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CONDITIONS AGAINST RAPID DECREASE OF OSCILLATORY INTEGRALS AND THEIR APPLICATIONS TO INVERSE SCATTERING PROBLEMS

HIDEO SOGA

(Received March 22, 1985)

Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter \( \sigma > 0 \):

\[
I(\sigma) = \int e^{-i\varphi(x)} \rho(x; \sigma) dx \quad \text{or} \quad \int e^{i\varphi(x)} \rho(x; \sigma) dx,
\]

where \( \varphi(x) \) is a real-valued \( C^\infty \) function and \( \rho(x; \sigma) \) is a \( C^\infty \) function with an asymptotic expansion

\[
\rho(x; \sigma) \sim \rho_0(x) + \rho_1(x)(i\sigma)^{-1} + \rho_2(x)(i\sigma)^{-2} + \cdots \quad \text{as } \sigma \to \infty.
\]

In this paper we study conditions for the integral \( I(\sigma) \) not to decrease rapidly as \( \sigma \to \infty \), and solve some inverse scattering problems.

As is well known, if stationary points of \( \varphi(x) \) are non-degenerate (i.e. \( \det (\partial^2_x \varphi(x)) \neq 0 \)) and \( \partial_x \varphi(x) = 0 \), \( I(\sigma) \) is expanded asymptotically as \( \sigma \to \infty \), and we can know whether \( I(\sigma) \) decreases rapidly as \( \sigma \to \infty \). Also when the stationary points are degenerate, the asymptotic expansion of \( I(\sigma) \) is obtained if \( \varphi(x) \) is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of \( \varphi(x) \) vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of \( I(\sigma) \). In the previous paper [13], the author examined the case that \( n=2 \) and \( \rho_1(x) = 0 \) \((j \geq 1)\): If \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^2 \) and \( \rho_0(x_0) > 0 \) for a degenerate stationary point \( x_0 \) of \( \varphi(x) \), then \((1+|\sigma|)^m I(\sigma) \in L^2(\mathbb{R}^2) \) for some \( m \leq 2^{-1} \) (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of \( n \geq 3 \).

Let \( \text{supp} [\rho(\cdot; \sigma)] \) and \( \text{supp} [\rho_j] \) \((j \geq 0)\) be contained in a compact set \( D \) in \( \mathbb{R}^n \). We set

\[
E(s) = \{x: \varphi(x) \leq s\} \quad (s \in \mathbb{R}),
\]
One of our main results is the following

**Theorem 1.** Let all \( \rho_j \) \((j \geq 0)\) be real-valued. Then, for every \( m \in \mathbb{R} \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

if and only if for every integer \( N(\geq 1) \)

\[
\bar{g}_N(s) = g_0(s) + \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} (t-t_j)^{j-1} \frac{g_j(t)}{(j-1)!} dt \in C^N(\mathbb{R}^1).
\]

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

**Theorem 2.** Let all \( \rho_j \) \((j \geq 0)\) be real-valued, and let \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \). If \( \rho_0 \) satisfies

\[
\rho_0(x) > 0 \quad \text{on} \quad E(\min_{x \in D} \varphi(x)),
\]

then for some \( m(\in \mathbb{R}) \) depending only on the dimension \( n \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty).
\]

Theorem 1 implies that decreasingness of \( I(\sigma) \) is connected with smoothness of the measure \( |E(s)| \). This is seen also from the discussions in Vasil’ev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of \( |E(s)| \).

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle \( \mathcal{O} \subset \mathbb{R}^n \) with a \( C^\infty \) boundary \( \partial \mathcal{O} \). Assume that the domain \( \Omega = \mathbb{R}^n - \mathcal{O} \) is connected, and consider the initial-boundary value problem

\[
\begin{aligned}
\Box u(t, x) &= 0 \quad \text{in} \quad \mathbb{R}^1 \times \Omega \quad (\Box = \partial_t^2 - \Delta), \\
u(t, x') &= 0 \quad \text{on} \quad \mathbb{R}^1 \times \partial \Omega \quad (\partial \Omega = \partial \mathcal{O}), \\
u(0, x) &= f_1(x) \quad \text{on} \quad \Omega, \\
\partial_t u(0, x) &= f_2(x) \quad \text{on} \quad \Omega.
\end{aligned}
\]

We denote by \( k_-(s, \omega) (k_+(s, \omega)) \in L^2(\mathbb{R}^1 \times S^{n-1}) \) the incoming (outgoing) translation representation of the data \((f_1, f_2)\) (cf. Lax and Phillips [6], [7]). The operator \( S: k_- \rightarrow k_+ \) is called the scattering operator and represented by a distribution kernel \( S(s, \theta, \omega) \) called the scattering kernel.
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\[(Sk_\sigma)(s, \theta) = \iint S(s, \theta, \omega)k_\sigma(t, \omega)dt d\omega\]

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of \(O \subset R^3\) (i.e. \(n=3\)) that for any fixed \(\omega \in S^2\)

\[(0.2)\]

\begin{itemize}
  \item[(i)] \(\text{supp } S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)]\),
  \item[(ii)] \(S(s, -\omega, \omega) \text{ is singular (not } C^\infty\text{) at } s = -2r(\omega)\),
\end{itemize}

where \(r(\omega) = \min x \cdot \omega\). He reduced proof of the above (ii) to verifying that the integral of the form

\[\int_{R^n} e^{-i\sigma} \rho(x; \sigma) dx\]

does not decrease rapidly as \(\sigma \to \infty\) (cf. §2 of Majda [8] or §4 of the author [14]).

His methods are not applicable to the case of \(n>3\), one of whose reasons is that the stationary points of the phase function \(\phi(x)\) are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when \(n>3\):

**Theorem 3.** For any fixed \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), we have

\begin{itemize}
  \item[(i)] \(\text{supp } S(\cdot, \theta, \omega) \subset (-\infty, -r(\omega-\theta)]\),
  \item[(ii)] \(S(s, \theta, \omega) \text{ is singular at } s = -r(\omega-\theta)\).
\end{itemize}

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

\[(0.3)\]

\[\begin{cases}
  \partial_t u(t, x) - \sum_{i,j=1}^n \partial_x(a_{ij}(x)\partial_x u(t, x)) = 0 & \text{in } R^1 \times R^n, \\
  u(0, x) = f_1(x) & \text{on } R^n, \\
  \partial_t u(0, x) = f_2(x) & \text{on } R^n,
\end{cases}\]

where \(a_{ij}(x)\) are real-valued \(C^\infty\) functions satisfying

\[
  a_{ij}(x) = a_{ji}(x), \quad x \in R^n, \\
  a_{ij}(x) = 0 \quad (i \neq j), \quad a_{ii}(x) = 1 \quad \text{when } |x| \geq r_0,
\]

\[
  \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta |\xi|^2, \quad x \in R^n, \quad \xi \in R^n.
\]

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of \(n=3\). By means of Theorem 2 we get rid of the restriction to the dimension \(n\).

