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CONDITIONS AGAINST RAPID DECREASE OF OSCILLATORY INTEGRALS AND THEIR APPLICATIONS TO INVERSE SCATTERING PROBLEMS

HIDEO SOGA

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Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter $\sigma \geq 0$:

$$I(\sigma) = \int e^{-i\phi(x)} \rho(x; \sigma) dx \quad \text{or} \quad \int e^{i\phi(x)} \rho(x; \sigma) dx,$$

where $\phi(x)$ is a real-valued $C^\infty$ function and $\rho(x; \sigma)$ is a $C^\infty$ function with an asymptotic expansion

$$\rho(x; \sigma) \sim \rho_0(x) + \rho_1(x) (i\sigma)^{-1} + \rho_2(x) (i\sigma)^{-2} + \cdots \quad \text{as } \sigma \to \infty.$$

In this paper we study conditions for the integral $I(\sigma)$ not to decrease rapidly as $\sigma \to \infty$, and solve some inverse scattering problems.

As is well known, if stationary points of $\phi(x)$ are non-degenerate (i.e. $\det (\partial^2 \phi(x)) \neq 0$ when $\partial \phi(x) = 0$), $I(\sigma)$ is expanded asymptotically as $\sigma \to \infty$, and we can know whether $I(\sigma)$ decreases rapidly as $\sigma \to \infty$. Also when the stationary points are degenerate, the asymptotic expansion of $I(\sigma)$ is obtained if $\phi(x)$ is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of $\phi(x)$ vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of $I(\sigma)$. In the previous paper [13], the author examined the case that $n=2$ and $\rho_1(x) = 0$ ($j \geq 1$): If $\rho_0(x) \geq 0$ on $\mathbb{R}^2$ and $\rho_0(x_0) > 0$ for a degenerate stationary point $x_0$ of $\phi(x)$, then $(1+|\sigma|)^m I(\sigma) \in L^1(\mathbb{R}^2)$ for some $m < 2^{-1}$ (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of $n \geq 3$.

Let $\text{supp}\{\rho(\cdot; \sigma)\}$ and $\text{supp}\{\rho_j\}$ ($j \geq 0$) be contained in a compact set $D$ in $\mathbb{R}^n$. We set

$$E(s) = \{x : \phi(x) \leq s\} \quad (s \in \mathbb{R}),$$
One of our main results is the following

**Theorem 1.** Let all \( \rho_j \) \((j \geq 0)\) be real-valued. Then, for every \( m \in \mathbb{R} \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

if and only if for every integer \( N(\geq 1) \)

\[
g_N(s) = g_0(s) + \sum_{j=1}^{N} \int_{0}^{1} \frac{(t-t)^{j-1}}{(j-1)!} g_j(t) \, dt \in C^N(\mathbb{R}^d) .
\]

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

**Theorem 2.** Let all \( \rho_j \) \((j \geq 0)\) be real-valued, and let \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \). If \( \rho_0 \) satisfies

\[
\rho_0(x) > 0 \quad \text{on } E(\min \phi(x)) ,
\]

then for some \( m(\in \mathbb{R}) \) depending only on the dimension \( n \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty) .
\]

Theorem 1 implies that decreasingness of \( I(\sigma) \) is connected with smoothness of the measure \( |E(s)| \). This is seen also from the discussions in Vasil'ev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of \( |E(s)| \).

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle \( \mathcal{O} \subset \mathbb{R}^n \) with a \( C^\infty \) boundary \( \partial \mathcal{O} \). Assume that the domain \( \Omega = \mathbb{R}^n - \mathcal{O} \) is connected, and consider the initial-boundary value problem

\[
\begin{align*}
\Box u(t, x) &= 0 \quad \text{in } \mathbb{R}^d \times \Omega \quad (\Box = \partial^2_t - \Delta) , \\
u(t, x') &= 0 \quad \text{on } \mathbb{R}^d \times \partial \Omega \quad (\partial \Omega = \partial \mathcal{O}) , \\
u(0, x) &= f_1(x) \quad \text{on } \Omega , \\
\partial_t \nu(0, x) &= f_2(x) \quad \text{on } \Omega .
\end{align*}
\]

We denote by \( k_-(s, \omega) \) \((k_+(s, \omega)) \in L^2(\mathbb{R}^d \times \mathbb{S}^{n-1}) \) the incoming (outgoing) translation representation of the data \((f_1, f_2)\) (cf. Lax and Phillips [6], [7]). The operator \( S: k_- \to k_+ \) is called the scattering operator and represented by a distribution kernel \( S(s, \theta, \omega) \) called the scattering kernel:
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\[ (Sk_\omega)(s, \theta) = \iint S(s-t, \theta, \omega) k_\omega(t, \omega) dt d\omega \]

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of \( \Omega \subset \mathbb{R}^3 \) (i.e. \( n=3 \)) that for any fixed \( \omega \in S^2 \)

(0.2)

(i) \( \text{supp } S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)) \),

(ii) \( S(s, -\omega, \omega) \) is singular (not \( C^\infty \)) at \( s = -2r(\omega) \),

where \( r(\omega) = \min x \cdot \omega \). He reduced proof of the above (ii) to verifying that the integral of the form

\[ \int_{\mathbb{R}^n} e^{-i \sigma \varphi(x)} \rho(x; \sigma) dx \]

does not decrease rapidly as \( \sigma \to \infty \) (cf. §2 of Majda [8] or §4 of the author [14]).

His methods are not applicable to the case of \( n>3 \), one of whose reasons is that the stationary points of the phase function \( \varphi(x) \) are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when \( n>3 \):

**Theorem 3.** For any fixed \( \omega \) and \( \theta \in S^{n-1} \) with \( \omega \pm \theta \), we have

(i) \( \text{supp } S(\cdot, \theta, \omega) \subset (-\infty, -r(\omega-\theta)) \),

(ii) \( S(s, \theta, \omega) \) is singular at \( s = -r(\omega-\theta) \).

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

\[
\begin{aligned}
\partial_t u(t, x) - \sum_{i,j=1}^n \partial_x (a_{ij}(x) \partial_{x_j} u(t, x)) &= 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n , \\
u(0, x) &= f_1(x) & \text{on } \mathbb{R}^n , \\
\partial_t u(0, x) &= f_2(x) & \text{on } \mathbb{R}^n ,
\end{aligned}
\]

(0.3)

where \( a_{ij}(x) \) are real-valued \( C^\infty \) functions satisfying

\[
a_{ij}(x) = a_{ij}(x), \quad x \in \mathbb{R}^n ,
a_{ij}(x) = 0 \ (i \neq j), \quad a_{ii}(x) = 1 \quad \text{when } |x| \geq r_0 ,
\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \delta |\xi|^2 , \quad x \in \mathbb{R}^n , \quad \xi \in \mathbb{R}^n .
\]

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of \( n=3 \). By means of Theorem 2 we get rid of the restriction to the dimension \( n \).

