_j(t) = \alpha_j(t)I + \sum_{p:\ \mathit{finite}} \beta_{jp}(t)M_{jp} \tag{3.4}$$

for $j = 0, 1, \dots, d$, *with* $[M_{jp}, M_{lq}] = 0$ *for all pairs* (j, p) *and* (l, q). *Suppose that for each* $j = 1, 2, \dots, d$, *the eigenvalues of* M_{jp}*'s are real. Suppose, in addition, that for each* $j = 1, 2, \dots, d$, $\alpha_j(t)$ *and* $\beta_{jp}(t)$*'s are real-valued continuous functions, and* $\alpha_0(t)$ *and* $\beta_{0p}(t)$*'s are possibly complex-valued continuous functions. Furthermore, suppose that* $f(t, x)$ *and* $\eta(x)$ *satisfy the same assumptions as in* Theorem 1.1, *where* $m = 1$ *if all the* M_{jp}*'s are semisimple, and otherwise with* m *in* (1.5) *replaced by*

$$\begin{aligned} m := \max\{n \mid n \text{ equals the algebraic multiplicity of} \\ \text{an eigenvalue of some } M_{jp},\ 1 \le j \le d,\ p,\}. \end{aligned} \tag{3.5}$$

Then the solution of the Cauchy problem (1.1)–(1.2) *is given by* (1.6), *which is a* C^1-*function in* $[0, T] \times \mathbb{R}^d$.

EXAMPLE 3.2 ($d = 1$). We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = A(t)\frac{\partial u}{\partial x} \tag{3.6}$$

$$u(0, x) = \eta(x) \tag{3.7}$$

where

$$A(t) = \alpha(t)I + \beta_1(t)M_1 + \beta_2(t)M_2 \tag{3.8}$$

$$M_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & J_2 \end{bmatrix} \tag{3.9}$$

$$J_l = \begin{bmatrix} \lambda_l & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \lambda_l & 1 \\ \mathbf{0} & & & \lambda_l \end{bmatrix}, \quad (l = 1, 2) \tag{3.10}$$

I being the k by k identity matrix, J_1 and J_2 being k_1 by k_1 and k_2 by k_2 matrices respectively, and $k_1 + k_2 = k$. We suppose that $\alpha(t)$, $\beta_1(t)$ and $\beta_2(t)$ are real-valued continuous functions. It is easy to see that all the assumptions of Theorem 3.1 are verified. Thus, if the initial data is assumed to satisfy that $\langle\xi\rangle^m\hat{\eta} \in L^1(\mathbb{R})$, where $m = \max\{k_1, k_2\}$, then Theorem 3.1 gives the C^1-solution $u(t, x)$ to the Cauchy problem (3.6), (3.7).

We shall compute $u(t, x)$. To this end, we write

$$J_1 = \lambda_1 I_1 + N_1, \qquad J_2 = \lambda_2 I_2 + N_2$$

where I_l is the k_l by k_l identity matrix ($l = 1, 2$). Noting that $N_l^{k_l} = 0$, we have

$$e^{i\xi\widetilde{\beta}_l(t)J_l} = e^{i\xi\widetilde{\beta}_l(t)\lambda_l}\sum_{q=0}^{k_l-1}\frac{1}{q!}\big(i\xi\widetilde{\beta}_l(t)N_l\big)^q, \quad l = 1, 2. \tag{3.11}$$

Here $\widetilde{\beta}_l(t)$ is defined similarly to (2.11). Since

$$e^{B(t,\xi)} = e^{i\xi\widetilde{\alpha}(t)} \begin{bmatrix} e^{i\xi\widetilde{\beta}_1(t)J_1} & 0 \\ 0 & e^{i\xi\widetilde{\beta}_2(t)J_2} \end{bmatrix},$$

the formula (1.6), together with (3.11), leads to the representation

$$u(t,x) = \begin{bmatrix} \displaystyle\sum_{q=0}^{k_1-1} \frac{1}{q!} \widetilde{\beta}_1(t)^q N_1^q (\partial_x^q \eta^*)\big(x + \widetilde{\alpha}(t) + \lambda_1 \widetilde{\beta}_1(t)\big) \\ \displaystyle\sum_{q=0}^{k_2-1} \frac{1}{q!} \widetilde{\beta}_2(t)^q N_2^q (\partial_x^q \eta^{**})\big(x + \widetilde{\alpha}(t) + \lambda_2 \widetilde{\beta}_2(t)\big) \end{bmatrix},$$

where $\eta^*(x) = {}^t[\eta_1(x), \dots, \eta_{k_1}(x)]$, $\eta^{**}(x) = {}^t[\eta_{k_1+1}(x), \dots, \eta_k(x)]$ and

$$\eta(x) = \begin{bmatrix} \eta^*(x) \\ \eta^{**}(x) \end{bmatrix}.$$

REMARK. Neither of Theorems 1.1 and 3.1 is applicable to the system of the form

$$\frac{\partial u}{\partial t} = \begin{bmatrix} a(t) & b(t) \\ c(t) & -a(t) \end{bmatrix} \frac{\partial u}{\partial x}$$

which is a special case of the system that was studied in Nishitani [4].

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