Let us review the results of [15]. We set

\[
  \lambda_0(x, \xi) = -\left\{\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\right\}^{1/2}.
\]
Denote by \((q^-(t; s, x, \xi), p^-(t; s, x, \xi))\) the solution of the equation
\[
\begin{cases}
\frac{dq^-}{dt} = -\partial_x \lambda_5(q^-, p^-), \\
\frac{dp^-}{dt} = \partial_x \lambda_5(q^-, p^-), \\
q^-|_{t=-1} = x, \\
p^-|_{t=-1} = \xi,
\end{cases}
\]
and for \(\omega, \theta \in S^{n-1}\) set
\[
M^\omega(\theta) = \{y: y \cdot \omega = -r_0, \lim_{t \to \infty} p^-(t; -r_0, y, \omega) = \theta\},
\]
\[
s^\omega(\theta) = \sup_{\zeta \in \Omega_\omega(\theta)} \{\lim_{t \to \infty} (q^-(t; -r_0, y, \omega) \cdot \theta - t)\},
\]
\[
\tilde{M}^\omega(\theta) = \{y \in M^\omega(\theta): s^\omega(\theta) = \lim_{t \to \infty} (q^-(t; -r_0, y, \omega) \cdot \theta - t)\}.
\]
We assume that for any \(y (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\)
\[
(0.4) \quad \lim_{t \to \infty} |q^-(t; -r_0, y, \omega)| = \infty.
\]
Then singular support of the scattering kernel \(S(\cdot, \theta, \omega)\) for the equation (0.3) is contained in the interval \((-\infty, s^\omega(\theta)]\) (cf. Theorem 2 in the author [15]); furthermore, when \(n=2\), it is proved under some assumptions that \(S(s, \theta, \omega)\) is singular at \(s=s^\omega(\theta)\) (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of \(n>2\):

**Theorem 4.** Assume (0.4) for any \(y (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\). Fix \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), and let the assumption
\[
(0.5) \quad \det[\partial_s q^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \tilde{M}^\omega(\theta)
\]
be satisfied. Then \(S(s, \theta, \omega)\) is singular at \(s=s^\omega(\theta)\).

The assumption (0.5) means that there is no caustic on \(\{(t, x): x = q^-(t; -r_0, y, \omega), -r_0 \leq t < \infty, y \in \tilde{M}^\omega(\theta)\}\), namely, the mapping: \((t, y) \to q^-(t; -r_0, y, \omega) (-r_0 \leq t < \infty, y \cdot \omega = -r_0)\) is diffeomorphic on \([-r_0, \infty) \times \tilde{M}^\omega(\theta)\). In the previous paper [15] we added the assumption
\[
\det[\partial_t p^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \tilde{M}^\omega(\theta),
\]
but this is not necessary.

**1. Proofs of Theorem 1 and Theorem 2**

We denote by \(H^m(M)\) the Sobolev space of order \(m\) on \(M\), and by \(H^m_{\text{loc}}(M)\) the space of functions \(g(x)\) satisfying \(\alpha(x)g(x) \in H^m(M)\) for any \(\alpha(x) \in C^\infty_0(M)\) (\(C^\infty_0(M)\) is the space of \(C^\infty\) functions on \(M\) with compact support).

**Lemma 1.1.** Let \(\varphi(x)\) be a real-valued \(C^\infty\) function on \(\mathbb{R}^n\), and let \(\rho(x)\) be a \(C^\infty\) function on \(\mathbb{R}^n\) with compact support. Then the function
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\[ g(s) \equiv \int_{E(s)} \rho(x) \, dx \]

(where \( E(s) = \{ x : \varphi(x) \leq s \} \) satisfies

(i) \( g(s) = 0 \) if \( s < \min_{x \in \text{supp}[\rho]} \varphi(x) \),

(ii) \( g(s) \) is constant if \( s > \max_{x \in \text{supp}[\rho]} \varphi(x) \),

(iii) \( g(s) \in H^{m}_{loc}(\mathbb{R}^{1}) \) for any \( m < 2^{-1} \).

Proof. Set

\[ H(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \]

Then it follows that \( H(s) \in H^{m}_{loc}(\mathbb{R}^{1}) \) for any \( m < 2^{-1} \), and so \( H(s - \varphi(x)) \) becomes a \( H^{m}_{loc}(\mathbb{R}^{1}) \)-valued continuous function on \( \mathbb{R}^{1} \). Therefore, noting that \( g(s) = \int_{\mathbb{R}^{1}} \rho(x) H(s - \varphi(x)) \, dx \), we obtain (iii). If \( s < \min_{x \in \text{supp}[\rho]} \varphi(x) \) we have \( E(s) \cap \text{supp}[\rho] = \emptyset \), which proves (i). If \( s > \max_{x \in \text{supp}[\rho]} \varphi(x) \), \( E(s) \) contains \( \text{supp}[\rho] \), which yields (ii). The proof is complete.

Proof of Theorem 1. It follows from (iii) of Lemma 1.1 that the function \( g_j(s) \) defined in (0.1) belongs to \( L^{2}_{loc}(\mathbb{R}^{1}) \). Therefore we have

\[ \int_{t_{0}}^{t} \frac{(s-t)^{i-1}}{(j-1)!} g_j(t) \, dt \in H^{i}_{loc}(\mathbb{R}^{1}) \quad (i \geq 1), \]

\[ \partial^{i}_{s} \int_{t_{0}}^{t} \frac{(s-t)^{i-1}}{(j-1)!} g_j(t) \, dt = g_j(s). \]

Hence the function \( \bar{g}_{N}(s) = g_j(s) + \sum_{j=1}^{N} \int_{t_{0}}^{t} \frac{(s-t)^{j-1}}{(j-1)!} g_j(t) \, dt \) satisfies

\[ \partial^{i}_{s} \bar{g}_{N}(s) = \sum_{j=0}^{N} \partial^{N-i}_{s} g_j(s). \]

We define \( \bar{I}(\sigma) \) by

\[ \bar{I}(\sigma) = \begin{cases} I(\sigma) & \text{for } \sigma > 0, \\ I(-\sigma) & \text{for } \sigma < 0. \end{cases} \]

Then \( \sigma^{n}I(\sigma) \in L^{2}(1, \infty) \) if and only if \( (1+|\sigma|)^{n}\bar{I}(\sigma) \in L^{2}(\mathbb{R}^{1}) \). Furthermore, since \( \rho_j(x) \) are assumed real-valued, it follows that for any integer \( N \geq 0 \)

\[ \bar{I}(\sigma) = \sum_{j=0}^{N} \int_{\mathbb{R}^{1}} e^{-i\sigma \varphi(x)} \rho_j(x) \, dx |i\sigma|^{-j} + O(|\sigma|^{-N-1}). \]

