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CONDITIONS AGAINST RAPID DECREASE OF
OSCILLATORY INTEGRALS AND THEIR APPLICATIONS
TO INVERSE SCATTERING PROBLEMS

HIDEO SOGA

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Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter $\sigma > 0$:

$$I(\sigma) = \int_{\mathbb{R}^n} e^{-i\varphi(x)} \rho(x; \sigma) dx \quad \text{and} \quad \int e^{i\varphi(x)} \rho(x; \sigma) dx,$$

where $\varphi(x)$ is a real-valued $C^\infty$ function and $\rho(x; \sigma)$ is a $C^\infty$ function with an asymptotic expansion

$$\rho(x; \sigma) \sim \rho_0(x) + \rho_1(x) (i\sigma)^{-1} + \rho_2(x) (i\sigma)^{-2} + \cdots \quad (\text{as } \sigma \to \infty).$$

In this paper we study conditions for the integral $I(\sigma)$ not to decrease rapidly as $\sigma \to \infty$, and solve some inverse scattering problems.

As is well known, if stationary points of $\varphi(x)$ are non-degenerate (i.e. $\det (\partial^2 \varphi(x)) \neq 0$ when $\partial_x \varphi(x) = 0$), $I(\sigma)$ is expanded asymptotically as $\sigma \to \infty$, and we can know whether $I(\sigma)$ decreases rapidly as $\sigma \to \infty$. Also when the stationary points are degenerate, the asymptotic expansion of $I(\sigma)$ is obtained if $\varphi(x)$ is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of $\varphi(x)$ vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of $I(\sigma)$. In the previous paper [13], the author examined the case that $n=2$ and $\rho_j(x) = 0$ $(j \geq 1)$: If $\rho_0(x) \geq 0$ on $\mathbb{R}^2$ and $\rho_0(x_0) > 0$ for a degenerate stationary point $x_0$ of $\varphi(x)$, then $(1 + |\sigma|)^m I(\sigma) \in L^2(\mathbb{R}^2)$ for some $m < 2^{-1}$ (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of $n \geq 3$.

Let $\text{supp} [\rho(\cdot; \sigma)]$ and $\text{supp} [\rho_j] (j \geq 0)$ be contained in a compact set $D$ in $\mathbb{R}^n$. We set

$$E(s) = \{ x : \varphi(x) \leq s \} \quad (s \in \mathbb{R}),$$
One of our main results is the following

**Theorem 1.** Let all \( \rho_j (j \geq 0) \) be real-valued. Then, for every \( m \in \mathbb{R} \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

if and only if for every integer \( N( \geq 1) \)

\[
g_N(s) = g_0(s) + \sum_{j=1}^{N} \int_{0}^{s} \frac{(s-t)^j}{j!} g_j(t) dt \in C^N(\mathbb{R}^1).
\]

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

**Theorem 2.** Let all \( \rho_j (j \geq 0) \) be real-valued, and let \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \). If \( \rho_0 \) satisfies

\[
\rho_0(x) > 0 \quad \text{on} \quad \bigcup_{x \in \partial D} \varphi(x),
\]

then for some \( m(\in \mathbb{R}) \) depending only on the dimension \( n \) we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty).
\]

Theorem 1 implies that decreasingness of \( I(\sigma) \) is connected with smoothness of the measure \( |E(s)| \). This is seen also from the discussions in Vasil’ev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of \( |E(s)| \).

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle \( O \subset \mathbb{R}^n \) with a \( C^\infty \) boundary \( \partial O \). Assume that the domain \( \Omega = \mathbb{R}^n - O \) is connected, and consider the initial-boundary value problem

\[
\begin{cases}
\Box u(t, x) = 0 & \text{in } \mathbb{R}^1 \times \Omega \quad (\Box = \partial^2 t - \Delta), \\
u(t, x') = 0 & \text{on } \mathbb{R}^1 \times \partial \Omega \quad (\partial \Omega = \partial O), \\
u(0, x) = f_1(x) & \text{on } \Omega, \\
\partial_t u(0, x) = f_2(x) & \text{on } \Omega.
\end{cases}
\]

We denote by \( k_-(s, \omega) \) \((k_+(s, \omega)) \in L^2(\mathbb{R}^1 \times S^{n-1})\) the incoming (outgoing) translation representation of the data \((f_1, f_2)\) (cf. Lax and Phillips [6], [7]). The operator \( S: k_- \rightarrow k_+ \) is called the scattering operator and represented by a distribution kernel \( S(s, \theta, \omega) \) called the scattering kernel.
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\[(Sk_{-})(s, \theta) = \iiint S(s-t, \theta, \omega)k_{-}(t, \omega)dt d\omega\]

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of \(\mathcal{O} \subset \mathbb{R}^3\) (i.e. \(n=3\)) that for any fixed \(\omega \in S^2\)

\[(0.2)\]
(i) \(\text{supp } S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)]\),
(ii) \(S(s, -\omega, \omega) \text{ is singular (not } C^\infty\) at \(s = -2r(\omega)\),

where \(r(\omega) = \min x \cdot \omega\). He reduced proof of the above (ii) to verifying that the integral of the form

\[\int_{\mathbb{R}^n} e^{-i\sigma \varphi(x)} \rho(x; \sigma)dx\]

does not decrease rapidly as \(\sigma \to \infty\) (cf. §2 of Majda [8] or §4 of the author [14]).

His methods are not applicable to the case of \(n > 3\), one of whose reasons is that the stationary points of the phase function \(\varphi(x)\) are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when \(n > 3\):

**Theorem 3.** For any fixed \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), we have

(i) \(\text{supp } S(\cdot, \theta, \omega) \subset (-\infty, -r(\omega-\theta)]\),
(ii) \(S(s, \theta, \omega) \text{ is singular at } s = -r(\omega-\theta)\).

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

\[(0.3)\]

\[
\begin{aligned}
\partial_t u(t, x) - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x) \partial_{x_j} u(t, x)) &= 0 \quad \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
u(0, x) &= f_1(x) \quad \text{on } \mathbb{R}^n, \\
\partial_t u(0, x) &= f_2(x) \quad \text{on } \mathbb{R}^n,
\end{aligned}
\]

where \(a_{ij}(x)\) are real-valued \(C^\infty\) functions satisfying

\[
\begin{aligned}
a_{ij}(x) &= a_{ji}(x), \quad x \in \mathbb{R}^n, \\
a_{ij}(x) &= 0 \quad (i \neq j), \quad a_{ii}(x) = 1 \quad \text{when } |x| \geq r_0, \\
\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j &\geq \delta |\xi|^2, \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n.
\end{aligned}
\]

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of \(n \geq 3\). By means of Theorem 2 we get rid of the restriction to the dimension \(n\).