Let us review the results of [15]. We set

\[ \lambda_\omega(x, \xi) = -\{ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \}^{1/2} . \]
Denote by \((q^-(t; s, x, \xi), p^-(t; s, x, \xi))\) the solution of the equation
\[
\begin{align*}
\frac{dq^-}{dt} &= -\partial_t\lambda_\diamond(q^-, p^-), \\
\frac{dp^-}{dt} &= \partial_x\lambda_\diamond(q^-, p^-), \\
q^-|_{t=0} &= x, \\
p^-|_{t=0} &= \xi,
\end{align*}
\]
and for \(\omega, \theta \in S^{n-1}\) set
\[
\begin{align*}
M_{\omega}(\theta) &= \{y: y \cdot \omega = -r_0, \lim_{t \to \infty} p^-(t; -r_0, y, \omega) = \theta\}, \\
s_{\omega}(\theta) &= \sup_{y \in M_{\omega}(\theta)} \{\lim_{t \to \infty} q^-(t; -r_0, y, \omega) \cdot \theta - t\}, \\
\bar{M}_{\omega}(\theta) &= \{y \in M_{\omega}(\theta): s_{\omega}(\theta) = \lim_{t \to \infty} q^-(t; -r_0, y, \omega) \cdot \theta - t\}.
\end{align*}
\]
We assume that for any \(y \ (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\)
\begin{equation}
\lim_{t \to \infty} |q^-(t; -r_0, y, \omega)| = \infty.
\end{equation}
Then singular support of the scattering kernel \(S(\cdot, \theta, \omega)\) for the equation (0.3) is contained in the interval \((-\infty, s_{\omega}(\theta)]\) (cf. Theorem 2 in the author [15]); furthermore, when \(n=2\), it is proved under some assumptions that \(S(s, \theta, \omega)\) is singular at \(s=s_{\omega}(\theta)\) (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of \(n>2\):

**Theorem 4.** Assume (0.4) for any \(y \ (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\). Fix \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), and let the assumption
\begin{equation}
\det[\partial_t q^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_{\omega}(\theta)
\end{equation}
be satisfied. Then \(S(s, \theta, \omega)\) is singular at \(s=s_{\omega}(\theta)\).

The assumption (0.5) means that there is no caustic on \(\{(t, x): x = q^-(t; -r_0, y, \omega), \ \ -r_0 \leq t < \infty, y \in \bar{M}_{\omega}(\theta)\}\), namely, the mapping: \((t, y) \rightarrow q^-(t; -r_0, y, \omega) (-r_0 \leq t < \infty, y \cdot \omega = -r_0)\) is diffeomorphic on \([-r_0, \infty) \times \bar{M}_{\omega}(\theta)\). In the previous paper [15] we added the assumption
\[
\det[\partial_t p^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_{\omega}(\theta),
\]
but this is not necessary.

1. **Proofs of Theorem 1 and Theorem 2**

We denote by \(H^m(M)\) the Sobolev space of order \(m\) on \(M\), and by \(H^m_{\text{loc}}(M)\) the space of functions \(g(x)\) satisfying \(\alpha(x)g(x) \in H^m(M)\) for any \(\alpha(x) \in C_0^\infty(M)\) (\(C_0^\infty(M)\) is the space of \(C^\infty\) functions on \(M\) with compact support).

**Lemma 1.1.** Let \(\varphi(x)\) be a real-valued \(C^m\) function on \(\mathbb{R}^n\), and let \(\rho(x)\) be a \(C^m\) function on \(\mathbb{R}^n\) with compact support. Then the function
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$g(s) \equiv \int_{E(s)} \rho(x)dx$

(where $E(s) = \{ x : \varphi(x) \leq s \}$) satisfies

(i) $g(s) = 0$ if $s < \min_{x \in \text{supp}[\rho]} \varphi(x)$,

(ii) $g(s)$ is constant if $s > \max_{x \in \text{supp}[\rho]} \varphi(x)$,

(iii) $g(s) \in H^m_{\text{loc}}(\mathbb{R}^d)$ for any $m < 2^{-1}$.

Proof. Set

$H(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases}$

Then it follows that $H(s) \in H^m_{\text{loc}}(\mathbb{R}^d)$ for any $m < 2^{-1}$, and so $H(s - \varphi(x))$ becomes a $H^m_{\text{loc}}(\mathbb{R}^d)$-valued continuous function on $\mathbb{R}^d$. Therefore, noting that $g(s) = \int_{\mathbb{R}^d} \rho(x)H(s - \varphi(x))dx$, we obtain (iii). If $s < \min_{x \in \text{supp}[\rho]} \varphi(x)$ we have $E(s) \cap \text{supp}[\rho] = \emptyset$, which proves (i). If $s > \max_{x \in \text{supp}[\rho]} \varphi(x)$, $E(s)$ contains $\text{supp}[\rho]$, which yields (ii). The proof is complete.

Proof of Theorem 1. It follows from (iii) of Lemma 1.1 that the function $g_j(s)$ defined in (0.1) belongs to $L^2_{\text{loc}}(\mathbb{R}^d)$. Therefore we have

$\int_{t_0}^{t} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t)dt \in H^j_{\text{loc}}(\mathbb{R}^d) \quad (j \geq 1),$

$\partial_t^j \int_{t_0}^{t} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t)dt = g_j(s).$

Hence the function $\bar{g}_N(s) = g_0(s) + \sum_{j=1}^{N} \int_{t_0}^{t} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t)dt$ satisfies

(1.1) $\partial_t^j \bar{g}_N(s) = \sum_{j=0}^{N} \partial_t^j g_j(s).$

We define $\bar{I}(\sigma)$ by

$\bar{I}(\sigma) = \begin{cases} I(\sigma) & \text{for } \sigma > 0, \\ \frac{I(-\sigma)}{I(-\sigma)} & \text{for } \sigma < 0. \end{cases}$

Then $\sigma^n \bar{I}(\sigma) \in L^2(1, \infty)$ if and only if $(1 + |\sigma|)^n \bar{I}(\sigma) \in L^2(\mathbb{R}^d)$. Furthermore, since $\rho_j(x)$ are assumed real-valued, it follows that for any integer $N \geq 0$

(1.2) $\bar{I}(\sigma) = \sum_{j=0}^{N} \int_{\mathbb{R}^d} e^{-i\sigma x} \rho_j(x)dx(i\sigma)^{-1} + 0(|\sigma|^{-N-1}).$