Here \( O(|\sigma|^{k}) \) means that \( |O(|\sigma|^{k})| \leq C |\sigma|^{k} (|\sigma| \geq 1) \) for some constant \( C \) independent of \( \sigma \).
Noting that \( \delta(s-\varphi(x)) \) is a \( H^m(\mathbb{R}) \)-valued continuous function of \( x \) \((m<-2^{-i}) \) and equal to \( \partial_s H(s-\varphi(x)) \), we obtain

\[
e^{-i\sigma \varphi(x)} = \int e^{-i\sigma \delta(s-\varphi(x))}ds = F[\partial_s H(s-\varphi(x))] (\sigma),
\]

where \( F \) is the Fourier transformation in \( s \) (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum \( \int_{\mathbb{R}} e^{-i\sigma \phi(x)} \rho_i(x)dx \) in the following way:

\[
\int_{\mathbb{R}} e^{-i\sigma \phi(x)} \rho_i(x)dx = F[\partial_s \int_{\mathbb{R}} H(s-\varphi(x))\rho_i(x)dx] (\sigma)
= F[\partial_s \tilde{g}_N(s)] (\sigma).
\]

(1.1), (1.2) and (1.3) yield that

\[
(i\sigma)^{N-1} \tilde{I}(\sigma) = F[\partial_s^{N} \tilde{g}_N(s)] (\sigma) + 0(|\sigma|^{-2}).
\]

Let \( (1+|\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^3) \) for every \( m \in \mathbb{R} \). Then it follows from (1.4) that

\[
\partial_s^{N} \tilde{g}_N(s) \in H^1(\mathbb{R}^3),
\]

which implies

\[
\tilde{g}_N(s) \in C^N(\mathbb{R}^3).
\]

Conversely, let \( \tilde{g}_N(s) \in C^N \) for every non-negative integer \( N \). Then we have \( \partial_s^{N+1} \tilde{g}_N(s) \in H^{-1}(\mathbb{R}^3) \), which means that \( \partial_s^{N+1} \tilde{g}_N(s) \in H^{-1}(\mathbb{R}^3) \) since \( \partial_s^{N+1} \tilde{g}_N(s) = 0 \) for large \( |s| \) (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain \( (1+|\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^3) \) for every integer \( N(\geq 1) \). This shows that

\[
(1+|\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^3) \quad \text{for every} \quad m \in \mathbb{R}.
\]

The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that \( s_0 = \min_{x \in D} \varphi(x) = 0 \). Since \( \max_{0 \leq t \leq s} |g_f(t)| \leq |E(s)| \max_{x \in D} |\rho_f(x)| \) \((|E(s)| = \int_{E(s)} dx)\), there is a constant \( C \) independent of \( s \) such that

\[
\left| \int_0^s \frac{(s-t)^{j-1}}{(j-1)!} g_f(t)dt \right| \leq C |s|^j |E(s)| \quad (j \geq 1).
\]

Therefore we have

\[
|\tilde{g}_N(s)| \geq |g_0(s)| - \sum_{j=1}^\infty \left| \int_0^s \frac{(s-t)^{j-1}}{(j-1)!} g_f(t)dt \right|
\geq (\min_{x \in \Omega} \rho_0(x) - C \sum_{j=1}^\infty |s|^j |E(s)|).
\]
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Since \( \min_{x \in \mathbb{R}} \rho_0(x) > 0 \), we obtain \( \min_{x \in \mathbb{R}} \rho_0(x) \geq 2\delta \) for a constant \( \delta > 0 \) independent of \( s \) if \( |s| \) is small enough. Therefore, if \( |s| \) is small enough, it follows that

\[
|\hat{g}_N(s)| \geq \delta |E(s)|.
\]

Take a point \( x_0 \) satisfying \( \varphi(x_0) = 0 \) \((=\min_{x \in \mathbb{R}} \varphi(x))\). Then there is a constant \( d > 0 \) such that

\[
E(s) \geq \hat{E}(s) = \{ x : d|x-x_0| \leq s \},
\]

which yields \( |E(s)| \geq |\hat{E}(s)| = \delta' s^n \) for \( s \geq 0 \) (the constant \( \delta' \) does not depend on \( s \)). Thus, for any sufficiently small \( s \geq 0 \) we have

\[
(1.5) \quad |\hat{g}_N(s)| \geq \delta' s^n.
\]

Now, assume that \( \sigma^m I(\sigma) \in L^2(1, \infty) \) for every \( m \in \mathbb{R} \). Then it follows from Theorem 1 that \( \hat{g}_N(s) \in C^N \) for any integer \( N \geq 0 \). Take the \( N \) so that \( N \geq n+1 \). All the derivatives \( g_N(0), \partial_\gamma g_N(0), \cdots, \partial^{n}_\gamma g_N(0) \) vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

\[
|\hat{g}_N(s)| \leq C |s|^{n+1}.
\]

This is not consistent with (1.5) as \( s \to +0 \). Therefore we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

for some constant \( m \in \mathbb{R} \) depending only on \( n \).

2. Proof of Theorem 3

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let \( v(t, x; \omega) \) be the solution of the equation

\[
\begin{cases}
\square v(t, x) = 0 & \text{in } \mathbb{R}^4 \times \Omega, \\
v(t, x') = -2^{-1}(-2\pi i)^{1-n}\delta(t-x' \cdot \omega) & \text{on } \mathbb{R}^4 \times \partial \Omega, \\
v(t, x) = 0 & \text{for } t < r(\omega).
\end{cases}
\]

Then \( v(t, x; \omega) \) is a \( C^m \) function of \( x \) and \( \omega \) with the value \( S'(R^1) \).

**Proposition 2.1.** \( S(s, \theta, \omega) \) is represented of the form

\[
S(s, \theta, \omega) = \int_{\partial \Omega} \{ \partial_t^{m-2} \partial_{s} v(x \cdot \theta - s, x; \omega) - v \cdot \theta \partial_t^{m-1} v(x \cdot \theta - s, x; \omega) \} dS_x \quad (\omega \neq \theta),
\]

where \( v \) is the outer unit vector normal to \( \partial \Omega \) (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral \( \int \cdot dS_x \) is in the sense of the Riemann
integral with the value $S'(\mathbb{R}^1)$. For the proof see Majda [8] and the author [14].

