<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Conditions against rapid decrease of oscillatory integrals and their applications to inverse scattering problems</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Soga, Hideo</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 1986, 23(2), p. 441–456</td>
</tr>
<tr>
<td><strong>Version Type</strong></td>
<td>VoR</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/4375">https://doi.org/10.18910/4375</a></td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>
Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter $\sigma > 0$:

$$ I(\sigma) = \int_{\mathbb{R}^n} e^{-i\varphi(x)} \rho(x; \sigma) dx \ (\text{or} \ int_{\mathbb{R}^n} e^{i\varphi(x)} \rho(x; \sigma) dx), $$

where $\varphi(x)$ is a real-valued $C^\infty$ function and $\rho(x; \sigma)$ is a $C^\infty$ function with an asymptotic expansion

$$ \rho(x; \sigma) \sim \rho_0(x) + \rho_1(x)(i\sigma)^{-1} + \rho_2(x)(i\sigma)^{-2} + \cdots \ (\text{as } \sigma \to \infty). $$

In this paper we study conditions for the integral $I(\sigma)$ not to decrease rapidly as $\sigma \to \infty$, and solve some inverse scattering problems.

As is well known, if stationary points of $\varphi(x)$ are non-degenerate (i.e. $\det (\partial^2_x \varphi(x)) \neq 0$ when $\partial_x \varphi(x) = 0$), $I(\sigma)$ is expanded asymptotically as $\sigma \to \infty$, and we can know whether $I(\sigma)$ decreases rapidly as $\sigma \to \infty$. Also when the stationary points are degenerate, the asymptotic expansion of $I(\sigma)$ is obtained if $\varphi(x)$ is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of $\varphi(x)$ vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of $I(\sigma)$. In the previous paper [13], the author examined the case that $n=2$ and $\rho_j(x) = 0 \ (j \geq 1)$: If $\rho_0(x) \geq 0$ on $\mathbb{R}^2$ and $\rho_0(x_0) > 0$ for a degenerate stationary point $x_0$ of $\varphi(x)$, then $(1+|\sigma|)^m I(\sigma) \in L^2(\mathbb{R}^2)$ for some $m < 2^{-1}$ (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of $n \geq 3$.

Let $supp[\rho(\cdot; \sigma)]$ and $supp[\rho_j] (j \geq 0)$ be contained in a compact set $D$ in $\mathbb{R}^n$. We set

$$ E(s) = \{x: \varphi(x) \leq s\} \ (s \in \mathbb{R}), $$
One of our main results is the following

**Theorem 1.** Let all \( \rho_j \) \((j \geq 0)\) be real-valued. Then, for every \( m \in \mathbb{R} \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

if and only if for every integer \( N(\geq 1) \)

\[
g_N(s) = g_0(s) + \sum_{j=1}^{N} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) dt \in C^N(\mathbb{R}^d)
\]

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

**Theorem 2.** Let all \( \rho_j \) \((j \geq 0)\) be real-valued, and let \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \). If \( \rho_0 \) satisfies

\[
\rho_0(x) > 0 \quad \text{on} \quad E(\min_{x \in D} \varphi(x)),
\]

then for some \( m(\in \mathbb{R}) \) depending only on the dimension \( n \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty).
\]

Theorem 1 implies that decreasingness of \( I(\sigma) \) is connected with smoothness of the measure \( |E(s)| \). This is seen also from the discussions in Vasiľev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of \( |E(s)| \).

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle \( \mathcal{O} (\subset \mathbb{R}^n, n \geq 2) \) with a \( C^\infty \) boundary \( \partial \mathcal{O} \). Assume that the domain \( \Omega = \mathbb{R}^n - \mathcal{O} \) is connected, and consider the initial-boundary value problem

\[
\begin{cases}
\square u(t, x) = 0 & \text{in} \quad \mathbb{R}^1 \times \Omega \quad (\square = \partial^2_t - \Delta), \\
u(t, x') = 0 & \text{on} \quad \mathbb{R}^1 \times \partial \Omega \quad (\partial \Omega = \partial \mathcal{O}), \\
\partial_{t} u(0, x) = f_1(x) & \text{on} \quad \Omega, \\
\partial_{x} u(0, x) = f_2(x) & \text{on} \quad \Omega.
\end{cases}
\]

We denote by \( k_-(s, \omega) (k_+(s, \omega)) \in L^2(\mathbb{R}^1 \times S^{n-1}) \) the incoming (outgoing) translation representation of the data \((f_1, f_2)\) (cf. Lax and Phillips [6], [7]). The operator \( S: k_- \to k_+ \) is called the scattering operator and represented by a distribution kernel \( S(s, \theta, \omega) \) called the scattering kernel:
RAPID DECREASE OF OSCILLATORY INTEGRALS

\[ (S_{(s, \theta)}(s, \theta) = \int_{\mathbb{R}^d} S(s-t, \theta, \omega)k_\perp(t, \omega)dtd\omega \]

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of \( C \subseteq \mathbb{R}^3 \) (i.e. \( n = 3 \)) that for any fixed \( \omega \in S^2 \)

\[(0.2) \begin{align*}
(i) \quad & \text{supp } S(\cdot, -\omega, \omega) \subseteq (-\infty, -2r(\omega)), \\
(ii) \quad & S(s, -\omega, \omega) \text{ is singular (not } C^\infty \text{) at } s = -2r(\omega),
\end{align*}\]

where \( r(\omega) = \min x \cdot \omega \). He reduced proof of the above (ii) to verifying that the integral of the form

\[ \int_{\mathbb{R}^d} e^{-\iota \sigma \varphi(x)} \rho(x; \sigma) dx \]

does not decrease rapidly as \( \sigma \to \infty \) (cf. §2 of Majda [8] or §4 of the author [14]). His methods are not applicable to the case of \( n > 3 \), one of whose reasons is that the stationary points of the phase function \( \varphi(x) \) are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when \( n > 3 \):

**Theorem 3.** For any fixed \( \omega \) and \( \theta \in S^{n-1} \) with \( \omega \not\equiv \theta \), we have

(i) \( \text{supp } S(\cdot, \theta, \omega) \subseteq (-\infty, -r(\omega-\theta)) \),

(ii) \( S(s, \theta, \omega) \) is singular at \( s = -r(\omega-\theta) \).

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

\[
\begin{cases}
\partial_t u(t, x) - \sum_{i,j=1}^n \partial_{x_i} (a_{ij}(x) \partial_{x_j} u(t, x)) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
u(0, x) = f_1(x) & \text{on } \mathbb{R}^n, \\
\partial_{x_i} u(0, x) = f_2(x) & \text{on } \mathbb{R}^n,
\end{cases}
\]

where \( a_{ij}(x) \) are real-valued \( C^\infty \) functions satisfying

\[
\begin{align*}
a_{ij}(x) &= a_{ji}(x), \quad x \in \mathbb{R}^n, \\
a_{ij}(x) &= 0 \ (i \neq j), \quad a_{ii}(x) = 1 \quad \text{when } |x| \geq r_0, \\
\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j &\geq \delta |\xi|^2, \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n.
\end{align*}
\]

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of \( n \geq 3 \). By means of Theorem 2 we get rid of the restriction to the dimension \( n \).