Let us review the results of [15]. We set

\[
\lambda_{\delta}(x, \xi) = -\left\{ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \right\}^{1/2}.
\]
Denote by \((q^-(t; s, x, \xi), p^-(t; s, x, \xi))\) the solution of the equation
\[
\begin{aligned}
\frac{dq^-}{dt} &= -\partial_t \lambda_0(q^-, p^-), \\
\frac{dp^-}{dt} &= \partial_x \lambda_0(q^-, p^-), \\
q^-|_{t=1} &= s, \\
p^-|_{t=1} &= \xi,
\end{aligned}
\]
and for \(\omega, \theta \in S^{n-1}\) set
\[
\begin{aligned}
M_\omega(\theta) &= \{y: y \cdot \omega = -r_0, \lim_{t \to \infty} p^-(t; -r_0, y, \omega) = \theta\}, \\
\bar{s}_\omega(\theta) &= \sup_{x \in \Omega(\theta)} \{\lim_{t \to \infty} (q^-(t; -r_0, y, \omega) \cdot \theta - t)\}, \\
\bar{M}_\omega(\theta) &= \{y \in M_\omega(\theta): \bar{s}_\omega(\theta) = \lim_{t \to \infty} (q^-(t; -r_0, y, \omega) \cdot \theta - t)\}.
\end{aligned}
\]
We assume that for any \(y \ (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\)
\[
\lim_{t \to \infty} |q^-(t; -r_0, y, \omega)| = \infty.
\]
Then singular support of the scattering kernel \(S(\cdot, \theta, \omega)\) for the equation (0.3) is contained in the interval \((-\infty, \bar{s}_\omega(\theta)]\) (cf. Theorem 2 in the author [15]); furthermore, when \(n=2\), it is proved under some assumptions that \(S(s, \theta, \omega)\) is singular at \(s = \bar{s}_\omega(\theta)\) (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of \(n>2\):

**Theorem 4.** Assume (0.4) for any \(y \ (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\). Fix \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), and let the assumption
\[
\det[\partial_t q^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_\omega(\theta)
\]
be satisfied. Then \(S(s, \theta, \omega)\) is singular at \(s = \bar{s}_\omega(\theta)\).

The assumption (0.5) means that there is no caustic on \(\{(t, x): x = q^-(t; -r_0, y, \omega), -r_0 \leq t < \infty, y \in \bar{M}_\omega(\theta)\}\), namely, the mapping: \((t, y) \to q^-(t; -r_0, y, \omega) (-r_0 \leq t < \infty, y \cdot \omega = -r_0)\) is diffeomorphic on \([-r_0, \infty) \times \bar{M}_\omega(\theta)\). In the previous paper [15] we added the assumption
\[
\det[\partial_t p^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_\omega(\theta),
\]
but this is not necessary.

1. **Proofs of Theorem 1 and Theorem 2**

We denote by \(H^m(M)\) the Sobolev space of order \(m\) on \(M\), and by \(H_{loc}^m(M)\) the space of functions \(g(x)\) satisfying \(\alpha(x)g(x) \in H^m(M)\) for any \(\alpha(x) \in C_0^\infty(M)\) \((C_0^\infty(M)\) is the space of \(C^\infty\) functions on \(M\) with compact support).

**Lemma 1.1.** Let \(\varphi(x)\) be a real-valued \(C^m\) function on \(\mathbb{R}^n\), and let \(\rho(x)\) be a \(C^m\) function on \(\mathbb{R}^n\) with compact support. Then the function

\[
\begin{aligned}
\tilde{\varphi}(x) &= \frac{\varphi(x)}{\rho(x)}, \\
\tilde{\rho}(x) &= \rho(x), \\
\tilde{\rho}^m(x) &= \rho^m(x).
\end{aligned}
\]
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Here \( E(s) = \{ x : \varphi(x) \leq s \} \) satisfies

(i) \( g(s) = 0 \) if \( s < \min_{x \in \text{supp}[\rho]} \varphi(x) \),

(ii) \( g(s) \) is constant if \( s > \max_{x \in \text{supp}[\rho]} \varphi(x) \),

(iii) \( g(s) \in H_{10}^m(\mathbb{R}^1) \) for any \( m < 2^{-1} \).

**Proof.** Set

\[
H(s) = \begin{cases} 
1 & \text{for } s \geq 0, \\
0 & \text{for } s < 0.
\end{cases}
\]

Then it follows that \( H(s) \in H_{10}^m(\mathbb{R}^1) \) for any \( m < 2^{-1} \), and so \( H(s - \varphi(x)) \) becomes a \( H_{10}^m(\mathbb{R}^1) \)-valued continuous function on \( \mathbb{R}^1 \). Therefore, noting that \( g(s) = \int_{\mathbb{R}^1} \rho(x) H(s - \varphi(x)) dx \), we obtain (iii). If \( s < \min_{x \in \text{supp}[\rho]} \varphi(x) \) we have \( E(s) \cap \text{supp}[\rho] = \emptyset \), which proves (i). If \( s > \max_{x \in \text{supp}[\rho]} \varphi(x) \), \( E(s) \) contains \( \text{supp}[\rho] \), which yields (ii). The proof is complete.

**Proof of Theorem 1.** It follows from (iii) of Lemma 1.1 that the function \( g_j(s) \) defined in (0.1) belongs to \( L^1_{10}(\mathbb{R}^1) \). Therefore we have

\[
\int_{\mathbb{R}^1} (s-t)^{j-1} g_j(t) dt \in H^j_{10}(\mathbb{R}^1) \quad (j \geq 1),
\]

\[
\partial_s^j \int_{\mathbb{R}^1} (s-t)^{j-1} g_j(t) dt = g_j(s).
\]

Hence the function \( \tilde{g}_N(s) = g_0(s) + \sum_{j=1}^N \int_{\mathbb{R}^1} (s-t)^{j-1} g_j(t) dt \) satisfies

\[
(1.1) \quad \partial_s^j \tilde{g}_N(s) = \sum_{j=0}^N \partial_s^{N-j} g_j(s).
\]

We define \( \hat{I}(\sigma) \) by

\[
\hat{I}(\sigma) = \begin{cases} 
I(\sigma) & \text{for } \sigma > 0, \\
I(-\sigma) & \text{for } \sigma < 0.
\end{cases}
\]