It is seen that the wave front set of $\delta(t-x \cdot \omega)|_{\mathbb{R}^1 \times \partial \Omega}$ is non-glancing in $\{(t, x): -r(\omega-\theta)-2\eta \leq x \cdot \theta - t \}$ if $\eta(>0)$ is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution $v(t, x; \omega)$ of (2.1) mod $C^\infty$ by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about $\partial_v v|_{\mathbb{R}^1 \times \partial \Omega}$. This is indicated by Majda [8] in the case of $\theta = -\omega$ (cf. Lemma 2.1 of [8]). We have

**Lemma 2.2.** There exists a first order pseudo-differential operator $B$ on $\mathbb{R}^1 \times \Omega$ independent of $t$ such that

(i) its symbol $B(x'; \tau, \xi')$ represented near $N(\omega-\theta) = \{x: x \cdot (\theta-\omega) = r(\omega-\theta)\} \cap \partial \Omega$

by local coordinates $(t, x')$, has a homogeneous asymptotic expansion $\sum_{j=0}^\infty B_j(x'; \tau, \xi')$

satisfying

\begin{align}
-ib_0(x'; \pm 1, \pm \bar{\theta}') &> 0 \text{ on } N(\omega-\theta) \ (\bar{\theta}' \text{ is the tangential component of } \theta \\
B_j(x'; \tau, \xi') & \text{ are purely imaginary-valued for even } j \text{ and real-valued for odd } j,
\end{align}

(ii) $\partial_v v|_{\mathbb{R}^1 \times \partial \Omega}$ is equal to $B(v|_{\mathbb{R}^1 \times \partial \Omega})$ mod $C^\infty$ in $\{(t, x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \cap \mathbb{R}^1 \times \partial \Omega$ for some small constant $\eta > 0$.

In the above lemma, “a homogeneous asymptotic expansion $\sum_{j=0}^\infty B_j(x'; \tau, \xi')$" means that $B_j(x'; \mu \tau, \mu \xi') = \mu^{-j} B_j(x'; \tau, \xi')$ for $\mu \geq 1$, $|\tau| + |\xi'| \geq 1$ and that $|B(x'; \tau, \xi') - \sum_{j=1}^{N-1} B_j(x'; \tau, \xi')| \leq C_N (|\tau| + |\xi'| + 1)^{-N-1}$ for any non-negative integer $N$ (for detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.); (ii) in the lemma states that $\alpha(t, x') (\partial_v v|_{\mathbb{R}^1 \times \partial \Omega} - B(v|_{\mathbb{R}^1 \times \partial \Omega})) \in C^\infty$ for any $\alpha(t, x') \in C^\infty(\mathbb{R}^1 \times \partial \Omega)$ with supp $[\alpha] \subset \{(t, x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \}$.

**Proof of Lemma 2.2.** Let $\sum_{i=1}^I \chi_i(x)$ be a partition of unity on a neighborhood of $N(\omega-\theta)$ satisfying $\max |\text{supp} [\chi_i]| \leq \varepsilon_0$ ($\varepsilon_0$ is a sufficiently small positive constant). Then there is a constant $\varepsilon_i > 0$ such that $\sum_{i=1}^I \chi_i(x) = 1$ for any $x \in \partial \Omega$ satisfying $-r(\omega-\theta)-\varepsilon_i \leq x \cdot \theta - x \cdot \omega$. Let $v_i(t, x)$ be the solution of the equation

\[
\begin{align*}
\Box v_i(t, x) &= 0 \quad \text{in } \mathbb{R}^1 \times \Omega, \\
v_i(t, x') &= \chi_i(x') v(t, x'; \omega) \quad \text{on } \mathbb{R}^1 \times \partial \Omega, \\
v_i(t, x) &= 0 \quad \text{for } t < r(\omega).
\end{align*}
\]
RAPID DECREASE OF OSCILLATORY INTEGRALS

Since $\text{supp}[\psi_{|R^1 \times \partial \Omega}] \subset \{(t, x'): x' \cdot \omega = 0\}$, $\frac{1}{t} \sum_{i=1}^{l} v_i(t, x')$ is equal to $v(t, x'; \omega)$ on $(R^1 \times \partial \Omega) \cap \{(t, x'): -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\}$, and so, noting that the propagation speed is less than one, we have

$$v(t, x; \omega) = \sum_{i=1}^{l} v_i(t, x) \quad \text{in} \ (R^1 \times \Omega) \cap \{(t, x): -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\}.$$ 

We denote by $\text{WF}[f(t, x)]$ the wave front set of $f(t, x)$. It is seen that $\text{WF}[\psi_{|R^1 \times \partial \Omega}] = \text{WF}[\delta(x' \cdot \omega - t)|_{R^1 \times \partial \Omega}] = \{(t, x'; \tau, \xi') \in R^1 \times \partial \Omega, x' \cdot \omega - t = 0, \xi' = -\tau(\omega - (\omega \cdot \nu)\nu), \tau \neq 0\} \ (\nu$ is the outer unit normal to $\partial \Omega).$ Hence, for any $(t, x'; \tau, \xi') \in \text{WF}[v_i_{|R^1 \times \partial \Omega}]$ the equation $\tau^2 - |\xi' + \lambda\nu|^2 = 0$ in $\lambda$ has real roots, and the null-bicharacteristics associated with $\partial_t - \Delta$ through $\text{WF}[v_i_{|R^1 \times \partial \Omega}]$ are transversal to $R^1 \times \partial \Omega$ (non-glancing). This implies that $\text{sing supp}[\partial_t v_i_{|R^1 \times \partial \Omega}] \subset \text{sing supp}[\psi_{|R^1 \times \partial \Omega}]$ (cf. Theorem 7 in §9 of (Lax and) Nirenberg [10]), and so it suffices to examine $v_i(t, x)$ only in a neighborhood $(t_i - \varepsilon_0, t_i + \varepsilon_0) \times \Omega_i$ of $(t_i, x') \ (x' \in \text{supp}[\psi_i] \cap N(\omega - \theta)$ and $t_i = x' \cdot \omega$).