Let us review the results of [15]. We set

\[ \lambda_0(x, \xi) = -\left\{ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \right\}^{1/2}. \]
Denote by $(q^{-}(t; s, x, \xi), p^{-}(t; s, x, \xi))$ the solution of the equation
\[
\begin{align*}
\frac{dq^{-}}{dt} &= -\partial_{t} \lambda_{0}(q^{-}, p^{-}), \\
\frac{dp^{-}}{dt} &= \partial_{x} \lambda_{0}(q^{-}, p^{-}), \\
q^{-}|_{t=-} &= x, \\
p^{-}|_{t=-} &= \xi,
\end{align*}
\]
and for $\omega, \theta \in S^{n-1}$ set
\[
\begin{align*}
M_{\omega}(\theta) &= \{y: y \cdot \omega = -r_{0}, \lim_{t \to \infty} p^{-}(t; -r_{0}, y, \omega) = \theta\}, \\
s_{\omega}(\theta) &= \sup_{y \in M_{\omega}(\theta)} \{\lim_{t \to \infty}(q^{-}(t; -r_{0}, y, \omega) \cdot \theta - t)\}, \\
\tilde{M}_{\omega}(\theta) &= \{y \in M_{\omega}(\theta): s_{\omega}(\theta) = \lim_{t \to \infty}(q^{-}(t; -r_{0}, y, \omega) \cdot \theta - t)\}.
\end{align*}
\]
We assume that for any $y$ ($y \cdot \omega = -r_{0}$) and $\omega \in S^{n-1}$
\begin{equation}
\lim_{t \to \infty} |q^{-}(t; -r_{0}, y, \omega)| = \infty.
\end{equation}
Then singular support of the scattering kernel $S(\cdot, \theta, \omega)$ for the equation (0.3) is contained in the interval $(-\infty, s_{\omega}(\theta)]$ (cf. Theorem 2 in the author [15]); furthermore, when $n=2$, it is proved under some assumptions that $S(s, \theta, \omega)$ is singular at $s = s_{\omega}(\theta)$ (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of $n > 2$:

**Theorem 4.** Assume (0.4) for any $y$ ($y \cdot \omega = -r_{0}$) and $\omega \in S^{n-1}$. Fix $\omega$ and $\theta \in S^{n-1}$ with $\omega \neq \theta$, and let the assumption
\begin{equation}
\det[\partial_{x} q^{-}(t; -r_{0}, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_{0}, \infty) \times \tilde{M}_{\omega}(\theta)
\end{equation}
be satisfied. Then $S(s, \theta, \omega)$ is singular at $s = s_{\omega}(\theta)$.

The assumption (0.5) means that there is no caustic on $\{(t, x): x = q^{-}(t; -r_{0}, y, \omega), -r_{0} \leq t < \infty, y \in M_{\omega}(\theta)\}$, namely, the mapping: $(t, y) \rightarrow q^{-}(t; -r_{0}, y, \omega)$ ($-r_{0} \leq t < \infty, y \cdot \omega = -r_{0}$) is diffeomorphic on $[-r_{0}, \infty) \times \tilde{M}_{\omega}(\theta)$. In the previous paper [15] we added the assumption
\[
\det[\partial_{t} p^{-}(t; -r_{0}, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_{0}, \infty) \times \tilde{M}_{\omega}(\theta),
\]
but this is not necessary.

**1. Proofs of Theorem 1 and Theorem 2**

We denote by $H^{m}(M)$ the Sobolev space of order $m$ on $M$, and by $H^{m}_{loc}(M)$ the space of functions $g(x)$ satisfying $\alpha(x)g(x) \in H^{m}(M)$ for any $\alpha(x) \in C^{\infty}_{0}(M)$ ($C^{\infty}_{0}(M)$ is the space of $C^{\infty}$ functions on $M$ with compact support).

**Lemma 1.1.** Let $\varphi(x)$ be a real-valued $C^{\infty}$ function on $\mathbb{R}^{n}$, and let $\rho(x)$ be a $C^{\infty}$ function on $\mathbb{R}^{n}$ with compact support. Then the function
\[
\begin{align*}
\varphi(x) &\rightarrow \rho(x) \varphi(x), \\
\varphi(x) &\rightarrow \varphi(x) \rho(x), \\
\partial_{x} \varphi(x) &\rightarrow \partial_{x} \rho(x) \varphi(x), \\
\partial_{x} \varphi(x) &\rightarrow \varphi(x) \partial_{x} \rho(x),
\end{align*}
\]
\[ g(s) \equiv \int_{E(s)} \rho(x) dx \]

(where \( E(s) = \{ x : \varphi(x) \leq s \} \) satisfies

(i) \( g(s) = 0 \) if \( s < \min_{x \in \text{supp} \rho} \varphi(x) \),

(ii) \( g(s) \) is constant if \( s > \max_{x \in \text{supp} \rho} \varphi(x) \),

(iii) \( g(s) \in H^m_{\text{loc}}(\mathbf{R}^n) \) for any \( m < 2^{-1} \).

Proof. Set

\[ H(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \]

Then it follows that \( H(s) \in H^m_{\text{loc}}(\mathbf{R}^n) \) for any \( m < 2^{-1} \), and so \( H(s - \varphi(x)) \) becomes a \( H^m_{\text{loc}}(\mathbf{R}^n) \)-valued continuous function on \( \mathbf{R}^n \). Therefore, noting that \( g(s) = \int_{\mathbf{R}^n} \rho(x) H(s - \varphi(x)) dx \), we obtain (iii). If \( s < \min_{x \in \text{supp} \rho} \varphi(x) \) we have \( E(s) \cap \text{supp} \rho = \emptyset \), which proves (i). If \( s > \max_{x \in \text{supp} \rho} \varphi(x) \), \( E(s) \) contains \( \text{supp} \rho \), which yields (ii). The proof is complete.