Then \( \sigma^N I(\sigma) \in L^2(1, \infty) \) if and only if \( (1 + |\sigma|)^N \hat{I}(\sigma) \in L^2(\mathbb{R}^1) \). Furthermore, since \( \rho_j(x) \) are assumed real-valued, it follows that for any integer \( N \geq 0 \)

\[
(1.2) \quad \hat{I}(\sigma) = \sum_{j=0}^N \int_{\mathbb{R}^1} e^{-i\sigma \varphi(x)} \rho_j(x) dx (i\sigma)^{-j} + 0(|\sigma|^{-N-1}).
\]

Here \( 0(|\sigma|^k) \) means that \( |0(|\sigma|^k)| \leq C |\sigma|^k (|\sigma| \geq 1) \) for some constant \( C \) independent of \( \sigma \).
Noting that \( \delta(s-\varphi(x)) \) is a \( H^m(\mathbb{R}_1) \)-valued continuous function of \( x \) \((m< -2)\) and equal to \( \partial_x H(s-\varphi(x)) \), we obtain
\[
\int e^{-i\sigma x} \delta(s-\varphi(x)) ds = F[\partial_x H(s-\varphi(x))](\sigma) ,
\]
where \( F \) is the Fourier transformation in \( s \) (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum \( \int_{\mathbb{R}^n} e^{-i\pi H(s-\varphi(x))} \rho_j(x) dx \) in the following way:

\[
\int_{\mathbb{R}^n} e^{-i\pi H(s-\varphi(x))} \rho_j(x) dx = F[\partial_x H(s-\varphi(x)) \rho_j(x) dx](\sigma) = F[\partial_x \mathcal{G}(\sigma)](\sigma).
\]

(1.1), (1.2) and (1.3) yield that

\[
(i\sigma)^{N-1} \mathcal{I}(\sigma) = F[\partial_x^N \mathcal{G}(\sigma)](\sigma) + O(|\sigma|^{-2}).
\]

Let \( (1 + |\sigma|)^N \mathcal{I}(\sigma) \in L^2(\mathbb{R}^1) \) for every \( m \in \mathbb{R} \). Then it follows from (1.4) that

\[
\partial_x^N \mathcal{G}(\sigma) \in H^1(\mathbb{R}^1) ,
\]

which implies

\[
\mathcal{G}(\sigma) \in C^N(\mathbb{R}^1) .
\]

Conversely, let \( \mathcal{G}(\sigma) \in C^N \) for every non-negative integer \( N \). Then we have \( \partial_x^{N+1} \mathcal{G}(\sigma) \in H^{-1}(\mathbb{R}^1) \), which means that \( \partial_x^{N+1} \mathcal{G}(\sigma) \in H^{-1}(\mathbb{R}^1) \) since \( \partial_x^{N+1} \mathcal{G}(\sigma) = 0 \) for large \( |s| \) (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain

\[
(1 + |\sigma|)^{N-1} \mathcal{I}(\sigma) \in L^2(\mathbb{R}^1) \text{ for every integer } N(\geq 1) .
\]

This shows that

\[
(1 + |\sigma|)^N \mathcal{I}(\sigma) \in L^2(\mathbb{R}^1) \text{ for every } m \in \mathbb{R} .
\]

The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that \( s_0 = \min_{x \in D} \varphi(x) = 0 \). Since \( \max_{0 \leq t \leq s} g_j(t) \leq |E(s)| \max_{x \in D} \rho_j(x) \) \((|E(s)| = \int_{E(s)} dx)\), there is a constant \( C \) independent of \( s \) such that

\[
\left| \int_0^s \left( \frac{s-t}{j-1} \right)^{j-1} g_j(t) dt \right| \leq C |s|^j |E(s)| \quad (j \geq 1) .
\]

Therefore we have

\[
|\mathcal{G}(\sigma)| \geq |g_0(\sigma)| - \sum_{j=1}^{N} \left| \int_0^s \left( \frac{s-t}{j-1} \right)^{j-1} g_j(t) dt \right| \\
\geq (\min_{x \in D} \rho_0(x) - C \sum_{j=1}^{N} |s|^j |E(s)|) .
\]
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Since \( \min_{x \in \mathbb{R}^d} \rho_0(x) > 0 \), we obtain \( \min_{x \in \mathbb{R}^d} \rho(x) \geq 2\delta \) for a constant \( \delta > 0 \) independent of \( s \) if \( |s| \) is small enough. Therefore, if \( |s| \) is small enough, it follows that

\[
|\tilde{g}_N(s)| \geq \delta |E(s)|.
\]

Take a point \( x_0 \) satisfying \( \varphi(x_0) = 0 \) (\( = \min_{x \in \mathbb{R}^d} \varphi(x) \)). Then there is a constant \( d > 0 \) such that

\[
E(s) \supset \tilde{E}(s) = \{ x : d |x - x_0| \leq s \},
\]

which yields \( |E(s)| \geq |\tilde{E}(s)| = \delta' s^n \) for \( s \geq 0 \) (the constant \( \delta' \) does not depend on \( s \)). Thus, for any sufficiently small \( s \geq 0 \) we have

\[
(1.5) \quad |\tilde{g}_N(s)| \geq \delta' s^n.
\]

Now, assume that \( \sigma_m I(\sigma) \in L^2(1, \infty) \) for every \( m \in \mathbb{R} \). Then it follows from Theorem 1 that \( \tilde{g}_N(s) \in C^N \) for any integer \( N \geq 0 \). Take the \( N \) so that \( N \geq n+1 \). All the derivatives \( g_N(0), \partial_s g_N(0), \cdots, \partial_s^N g_N(0) \) vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

\[
|\tilde{g}_N(s)| \leq C |s|^{n+1}.
\]

This is not consistent with (1.5) as \( s \to 0^+ \). Therefore we have

\[
\sigma_m I(\sigma) \in L^2(1, \infty)
\]

for some constant \( m \in \mathbb{R} \) depending only on \( n \).

2. Proof of Theorem 3

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let \( v(t, x; \omega) \) be the solution of the equation

\[
\begin{aligned}
\square v(t, x) &= 0 \quad \text{in } \mathbb{R}^d \times \Omega, \\
v(t, x') &= -2^{-1}(-2\pi i)^{1-n} \delta(t-x' \cdot \omega) \quad \text{on } \mathbb{R}^d \times \partial \Omega, \\
v(t, x) &= 0 \quad \text{for } t < r(\omega).
\end{aligned}
\] (2.1)

Then \( v(t, x; \omega) \) is a \( C^m \) function of \( x \) and \( \omega \) with the value \( S'(\mathbb{R}^d) \).