To analyze $v_i$ more precisely, we transform $\Omega$ in $C/_{\xi}$ into the half-space $\Omega_+ = \{\xi = (x', \xi): x_0 > 0\}$. Let the derivative $\partial_\theta$ be transformed in $U_i$ into $-\partial_\xi_0$. For any set $M$ in $R^1$, we denote by $M$ the set transformed by the coordinates $\xi$. Let $-\Delta_\xi$ be represented by $\xi$ of the form $A = \sum_{\xi |\xi | = 2} a_\xi(\xi) \delta_\xi^2$. Here we can assume that the coefficients $a_\xi(\xi)$ are real-valued $C^\infty$ functions defined on $R^n$ and constant out of $U_i$. Let us examine the solution $\theta(t, \xi)$ of the following equation instead of $v_i(t, x)$:

\[
\begin{cases}
(\partial_t^2 + A) \theta(t, \xi) = 0 & \text{in} \ R^1 \times R^1_+,

\theta(t, \xi_0) = g(t, \xi_0) & \text{on} \ R^1 \times R_0^{-1},

\theta(t, \xi) = 0 & \text{for} \ t < t_i - \varepsilon_0,
\end{cases}
\]

where $g(t, \xi_0) = -2^{-1/2}(-2\pi t)^{-1} \delta(x(\xi_0) \cdot \omega - t)\chi_i(x(\xi_0))$. Note that $\text{WF}[g(t, \xi')]$ is contained in a sufficiently small conic neighborhood of $(t_i, x'; \pm 1, \mp \theta')$ ($\theta'$ is the component of $\theta$ (transformed by the coordinates $\xi$) tangent to the plane $\xi_0 = 0$), and that if $|(\tau, \xi_0)|^{-1}(|(\tau, \xi_0)| - 1(\pm 1, \mp \theta')$ the equation

\[(2.4) \quad \tau^2 + A_0(\xi, \xi' \cdot \xi_0) = 0 \]

for $A_0(\xi, \xi') = \sum_{|\xi| = 2} a_\xi(\xi) \xi_{\xi'}^2$ in $\xi_0$ has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators $\xi \cdot (x_0; D_{\theta}, D_\xi)$ on $R^1 \times R^1_+$ (independent of $t$) with homogeneous asymptotic expansions $\sum_{j=0}^{\infty} \xi_0^j(\xi; \tau, \xi')$ such that

(i) $\xi_0^j(\xi; \tau, \xi')$ are real-valued for even $j$ and purely imaginary-valued for odd $j$,
(ii) if $|\tau, \xi'|^{-1}(\tau, \xi')$ is near $|(-1, \theta')|^{-1}(-1, \theta')$ or $|(1, -\theta')|^{-1}(1, -\theta')$, 
$\xi^0(\mathfrak{T}, \tau, \xi')$ are equal to the roots of the equation (2.4), and

$$\xi^0(\mathfrak{T}; \pm 1, \mp \theta') = \mp(1 - |\theta'|^2)^{1/2},$$

(iii) all the null-bicharacteristic curves associated with $D\mathcal{T}_0 - \xi^0(\mathfrak{T}; D_t, D_x)$ through $\mathrm{WF}[g(\mathfrak{T}, \mathfrak{T}')]$ are transversal to the boundary $\{\mathfrak{T}_0 = 0\}$ and proceed in the direction $t > 0$ as they leave the boundary,

(iv) if the wave front set of $u(t, x)$ is near the bicharacteristic curves stated in the above (iii), then we have

$$(D\mathcal{T}_0 - \xi^0(\mathfrak{T}; D_t, D_x))(D\mathcal{T}_0 - \xi^0(D_t, D_x))u = \zeta(\mathfrak{T})(\partial^2_t + A)u \mod C^{\infty},$$

where $\zeta(\mathfrak{T})$ is a $C^{\infty}$ function on $\mathbb{R}^n$ satisfying $\zeta(\mathfrak{T}) < 0$ for every $\mathfrak{T}$.

(iii) and (iv) imply that $\theta(t, \mathfrak{T}', \mathfrak{T}_0)$ is approximated mod $C^\infty$ by the solution $w(\mathfrak{T}_0; t, \mathfrak{T}')$ of the equation

$$\left\{ (D\mathcal{T}_0 - \xi^0(\mathfrak{T}; D_t,D_x))w = 0 , \quad \mathfrak{T}_0 > 0 , \right. \left. \right\} w|_{\mathfrak{T}_0 = 0} = h(t, \mathfrak{T}').$$

Therefore we have

$$-\partial\mathcal{T}_0 \theta|_{\mathfrak{T}_0 = 0} = -i\xi^0(\mathfrak{T}', 0; D_t, D_x)(\theta|_{\mathfrak{T}_0 = 0}) \mod C^\infty.$$

Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution $v(t, x; \omega)$ in (2.1) satisfies $\text{supp}[v|_{\mathfrak{T} \times 3\mathbb{Q}}] \subset \{(t, x) : x \cdot \mathfrak{T} = t\}$. Therefore, noting that the propagation speed is less than one, we see that $\text{supp}[v(t, x; \omega)] \subset \{(t, x) : x \cdot \mathfrak{T} \leq t\}$, which yields

$$v(x \cdot \theta - s, x; \omega) = 0 \quad \text{if} \quad s > x \cdot (\theta - \omega).$$

Hence, if $s > \max x \cdot (\theta - \omega) = -r(\omega - \theta) (\omega \neq \theta)$, we obtain $S(s, \theta, \omega) = 0$ from Proposition 2.1.

Next, let us prove that $S(s, \theta, \omega)$ is singular at $s = -r(\omega - \theta)$. Take $\alpha(s) \in C^\infty(\mathbb{R})$ such that $0 \leq \alpha \leq 1$ on $\mathbb{R}$, $\alpha(s) = 1$ for $|s| \leq 2^{-1}$ and $\alpha(s) = 0$ for $|s| \geq 1$. For any $\varepsilon > 0$ set

$$\alpha_\varepsilon(s) = \alpha\left(\frac{s + r(\omega - \theta)}{2\varepsilon}\right).$$

Then we have only to prove that $\alpha_{\varepsilon}(s)S(s, \theta, \omega)$ is not $C^\infty$ for any small $\varepsilon > 0$. Proposition 2.1 yields

$$\alpha_\varepsilon(s)S(s, \theta, \omega) = \int_{\partial \mathcal{Q}} \alpha_\varepsilon(s)(\partial_i t^{-2} \partial_j v)(x \cdot \theta - s, x; \omega) dS_x$$

$$-\int_{\partial \mathcal{Q}} v \cdot \theta \alpha_\varepsilon(s)(\partial_i t^{-1} v)(x \cdot \theta - s, x; \omega) dS_x \equiv f_1(s) + f_2(s).$$
Let $F(k(s))(\sigma) = \int e^{i\sigma k(s)} ds$. As is readily seen, it follows that

$$F[J(s)](\sigma) = -2^{1-n}/2 \pi i^{n-1} \sum_{j=0}^{n-1} C_j^{n-1}(i\sigma)^{n-1-j} \int_{\mathbb{R}^n} e^{i\sigma x \cdot (\theta - \omega)} (-v \cdot \theta) \cdot \alpha_j^{(j)}(x \cdot (\theta - \omega)) dS_x$$

(where $C_j^{n-1} = (n-1)!/(n-1-j)!j!$). Taking the $\varepsilon(>0)$ so that $2\varepsilon \leq \eta$, by Lemma 2.2 we have

$$F[J(s)](\sigma) = \int_{\mathbb{R}^n \times \partial\Omega} \int e^{i\sigma (x \cdot \theta - s)} |\partial^2 [B_\varepsilon]|_{\mathbb{R}^n \times \partial\Omega} (s, x) dsdS_x$$

$\varepsilon = 0$.