Proof of Theorem 1. It follows from (iii) of Lemma 1.1 that the function \( g_j(s) \) defined in (0.1) belongs to \( L^2_{\text{loc}}(\mathbf{R}^n) \). Therefore we have

\[
\int_{t_0}^t \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) dt \in H^j_{\text{loc}}(\mathbf{R}^n) \quad (j \geq 1),
\]

\[ \partial_t^j \int_{t_0}^t \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) dt = g_j(s). \]

Hence the function \( \tilde{g}_N(s) = \sum_{j=0}^N (1 - i \sigma)^{-j} g_j(s) \) satisfies

\[ \partial_t^j \tilde{g}_N(s) = \sum_{j=0}^N \partial_t^{N-j} g_j(s). \]

We define \( \tilde{I}(\sigma) \) by

\[ \tilde{I}(\sigma) = \begin{cases} I(\sigma) & \text{for } \sigma > 0, \\ I(-\sigma) & \text{for } \sigma < 0. \end{cases} \]

Then \( \sigma^m \tilde{I}(\sigma) \in L^2(1, \infty) \) if and only if \( (1 + |\sigma|^N) \tilde{I}(\sigma) \in L^2(\mathbf{R}^n) \). Furthermore, since \( \rho_j(x) \) are assumed real-valued, it follows that for any integer \( N \geq 0 \)

\[ \tilde{I}(\sigma) = \sum_{j=0}^N \int_{\mathbf{R}^n} e^{-i \sigma \varphi(x)} \rho_j(x) dx |i\sigma|^{-j} + O(|\sigma|^{-N-1}). \]

Here \( O(|\sigma|^k) \) means that \( |O(|\sigma|^k)| \leq C |\sigma|^k \) for some constant \( C \) independent of \( \sigma \).
Noting that \( \delta(s-\varphi(x)) \) is a \( \mathcal{H}(\mathbb{R}^1) \)-valued continuous function of \( x (m<-2^{-i}) \) and equal to \( \partial_s H(s-\varphi(x)) \), we obtain

\[
e^{-i\sigma \varphi(x)} = \int e^{-i\sigma \delta(s-\varphi(x))} ds = F[\partial_s H(s-\varphi(x))](\sigma),
\]

where \( F \) is the Fourier transformation in \( s \) (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum \( \int_{\mathbb{R}^d} e^{-i\sigma \varphi(x)} \rho_j(x) dx \) in the following way:

\[
\int_{\mathbb{R}^d} e^{-i\sigma \varphi(x)} \rho_j(x) dx = F[\partial_s \int_{\mathbb{R}^d} H(s-\varphi(x)) \rho_j(x) dx] (\sigma) = F[\partial_s \tilde{g}_N(s)](\sigma).
\]

(1.1), (1.2) and (1.3) yield that

\[
(i\sigma)^{N-1} \tilde{I}(\sigma) = F[\partial_s \tilde{g}_N(s)] (\sigma) + 0(1/|\sigma|^{-2}).
\]

Let \( (1 + |\sigma|^N) \tilde{I}(\sigma) \in L^2(\mathbb{R}^d) \) for every \( m \in \mathbb{R} \). Then it follows from (1.4) that

\[
\partial_s \tilde{g}_N(s) \in H^1(\mathbb{R}^d),
\]

which implies

\[
\tilde{g}_N(s) \in C^N(\mathbb{R}^d).
\]

Conversely, let \( \tilde{g}_N(s) \in C^N \) for every non-negative integer \( N \). Then we have \( \partial_s \tilde{g}_N(s) \in H^{1-1}_0(\mathbb{R}^d) \), which means that \( \partial_s^{N+1} \tilde{g}_N(s) \in H^{-1}(\mathbb{R}^d) \) since \( \partial_s^{N+1} \tilde{g}_N(s) = 0 \) for large \( |s| \) (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain

\[
(1 + |\sigma|^N) \tilde{I}(\sigma) \in L^2(\mathbb{R}^d) \text{ for every integer } N(\geq 1).
\]

This shows that

\[
(1 + |\sigma|^N) \tilde{I}(\sigma) \in L^2(\mathbb{R}^d) \text{ for every } m \in \mathbb{R}.
\]

The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that \( s_0 = \min_{x \in D} \varphi(x) = 0 \). Since \( \max_{0 \leq t \leq s} |g_j(t)| \leq \|E(s)\| \max_{x \in D} |\rho_j(x)| \) (\( \|E(s)\| = \int_{E(s)} dx \)), there is a constant \( C \) independent of \( s \) such that

\[
|\int_0^t (s-t)^{j-1} g_j(t) dt| \leq C |s|^j |E(s)| \quad (j \geq 1).
\]

Therefore we have

\[
|\tilde{g}_N(s)| \geq |g_0(s)| - \sum_{j=1}^N \left| \int_0^t (s-t)^{j-1} g_j(t) dt \right| \\
\geq (\min_{x \in D} \rho_0(x) - C \sum_{j=1}^N |s|^j |E(s)|).
\]
Since \( \min_{x \in R^d} \rho_0(x) > 0 \), we obtain \( \min_{x \in R^d} \rho_0(x) \geq 2\delta \) for a constant \( \delta > 0 \) independent of \( s \) if \( |s| \) is small enough. Therefore, if \( |s| \) is small enough, it follows that

\[
|\tilde{g}_N(s)| \geq \delta |E(s)|.
\]

Take a point \( x_0 \) satisfying \( \varphi(x_0) = 0 \) \((= \min_{x \in \Omega} \varphi(x))\). Then there is a constant \( d (> 0) \) such that

\[
E(s) \geq \tilde{E}(s) = \{x: d|x-x_0| \leq s\},
\]

which yields \( |E(s)| \geq |\tilde{E}(s)| = \delta's^n \) for \( s \geq 0 \) (the constant \( \delta' \) does not depend on \( s \)). Thus, for any sufficiently small \( s \geq 0 \) we have

\[
(1.5) \quad |\tilde{g}_N(s)| \geq \delta s^n.
\]

Now, assume that \( \sigma^mI(\sigma) \in L^2(1, \infty) \) for every \( m \in R \). Then it follows from Theorem 1 that \( \tilde{g}_N(s) \in C^N \) for any integer \( N \geq 0 \). Take the \( N \) so that \( N \geq n+1 \). All the derivatives \( g_N(0), \partial_xg_N(0), \cdots, \partial_x^n g_N(0) \) vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

\[
|\tilde{g}_N(s)| \leq C |s|^{n+1}.
\]

This is not consistent with (1.5) as \( s \to +0 \). Therefore we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

for some constant \( m \in R \) depending only on \( n \).

2. Proof of Theorem 3

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let \( v(t, x; \omega) \) be the solution of the equation

\[
\begin{cases}
\square v(t, x) = 0 & \text{in } R^d \times \Omega, \\
v(t, x') = -2^{-1}(-2\pi i)^{1-n} \delta(t-x' \cdot \omega) & \text{on } R^d \times \partial \Omega, \\
v(t, x) = 0 & \text{for } t < r(\omega).
\end{cases}
\]

Then \( v(t, x; \omega) \) is a \( C^m \) function of \( x \) and \( \omega \) with the value \( S'(R^d) \).

**Proposition 2.1.** \( S(s, \theta, \omega) \) is represented of the form

\[
S(s, \theta, \omega) = \int_{\partial \Omega} \{ \partial_x^{t-2} \partial_s v(x \cdot \theta - s, x; \omega) - v \partial_x^{t-1} v(x \cdot \theta - s, x; \omega) \} dS_x \quad (\omega \neq \theta),
\]

where \( v \) is the outer unit vector normal to \( \partial \Omega \) (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral \( \int \cdot dS_x \) is in the sense of the Riemann
integral with the value \( S'(R^1) \). For the proof see Majda [8] and the author [14].