Proposition 2.1. \( S(s, \theta, \omega) \) is represented of the form

\[
S(s, \theta, \omega) = \int_{\partial \Omega} \{ \partial_t \tilde{g}_N(x \cdot \theta - s, x; \omega) - v \cdot \theta \partial_t^{-1} v(x' \cdot \theta - s, x; \omega) \} dS_x, \quad (\omega \neq \theta),
\]

where \( v \) is the outer unit vector normal to \( \partial \Omega \) (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral \( \int \cdot dS_x \) is in the sense of the Riemann
integral with the value \( S'(R^1) \). For the proof see Majda [8] and the author [14].

It is seen that the wave front set of \( \delta(t-x \cdot \omega)|_{R^1 \times \partial \Omega} \) is non-glancing in \( \{(t, x): -r(\omega - \theta) - 2 \eta \leq x \cdot \theta - t \} \cap (R^1 \times \partial \Omega) \) if \( \eta > 0 \) is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution \( v(t, x; \omega) \) of (2.1) mod \( C^\infty \) by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about \( \partial_x v|_{R^1 \times \partial \Omega} \). This is indicated by Majda [8] in the case of \( \theta = -\omega \) (cf. Lemma 2.1 of [8]). We have

**Lemma 2.2.** There exists a first order pseudo-differential operator \( B \) on \( R^1 \times \partial \Omega \) independent of \( t \) such that

(i) its symbol \( B(\bar{x}'; \tau, \bar{\xi}') \) represented near

\[
N(\omega - \theta) = \{x: x \cdot (\theta - \omega) = r(\omega - \theta)\} \cap \partial \Omega
\]

by local coordinates \((t, \bar{x}')\), has a homogeneous asymptotic expansion \( \sum_{j=0} B_j(\bar{x}'; \tau, \bar{\xi}') \)

satisfying

\[
-iB_0(x'; \tau, \bar{\xi}') > 0 \text{ on } N(\omega - \theta) (\bar{\theta}' \text{ is the tangential component of } \theta \text{ to the plane } \{x: x \cdot (\omega - \theta) = r(\omega - \theta)\}),
\]

\[
B_j(x'; \tau, \bar{\xi}') \text{ are purely imaginary-valued for even } j \text{ and real-valued for odd } j,
\]

(ii) \( \partial_x v|_{R^1 \times \partial \Omega} \) is equal to \( B(v|_{R^1 \times \partial \Omega}) \) mod \( C^\infty \) in \( \{(t, x): -r(\omega - \theta) - \eta \leq x \cdot \theta - t \} \cap R^1 \times \partial \Omega \) for some small constant \( \eta > 0 \).

In the above lemma, “a homogeneous asymptotic expansion \( \sum_{j=0} B_j(\bar{x}'; \tau, \bar{\xi}') \)” means that \( B_j(\bar{x}'; \mu \tau, \mu \bar{\xi}') = \mu^{-j} B_j(\bar{x}'; \tau, \bar{\xi}') \) for \( \mu \geq 1 \), \( |\tau| + |\bar{\xi}'| \geq 1 \) and that

\[
|B(\bar{x}'; \tau, \bar{\xi}') - \sum_{j=1}^N B_j(\bar{x}'; \tau, \bar{\xi}')| \leq C_N(|\tau| + |\bar{\xi}'|^{-1})^{-N-1}
\]

for any non-negative integer \( N \) (for detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.); (ii) in the lemma states that

\[
\alpha(t, x') (\partial_x v|_{R^1 \times \partial \Omega} - B(v|_{R^1 \times \partial \Omega})) \in C^\infty \text{ for any } \alpha(t, x') \in C^\infty(R^1 \times \partial \Omega) \text{ with supp}[\alpha] \subset \{(t, x): -r(\omega - \theta) - \eta \leq x \cdot \theta - t\}.
\]

Proof of Lemma 2.2. Let \( \sum_{i=1}^I \chi_i(x) \) be a partition of unity on a neighborhood of \( N(\omega - \theta) \) satisfying \( \max_{1 \leq i \leq I} |\text{supp}[\chi_i]| \leq \varepsilon_0 \) (\( \varepsilon_0 \) is a sufficiently small positive constant). Then there is a constant \( \varepsilon > 0 \) such that \( \sum_{i=1}^I \chi_i(x) = 1 \) for any \( x \in \partial \Omega \) satisfying \( -r(\omega - \theta) - \varepsilon \leq x \cdot \theta - x \cdot \omega \). Let \( v_i(t, x) \) be the solution of the equation

\[
\begin{cases}
\Box v_i(t, x) = 0 & \text{in } R^1 \times \Omega, \\
v_i(t, x') = \chi_i(x') v(t, x'; \omega) & \text{on } R^1 \times \partial \Omega, \\
v_i(t, x) = 0 & \text{for } t < r(\omega).
\end{cases}
\]
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Since \( \text{supp}[\psi_{\Omega}] \subset \{(t, x'): x' \cdot \omega = t\} \), \( \sum_{i=1}^{I} v_i(t, x') \) is equal to \( v(t, x'; \omega) \) on \( (R^i \times \partial \Omega) \cap \{(t, x'): -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\} \), and so, noting that the propagation speed is less than one, we have

\[
v(t, x; \omega) = \sum_{i=1}^{I} v_i(t, x) \quad \text{in} \quad (R^i \times \Omega) \cap \{(t, x'): -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\}.
\]

We denote by \( \text{WF}[f(t, x)] \) the wave front set of \( f(t, x) \). It is seen that \( \text{WF}[\psi_{\Omega}] = \text{WF}[\delta(x' \cdot \omega - t)] \subset \{(t, x'; \tau, \xi') \in R^i \times \partial \Omega, x' \cdot \omega = t = 0, \xi' = -\tau(\omega - \nu(\nu)) \}, \tau \neq 0 \) \( (\nu \) is the outer unit normal to \( \partial \Omega) \). Hence, for any \( (t, x'; \tau, \xi') \in \text{WF}[\psi_{\Omega}] \) the equation \( \tau^2 - |\xi' + \lambda \nu|^2 = 0 \) in \( \lambda \) has real roots, and the null-bicharacteristics associated with \( \partial^2_\tau - \Delta \) through \( \text{WF}[\psi_{\Omega}] \) are transversal to \( R^i \times \partial \Omega \) (non-glancing). This implies that \( \text{sing suppf} \subset \text{WF}[\psi_{\Omega}] \) (cf. Theorem 7 in §9 of (Lax and) Nirenberg [10]), and so it suffices to examine \( v_i(t, x) \) only in a neighborhood \( (t_i - \varepsilon_0, t_i + \varepsilon_0) \times \Omega \) of \( (t_i, x_i) \) \( (x' \in \text{supp}[\chi] \cap \Omega(\omega - \theta) \) and \( t_i = x_i \cdot \omega) \).