Here $B^\ell$ denotes the transposed operator of $B$ (i.e. $\langle Bf, g \rangle = \langle f, Bg \rangle$ for any $f$ and $g \in C^\infty_0(\mathbb{R}^n \times \partial\Omega)$). Let us note that the symbol of $B^\ell$ expressed near $\text{supp} \left[ \partial_\varepsilon (x \cdot \theta - t) \right] \cap (\mathbb{R}^n \times \partial\Omega)$ by the local coordinates $(t, x')$, has a homogeneous asymptotic expansion $\sum_{\beta \in \mathbb{N}} B^\ell(x'; \tau, \xi')$ such that $B^\ell(x'; \tau, \xi')$ are real-valued for odd $j$ and purely imaginary valued for even $j$ and that $-iB^\ell(x'; \tau, \xi') = -iB(x'; \tau, \xi') \equiv 0$ for $x' \in N(\omega - \theta)$, which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand $B^\ell e^{i\sigma (x \cdot \theta - s)} \alpha_j^{(j)}(x \cdot (\theta - s))$ asymptotically (as $\sigma \rightarrow \infty$) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion

$$F[J(s)](\sigma) \sim -2^{1-n}/2 \pi i^{n-1} \sum_{\beta \in \mathbb{N}} (i\sigma)^{n-1-j} \int_{\mathbb{R}^n} e^{i\sigma x \cdot (\theta - \omega)} \beta_j(x) dS_x \quad (\text{as } \sigma \rightarrow \infty),$$

where $\beta_j(x)$ are real-valued $C^\infty$ functions on $\partial\Omega$ with $\text{supp} [\beta_j] \subset \text{supp} [\partial_\varepsilon (x \cdot (\theta - \omega))] \cap \partial\Omega$, and $\beta_0(x)$ is non-negative valued and satisfies

$$\beta_0(x) = -iB(x; \tau, \xi') \alpha_\varepsilon(x \cdot (\theta - \omega)) > 0 \quad \text{for } x \in N(\omega - \theta).$$

Combining (2.5) and (2.6) yields that for any integer $N(>0)$

$$F[\partial_\varepsilon (x \cdot (\theta - s)) \partial_\varepsilon (\omega - \theta)](\sigma) = \sigma^{-N} F[\partial_\varepsilon (x \cdot (\omega - s)) \partial_\varepsilon (\omega - \theta)](\sigma)$$

Here $x'$ is the local coordinates on $\partial\Omega$ near $N(\omega - \theta)$ and

$$\rho_j(x') = \beta_j(x(x')) + (-v \cdot \theta) \alpha_j^{(j)}(x(x') \cdot (\theta - \omega)) \quad (\alpha_j^{(j)} = 0, j \geq n).$$

Noting that $\rho_j(x') > 0$ when the phase function $x(x') \cdot (\omega - \theta)$ is minimum, and applying Theorem 2, we obtain for some constant $m \in \mathbb{R}$

$$\sigma^m F[\partial_\varepsilon (x \cdot s)(\theta, \omega)](\sigma) \in L^2(1, \infty),$$
which shows that $\alpha(s)S(s, \theta, \omega)$ is not $C^\infty$. The proof is complete.

3. Proof of Theorem 4

We use the same notations as for the scattering by obstacles in §2. The scattering operator $S$ for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

**Proposition 3.1.** Set

$$
S_0(s, \theta, \omega) = \int_{\mathbb{R}^n} (\partial_i^2 - A)w(x, \theta - s, x) dx,
$$

$$
Kk = F^{-1}[(\text{sgn } \sigma)^{n-1}(Fk)(\sigma)],
$$

where $w(t, x)$ is the solution of the equation

$$
\begin{align*}
\left\{
\begin{array}{ll}
(\partial_i^2 - A)w(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
\sigma w(-r_0, x) = -2^{-i(1-2\pi)^{n-1}}\delta(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n, \\
\partial_t w(-r_0, x) = -2^{-i(1-2\pi)^{n-1}}\delta'(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n.
\end{array}
\right.
\end{align*}
$$

Then we have

$$
(Sk)(s, \theta) = \int \int S_0(s-t, \theta, \omega)k(t, \omega) dt d\omega + (Kk)(s, \theta).
$$

Note that $S_0(s, \theta, \omega) = S(s, \theta, \omega)$ if $\omega \neq \theta$.

To prove Theorem 4, we have only to show that for any small $\varepsilon(>0)$ there exist a real number $m$ and a function $\rho(s) \in C_0^\infty(s_\omega(\theta) - 2\varepsilon, s_\omega(\theta) + 2\varepsilon)$ such that

$$
(1 + |\sigma|)^m \mathcal{F}[\rho(s)S(s, \theta, \omega)](\sigma) \in L^2(\mathbb{R}^1).
$$

Let $\gamma(x) \in C_0^\infty(\mathbb{R}^n)$ with $\gamma(x) = 1$ in a neighborhood of $\bar{M}_\omega(\theta)$, and denote by $\bar{w}(t, x)$ the solution of the equation

$$
\begin{align*}
\left\{
\begin{array}{ll}
(\partial_i^2 - A)\bar{w}(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
\bar{w}(-r_0, x) = \gamma(x)w(-r_0, x) & \text{on } \mathbb{R}^n, \\
\partial_t \bar{w}(-r_0, x) = \gamma(x)\partial_t w(-r_0, x) & \text{on } \mathbb{R}^n.
\end{array}
\right.
\end{align*}
$$

The author [15] showed that if $\tilde{t}$ is large enough we have for any integer $N(>0)$

$$
\begin{align*}
\mathcal{F}[\rho(s)S(s, \theta, \omega)](\sigma) &= 2^{-i e^{-i\sigma\tilde{t}}} \sum_{j=0}^{N-1} (i\sigma)^{n-1-j} \mathcal{F}^\prime[\beta_j(x)\{\bar{w}(\tilde{t}, x) + (i\sigma)^{-1} \partial_t \bar{w}(\tilde{t}, x)\}](\sigma \theta) + O(\sigma^{-N+N_0})
\end{align*}
$$

as $\sigma \to \infty$ ($N_0$ is an integer independent of $N$) (cf. (4.5) in [15]). Here, $\mathcal{F}^\prime$ denotes the Fourier transformation in $x$, and the functions $\beta_j(x) \in C_0^\infty(\mathbb{R}^n)$ are all real-valued.
We take $\ell$ so large as to have (i) and (ii) stated in the following

**Lemma 3.2.** Let $r_1$ be an arbitrary constant ($\geq r_0$), and set

$$
\psi(x; t) = q^{-t}(-r_0, x, \omega) \cdot \theta.
$$

Then, for any $\varepsilon(> 0)$ there is a constant $\ell_0$ such that for any fixed $\ell \geq \ell_0$