It is seen that the wave front set of \( \delta(t-x-\omega)|_{R^1 \times \partial \Omega} \) is non-glancing in \( \{(t, x): -r(\omega-\theta)-2\eta \leq x \cdot \theta - t \} \cap (R^1 \times \partial \Omega) (\omega \neq \theta) \) if \( \eta (>0) \) is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution \( v(t, x; \omega) \) of (2.1) mod \( C^\infty \) by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about \( \partial_v |_{R^1 \times \partial \Omega} \). This is indicated by Majda [8] in the case of \( \theta = -\omega \) (cf. Lemma 2.1 of [8]). We have

**Lemma 2.2.** There exists a first order pseudo-differential operator \( B \) on \( R^1 \times \partial \Omega \) independent of \( t \) such that

(i) its symbol \( B(\xi'; \tau, \xi') \) represented near

\[
N(\omega-\theta) = \{ x: x \cdot (\theta - \omega) = r(\omega-\theta) \} \cap \partial \Omega
\]

by local coordinates \( (t, \xi') \), has a homogeneous asymptotic expansion \( \sum_{j=0} B_j(\xi'; \tau, \xi') \) satisfying

\[
-iB_0(\xi'; \pm 1, \mp \theta') > 0 \text{ on } N(\omega-\theta) (\theta' \text{ is the tangential component of } \theta \text{ to the plane } \{ x: x \cdot (\omega-\theta) = r(\omega-\theta) \}),
\]

(2.2) \( B_j(\xi'; \tau, \xi') \) are purely imaginary-valued for even \( j \) and real-valued for odd \( j \),

(ii) \( \partial_v |_{R^1 \times \partial \Omega} \) is equal to \( B(v |_{R^1 \times \partial \Omega}) \) mod \( C^\infty \) in \( \{(t, x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \cap R^1 \times \partial \Omega \) for \( \eta > 0 \).

In the above lemma, "a homogeneous asymptotic expansion \( \sum_{j=0} B_j(\xi'; \tau, \xi') \)" means that \( B_j(\xi'; \mu \tau, \mu \xi') = \mu^{-j} B_j(\xi'; \tau, \xi') \) for \( \mu \geq 1, |\tau| + |\xi'| \geq 1 \) and that \( |B(\xi'; \tau, \xi') - \sum_{j=1}^N B_j(\xi'; \tau, \xi')| \leq C_N(|\tau| + |\xi'| + 1)^{-N+1} \) for any non-negative integer \( N \) (for detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.); (ii) in the lemma states that \( \alpha(t, x') (\partial_v |_{R^1 \times \partial \Omega} - B(v |_{R^1 \times \partial \Omega})) \in C^\infty \) for any \( \alpha(t, x') \in C^\infty(R^1 \times \partial \Omega) \) with \( \text{supp } [\alpha] \subset \{(t, x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \).

**Proof of Lemma 2.2.** Let \( \sum_{i=1}^l \chi_i(x) \) be a partition of unity on a neighborhood of \( N(\omega-\theta) \) satisfying \( \max_{1 \leq i \leq l} |\text{supp } [\chi_i]| \leq \epsilon_0 \) (\( \epsilon_0 \) is a sufficiently small positive constant). Then there is a constant \( \epsilon_i > 0 \) such that \( \sum_{i=1}^l \chi_i(x) = 1 \) for any \( x \in \partial \Omega \) satisfying \( -r(\omega-\theta)-\epsilon_i \leq x \cdot \theta - x \cdot \omega \). Let \( v_i(t, x) \) be the solution of the equation

\[
\begin{align*}
\Box v_i(t, x) &= 0 & \text{ in } R^1 \times \Omega, \\
v_i(t, x') &= \chi_i(x') v(t, x'; \omega) & \text{ on } R^1 \times \partial \Omega, \\
v_i(t, x) &= 0 & \text{ for } t < r(\omega).
\end{align*}
\]
RAPID DECREASE OF OSCILLATORY INTEGRALS

Since \(\text{supp}[v|_{R^1 \times \Omega}] \subset \{(t, x'): x' \cdot \omega = t\} \), \(\sum_{i=1}^{1} v_i(t, x')\) is equal to \(v(t, x'\cdot \omega)\) on \((R^1 \times \partial \Omega) \cap \{(t, x')$: \(-r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\}\), and so, noting that the propagation speed is less than one, we have

\[
v(t, x; \omega) = \sum_{i=1}^{1} v_i(t, x) \quad \text{in } (R^1 \times \Omega) \cap \{(t, x')$: \(-r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\}\).
\]

We denote by \(WF[f(t, x)]\) the wave front set of \(f(t, x)\). It is seen that \(WF[f|_{R^1 \times \Omega}] = WF[\delta(x' \cdot \omega - t)|_{R^1 \times \Omega}] = \{(t, x'); \tau, \xi') \in R^1 \times \partial \Omega, x' \cdot \omega - t = 0, \xi' = -r(\omega - (\omega \cdot v)\nu), \tau = 0\} (v \text{ is the outer unit normal to } \partial \Omega).\) Hence, for any \((t, x'); \tau, \xi') \in WF[v|_{R^1 \times \Omega}]\) the equation \(\tau^2 - |\xi' + \lambda \nu|^2 = 0\) in \(\lambda\) has real roots, and the null-bicharacteristics associated with \(d^2 - \Delta\) through \(WF[v|_{R^1 \times \Omega}]\) are transversal to \(R^1 \times \partial \Omega\) (non-glancing). This implies that sing \(\text{supp}[\partial \nu v|_{R^1 \times \Omega}]\) \(\subset \text{sing supp}[v|_{R^1 \times \Omega}]\) (cf. Theorem 7 in §9 of (Lax and) Nirenberg [10]), and so it suffices to examine \(v_i(t, x)\) only in a neighborhood \((t_i - \varepsilon_0, t_i + \varepsilon_0) \times \Omega\) of \((t_i, x^i)\) \((x^i \in \text{supp}[\chi_i] \cap N(\omega - \theta))\) and \(t_i = x^i \cdot \omega)\).

To analyze \(v_i\) more precisely, we transform \(\Omega\) in \(U_i\) into the half-space \(\Omega^+ = \{x \in \Omega^i; x_0 > 0\}\). Let the derivative \(\partial \nu\) be transformed in \(U_i\) into \(\partial \nu^\nu\). For any set \(M\) in \(R^1\) we denote by \(\hat{M}\) the set transformed by the coordinates \(\tilde{x}\).