Here $0(|\sigma|^k)$ means that $|0(|\sigma|^k)| \leq C |\sigma|^k (|\sigma| \geq 1)$ for some constant $C$ independent of $\sigma$. 
Noting that \( \delta(s-\varphi(x)) \) is a \( H^n(R^1) \)-valued continuous function of \( x (m< -2^{-i}) \) and equal to \( \partial_s H(s-\varphi(x)) \), we obtain
\[
e^{-i\sigma\varphi(x)} = \int e^{-i\sigma\varphi(x)} ds = F[\partial_s H(s-\varphi(x))] (\sigma),
\]
where \( F \) is the Fourier transformation in \( s \) (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum \( \int_{R^n} e^{-i\sigma\varphi(x)} \rho_j(x) dx \) in the following way:
\[
\int_{R^n} e^{-i\sigma\varphi(x)} \rho_j(x) dx = F[\int_{R^n} H(s-\varphi(x)) \rho_j(x) dx] (\sigma) = F[\partial_s \tilde{g}_N(s)] (\sigma).
\]
(1.1), (1.2) and (1.3) yield that
\[
(i\sigma)^{N-1} \tilde{I}(\sigma) = F[\partial_s \tilde{g}_N(s)] (\sigma) + O(1)^{N-2}).
\]
Let \( (1+|\sigma|)^{N-1} \tilde{I}(\sigma) \in L^2(R^d) \) for every \( m \in R^1 \). Then it follows from (1.4) that
\[
\partial_s \tilde{g}_N(s) \in H^1(R^1),
\]
which implies
\[
\tilde{g}_N(s) \in C^N(R^1).
\]
Conversely, let \( \tilde{g}_N(s) \in C^N \) for every non-negative integer \( N \). Then we have \( \partial_s^{N+1} \tilde{g}_N(s) \in H^{-1}(R^1) \), which means that \( \partial_s^{N+1} \tilde{g}_N(s) \in H^{-1}(R^1) \) since \( \partial_s^{N+1} \tilde{g}_N(s) = 0 \) for large \( |s| \) (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain
\[
(1+|\sigma|)^{N-1} \tilde{I}(\sigma) \in L^2(R^d) \] for every integer \( N(\geq 1) \). This shows that
\[
(1+|\sigma|)^m \tilde{I}(\sigma) \in L^2(R^d) \] for every \( m \in R^1 \).
The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that \( s_0 = \min_{x \in D} \varphi(x) = 0 \). Since \( \max_{0 \leq t \leq \epsilon} |g_\epsilon(t)| \leq |E(s)| \max_{x \in D} |\rho_j(x)| \), there is a constant \( C \) independent of \( s \) such that
\[
\int_0^\epsilon (s-t)^{j-1} g_\epsilon(t) dt \leq C |s|^j |E(s)| \] for \( j \geq 1 \).
Therefore we have
\[
|\tilde{g}_N(s)| \geq \frac{1}{N!} \sum_{j=1}^N (s-t)^{j-1} g_\epsilon(t) dt \geq (\min_{x \in D} \rho_j(x) - C \sum_{j=1}^N |s|^j |E(s)|).
Since $\min_{x \in \mathbb{R}^d} \rho_0(x) > 0$, we obtain $\min_{x \in \mathbb{R}^d} \rho_0(x) \geq 2\delta$ for a constant $\delta > 0$ independent of $s$ if $|s|$ is small enough. Therefore, if $|s|$ is small enough, it follows that

$$|\tilde{g}_N(s)| \geq \delta |E(s)|.$$  

Take a point $x_0$ satisfying $\varphi(x_0) = 0 (= \min_{x \in \mathbb{R}^d} \varphi(x))$. Then there is a constant $d > 0$ such that

$$E(s) \supseteq \bar{E}(s) = \{ x : d|x-x_0| \leq s \},$$

which yields $|E(s)| \geq |\bar{E}(s)| = \delta's^n$ for $s \geq 0$ (the constant $\delta'$ does not depend on $s$). Thus, for any sufficiently small $s \geq 0$ we have

$$(1.5) \quad |\tilde{g}_N(s)| \geq \delta's^n.$$  

Now, assume that $\sigma^m I(\sigma) \in L^2(1, \infty)$ for every $m \in \mathbb{R}$. Then it follows from Theorem 1 that $\tilde{g}_N(s) \in C^N$ for any integer $N \geq 0$. Take the $N$ so that $N \geq n + 1$. All the derivatives $g_N(0), \partial_t g_N(0), \cdots, \partial^{n}_t g_N(0)$ vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

$$|\tilde{g}_N(s)| \leq C |s|^{n+1}.$$  

This is not consistant with (1.5) as $s \to +0$. Therefore we have

$$\sigma^m I(\sigma) \notin L^2(1, \infty)$$

for some constant $m \in \mathbb{R}$ depending only on $n$.

### 2. Proof of Theorem 3

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let $v(t, x; \omega)$ be the solution of the equation

$$(2.1) \quad \begin{cases} 
\square v(t, x) = 0 & \text{in } \mathbb{R}^1 \times \Omega, \\
v(t, x') = -2^{-1}(-2\pi i)^{1-n} \delta(t-x' \cdot \omega) & \text{on } \mathbb{R}^1 \times \partial \Omega, \\
v(t, x) = 0 & \text{for } t < r(\omega).
\end{cases}$$

Then $v(t, x; \omega)$ is a $C^m$ function of $x$ and $\omega$ with the value $S'(\mathbb{R}_1)$.

**Proposition 2.1.** $S(s, \theta, \omega)$ is represented of the form

$$S(s, \theta, \omega) = \int_{\partial \Omega} \{ \partial_t^2 - \partial_x v(x \cdot \theta - s, x; \omega) - v \cdot \partial_t v(x \cdot \theta - s, x; \omega) \} \cdot dS_x \quad (\omega \neq \theta),$$

where $v$ is the outer unit vector normal to $\partial \Omega$ (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral $\int \cdot dS_x$ is in the sense of the Riemann
integral with the value \( S'(R^1) \). For the proof see Majda [8] and the author [14].