(i) $\max_{|x| \leq r_1} |\psi(x; t)| \leq \varepsilon(t) + \ell + \varepsilon,$

(ii) all points at which $\psi(x; t)$ is maximum $(x \cdot \omega = -r_0, |x| \leq r_1)$, are contained in $\varepsilon$-neighborhood $(\mathcal{M}_\omega(\theta))_\varepsilon$ of $\mathcal{M}_\omega(\theta)$ $((\mathcal{M})_\varepsilon = \{x: \text{dis}(x, \mathcal{M}) < \varepsilon\}).$

This lemma will be proved later. Choose the $p(s)$ so that $p(s) = 0$ on $\mathbb{R}^1$ and $p(s) > 0$ on $[s_\omega(\theta) - \varepsilon, s_\omega(\theta) + \varepsilon]$. Then it is seen from the form of $\beta_0(x)$ (cf. (4.4) and (4.6) in [15]) and the above lemma that

$$
(3.1) \quad \beta_0(x) \leq 0 \quad \text{on } \mathbb{R}^n \quad \text{and} \quad \beta_0(q^-((\ell; -r_0, y, \omega))) > 0
$$

for any $y \in (\mathcal{M}_\omega(\theta))_\varepsilon$ $(y \cdot \omega = -r_0).$

We take the $\gamma(x)$ so that $\gamma(x) \geq 0$ on $\mathbb{R}^n$, $\gamma(x) > 0$ on $(\mathcal{M}_\omega(\theta))_\varepsilon$ and $\text{supp}[\gamma] \subset (\mathcal{M}_\omega(\theta))_{2\varepsilon}$. By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. §5 of [10] or Appendix II of [5]), we can construct a symbol $\lambda(x, \xi)$ with a homogeneous asymptotic expansion $\sum_{j=0}^\infty \lambda_j(x, \xi)$ such that

$$
\lambda_0(x, \xi) = \left\{ \sum_{i=1}^n a_{ij}(x) \xi_i \xi_j \right\}^{1/2},
$$

$$
-\partial_{\xi}^2 + A = (D_1 + \lambda(x, D_2)) (D_1 - \lambda(x, D_2)) \text{ modulo a smoothing operator (cf. Corollary 2.5 in the author [15] also).}$

Furthermore we see that $\lambda_j(x, \xi)$ are real-valued for even $j$ and purely imaginary valued for odd $j$ since the coefficients $a_{ij}(x)$ are all real-valued (recall the construction of $\xi^\pm(x'; \tau, \xi')$ in §2). Consider the Cauchy problem

$$
\begin{cases}
(D_1 - \lambda(x, D_2)) u(t, x) = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^n, \\
|_{t=0} = u_0(x) & \text{on } \mathbb{R}^n,
\end{cases}
$$

and denote by $E(t)$ the operator: $u_0 \rightarrow u(t, \cdot)$. Then $\partial_t \tilde{w}(\ell, x)$ and $\partial_t \tilde{w}(\ell, x)$ are represented as follows:

$$
\begin{align*}
\tilde{w}(\ell, x) &= 2^{-1}E(\ell + r_0) \left( \tilde{w}(-r_0, \cdot) - i\tilde{\mu} \partial_t \tilde{w}(-r_0, \cdot) \right)(x) \\
&\quad + 2^{-1}E(-\ell - r_0) \left( \tilde{w}(-r_0, \cdot) + i\tilde{\mu} \partial_t \tilde{w}(-r_0, \cdot) \right)(x), \\
\partial_t \tilde{w}(\ell, x) &= 2^{-1}E(\ell + r_0) \tilde{\lambda}(\tilde{w}(-r_0, \cdot) - i\tilde{\mu} \partial_t \tilde{w}(-r_0, \cdot))(x) \\
&\quad + 2^{-1}E(-\ell - r_0) i\tilde{\lambda}(\tilde{w}(-r_0, \cdot) + i\tilde{\mu} \partial_t \tilde{w}(-r_0, \cdot))(x),
\end{align*}
$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are pseudo-differential operators whose symbols coincide with
\( \lambda(x, \xi) \) and \( \mu(x, \xi) \) (\( \mu(x, D_x) \) is the parametrix of \( \lambda(x, D_x) \)) respectively in a neighborhood of \( \text{supp}[\gamma(x)] \) and vanish for large \( |x| \). Therefore, noting that

\[
\begin{align*}
\mathcal{L}'[\beta, E(-\tilde{t}-r_0)(\tilde{w}(-r_0, \cdot) + i\bar{\mu}\partial_\xi \tilde{w}(-r_0, \cdot))](\sigma \theta) &= 0(\sigma^{-\infty}), \\
\mathcal{L}'[\beta, E(-\tilde{t}-r_0)(\tilde{\lambda}(\tilde{w}(-r_0, \cdot) + i\bar{\mu}\partial_\xi \tilde{w}(-r_0, \cdot))](\sigma \theta) &= 0(\sigma^{-\infty})
\end{align*}
\]

as \( \sigma \to \infty \) (cf. §4 of the author [15]), we have

\[
F[p(s)S(s, \theta, \omega)](\sigma) = 2^{-1}e^{-i\sigma \tau} \sum_{j=0}^{N-1} (i\sigma)^{\tau-j} \mathcal{L}'[2^{-1}\beta, E(\tilde{t}+r_0)(1+\sigma^{-1}\tilde{\lambda})]
\]

\[
\cdot (\tilde{w}(-r_0, \cdot) - i\bar{\mu}\partial_\xi \tilde{w}(-r_0, \cdot)) (-\sigma \theta) + 0(\sigma^{-N+N_0}).
\]

The assumption (0.5) implies that if \( \text{WF}[\mu_0] \) is contained in a conic neighborhood of \( \tilde{M}_\omega(\theta) \times \{0\} \) (\( \text{WF}[\tilde{w}(\tilde{t}-r_0, \cdot) - i\bar{\mu}\partial_\xi \tilde{w}(-r_0, \cdot)] \) is contained there) \( E(\tilde{t}+r_0)u_0 \) is represented by the Fourier integral operator:

\[
E(\tilde{t}+r_0)u_0(x) = \frac{1}{(2\pi)^{\tau}} \int e^{i\phi(\tilde{t}+r_0, x, \xi)} a(\tilde{t}+r_0, x, \xi) \tilde{\Phi}(\xi)d\xi \mod C^\infty
\]