Let \(-\Delta\) be represented by \(\tilde{x}\) of the form \(\tilde{A} = \sum a_{\nu}(\tilde{x}) \partial_{\nu}^2\). Here we can assume that the coefficients \(a_{\nu}(\tilde{x})\) are real-valued \(C^\infty\) functions defined on \(R^1\) and constant out of \(\tilde{U}_i\). Let us examine the solution \(\tilde{v}(t, \tilde{x})\) of the following equation instead of \(v_i(t, x)\):

\[
\begin{aligned}
\begin{cases}
(\partial_{\tilde{t}}^2 + \tilde{A})\tilde{v}(t, \tilde{x}) = 0 & \text{in } R^1 \times R^1,
\tilde{v}(t, \tilde{x}) = g(t, \tilde{x}) & \text{on } R^1 \times R^{-1},
\tilde{v}(t, \tilde{x}) = 0 & \text{for } t < t_i - \varepsilon_0,
\end{cases}
\end{aligned}
\]

where \(g(t, \tilde{x}) = -2^{-1}(-2\pi i)^{-n}\delta(x(\tilde{x}) \cdot \omega - t)\chi_i(x(\tilde{x}))\). Note that \(WF[g(t, \tilde{x})]\) is contained in a sufficiently small conic neighborhood of \((t_i, x^i; \pm 1, \mp \tilde{\theta}^i)\) \(\tilde{\theta}^i\) is the component of \(\theta\) (transformed by the coordinates \(\tilde{x}\)) tangent to the plane \(x_0 = 0)\), and that if \(|(\tau, \xi^i)|^{-1}(|(\tau, \xi^i)|^{-1}) = |(\pm 1, \mp \tilde{\theta}^i)|^{-1}(|(\pm 1, \mp \tilde{\theta}^i)|\) the equation

\[
(2.4)
\tau^2 + \tilde{A}_\varphi(\tilde{x}; \xi^i, \xi_0) = 0
\]

\((\tilde{A}_\varphi(\tilde{x}, \xi^i) = \sum_{|\alpha| = 2} a_{\nu}(\tilde{x}) \partial_{\nu}^2\) in \(\xi_0\) has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators \(\tilde{\xi}^i(\tilde{x}; D_\tilde{t}, D_x)\) on \(R^1 \times R^1\) (independent of \(t\)) with homogeneous asymptotic expansions \(\sum_{j=0}^{\infty} \tilde{\xi}^i_j(\tilde{x}; \tau, \xi^i)\) such that

\[
(i) \quad \tilde{\xi}^i_j(\tilde{x}; \tau, \xi^i) \text{ are real-valued for even } j \text{ and purely imaginary-valued for odd } j.
\]
(ii) if \(|(\tau, \xi')(\tau, \xi')|^{-1}(\tau, \xi')|^{-1}(\tau, \xi')\) or \(|(1, -\bar{\theta}')|^{-1}(1, -\bar{\theta}')\), 
\(\xi_0(x; \tau, \xi')\) are equal to the roots of the equation (2.4), and

\[\xi_0(x; \pm 1, \mp \bar{\theta}') = \mp(1 - |\bar{\theta}'|^2)^{1/2},\]

(iii) all the null-bicharacteristic curves associated with \(Dx_0 - \xi_0(x; D, D_x)\) through \(WF[g(t, x')]\) are transversal to the boundary \(\{x_0 = 0\}\) and proceed in the direction \(t > 0\) as they leave the boundary,

(iv) if the wave front set of \(u(t, x)\) is near the bicharacteristic curves stated in the above (iii), then we have

\[(Dx_0 - \xi_0(x; D, D_x))(Dx_0 - \xi_0(x; D, D_x))\cdot \zeta(x)(\partial_x^2 + A)u \mod C^\infty,\]

where \(\zeta(x)\) is a \(C^\infty\) function on \(\mathbb{R}^n\) satisfying \(\zeta(x) < 0\) for every \(x\).

(iii) and (iv) imply that \(\vartheta(t, x', x_0)\) is approximated \(\mod C^\infty\) by the solution \(w(x_0; t, x')\) of the equation

\[\{ (Dx_0 - \xi_0(x; D, D_x))w = 0, \text{ } x_0 > 0, \text{ } \}

\[-w|_{x_0=0} = h(t, x') .\]

Therefore we have

\[\partial_{x_0}\vartheta|_{x_0=0} = -i\xi_0(x', 0; D, D_x)\vartheta|_{x_0=0} \mod C^\infty .\]

Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution \(v(t, x; \omega)\) in (2.1) satisfies \(\text{supp}[v|_{\mathbb{R}^n \times \Omega}] \subset \{(t, x): x \cdot \omega = t\}\). Therefore, noting that the propagation speed is less than one, we see that \(\text{supp}[v(t, x; \omega)] \subset \{(t, x): x \cdot \omega \leq t\}\), which yields

\[v(x \cdot \omega - s, x; \omega) = 0 \text{ } \text{if } s > x \cdot (\omega - \omega) .\]

Hence, if \(s > \max x \cdot (\omega - \omega) = -r(\omega - \omega) (\omega \equiv \theta)\), we obtain \(S(s, \theta, \omega) = 0\) from Proposition 2.1.

Next, let us prove that \(S(s, \theta, \omega)\) is singular at \(s = -r(\omega - \theta)\). Take \(\alpha(s) \in C^\infty(\mathbb{R}^n)\) such that \(0 \leq \alpha \leq 1\) on \(\mathbb{R}^n\), \(\alpha(s) = 1\) for \(|s| \leq 2^{-1}\) and \(\alpha(s) = 0\) for \(|s| \geq 1\). For any \(\varepsilon > 0\) set

\[\alpha_\varepsilon(s) = \alpha\left(\frac{s + r(\omega - \theta)}{2\varepsilon}\right) .\]

Then we have only to prove that \(\alpha_\varepsilon(s)S(s, \theta, \omega)\) is not \(C^\infty\) for any small \(\varepsilon > 0\). Proposition 2.1 yields

\[\alpha_\varepsilon(s)S(s, \theta, \omega) = \int_{\partial \Omega} \alpha_\varepsilon(s)(\partial_{\nu}^q(\partial_{x}^p v))(x \cdot \theta - s, x; \omega) dS_x - \int_{\partial \Omega} v \cdot \theta \alpha_\varepsilon(s)(\partial_{\nu}^q v)(x \cdot \theta - s, x; \omega) dS_x \equiv \int_f(s) + \int_s(s) .\]
Let $F[k](s)(\sigma)=\int e^{i\sigma s}k(ds)$. As is readily seen, it follows that

\begin{equation}
F[J_2(s)](\sigma) = -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} C_{j-1}^{n-1}(i\sigma)^{n-1-j} \int_{\partial\Omega} e^{i\sigma x\cdot(\theta-w)}(-n\cdot\theta) \cdot \alpha^{(j)}(x\cdot(\theta-w)) dS
\end{equation}