To analyze \( v_i \) more precisely, we transform \( \Omega \) in \( \mathbb{C}/\mathbb{Z} \) into the half-space \( \Omega^+ = \{x = (x', x_0): x_0 > 0\} \). Let the derivative \( \partial_\nu \) be transformed in \( U_i \) into \( -\partial_{x_0} \). For any set \( M \) in \( \mathbb{R}^n \) we denote by \( M \) the set transformed by the coordinates \( x \). Let \( \Delta \) be represented by \( \xi \) of the form \( \Delta = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha \). Here we can assume that the coefficients \( a_\alpha(x) \) are real-valued \( C^m \) functions defined on \( \mathbb{R}^n \) and constant out of \( U_i \). Let us examine the solution \( \vartheta(t, x) \) of the following equation instead of \( v_i(t, x) \):

\[
\begin{cases}
(\partial_\tau^2 + A)\vartheta(t, x) = 0 & \text{in} \quad \mathbb{R}^i \times \mathbb{R}^*_+, \\
\vartheta(t, x) = g(t, x') & \text{on} \quad \mathbb{R}^i \times \mathbb{R}^-^i, \\
\vartheta(t, x) = 0 & \text{for} \quad t < t_i - \varepsilon_0,
\end{cases}
\]

where \( g(t, x') = -2^{-\frac{i}{2}}(-2\pi i)^{-n} \delta(x(x') \cdot \omega - t) \chi_i(x(x')) \). Note that \( \text{WF}[g(t, x')] \) is contained in a sufficiently small conic neighborhood of \( (t_i, x'; \pm 1, \mp \partial_\theta') \) \( (\partial_\theta' \) is the component of \( \theta \) (transformed by the coordinates \( x \)) tangent to the plane \( x_0 = 0 \), and that if \( |(\tau, \xi')|^{-1}|(\tau, \xi')|^{-1}(\pm 1, \mp \partial_\theta') \) the equation

\[
\tau^2 + A_\delta(x; \xi', \xi_0) = 0
\]

\( (A_\delta(x, \xi) = \sum_{|\alpha| = 2} a_\alpha(x) \xi^\alpha) \) in \( \xi_0 \) has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators \( \xi^\pm(x; D_\tau, D_\theta) \) on \( \mathbb{R}^i \times \mathbb{R}^*_x \) (independent of \( t \)) with homogeneous asymptotic expansions \( \sum_{j=0}^{\infty} \xi^j(x; \tau, \xi') \) such that

\[
\begin{align*}
(1) & \quad \xi^j(x; \tau, \xi') \text{ are real-valued for even} \quad j \quad \text{and purely imaginary-valued for odd} \quad j,
\end{align*}
\]
(ii) if \(|(r, \xi')|^{-1}(r, \xi')|^{-1}(r, \xi') = |(1, -\theta')|^{-1}(1, -\theta')\),
\(\xi_0^\pm(\vec{x}; r, \xi')\) are equal to the roots of the equation (2.4), and
\[
\xi_0^\pm(\vec{x}; \pm 1, \mp \theta') = \mp (1 - |\theta'|^2)^{1/2},
\]

(iii) all the null-bicharacteristic curves associated with \(D_{\vec{x}_0} - \xi_0^\pm(\vec{x}; D_t, D_{\vec{x}})\)
through \(\text{WF}[g(t, \vec{x}')]\) are transversal to the boundary \(\{\vec{x}_0 = 0\}\) and proceed in the
direction \(t > 0\) as they leave the boundary,

(iv) if the wave front set of \(u(t, x)\) is near the bicharacteristic curves stated
in the above (iii), then we have
\[
(D_{\vec{x}_0} - \xi(\vec{x}; D_t, D_{\vec{x}}))(D_{\vec{x}_0} - \xi^+(\vec{x}; 0, D_t, D_{\vec{x}}))u = \zeta(\vec{x})(\partial_\vec{x}^2 + A)u \mod C^\infty,
\]
where \(\zeta(\vec{x})\) is a \(C^\infty\) function on \(\mathbb{R}^n\) satisfying \(\zeta(\vec{x}) < 0\) for every \(\vec{x}\).

(iii) and (iv) imply that \(v(t, x, \omega)\) is approximated \(\mod C^\infty\) by the solution
\(w(\vec{x}_0; t, \vec{x}')\) of the equation
\[
(D_{\vec{x}_0} - \xi(\vec{x}; D_t, D_{\vec{x}}))w = 0, \quad \vec{x}_0 > 0,
\]
\[
\{ w |_{\vec{x}_0 = 0} = h(t, \vec{x}') \}.
\]

Therefore we have
\[
-\partial_{\vec{x}_0} \vec{v} \big|_{\vec{x}_0 = 0} = -i \xi^+(\vec{x}', 0; D_t, D_{\vec{x}}) (\vec{v} \big|_{\vec{x}_0 = 0}) \mod C^\infty.
\]
Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution \(v(t, x; \omega)\) in (2.1) satisfies \(\text{supp}[v |_{\mathbb{R}^n \times \mathbb{R}^n}] \subset \{(t, x): x \cdot \omega < t\}\). Therefore, noting that the propagation speed is less than one, we see that \(\text{supp}[v(t, x; \omega)] \subset \{(t, x); x \cdot \omega < t\}\), which yields
\[
v(x \cdot \theta - s, x; \omega) = 0 \quad \text{if } s > x \cdot (\theta - \omega).
\]
Hence, if \(s > \max x \cdot (\theta - \omega) = -r(\omega - \theta)(\omega \neq \theta)\), we obtain \(S(s, \theta, \omega) = 0\) from Proposition 2.1.