(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that \( \mathcal{L}'[\delta^{(k)}(\tilde{t}-r_0-x \cdot \omega)](B\eta) = (-i\tau \gamma l) e^{\tau \gamma l} \tilde{\Phi}(\eta) \) (\( \eta = (\eta_1, \eta') \)), where \( B = (b_1, \ldots, b_n) \) is an orthogonal matrix with \( b_1 = \omega \). Then, introducing change of the variables \( x = \tilde{q}(\tilde{t}; -r_0, y, \omega) \) near \( x = \tilde{q}(\tilde{t}; -r_0, \tilde{M}_\omega(\theta), \omega) \) (\( y = (y_0, y') \) is orthogonal coordinates with \( y_0 = x \cdot \omega \)), we obtain

\[
\begin{align*}
\mathcal{L}'[2^{-1}\beta, E(\tilde{t}+r_0)(1+\sigma^{-1}\tilde{\lambda})](\tilde{w}(-r_0, \cdot) - i\bar{\mu}\partial_\xi \tilde{w}(-r_0, \cdot))(-\sigma \theta) &= \int e^{i\sigma \phi(\tilde{t}+r_0, x, \omega)} e^{i\phi(\tilde{t}+r_0, \cdot, \omega)} a(\tilde{t}+r_0, x, \tau \omega, \omega) e^{-i\tau \omega d\tau dx} + 0(\sigma^{-\infty})
\end{align*}
\]

\[
\begin{align*}
= \int \int e^{i\phi(\tilde{t}+r_0, \cdot, \omega)} a(\tilde{t}+r_0, q^{-}(y), -\sigma \tau \omega) \left| \det \frac{\partial q^{-}}{\partial y} \right| + 0(\sigma^{-\infty}) & (\text{as } \sigma \to \infty)
\end{align*}
\]

\( (\tilde{\gamma}(x) \in C^\infty_0(R^n), \tilde{\gamma}(x) = 1 \) on a neighborhood of \( q^{-} \) (\( \text{supp}[\gamma] \)), and \( \tau \) is a positive constant independent of \( \sigma \). The function \( \Phi(y_0, \tau) = q^{-}(y_0, y') \cdot \theta \cdot \tau \cdot (y_0 + r_0) \) has the stationary point \( (y_0, \tau) = (-r_0, \bar{p}^{-}(r_0, y') \cdot \theta) \), at which its Hesse matrix equals

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

Expanding \( \int_{-\infty}^{\infty} e^{i\phi(\tilde{t}+r_0, \tau, \omega)} \beta \gamma \cdots d\tau \) (as \( \sigma \to \infty \)) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion

\[
(3.2) \quad F[p(s)S(s, \theta, \omega)](\sigma) = e^{-i\sigma \tilde{\gamma}(i\sigma)^{\tau-1}} \int_{\gamma_0} e^{i\sigma \tilde{\gamma}(\tilde{t}; -r_0, x, \omega, \theta)} \cdot \left\{ \sum_{j=0}^{N-1} \rho_j(x)(i\sigma)^{-1} \right\} dx + 0(\sigma^{-N+N_0})
\]

(\( N_0 \) is an integer independent of \( N=1, 2, \cdots \)). Here \( \rho_j \) are \( C^\infty \) functions with
supp[\rho_0] \subseteq \text{supp}[\gamma] and all real-valued, which follows from the fact that the symbol \( a(\xi, x, \xi) \) has a homogeneous asymptotic expansion \( \sum_{k=0}^{\infty} a_k(\xi, x, \xi) \) such that \( a_k(\xi, x, \xi) \) are real-valued for even \( k \) and purely imaginary valued for odd \( k \); furthermore \( \rho_0 \) is of the form

\[
\rho_0(y) = \gamma(y)\beta_0(q^-(\xi; -r_0, y, \omega))a_0(\xi + r_0, q^-(\xi; -r_0, y, \omega), -\omega)|\det \frac{\partial q^-}{\partial y}|.
\]

Combining this with (3.1) and (ii) of Lemma 3.2, we see that \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \) and \( \rho_0(x) > 0 \) for any \( x \) at which the function

\[
\varphi(x) = -q^-(\xi; -r_0, x, \omega) \cdot \theta \quad (x \cdot \omega = -r_0)
\]

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

\[
\sigma^m \mathcal{F}[\rho \mathcal{S}(\sigma)] \in L^2(1, \infty)
\]

for some constant \( m \in \mathbb{R} \), which proves Theorem 4.

Proof of Lemma 3.2. We denote by \( y \) the variables on \( \mathbb{R}^{n-1} = \{ x: x \cdot \omega = -r_0 \} \). It follows from (0.4) that for a large constant \( t_0 \) independent of \( t, y \) and \( \omega \)

\[
q^-(t; -r_0, y, \omega) = q^-(t_0; -r_0, y, \omega) + (t - t_0)p^-(t_0; -r_0, y, \omega), \quad t \geq t_0, \quad y \in \mathbb{R}^{n-1}.
\]

Fix \( \bar{y} \in \mathcal{M}_{\omega}(\theta) \) arbitrarily and take a neighborhood \( U(\bar{y}) \) of \( \bar{y} \) such that

\[
|q^-(t_0; -r_0, y, \omega) - q^-(t_0; -r_0, \bar{y}, \omega)| \leq \varepsilon/2 \quad \text{for any } y \in U(\bar{y}),
\]

\[
|t_0\{p^-(t_0; -r_0, y, \omega) - p^-(t_0; -r_0, \bar{y}, \omega)\}| \leq \varepsilon/2 \quad \text{for any } y \in U(\bar{y}).
\]

Then, in view of the definitions of \( \mathcal{M}_{\omega}(\theta) \) and \( \mathcal{S}_{\omega}(\theta) \) we have for any \( y \in U(\bar{y}) \) and \( \bar{t} \geq t_0 \)

\[
\psi(y; \bar{t}) \leq q^-(t_0; -r_0, \bar{y}, \omega) \cdot \theta - t_0p^-(t_0; -r_0, \bar{y}, \omega) \cdot \theta + \bar{t}p^-(t_0; -r_0, y, \omega) \cdot \theta + \varepsilon 
\]

\[
\leq \mathcal{S}_{\omega}(\theta) + \varepsilon + \bar{t}.
\]

On the other hand, for any neighborhood \( U \) of \( \mathcal{M}_{\omega}(\varepsilon) \) it follows that \( \delta = \inf \{1 - \frac{p^-(t_0; -r_0, y, \omega) \cdot \theta}{|y| \leq r_1, \bar{t}}\} > 0 \), which yields that \( \psi(y; t) \leq (C - \delta t) + t \) for any \( y \in U \) (\( |y| \leq r_1 \)) and \( t \geq t_0 \) (\( C \) is a constant independent of \( y \) and \( t \)). This means that

\[
\psi(y; \bar{t}) \leq \mathcal{S}_{\omega}(\theta) - 1 + \bar{t}
\]

if \( y \in U \), \( |y| \leq r_1 \) and \( \bar{t} \) is large enough. Therefore we obtain the lemma.

References


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