(see $C_j^{-1}=(n-1)!/(n-1-j)!j!$). Taking the $\epsilon(>0)$ so that $2\epsilon\leq \eta$, by Lemma 2.2 we have

\begin{equation}
F[J_1(s)](\sigma) = \int_{\mathbb{R}^1\times\partial\Omega} e^{i\sigma x\cdot(\theta-w)} \alpha^{(j)}(x\cdot(\theta-w)) dS
\end{equation}

(where $C_j^{n-1}=(n-1)!/(n-1-j)!j!$). Taking the $\epsilon(>0)$ so that $2\epsilon\leq \eta$, by Lemma 2.2 we have

\begin{equation}
\begin{aligned}
F[J_1(s)](\sigma) &= \int_{\mathbb{R}^1\times\partial\Omega} e^{i\sigma x\cdot(\theta-w)} \alpha^{(j)}(x\cdot(\theta-w)) dS \\
&= -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} C_{j-1}^{n-1}(i\sigma)^{n-1-j} \int_{\partial\Omega} e^{i\sigma x\cdot(\theta-w)} \alpha^{(j)}(x\cdot(\theta-w)) dS
\end{aligned}
\end{equation}

Here $'B$ denotes the transposed operator of $B$ (i.e. $\langle 'Bf, g\rangle = \langle f, Bg\rangle$ for any $f$ and $g\in C_0^\infty(\mathbb{R}^1\times\partial\Omega)$). Let us note that the symbol of $'B$ expressed near supp $[\alpha_4(x\cdot(\theta-w))\cap(\mathbb{R}^1\times\partial\Omega)]$ by the local coordinates $(t, x')$, has a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} 'B_j(x'; \tau, \xi')$ such that $'B_j(x'; \tau, \xi')$ are real-valued for odd $j$ and purely imaginary valued for even $j$ and that $-i'B_0(x'; \tau, \xi')= -iB_0(x'; \tau, \xi')=0$ for $x'\in N(\omega-\theta)$, which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand $'B_j(x'; \tau, \xi')$ asymptotically (as $\sigma\to\infty$) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion

\begin{equation}
F[J_1(s)](\sigma) \sim -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{\infty} (i\sigma)^{n-1-j} \int_{\partial\Omega} e^{i\sigma x\cdot(\theta-w)} \beta_j(x) dS
\end{equation}

where $\beta_j(x)$ are real-valued $C^\infty$ functions on $\partial\Omega$ with supp $[\beta_j] \subset$ supp $[\alpha_4(x\cdot(\theta-w))]\cap \partial\Omega$, and $\beta_0(x)$ is non-negative valued and satisfies

\begin{equation}
\beta_0(x) = -i'B_0(x'; \tau, \xi') \alpha_4(x\cdot(\theta-w)) > 0 \quad \text{for } x'\in N(\omega-\theta).
\end{equation}

Combining (2.5) and (2.6) yields that for any integer $N(>0)$

\begin{equation}
F[\alpha(s)S(s, \theta, \omega)](\sigma) = -2^{-1}(-2\pi i)^{1-n} (i\sigma)^{n-1} \int_{R^{n-1}} e^{-i\sigma x\cdot(\theta-w)} \cdot \{\sum_{j=0}^{\infty} \rho_j(x')(i\sigma)^{-j} d\mu + 0(\sigma^{-N})
\end{equation}

Here $x'$ is the local coordinates on $\partial\Omega$ near $N(\omega-\theta)$ and

\begin{equation}
\rho_j(x') = \beta_j(x(x')) + (-n\cdot\theta) \alpha^{(j)}(x(x')\cdot(\theta-w)) \quad (\alpha^{(j)}=0, j\geq n).
\end{equation}

Noting that $\rho_0(x')>0$ when the phase function $x(x')\cdot(\omega-\theta)$ is minimum, and applying Theorem 2, we obtain for some constant $m\in \mathbb{R}$

\begin{equation}
\sigma^m F[\alpha(s)S(s, \theta, \omega)](\sigma) \in L^2(1, \infty),
\end{equation}
which shows that $\alpha_\epsilon(s) S(s, \theta, \omega)$ is not $C^\infty$. The proof is complete.

3. **Proof of Theorem 4**

We use the same notations as for the scattering by obstacles in §2. The scattering operator $S$ for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

**Proposition 3.1.** Set

$$S_\delta(s, \theta, \omega) = \int_{R^n} (\overline{\partial}_t^{\alpha} w)(x, \theta - s, x) dx,$$

$$Kk = F^{-1}[(\text{sgn } \sigma)^{n-1}(Fk)(\sigma)],$$

where $w(t, x)$ is the solution of the equation

$$\begin{cases}
(\partial_t^2 - A)w(t, x) = 0 \quad (Aw = \sum_{j, j=1}^n \partial_j \partial_{x_j} w)) \quad \text{in } R^1 \times R^n, \\
w(-r_0, x) = -2^{-1}(-2\pi i)^{1-n} \delta(-r_0 - x \cdot \omega) \quad \text{on } R^n, \\
\partial_t w(-r_0, x) = -2^{-1}(-2\pi i)^{1-n} \delta'(0) \quad \text{on } R^n.
\end{cases}$$

Then we have

$$(Sk)(s, \theta) = \int \int S_\delta(s-t, \theta, \omega) k(t, \omega) dt d\omega + (Kk)(s, \theta).$$

Note that $S_\delta(s, \theta, \omega) = S(s, \theta, \omega)$ if $\omega \neq \theta$.

To prove Theorem 4, we have only to show that for any small $\epsilon(>0)$ there exist a real number $m$ and a function $\rho(s) \in C^\infty_0(s, \theta - 2\epsilon, s, \omega) + 2\epsilon$ such that

$$(1 + |\sigma|)^n F[\rho(s) S(s, \theta, \omega)](\sigma) \in L^2(R^n).$$

Let $\gamma(x) \in C^\infty_0(R^n)$ with $\gamma(x) = 1$ in a neighborhood of $\bar{M}_\omega(\theta)$, and denote by $\tilde{w}(t, x)$ the solution of the equation

$$\begin{cases}
(\partial_t^2 - A)\tilde{w}(t, x) = 0 \quad \text{in } R^1 \times R^n, \\
\tilde{w}(-r_0, x) = \gamma(x) w(-r_0, x) \quad \text{on } R^n, \\
\partial_t \tilde{w}(-r_0, x) = \gamma(x) \partial_t w(-r_0, x) \quad \text{on } R^n.
\end{cases}$$

The author [15] showed that if $\tilde{l}$ is large enough we have for any integer $N(>0)$

$$F[\rho(s) S(s, \theta, \omega)](\sigma) = 2^{-1} e^{-i\sigma l} \sum_{j=0}^{N-1} (i\sigma)^{n-1-j} F'[\beta_j(x) \{\tilde{w}(\tilde{l}, x) + (i\sigma)^{-1} \partial_t \tilde{w}(\tilde{l}, x)\} (-\sigma \theta) + 0(\sigma^{-N+\mu})$$

as $\sigma \to \infty$ ($N_\mu$ is an integer independent of $N$) (cf. (4.5) in [15]). Here, $F'$ denotes the Fourier transformation in $x$, and the functions $\beta_j(x) \in C^\infty_0(R^n)$ are all real-valued.
We take $\xi$ so large as to have (i) and (ii) stated in the following