Next, let us prove that \(S(s, \theta, \omega)\) is singular at \(s = -r(\omega - \theta)\). Take \(\alpha(s) \in C^\infty(\mathbb{R}^1)\) such that \(0 \leq \alpha \leq 1\) on \(\mathbb{R}^1\), \(\alpha(s) = 1\) for \(|s| \leq 2^{-1}\) and \(\alpha(s) = 0\) for \(|s| \geq 1\). For any \(\varepsilon > 0\) set
\[
\alpha_\varepsilon(s) = \alpha\left(\frac{s + r(\omega - \theta)}{2\varepsilon}\right).
\]
Then we have only to prove that \(\alpha_\varepsilon(s)S(s, \theta, \omega)\) is not \(C^\infty\) for any small \(\varepsilon > 0\). Proposition 2.1 yields
\[
\alpha_\varepsilon(s)S(s, \theta, \omega) = \int_{\partial \Omega} \alpha_\varepsilon(s) (\partial_t - \vec{v} \cdot \partial_x)v (x \cdot \theta - s, x; \omega) dS_x
\]
\[
- \int_{\partial \Omega} v \cdot \theta \alpha_\varepsilon(s) (\partial_t - \vec{v}) (x \cdot \theta - s, x; \omega) dS_x = f_1(s) + f_2(s).
\]
Let \( F[k(s)](\sigma) = \int e^{i\sigma k(s)} ds \). As is readily seen, it follows that

\[
F[J_2(s)](\sigma) = -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} C_j^{-1}(i\sigma)^{n-1-j} \int_{\Omega} e^{i\sigma(\theta-\omega)} (-\nu \cdot \theta) \cdot \alpha_j^{(j)}(x \cdot (\theta-\omega)) dS_x
\]

(where \( C_j^{-1} = (n-1)!/(n-1-j)! j! \)). Taking the \( \varepsilon(>0) \) so that \( 2\varepsilon \leq \eta \), by Lemma 2.2 we have

\[
F[J_1(s)](\sigma) = \int \int_{\Omega \times \Omega} e^{i\sigma(x'-\omega)} \alpha_4(x' \cdot \theta - s) \partial_{\sigma}^{-2} [B_0|_{\Omega \times \Omega}] (x, x) dS_x
\]

\[
= -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-2} C_j^{-2} \int_{\Omega} iB[e^{i\sigma(x'-\omega)} \alpha_j^{(j)}(x \cdot \theta - s)]_{x=\omega} dS_x
\]

Here \( B \) denotes the transposed operator of \( B \) (i.e. \( \langle Bf, g \rangle = \langle f, Bg \rangle \) for any \( f \) and \( g \in C^\infty_c(\mathbb{R}^n \times \partial \Omega) \)). Let us note that the symbol of \( B \) expressed near supp \([\alpha_4(x' \cdot \theta - t)] \cap (\Omega \times \partial \Omega)\) by the local coordinates \((t, x')\), has a homogeneous asymptotic expansion \( \sum_{j=0}^{n} iB_j(x'; \tau, \xi') \) such that \( B_j(x'; \tau, \xi') \) are real-valued for odd \( j \) and purely imaginary valued for even \( j \) and that \(-iB_0(x'; \pm 1, \mp \xi') = -iB_0(x'; \pm 1, \mp \xi') = 0\) for \( x' \in \partial \Omega \), which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand \( B[e^{i\sigma(x'-\omega)} \alpha_j^{(j)}(x \cdot \theta - s)] \) asymptotically (as \( \sigma \to \infty \)) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion

\[
F[J](\sigma) \sim -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n} (i\sigma)^{n-1-j} \int_{\Omega} e^{i\sigma(x'-\omega)} \beta_j(x) dS_x \quad (\sigma \to \infty),
\]

where \( \beta_j(x) \) are real-valued \( C^\infty \) functions on \( \partial \Omega \) with supp \([\beta_j] \subseteq \text{supp} [\alpha_4(x' \cdot (\theta - \omega))] \cap \partial \Omega \), and \( \beta_0(x) \) is non-negative valued and satisfies

\[
\beta_0(x) = -iB_0(x'; -1, \mp \xi') \alpha_4(x' \cdot (\theta - \omega)) > 0 \quad \text{for } x' \in \partial \Omega (\omega - \theta).
\]

Combining (2.5) and (2.6) yields that for any integer \( N(>0) \)

\[
F[\alpha_4(s)S(s, \theta, \omega)](\sigma) = -2^{-1}(-2\pi i)^{1-n} (i\sigma)^{n-1} \int_{\Omega} e^{-i\sigma(x' \cdot (\omega - \theta)} \cdot \{ \sum_{j=0}^{n-1} \rho_j(x') (i\sigma)^{-j} \} dS_x + O(\sigma^{-N}).
\]

Here \( x' \) is the local coordinates on \( \partial \Omega \) near \( N(\omega - \theta) \) and

\[
\rho_j(x') = \beta_j(x(x')) + (-\nu \cdot \theta) \alpha_j^{(j)}(x(x') \cdot (\theta - \omega)) \quad (\alpha_j^{(j)} = 0, j \geq n).
\]

Noting that \( \rho_0(x') > 0 \) when the phase function \( x(x') \cdot (\omega - \theta) \) is minimum, and applying Theorem 2, we obtain for some constant \( m \in \mathbb{R} \)

\[
\sigma^m F[\alpha_4(s)S(s, \theta, \omega)](\sigma) \in L^2(1, \infty),
\]
which shows that \( \alpha(s)S(s, \theta, \omega) \) is not \( C^\infty \). The proof is complete.

3. Proof of Theorem 4

We use the same notations as for the scattering by obstacles in §2. The scattering operator \( S \) for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

**Proposition 3.1.** Set

\[
S_0(s, \theta, \omega) = \int_{\mathbb{R}^n} \left( \partial_t^{n-2} \mathbb{H} \right)(x \cdot \theta - x, x) dx ,
\]

\[
Kk = F^{-1}[\text{sgn } \sigma]^{n-1}(Fk) \left( \sigma \right) ,
\]

where \( w(t, x) \) is the solution of the equation

\[
\begin{cases}
(\partial_t^2 - \mathbb{A})w(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n , \\
-w(-r_0, x) = -2^{-1}(-2\pi i)^{n-1}\delta(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n , \\
\partial_t w(-r_0, x) = -2^{-1}(-2\pi i)^{n-1}\delta'(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n .
\end{cases}
\]

Then we have

\[
(Sk)(s, \theta) = \int \int S_0(s-t, \theta, \omega)k(t, \omega) dtd\omega + (Kk)(s, \theta) .
\]

Note that \( S_0(s, \theta, \omega) = S(s, \theta, \omega) \) if \( \omega \neq \theta \).