It is seen that the wave front set of \( \delta(t-x_{\ast}^{\omega}) \mid R^1 \times \partial \Omega \) is non-glancing in \( \{(t,x): -r(\omega-\theta)-2\eta \leq x \cdot \theta - t \} \cap (R^1 \times \partial \Omega) (\omega \neq \theta) \) if \( \eta (\geq 0) \) is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution \( v(t,x; \omega) \) of (2.1) mod \( C^\infty \) by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about \( \partial_v(v \mid R^1 \times \partial \Omega) \). This is indicated by Majda [8] in the case of \( \theta = -\omega \) (cf. Lemma 2.1 of [8]). We have

**Lemma 2.2.** There exists a first order pseudo-differential operator \( B \) on \( R^1 \times \partial \Omega \) independent of \( t \) such that

1. its symbol \( B(\xi'; \tau, \xi') \) represented near

\[
N(\omega-\theta) = \{ x: x \cdot (\theta-\omega) = r(\omega-\theta) \} \cap \partial \Omega
\]

by local coordinates \( (t, \xi') \), has a homogeneous asymptotic expansion \( \sum_{j=0}^\infty B_j(\xi'; \tau, \xi') \) satisfying

\[
-ib_\omega(\xi'; \pm 1, \mp \bar{\theta}') > 0 \text{ on } N(\omega-\theta) (\bar{\theta}' \text{ is the tangential component of } \theta \text{ to the plane } \{ x: x \cdot (\omega-\theta) = r(\omega-\theta) \}),
\]

\[
B_j(\xi'; \tau, \xi') \text{ are purely imaginary-valued for even } j \text{ and real-valued for odd } j,
\]

2. \( \partial_v(v \mid R^1 \times \partial \Omega) \) is equal to \( B(v \mid R^1 \times \partial \Omega) \) mod \( C^\infty \) in \( \{(t,x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \cap R^1 \times \partial \Omega \) for some small constant \( \eta > 0 \).

In the above lemma, “a homogeneous asymptotic expansion \( \sum_{j=0}^\infty B_j(\xi'; \tau, \xi') \)” means that \( B_j(\xi'; \mu \tau, \mu \xi') = \mu^{-j} B_j(\xi'; \tau, \xi') \) for \( \mu \geq 1, |\tau| + |\xi'| \geq 1 \) and that \( |B(\xi'; \tau, \xi') - \sum_{j=1}^N B_j(\xi'; \tau, \xi')| \leq C_N (|\tau| + |\xi'| + 1)^{-N-1} \) for any non-negative integer \( N \) (for detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.); (ii) in the lemma states that \( \alpha(t,x') (\partial_v \mid R^1 \times \partial \Omega - B(v \mid R^1 \times \partial \Omega)) \in C^\infty \) for any \( \alpha(t,x') \in C^\infty (R^1 \times \partial \Omega) \) with \( \text{supp } [\alpha] \subset \{(t,x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \).

**Proof of Lemma 2.2.** Let \( \sum_{i=1}^J \chi_i(x) \) be a partition of unity on a neighborhood of \( N(\omega-\theta) \) satisfying max \( \text{supp } [\chi_i] \leq \varepsilon_0 \) (\( \varepsilon_0 \) is a sufficiently small positive constant). Then there is a constant \( \varepsilon_1 > 0 \) such that \( \sum_{i=1}^J \chi_i(x) = 1 \) for any \( x \in \partial \Omega \) satisfying \( -r(\omega-\theta)-\varepsilon_1 \leq x \cdot \theta - x_{\ast}^{\omega} \). Let \( v_i(t,x) \) be the solution of the equation

\[
\begin{align*}
\Box v_i(t,x) &= 0 \quad \text{in } R^1 \times \Omega, \\
v_i(t,x') &= \chi_i(x') v(t, x'; \omega) \quad \text{on } R^1 \times \partial \Omega, \\
v_i(t,x) &= 0 \quad \text{for } t < r(\omega).
\end{align*}
\]
Since \( \text{supp}[v | R^\times \Omega] \subset \{(t, x') : x' \cdot \omega = t\} \), \( \sum_{i=1}^{l} v_i(t, x') \) is equal to \( v(t, x'; \omega) \) on \((R^\times \Omega) \times \partial \Omega \cap \{(t, x') : -r(\omega - \theta) - \varepsilon_i \leq x' \cdot \theta - t\} \), and so, noting that the propagation speed is less than one, we have

\[
v(t, x; \omega) = \sum_{i=1}^{l} v_i(t, x) \quad \text{in} \quad (R^\times \Omega) \cap \{(t, x) : -r(\omega - \theta) - \varepsilon_i \leq x \cdot \theta - t\}.
\]

We denote by \( \text{WF}[f(t, x)] \) the wave front set of \( f(t, x) \). It is seen that \( \text{WF}[v | R^\times \Omega] = \text{WF}[\delta(x' \cdot \omega - t) | R^\times \Omega] = \{(t, x') : (t, x') \in R^3 \times \partial \Omega, x' \cdot \omega - t = 0, \xi' = -\tau(\omega - (\omega \cdot v)v), \tau \neq 0\} \) (\( v \) is the outer unit normal to \( \partial \Omega \)). Hence, for any \((t, x'; \tau, \xi') \in \text{WF}[v_i | R^\times \Omega] \) the equation \( \tau^2 - |\xi' + \lambda v|^2 = 0 \) in \( \lambda \) has real roots, and the null-bicharacteristics associated with \( \partial^\tau - \Delta \) through \( \text{WF}[v_i | R^\times \Omega] \) are transversal to \( R^3 \times \partial \Omega \) (non-glancing). This implies that \( \text{sing supf}[\delta v_i | R^\times \Omega] \subset \text{sing supf}[v_i | R^\times \Omega] \) (cf. Theorem 7 in §9 of (Lax and Nirenberg [10])), and so it suffices to examine \( v_i(t, x) \) only in a neighborhood \((t_i - \varepsilon_0, t_i + \varepsilon_0) \times U_i \) of \((t_i, x_i) \) \((x' \in \text{supp}[\chi_i] \cap N(\omega - \theta) \) and \( t_i = x' \cdot \omega) \).

To analyze \( v_i \) more precisely, we transform \( \Omega \) in \( R^3 \) into the half-space \( \Omega^+ = \{x = (x', \xi) : \xi > 0\} \). Let the derivative \( \partial_\theta \) be transformed in \( U_i \) into \(-\partial_{\xi'}\). For any set \( M \) in \( R^3 \) we denote by \( M^+ \) the set transformed by the coordinates \( \xi \). Let \(-\Delta_\theta \) be represented by \( \xi \) of the form \( \Delta_\theta = \sum |a_\alpha(\xi)\xi^\alpha| \). Here we can assume that the coefficients \( a_\alpha(\xi) \) are real-valued \( C^\infty \) functions defined on \( R^3 \) and constant out of \( U_i \). Let us examine the solution \( \theta(t, \xi) \) of the following equation instead of \( v_i(t, x) \):

\[
\begin{cases}
(\partial^\tau + A) \theta(t, \xi) = 0 & \text{in} \quad R^3 \times \Omega^+,