**Lemma 3.2.** Let $r_1$ be an arbitrary constant ($\geq r_0$), and set

$$\phi_r(x; t) = q^- (t; -r_0, x, \omega) \cdot \theta .$$

Then, for any $\varepsilon (>0)$ there is a constant $\xi_0$ such that for any fixed $\xi \geq \xi_0$

(i) $\max_{|x| \leq r_1} \phi_r(x; \xi) \leq \varepsilon \omega (\xi) + \xi + \varepsilon ,$

(ii) all points at which $\phi_r(x; \xi)$ is maximum ($x \cdot \omega = -r_0, |x| \leq r_1$), are contained in $\varepsilon$-neighborhood $(\mathcal{M}_\omega(\theta))_\varepsilon$ of $\mathcal{M}_\omega(\theta)$ ($(\mathcal{M})_\varepsilon = \{ x : \text{dis}(x, \mathcal{M}) < \varepsilon \}$).

This lemma will be proved later. Choose the $\rho(s)$ so that $\rho(s) \geq 0$ on $\mathbb{R}^1$ and $\rho(s) > 0$ on $[s_\omega(\theta) - \varepsilon, s_\omega(\theta) + \varepsilon]$. Then it is seen from the form of $\beta_0(x)$ (cf. (4.4) and (4.6) in [15]) and the above lemma that

$$\beta_0(x) \geq 0 \text{ on } \mathbb{R}^n \text{ and } \beta_0(q^- (\xi; -r_0, y, \omega)) > 0$$

for any $y \in (\mathcal{M}_\omega(\theta))_\varepsilon$ and $\text{supp}[\gamma] \subset (\mathcal{M}_\varepsilon(\theta))_{2\varepsilon}$.

We take the $\gamma(x)$ so that $\gamma(x) \geq 0$ on $\mathbb{R}^n$, $\gamma(x) > 0$ on $(\mathcal{M}_\omega(\theta))_\varepsilon$ and $\text{supp}[\gamma] \subset (\mathcal{M}_\varepsilon(\theta))_{2\varepsilon}$.

By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. §5 of [10] or Appendix II of [5]), we can construct a symbol $\lambda(x, \xi)$ with a homogeneous asymptotic expansion $\sum_{j=0}^{\infty} \lambda_j(x, \xi)$ such that

$$\lambda_0(x, \xi) = \left\{ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \right\}^{1/2},$$

$$-\partial^2_t + A = (D_t + \lambda(x, D_x)) (D_t - \lambda(x, D_x)) \ mod \ulo \ text{smoothing operator}$$

(cf. Corollary 2.5 in the author [15] also). Furthermore we see that $\lambda_j(x, \xi)$ are real-valued for even $j$ and purely imaginary valued for odd $j$ since the coefficients $a_{ij}(x)$ are all real-valued (recall the construction of $\xi^\pm(x'; \tau, \xi')$ in §2). Consider the Cauchy problem

$$\{ (D_t - \lambda(x, D_x)) u(t, x) = 0 \text{ in } \mathbb{R}^1 \times \mathbb{R}^n, \}

u|_{t=0} = u_0(x) \text{ on } \mathbb{R}^n ,$$

and denote by $E(t)$ the operator: $u_0 \rightarrow u(t, \cdot)$. Then $\bar{\omega}(\xi, x)$ and $\partial_t \bar{\omega}(\xi, x)$ are represented as follows:

$$\bar{\omega}(\xi, x) = 2^{-1} E(\xi + r_0) \left( \bar{\omega}(-r_0, \cdot) - i \bar{\mu} \partial_\xi \bar{\omega}(-r_0, \cdot) \right) (x)$$

$$+ 2^{-1} E(-\xi - r_0) \left( \bar{\omega}(-r_0, \cdot) + i \bar{\mu} \partial_\xi \bar{\omega}(-r_0, \cdot) \right) (x) ,$$

$$\partial_t \bar{\omega}(\xi, x) = 2^{-1} E(\xi + r_0) \partial_t \lambda(\bar{\omega}(-r_0, \cdot) - i \bar{\mu} \partial_\xi \bar{\omega}(-r_0, \cdot)) (x)$$

$$+ 2^{-1} E(-\xi - r_0) \partial_t \lambda(\bar{\omega}(-r_0, \cdot) + i \bar{\mu} \partial_\xi \bar{\omega}(-r_0, \cdot)) (x) ,$$

where $\lambda$ and $\bar{\mu}$ are pseudo-differential operators whose symbols coincide with
\(\lambda(x, \xi)\) and \(\mu(x, \xi)\) \((\mu(x, D_x)\) is the parametrix of \(\lambda(x, D_x)\) respectively in a neighborhood of \(\text{supp}[\gamma(x)]\) and vanish for large \(|x|\). Therefore, noting that

\[
\begin{align*}
\mathcal{F} \left[ \beta E(-\xi - r_0) \left( \bar{w}(-r_0, \cdot) + i\bar{\mu} \partial_\xi \bar{w}(-r_0, \cdot) \right) \right] (-\sigma \theta) &= 0(\sigma^{-\infty}), \\
\mathcal{F} \left[ \beta E(-\xi - r_0) \lambda \left( \bar{w}(-r_0, \cdot) + i\bar{\mu} \partial_\xi \bar{w}(-r_0, \cdot) \right) \right] (-\sigma \theta) &= 0(\sigma^{-\infty})
\end{align*}
\]

as \(\sigma \to \infty\) (cf. §4 of the author [15]), we have

\[
F[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1} e^{-i\sigma \gamma(q, t)} \sum_{j=0}^{\infty} (i\sigma)^{j-1} \mathcal{F}[2^{-1} \beta E(\xi + r_0) (1 + \sigma^{-1} \lambda)]
\]

\[
\cdot \left( \bar{w}(-r_0, \cdot) - i\bar{\mu} \partial_\xi \bar{w}(-r_0, \cdot) \right) (-\sigma \theta) + 0(\sigma^{-n+N_0}).
\]

The assumption (0.5) implies that if \(WF[\mu_0]\) is contained in a conic neighborhood of \(M_{\omega}(t) \times \{-\omega\}\) \((WF[\bar{w}(-r_0, \cdot) - i\bar{\mu} \partial_\xi \bar{w}(-r_0, \cdot)]\) is contained there) \(E(\xi + r_0)\mu_0\) is represented by the Fourier integral operator:

\[
E(\xi + r_0)\mu_0(x) = (2\pi)^{-\frac{n}{2}} \int e^{i\sigma \gamma(q, t)} a(\xi + r_0, x, \xi) \partial_\xi d\xi \mod C^\infty
\]