To prove Theorem 4, we have only to show that for any small \( \varepsilon > 0 \) there exist a real number \( m \) and a function \( \rho(s) \in \mathcal{C}_0^\infty(s_\omega(\theta) - 2\varepsilon, s_\omega(\theta) + 2\varepsilon) \) such that

\[
(1 + |\sigma|)^m F[\rho(s)S(s, \theta, \omega)](\sigma) \in L^2(\mathbb{R}^1) .
\]

Let \( \gamma(x) \in C_c^\infty(\mathbb{R}^n) \) with \( \gamma(x) = 1 \) in a neighborhood of \( \overline{M_\omega}(\theta) \), and denote by \( \bar{w}(t, x) \) the solution of the equation

\[
\begin{cases}
(\partial_t^2 - \mathbb{A})\bar{w}(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n , \\
\bar{w}(-r_0, x) = \gamma(x)w(-r_0, x) & \text{on } \mathbb{R}^n , \\
\partial_t \bar{w}(-r_0, x) = \gamma(x)\partial_t w(-r_0, x) & \text{on } \mathbb{R}^n .
\end{cases}
\]

The author [15] showed that if \( \tilde{t} \) is large enough we have for any integer \( N(>0) \)

\[
F[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1}e^{-i\sigma \tilde{t}} \sum_{j=0}^{N-1} (i\sigma)^{n-1-j} \mathcal{F}'[\beta_j(x) \left\{ \bar{w}(\tilde{t}, x) \right\}] (-\sigma\theta) + O(\sigma^{-N+N_0})
\]

as \( \sigma \to \infty \) (\( N_0 \) is an integer independent of \( N \)) (cf. (4.5) in [15]). Here, \( \mathcal{F}' \) denotes the Fourier transformation in \( x \), and the functions \( \beta_j(x) \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) are all real-valued.
We take \( t \) so large as to have (i) and (ii) stated in the following

**Lemma 3.2.** Let \( r_1 \) be an arbitrary constant (\( \geq r_0 \)), and set

\[
\varphi(x; t) = q^\prime(t; -r_0, x, \omega) \cdot \theta.
\]

Then, for any \( \epsilon(>0) \) there is a constant \( t_0 \) such that for any fixed \( t \geq t_0 \)

(i) \( \max_{|x| \leq r_1} |\varphi(x; t)| \leq \varphi_0(\theta) + t + \epsilon \),

(ii) all points at which \( \varphi(x; t) \) is maximum \( (x \cdot \omega = -r_0, |x| \leq r_1) \), are contained in \( \epsilon \)-neighborhood \( (M_\omega(\theta))_\epsilon \) of \( M_\omega(\theta) \) \( ((M_\omega(\theta) = \{ x : \text{dis}(x, M_\omega(\theta)) < \epsilon \}) \).

This lemma will be proved later. Choose the \( \rho(s) \) so that \( \rho(s) \geq 0 \) on \( R^1 \) and \( \rho(s) > 0 \) on \([s_\omega(\theta)-\epsilon, s_\omega(\theta)+\epsilon]\). Then it is seen from the form of \( \beta_0(x) \) (cf. (4.4) and (4.6) in [15]) and the above lemma that

\[
(3.1) \quad \beta_0(x) \geq 0 \quad \text{on} \quad R^n \quad \text{and} \quad \beta_0(q^\prime(t; -r_0, y, \omega)) > 0 \quad \text{for any} \quad y \in (M_\omega(\theta))_\epsilon \quad (y \cdot \omega = -r_0).
\]

We take the \( \gamma(x) \) so that \( \gamma(x) \geq 0 \) on \( R^n \), \( \gamma(x) > 0 \) on \( (M_\omega(\theta))_\epsilon \) and \( \text{supp}[\gamma] \subset (M_\omega(\theta))_\epsilon \).

By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. §5 of [10] or Appendix II of [5]), we can construct a symbol \( \lambda(x, \xi) \) with a homogeneous asymptotic expansion \( \sum_{j=0}^\infty \lambda_j(x, \xi) \) such that

\[
\lambda_0(x, \xi) = \left\{ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \right\}^{1/2},
\]

\[-\partial_t^2 + A = (D_t + \lambda(x, D_x))(D_t - \lambda(x, D_x)) \quad \text{modulo a smoothing operator}
\]

(cf. Corollary 2.5 in the author [15] also). Furthermore we see that \( \lambda_j(x, \xi) \) are real-valued for even \( j \) and purely imaginary valued for odd \( j \) since the coefficients \( a_{ij}(x) \) are all real-valued (recall the construction of \( \xi^\pm(x'; \tau, \xi') \) in §2).

Consider the Cauchy problem

\[
\begin{cases}
(D_t - \lambda(x, D_x))u(t, x) = 0 & \text{in} \quad R^1 \times R^n, \\
|_{t=0} = u_0(x) & \text{on} \quad R^n,
\end{cases}
\]

and denote by \( E(t) \) the operator: \( u_0 \rightarrow u(t, \cdot) \). Then \( \tilde{\omega}(\tilde{t}, x) \) and \( \partial_t \tilde{\omega}(\tilde{t}, x) \) are represented as follows:

\[
\tilde{\omega}(\tilde{t}, x) = 2^{-1}E(\tilde{t} + r_0) \left( \omega(-r_0, \cdot) - i\bar{\mu} \partial_t \omega(-r_0, \cdot) \right)(x)
\]

\[
+ 2^{-1}E(-\tilde{t} - r_0) \left( \omega(-r_0, \cdot) + i\mu \partial_t \omega(-r_0, \cdot) \right)(x),
\]

\[
\partial_t \tilde{\omega}(\tilde{t}, x) = 2^{-1}E(\tilde{t} + r_0)i\lambda(\omega(-r_0, \cdot) - i\bar{\mu} \partial_t \omega(-r_0, \cdot))(x)
\]

\[
+ 2^{-1}E(-\tilde{t} - r_0)i\bar{\lambda}(\omega(-r_0, \cdot) + i\mu \partial_t \omega(-r_0, \cdot))(x),
\]

where \( \lambda \) and \( \bar{\mu} \) are pseudo-differential operators whose symbols coincide with
\( \lambda(x, \xi) \) and \( \mu(x, \xi) \) (\( \mu(x, D_x) \)) is the parametrix of \( \lambda(x, D_x) \) respectively in a neighborhood of \( \text{supp}[\gamma(x)] \) and vanish for large \( |x| \). Therefore, noting that
\[
\mathcal{L}'[\beta, E(-\dot{t} - r_0) (\dot{w}(-r_0, \cdot) + i\dot{\mu}\partial_t \dot{w}(-r_0, \cdot))] \quad (\sigma \theta) = 0(\sigma^{-\infty}),
\]
\[
\mathcal{L}'[\beta, E(-\dot{t} - r_0) \dot{\lambda}(\dot{w}(-r_0, \cdot) + i\dot{\mu}\partial_t \dot{w}(-r_0, \cdot))] \quad (\sigma \theta) = 0(\sigma^{-\infty})
\]
as \( \sigma \to \infty \) (cf. §4 of the author [15]), we have
\[
F[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1}e^{-i\sigma \gamma} \sum_{j=0}^{\gamma-1} (i\sigma)^{j-1} \mathcal{L}'[2^{-1} \beta, E(\dot{t} + r_0)(1 + \sigma^{-1}\dot{\lambda})]
\]
\[
\cdot (\dot{w}(-r_0, \cdot) - i\dot{\mu}\partial_t \dot{w}(-r_0, \cdot)) \quad (\sigma \theta) + 0(\sigma^{-N+N_0}).
\]