\theta(t, \xi') = g(t, \xi') & \text{on} \quad R^3 \times \Omega^{+,-},

\theta(t, \xi) = 0 & \text{for} \quad t < t_i - \varepsilon_0,
\end{cases}
\]

where \( g(t, \xi') = -2^{-\frac{1}{2}}(-2\pi i)^{-n} \delta(x(\xi') \cdot \omega - t) \chi_i(x(\xi')) \). Note that \( \text{WF}[g(t, x')] \) is contained in a sufficiently small conic neighborhood of \((t_i, x'; \pm 1, \mp \theta') \) \( \theta' \) is the component of \( \theta \) (transformed by the coordinates \( \xi \)) tangent to the plane \( \xi_0 = 0 \), and that if \( |(\tau, \xi')|^{-1}|(\tau, \xi')|^{-1}(\pm 1, \mp \theta') \) the equation

\[
(\Delta_\theta(\xi, \xi') - \sum_{|\alpha| = 2} a_\alpha(\xi)\xi^\alpha) \xi_0 = 0
\]

(2.4) \( \xi_j(\xi; \tau, \xi') \) in \( \xi_0 \) has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators \( \xi_j(\xi; D_\tau, D_\xi) \) on \( R^3 \times \Omega^+ \) (independent of \( t \)) with homogeneous asymptotic expansions \( \sum_{j=0}^\infty \xi_j(\xi; \tau, \xi') \) such that

\[
(i) \quad \xi_j(\xi; \tau, \xi') \quad \text{are real-valued for even} \quad j \quad \text{and purely imaginary-valued for odd} \quad j.
\]

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(ii) if \(((\tau, \xi'))^{-1}(\tau, \xi')\) is near \((-1, \bar{\theta}')\) or \((1, -\bar{\theta}')\), \(\xi_0^\pm(x; \tau, \xi')\) are equal to the roots of the equation (2.4), and
\[
\xi_0^\pm(x; \pm 1, \mp \bar{\theta}') = \mp(1 - |\bar{\theta}'|)^{1/2},
\]

(iii) all the null-bicharacteristic curves associated with \(D_{\bar{\xi}_0-\xi_0^\pm}(x; D_t, D_{x'})\) through \(\text{WF}[g(t, x')]\) are transversal to the boundary \(\{x_0=0\}\) and proceed in the direction \(t > 0\) as they leave the boundary,

(iv) if the wave front set of \(u(t, x)\) is near the bicharacteristic curves stated in the above (iii), then we have
\[
(D_{\bar{\xi}_0-\xi_0^-(x; D_t, D_{x'})) (D_{\bar{\xi}_0-\xi_0^+(x; D_t, D_{x'}) u = \zeta(x) (\bar{\partial}_t + \bar{A}) u \mod C^\infty,
\]
where \(\zeta(x)\) is a \(C^\infty\) function on \(\mathbb{R}^n\) satisfying \(\zeta(x) < 0\) for every \(x\).

(iii) and (iv) imply that \(d(t, x', x_0)\) is approximated mod \(C^\infty\) by the solution \(w(x_0; t, x')\) of the equation
\[
(D_{\bar{\xi}_0-\xi_0^-(x; D_t, D_{x'})) w = 0, \quad x_0 > 0,
\]
\[
|w|_{\bar{\xi}_0=0} = h(t, x')
\]
Therefore we have
\[
-\partial_{x_0} d_{\bar{\xi}_0=0} = -i\xi_0^+(x', 0; D_t, D_{x'} \partial_{\bar{\xi}_0=0} \mod C^\infty.
\]
Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution \(v(t, x; \omega)\) in (2.1) satisfies \(\text{supp}[v|_{x'=\omega}] \subset \{(t, x): x \cdot \omega = t\}\). Therefore, noting that the propagation speed is less than one, we see that \(\text{supp}[v(t, x; \omega)] \subset \{(t, x): x \cdot \omega \leq t\}\), which yields
\[
v(x \cdot \theta - s, x; \omega) = 0 \quad \text{if } s > x \cdot (\theta - \omega).
\]
Hence, if \(s > \max x \cdot (\theta - \omega) = -r(\omega - \theta) (\omega \pm \theta)\), we obtain \(S(s, \theta, \omega) = 0\) from Proposition 2.1.