(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that \(\mathcal{F} \left[ \delta^{(k)} \right] (-r_0 - x \cdot \omega) \) \((B\eta) = -i\eta\gamma) \) \((\gamma = (\gamma_1, \gamma_2)\), where \(B = (b_1, \ldots, b_n)\) is an orthogonal matrix with \(\gamma_1 = \omega_0\). Then, introducing change of the variables \(x = q(t; -r_0, \omega)\) \((= q^-(y))\) near \(x = q^-(t; -r_0, M_{\omega}(t), \omega)\) \((y = (y_0, y')\) is orthogonal coordinates with \(y_0 = x \cdot \omega)\), we obtain

\[
\mathcal{F}[2^{-1} \beta E(\xi + r_0) (1 + \sigma^{-1} \lambda)] \left( \bar{w}(-r_0, \cdot) - i\bar{\mu} \partial_\xi \bar{w}(-r_0, \cdot) \right) (-\sigma \theta)
\]

\[
= \int e^{i\sigma \gamma(q, t)} \beta_j(\xi) \int_0^{\tau_0} e^{i\sigma \gamma(t + r_0, \omega, \tau)} a(\xi + r_0, x, -\tau \omega) e^{-i\tau \omega \partial_\xi} d\tau dx + 0(\sigma^{-\infty})
\]

\[
= \int_{\mathcal{R}^n} dy \int_0^{\tau_0} d\tau e^{i\sigma \gamma(t, \omega, \tau)} a(\xi + r_0, \omega, \tau) \beta_j(q(y), \gamma(y))
\]

\[
\cdot \left( \bar{w}(-r_0 + q^-(y), -\sigma \tau \omega) \right) + 0(\sigma^{-\infty}) \quad \text{(as} \sigma \to \infty)\]

\((\gamma(x) \in C^\infty(\mathcal{R}^n), \gamma(x) = 1)\) on a neighborhood of \(q^-(\text{supp}[\gamma])\), and \(\tau\) is a positive constant independent of \(\sigma\). The function \(\Phi(y_0, \tau) = q^-(y_0, y') \cdot \theta - \tau(y_0 + r_0)\) has the stationary point \((y_0, \tau) = (-r_0, p^-(y_0, y') \cdot \theta)\), at which its Hesse matrix equals

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

Expanding \(\int_{-\infty}^{\infty} e^{i\sigma \Phi(y_0, \tau)} \beta_j \gamma \cdots dy_0 d\tau\) (as \(\sigma \to \infty)\) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion

\[
(3.2) \quad F[\rho(s)S(s, \theta, \omega)](\sigma) = e^{-i\sigma \gamma(i\sigma)^{j-1} \int_{\mathcal{R}^n} e^{i\sigma \gamma(t; -r_0, \omega, \tau)} \beta_j(q(y), \gamma(y))
\]

\[
\cdot \left( \sum_{j=0}^{\infty} \rho_j(x) (i\sigma)^{-j} \right) dx + 0(\sigma^{-n+N_0})
\]

\((N_0)\) is an integer independent of \(N=1, 2, \ldots\). Here \(\rho_j\) are \(C^\infty\) functions with
supp[\rho_0] \subset \text{supp}[\gamma] and all real-valued, which follows from the fact that the symbol \( a(t, x, \xi) \) has a homogeneous asymptotic expansion \( \sum_{k=0}^\infty a_k(t, x, \xi) \) such that \( a_k(t, x, \xi) \) are real-valued for even \( k \) and purely imaginary valued for odd \( k \); furthermore \( \rho_0 \) is of the form

\[
\rho_0(y) = \gamma(y) \beta(y) q^-(\xi; -r_0, y, \omega) \tilde{a}_0(\tilde{t} + r_0, q^-(\xi; -r_0, y, \omega), -\omega) |\det \frac{\partial q^-}{\partial y}|.
\]

Combining this with (3.1) and (ii) of Lemma 3.2, we see that \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \) and \( \rho_0(x) > 0 \) for any \( x \) at which the function

\[
\varphi(x) = -q^-(\xi; -r_0, x, \omega) \cdot \theta \quad (x \cdot \omega = -r_0)
\]

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

\[
\sigma^m \mathcal{F}[\rho S] (\sigma) \subseteq L^\beta(1, \infty)
\]

for some constant \( m \in \mathbb{R} \), which proves Theorem 4.

Proof of Lemma 3.2. We denote by \( y \) the variables on \( \mathbb{R}^{n-1} = \{ x : x \cdot \omega = -r_0 \} \).

It follows from (0.4) that for a large constant \( t_0 \) independent of \( t, y \) and \( \omega \)

\[
q^-(t; -r_0, y, \omega) = q^-(t_0; -r_0, y, \omega) + (t-t_0) p^-(t_0; -r_0, y, \omega), \quad t \geq t_0, y \in \mathbb{R}^{n-1}.
\]

Fix \( \tilde{y} \in \mathcal{M}^u(\theta) \) arbitrarily and take a neighborhood \( U(\tilde{y}) \) of \( \tilde{y} \) such that

\[
|q^-(t_0; -r_0, y, \omega) - q^-(t_0; -r_0, \tilde{y}, \omega)| \leq \varepsilon/2 \quad \text{for any } y \in U(\tilde{y}),
\]

\[
|t_0 \{|p^-(t_0; -r_0, y, \omega) - p^-(t_0; -r_0, \tilde{y}, \omega)| \} | \leq \varepsilon/2 \quad \text{for any } y \in U(\tilde{y}).
\]

Then, in view of the definitions of \( \mathcal{M}^u(\theta) \) and \( s_\omega(\theta) \) we have for any \( y \in U(\tilde{y}) \) and \( \tilde{t} \geq t_0 \)

\[
\psi(y; \tilde{t}) \leq q^-(t_0; -r_0, \tilde{y}, \omega) \cdot \theta - t_0 p^-(t_0; -r_0, \tilde{y}, \omega) \cdot \theta + \tilde{t} p^-(t_0; -r_0, y, \omega) \cdot \theta + \varepsilon \leq s_\omega(\theta) + \varepsilon + \tilde{t}.
\]

On the other hand, for any neighborhood \( U \) of \( \mathcal{M}^u(\varepsilon) \) it follows that \( \delta = \inf \{ 1 - \frac{p^-(t_0; -r_0, y, \omega) \cdot \theta \} > 0 \), which yields that \( \psi(y; t) \leq (C - \delta t) + t \) for any \( y \in U \) \((|y| \leq r_1) \) and \( t \geq t_0 \) \((C \text{ is a constant independent of } y \text{ and } t) \). This means that

\[
\psi(y; \tilde{t}) \leq s_\omega(\theta) - 1 + \tilde{t}
\]

if \( y \in U \), \(|y| \leq r_1 \) and \( \tilde{t} \) is large enough. Therefore we obtain the lemma.

References


Faculty of Education
Ibaraki University
Mito, Ibaraki 310
Japan