Next, let us prove that \(S(s, \theta, \omega)\) is singular at \(s = -r(\omega - \theta)\). Take \(\alpha(s) \in C^\infty(\mathbb{R}^1)\) such that \(0 \leq \alpha \leq 1\) on \(\mathbb{R}^1\), \(\alpha(s) = 1\) for \(|s| \leq 2^{-1}\) and \(\alpha(s) = 0\) for \(|s| \geq 1\). For any \(\varepsilon > 0\) set
\[
\alpha_\varepsilon(s) = \alpha\left(\frac{s + r(\omega - \theta)}{2\varepsilon}\right).
\]
Then we have only to prove that \(\alpha_\varepsilon(s) S(s, \theta, \omega)\) is not \(C^\infty\) for any small \(\varepsilon > 0\). Proposition 2.1 yields
\[
\alpha_\varepsilon(s) S(s, \theta, \omega) = \int_{\partial \Omega} \alpha_\varepsilon(s) (\bar{\partial}_t \bar{\partial}_v) (x \cdot \theta - s, x; \omega) dS_x,
\]
\[
-\int_{\partial \Omega} v \cdot \bar{\partial} \alpha_\varepsilon(s) (\bar{\partial}_t \bar{\partial}_v) (x \cdot \theta - s, x; \omega) dS_x \equiv \int_1 + \int_2(s).
\]
Let $F[k(s)](\sigma) = \int e^{ixs}k(s)ds$. As is readily seen, it follows that
\begin{equation}
F[J_2(s)](\sigma) = -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} C_j^{-1}(i\sigma)^{n-1-j} \int_{\Omega} e^{i\sigma x \cdot (\theta - \omega)}(-v \cdot \theta) \cdot \alpha_{x}^{(j)}(x \cdot (\theta - \omega))ds_x
\end{equation}
(where $C_j^{-1} = (n-1)!(n-1-j)!j!$). Taking the $\varepsilon(>0)$ so that $2\varepsilon \leq \eta$, by Lemma 2.2 we have
\begin{align}
F[J_1(s)](\sigma) &= \int_{\mathbb{R}^1 \times \Omega} e^{i\sigma x \cdot (\theta - \omega)} \alpha_{x}(x \cdot \theta - s) \partial^{n-2}_{x} [B^\dagger|_{\mathbb{R}^1 \times \Omega}](s, x)dsdx_x
\end{align}
\begin{align}
&= -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-2} C_j^{-2} \int_{\Omega} \left| B^\dagger e^{i\sigma x \cdot (\theta - \omega)} \alpha_{x}^{(j)}(x \cdot \theta - s) \right|_{x = s, \omega} ds_x
\end{align}
\begin{align}
&\cdot (i\sigma)^{n-2-j}.
\end{align}
Here $B^\dagger$ denotes the transposed operator of $B$ (i.e. $\langle Bf, g \rangle = \langle f, Bg \rangle$ for any $f$ and $g \in C_0^\infty(\mathbb{R}^1 \times \partial \Omega)$). Let us note that the symbol of $B^\dagger$ expressed near supp $[\alpha_{x}(x \cdot \theta - t)] \cap (\mathbb{R}^1 \times \partial \Omega)$ by the local coordinates $(t, \xi)$, has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} B_j(\xi'; \tau, \xi')$ such that $B_j(\xi'; \tau, \xi')$ are real-valued for odd $j$ and purely imaginary valued for even $j$ and that $-i B_0(\xi'; t, \pm \varepsilon) = -i B_0(\xi'; t, \pm \varepsilon) = 0$ for $\xi' \in \mathcal{N}(\omega - \theta)$, which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand $B^\dagger e^{i\sigma x \cdot (\theta - \omega)} \alpha_{x}^{(j)}(x \cdot \theta - s)$ asymptotically (as $\sigma \to \infty$) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion
\begin{equation}
F[J_1(s)](\sigma) \sim -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} (i\sigma)^{n-1-j} \int_{\Omega} e^{i\sigma x \cdot (\theta - \omega)} \beta_{x}(x)ds_x
\end{equation}
(\text{as } \sigma \to \infty),
where $\beta_{x}(x)$ are real-valued $C^\infty$ functions on $\partial \Omega$ with supp $[\beta_{x}] \subset \text{supp} [\alpha_{x}(x \cdot (\theta - \omega)) \cap \partial \Omega$, and $\beta_{x}(x)$ is non-negative valued and satisfies
$$
\beta_{x}(x) = -i B_{0}(\xi'; -1, \pm \varepsilon) \alpha_{x}(x \cdot (\theta - \omega)) > 0 \quad \text{for } x \in \mathcal{N}(\omega - \theta).
$$
Combining (2.5) and (2.6) yields that for any integer $N(>0)$
\begin{equation}
F[\alpha_{x}(s)S(s, \theta, \omega)](\sigma) = -2^{-1}(-2\pi i)^{1-n} (i\sigma)^{n-1} \int_{R^{n-1}} e^{-i\sigma \xi (\omega' - \omega)} \cdot \left\{ \sum_{j=0}^{n-1} \rho_{x}(\xi') (i\sigma)^{-j} d\xi' + o(\sigma^{-N}) \right\}
\end{equation}
Here $\xi'$ is the local coordinates on $\partial \Omega$ near $\mathcal{N}(\omega - \theta)$ and
$$
\rho_{x}(\xi') = \beta_{x}(x(\xi')) + (-v \cdot \theta) \alpha_{x}^{(j)}(x(\xi') \cdot (\theta - \omega)) \quad (\alpha_{x}^{(j)} = 0, j \geq n).
$$
Noting that $\rho_{0}(\xi') > 0$ when the phase function $\chi(\xi') \cdot (\omega - \theta)$ is minimum, and applying Theorem 2, we obtain for some constant $m \in \mathbb{R}$
\begin{equation}
\sigma^m F[\alpha_{x}(s)S(s, \theta, \omega)](\sigma) \in L^1(1, \infty),
\end{equation}
which shows that $\alpha_s(s, \theta, \omega)$ is not $C^\infty$. The proof is complete.

3. Proof of Theorem 4

We use the same notations as for the scattering by obstacles in §2. The scattering operator $S$ for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

**Proposition 3.1.** Set

$$S_0(s, \theta, \omega) = \int_{\mathbb{R}^n} (\partial_t^2 - A)\omega(t, x) \, (s \cdot \theta - s, x) \, dx,$$

$$Kk = F^{-1}[(\text{sgn } \sigma)^{\nu-1}(Fk)(\sigma)],$$

where $\omega(t, x)$ is the solution of the equation

$$\begin{cases}
(\partial_t^2 - A)\omega(t, x) = 0 \\ (A\omega = \sum_{i,j=1}^n \partial_{x_i} (a_{ij} \partial_{x_j} \omega))
\end{cases} \text{ in } \mathbb{R}^1 \times \mathbb{R}^n,$$

$$\begin{cases}
\omega(-r_0, x) = -2^{-4}(\gamma(x)^{-1/2})^{-n}\delta(-r_0 - s \cdot \omega) \\ \partial_t \omega(-r_0, x) = -2^{-4}(\gamma(x)^{-1/2})^{-n}\delta'(-r_0 - s \cdot \omega)
\end{cases} \text{ on } \mathbb{R}^n.$$

Then we have

$$(Sk)(s, \theta) = \int S_0(s-t, \theta, \omega)k(t, \omega) dt d\omega + (Kk)(s, \theta).$$

Note that $S_0(s, \theta, \omega) = S(s, \theta, \omega)$ if $\omega \equiv \theta$.

To prove Theorem 4, we have only to show that for any small $\varepsilon(>0)$ there exist a real number $m$ and a function $\rho(s) \in C^\infty_0(s_\omega(\theta) - 2\varepsilon, s_\omega(\theta) + 2\varepsilon)$ such that

$$(1 + |\sigma|)^m F[\rho(s)S(s, \theta, \omega)](\sigma) \in L^2(\mathbb{R}^1).$$

Let $\gamma(x) \in C^\infty_0(\mathbb{R}^n)$ with $\gamma(x) = 1$ in a neighborhood of $\bar{M}_\omega(\theta)$, and denote by $\tilde{\omega}(t, x)$ the solution of the equation

$$\begin{cases}
(\partial_t^2 - A)\tilde{\omega}(t, x) = 0 \\ \tilde{\omega}(-r_0, x) = \gamma(x)\omega(-r_0, x) \\ \partial_t \tilde{\omega}(-r_0, x) = \gamma(x)\partial_t \omega(-r_0, x)
\end{cases} \text{ in } \mathbb{R}^1 \times \mathbb{R}^n,$$

$$\text{on } \mathbb{R}^n.$$

The author [15] showed that if $\tilde{I}$ is large enough we have for any integer $N(>0)$

$$F[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{n-1} e^{-i\sigma\tilde{I}} \sum_{j=0}^{N-1} (i\sigma)^{n-1-j} F'[\beta_j(x) \{\tilde{\omega}(\tilde{I}, x) \\ + (i\sigma)^{-1}\partial_{\tilde{I}} \tilde{\omega}(\tilde{I}, x)\}](\sigma - \sigma \theta) + 0(\sigma^{-N+N_0})$$

as $\sigma \to \infty$ ($N_0$ is an integer independent of $N$) (cf. (4.5) in [15]). Here, $F'$ denotes the Fourier transformation in $x$, and the functions $\beta_j(x) \in C^\infty(\mathbb{R}^n)$ are all real-valued.
We take $\tilde{t}$ so large as to have (i) and (ii) stated in the following

**Lemma 3.2.** Let $r_1$ be an arbitrary constant ($\geq r_0$), and set

$$
\psi(t; \tilde{t}) = q^-(t; -r_0, x, \omega) \cdot \theta.
$$

Then, for any $\varepsilon(>0)$ there is a constant $\tilde{t}_0$ such that for any fixed $\tilde{t} \geq \tilde{t}_0$

1. $\max_{|x| \leq r_1} \psi(t; \tilde{t}) \leq \varepsilon$,\[ (i) \]
2. all points at which $\psi(t; \tilde{t})$ is maximum $(x \cdot \omega = -r_0, |x| \leq r_1)$, are contained in $\varepsilon$-neighborhood $(\bar{W}_\varepsilon(\tilde{t}))$ of $\bar{W}_\varepsilon(t)$ ($(\bar{W}_\varepsilon) = \{x: \text{dis}(x, \bar{M}) < \varepsilon\}$).

This lemma will be proved later. Choose the $p(s)$ so that $p(s) = 0$ on $\mathbb{R}$ and $p(s) > 0$ on $[s_\omega(\theta) - \varepsilon, s_\omega(\theta) + \varepsilon]$.

Then it is seen from the form of $\beta_0(x)$ (cf. (4.4) and (4.6) in [15]) and the above lemma that

$$(3.1) \quad \beta_0(x) \geq 0 \text{ on } \mathbb{R}^n \text{ and } \beta_0(q^-(\tilde{t}; -r_0, y, \omega)) > 0$$

for any $y \in (\bar{W}_\varepsilon(\tilde{t}))$, $y \cdot \omega = -r_0$.

We take the $\gamma(x)$ so that $\gamma(x) \geq 0$ on $\mathbb{R}^n$, $\gamma(x) > 0$ on $(\bar{W}_\varepsilon(\tilde{t}))$, and $\supp[\gamma] \subset (\bar{W}_\varepsilon(\tilde{t}))_{2\varepsilon}$.

By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. S$§$ 5 of [10] or Appendix II of [5]), we can construct a symbol $\lambda(x, \xi)$ with a homogeneous asymptotic expansion $\sum_0^\infty \lambda_j(x, \xi)$ such that

$$
\lambda_0(x, \xi) = \left\{ \lambda(x, \xi) \right\}, \quad \lambda_0 = \left( D_{1} + \lambda(x, D_2) \right) (D_{1} - \lambda(x, D_2)) \text{ modulo a smoothing operator}
$$

(cf. Corollary 2.5 in the author [15] also). Furthermore we see that $\lambda_j(x, \xi)$ are real-valued for $\lambda$ even and purely imaginary valued for $\lambda$ odd since the coefficients $a_{ij}(x)$ are all real-valued (recall the construction of $\xi \pm (\tau, \xi')$ in $\S 2$). Consider the Cauchy problem

$$
\begin{cases}
(D_1 - \lambda(x, D_2)) u(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
u|_{\tilde{t} = 0} = u_0(x) & \text{on } \mathbb{R}^n,
\end{cases}
$$

and denote by $E(t)$ the operator: $u_0 \rightarrow u(t, \cdot)$. Then $\bar{\omega}(\tilde{t}, x)$ and $\partial_t \bar{\omega}(\tilde{t}, x)$ are represented as follows:

$$
\bar{\omega}(\tilde{t}, x) = 2^{-1}E(\tilde{t} + r_0) \left( \bar{\omega}(\tilde{t} + r_0, \cdot) + i\bar{\omega} \partial_t \bar{\omega}(\tilde{t} + r_0, \cdot) \right)(x)
$$

$$
+ 2^{-1}E(-\tilde{t} + r_0) \left( \bar{\omega}(\tilde{t} - r_0, \cdot) + i\bar{\omega} \partial_t \bar{\omega}(\tilde{t} - r_0, \cdot) \right)(x),
$$

$$
\partial_t \bar{\omega}(\tilde{t}, x) = 2^{-1}E(\tilde{t} + r_0) i\lambda(\bar{\omega}(\tilde{t} + r_0, \cdot) + i\bar{\omega} \partial_t \bar{\omega}(\tilde{t} + r_0, \cdot))(x)
$$

$$
+ 2^{-1}E(-\tilde{t} + r_0) i\lambda(\bar{\omega}(\tilde{t} - r_0, \cdot) + i\bar{\omega} \partial_t \bar{\omega}(\tilde{t} - r_0, \cdot))(x),
$$

where $\lambda$ and $\bar{\mu}$ are pseudo-differential operators whose symbols coincide with
\(\lambda(x, \xi)\) and \(\mu(x, \xi)\) \((\mu(x, D_x)\) is the parametrix of \(\lambda(x, D_x)\) respectively in a neighborhood of \(\text{supp}[\gamma(x)]\) and vanish for large \(|x|\). Therefore, noting that

\[
\mathcal{F}'[\beta, E(-i - r_0) (\bar{w}(-r_0, \cdot) + i \bar{\mu} \partial_x \bar{w}(-r_0, \cdot))] (-\sigma \theta) = 0 (\sigma^{-\infty}),
\]

\[
\mathcal{F}'[\beta, E(-i - r_0) \lambda(\bar{w}(-r_0, \cdot) + i \bar{\mu} \partial_x \bar{w}(-r_0, \cdot))] (-\sigma \theta) = 0 (\sigma^{-\infty})
\]
as \(\sigma \to \infty\) (cf. §4 of the author [15]), we have

\[
F[\rho(s) S(s, \theta, \omega)](\sigma) = 2^{-1} e^{-i \sigma \gamma} \sum_{j=0}^{\infty} (i \sigma)^{s-1} \mathcal{F}'[2^{-1} \beta, E(i + r_0) (1 + \sigma^{-1} \lambda)]
\]

\[
\cdot (\bar{w}(-r_0, \cdot) - i \bar{\mu} \partial_x \bar{w}(-r_0, \cdot)) (-\sigma \theta) + 0 (\sigma^{-N+N_0}).
\]

The assumption (0.5) implies that if \(\text{WF}[u_0]\) is contained in a conic neighborhood of \(\tilde{M}_\omega(\theta) \times \{\omega\}\) \((\text{WF}[\bar{w}(-r_0, \cdot) - i \bar{\mu} \partial_x \bar{w}(-r_0, \cdot)]\) is contained there) \(E(i + r_0) u_0\) is represented by the Fourier integral operator:

\[
E(i + r_0) u_0(x) = (2\pi)^{-1} \int e^{i \bar{\phi}(i + r_0, \xi)} a(i + r_0, x, \xi) \hat{u}_0(\xi) d\xi \mod C^\infty
\]

(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that \(\mathcal{F}'[\delta(\cdot) (-r_0 - x \cdot \omega)](B\eta) = (-i \eta \gamma) e^{i \eta \gamma} \delta(\cdot) (\eta = (\eta_1, \eta'))\), where \(B = (b_1, \ldots, b_n)\) is an orthogonal matrix with \(b_1 = \omega\). Then, introducing change of the variables \(x = q(\tilde{t} - r_0, y, \omega) = q(y)\) near \(x = q(\tilde{t} - r_0, M_\omega(\theta), \omega) = (y_0, y')\) is orthogonal coordinates with \(y_0 = x \cdot \omega\), we obtain

\[
\mathcal{F}'[2^{-1} \beta, E(i + r_0) (1 + \sigma^{-1} \lambda)] (i \bar{\mu} \partial_x \bar{w}(-r_0, \cdot)) (-\sigma \theta)
\]

\[
= \int e^{i \sigma \gamma} \gamma(x) \beta_j(x) \int_{-\infty}^{\infty} e^{i \bar{\phi}(i + r_0, x, t)} a(i + r_0, x, -\tau \omega) e^{-i \tau \omega} d\tau dx + 0 (\sigma^{-\infty})
\]

\[
= \int_{R^n} d y \int_{-\infty}^{\infty} dy_0 \int_{-\infty}^{\infty} d \tau e^{i \sigma \gamma} \cdot (x, y, \gamma(y)(y_0, y') \beta_j(q(y)) \gamma(y)
\]

\[
\cdot a(i + r_0, q(y), -\sigma \tau \omega) \cdot \det \frac{\partial q(y)}{\partial y} + 0 (\sigma^{-\infty}) \quad (\text{as } \sigma \to \infty)
\]

\((\gamma(x) \in C^\infty(R^n), \gamma(x) = 1)\) on a neighborhood of \(q^- (\text{supp}[\gamma])\), and \(\tau\) is a positive constant independent of \(\sigma\). The function \(\Phi(y_0, \tau) = q^- (y_0, y') \cdot \theta - \tau (y_0 + r_0)\) has the stationary point \((y_0, \tau) = (-r_0, p^- (r_0, y') \cdot \theta), \) at which its Hesse matrix equals

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

Expanding \(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \sigma \gamma} \beta_j y \cdot \omega d \tau d y\) (as \(\sigma \to \infty\)) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion

(3.2) \[
F[\rho(s) S(s, \theta, \omega)](\sigma) = e^{-i \sigma \gamma} (i \sigma)^{s-1} \int_{\Sigma \omega = -\gamma_0} e^{i \sigma \gamma(t - r_0, \xi, \omega)} \cdot \{ \sum_{j=0}^{N_0} \rho_j(x) (i \sigma)^{-1} \}
\]

\((N_0\) is an integer independent of \(N = 1, 2, \cdots\). Here \(\rho_j\) are \(C^\infty\) functions with
supp[\rho] \subset \text{supp}[\gamma] and all real-valued, which follows from the fact that the symbol $a(t, x, \xi)$ has a homogeneous asymptotic expansion $\sum_{k=0}^{\infty} a_k(t, x, \xi)$ such that $a_k(t, x, \xi)$ are real-valued for even $k$ and purely imaginary valued for odd $k$; furthermore $\rho_0$ is of the form

$$\rho_0(y) = \gamma(\eta, \beta) q^{-}(t; -r_0, y, \omega)a_0(t + r_0, q^{-}(t; -r_0, y, \omega), -\omega) \det \frac{\partial q^{-}}{\partial y}.$$ \hspace{1cm}

Combining this with (3.1) and (ii) of Lemma 3.2, we see that $\rho_0(x) \geq 0$ on $\mathbb{R}^n$ and $\rho_0(x) > 0$ for any $x$ at which the function

$$\varphi(x) = -q^{-}(t; -r_0, x, \omega) \cdot \theta \ (x \cdot \omega = -r_0)$$

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

$$\sigma^m F[\rho S](\sigma) \in L^2(1, \infty)$$

for some constant $m \in \mathbb{R}$, which proves Theorem 4.

Proof of Lemma 3.2. We denote by $y$ the variables on $\mathbb{R}^{n-1} = \{x: x \cdot \omega = -r_0\}$. It follows from (0.4) that for a large constant $t_0$ independent of $t, y$ and $\omega$

$$q^{-}(t; -r_0, y, \omega) = q^{-}(t_0; -r_0, y, \omega) + (t - t_0)p^{-}(t_0; -r_0, y, \omega), \quad t \geq t_0, \ y \in \mathbb{R}^{n-1}.$$  

Fix $\tilde{y} \in M_\omega(\theta)$ arbitrarily and take a neighborhood $U(\tilde{y})$ of $\tilde{y}$ such that

$$|q^{-}(t_0; -r_0, y, \omega) - q^{-}(t_0; -r_0, \tilde{y}, \omega)| \leq \varepsilon/2 \quad \text{for any } y \in U(\tilde{y}),$$

$$|t_0\{p^{-}(t_0; -r_0, y, \omega) - p^{-}(t_0; -r_0, \tilde{y}, \omega)\}| \leq \varepsilon/2 \quad \text{for any } y \in U(\tilde{y}).$$

Then, in view of the definitions of $M_\omega(\theta)$ and $s_\omega(\theta)$ we have for any $y \in U(\tilde{y})$ and $t \geq t_0$

$$\psi(y; \tilde{t}) \leq q^{-}(t_0; -r_0, \tilde{y}, \omega) \cdot \theta - t_0p^{-}(t_0; -r_0, \tilde{y}, \omega) \cdot \theta + t_0p^{-}(t_0; -r_0, y, \omega) \cdot \theta + \varepsilon$$

$$\leq s_\omega(\tilde{t}) + \varepsilon + \tilde{t}.$$  

On the other hand, for any neighborhood $U$ of $M_\omega(\varepsilon)$ it follows that $\delta = \inf_{r \in \mathbb{R}^{n-1}} \{1 - p^{-}(t_0; -r_0, y, \omega) \cdot \theta\} > 0$, which yields that $\psi(y; t) \leq (C - \delta t) + t$ for any $y \in U$ ($|y| \leq r_1$) and $t \geq t_0$ ($C$ is a constant independent of $y$ and $t$). This means that

$$\psi(y; \tilde{t}) \leq s_\omega(\theta) - 1 + \tilde{t}$$

if $y \in U$, $|y| \leq r_1$ and $\tilde{t}$ is large enough. Therefore we obtain the lemma.

References


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