The assumption (0.5) implies that if \( \text{WF}[u_0] \) is contained in a conic neighborhood of \( \text{supp} \gamma \times \{0\} \) (\( \text{WF}[\dot{w}(-r_0, \cdot) - i\dot{\mu}\partial_t \dot{w}(-r_0, \cdot)] \) is contained there) \( E(\dot{t} + r_0)u_0 \) is represented by the Fourier integral operator:
\[
E(\dot{t} + r_0)u_0(x) = (2\pi)^{-n} \int e^{i\dot{\Phi}(\dot{t} + r_0, x, \xi)} a(\dot{t} + r_0, x, \xi) d\xi \quad \text{mod } C^\infty
\]
(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that \( \mathcal{L}'[\delta^{(k)} (\dot{t} - r_0 - x \cdot \omega)](B\gamma) = (\dot{t} - r_0 - x \cdot \omega) \) where \( B = (b_1, \ldots, b_n) \) is an orthogonal matrix with \( b_1 = \omega \). Then, introducing change of the variables \( x = q(\dot{t}; -r_0, y, \omega) \) near \( x = q(\dot{t}; -r_0, M_\omega(\theta), \omega) \) \( y = (y_0, y') \) is orthogonal coordinates with \( y_0 = x \cdot \omega \), we obtain
\[
\mathcal{L}'[2^{-1} \beta, E(\dot{t} + r_0)(1 + \sigma^{-1}\dot{\lambda}) (\dot{w}(-r_0, \cdot) - i\dot{\mu}\partial_t \dot{w}(-r_0, \cdot))] \quad (\sigma \theta)
\]
\[
= \int e^{i(\dot{t} - r_0 - x \cdot \omega) \beta}(\dot{t} + r_0, x, -\tau \omega) e^{-i\tau \omega \partial_t \dot{r} dx + 0(\sigma^{-\infty})}
\]
\[
= \int_{\mathbb{R}^n} dy' \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\gamma \int_{-\infty}^{\infty} d\gamma \beta_j(q(\gamma)) \gamma(y)
\]
\[
\cdot a(\dot{t} + r_0, q(\gamma), -\sigma \tau \omega) |\det \partial q/\partial y| + 0(\sigma^{-\infty}) \quad (\text{as } \sigma \to \infty)
\]
(\( \gamma(x) \subseteq C^\infty(\mathbb{R}^n) \), \( \gamma(x) = 1 \) on a neighborhood of \( q \) (\( \text{supp} \gamma \)), and \( \tau \) is a positive constant independent of \( \sigma \). The function \( \Phi(y_0, \tau) = q(y_0, y') \cdot \tau - \tau(y_0 + r_0) \) has the stationary point \( (y_0, \tau) = (-r_0, \dot{t} - r_0, y') \), at which its Hesse matrix equals
\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]
Expanding \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\dot{t} - r_0 - x \cdot \omega) \beta}(\dot{t} + r_0, y, \omega) dy d\tau \) (as \( \sigma \to \infty \)) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion
\[
(3.2) \quad F[\rho(s)S(s, \theta, \omega)](\sigma) = e^{-i\sigma \dot{t}(i\sigma)^{n-1} \sum_{s=0}^{\gamma-1} e^{i\sigma \dot{t}(-r_0, s \cdot \omega) \beta} \sum_{j=0}^{\gamma-1} \rho_j(x) (i\sigma)^{-j} dx + 0(\sigma^{-N+N_0})
\]
(\( N_\gamma \) is an integer independent of \( N = 1, 2, \ldots \)). Here \( \rho_j \) are \( C^\infty \) functions with
supp[ρ] ⊆ supp[γ] and all real-valued, which follows from the fact that the symbol \( a(\tilde{t}, x, \xi) \) has a homogeneous asymptotic expansion \( \sum_{k=0}^{\infty} a_k(\tilde{t}, x, \xi) \) such that

\[ a_k(\tilde{t}, x, \xi) \] are real-valued for even \( k \) and purely imaginary valued for odd \( k \); furthermore \( \rho_0 \) is of the form

\[ \rho_0(y) = \gamma(y) \beta_0(q^-(\tilde{t}; -r_0, y, \omega))a_0(\tilde{t} + r_0, q^-(\tilde{t}; -r_0, y, \omega), -\omega) |\det \frac{\partial q^-}{\partial y}|. \]

Combining this with (3.1) and (ii) of Lemma 3.2, we see that \( \rho_0(x) \geq 0 \) on \( \mathbb{R}^n \) and \( \rho_0(x) > 0 \) for any \( x \) at which the function

\[ \varphi(x) = -q^-(\tilde{t}; -r_0, x, \omega) \cdot \theta \quad (x \cdot \omega = -r_0) \]

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

\[ \sigma^m F[pS](\sigma) \in L^2(1, \infty) \]

for some constant \( m \in \mathbb{R} \), which proves Theorem 4.

Proof of Lemma 3.2. We denote by \( y \) the variables on \( \mathbb{R}^{n-1} = \{ x: x \cdot \omega = -r_0 \} \).

It follows from (0.4) that for a large constant \( t_0 \) independent of \( t, y \) and \( \omega \)

\[ q^-(t; -r_0, y, \omega) = q^-(t_0; -r_0, y, \omega) + (t-t_0)p^-(t_0; -r_0, y, \omega), \quad t \geq t_0, y \in \mathbb{R}^{n-1}. \]

Fix \( \bar{y} \in M_\omega(\theta) \) arbitrarily and take a neighborhood \( U(\bar{y}) \) of \( \bar{y} \) such that

\[ |q^-(t_0; -r_0, y, \omega) - q^-(t_0; -r_0, \bar{y}, \omega)| \leq \varepsilon/2 \quad \text{for any } y \in U(\bar{y}), \]

\[ |t_0(p^-(t_0; -r_0, y, \omega) - p^-(t_0; -r_0, \bar{y}, \omega))| \leq \varepsilon/2 \quad \text{for any } y \in U(\bar{y}). \]

Then, in view of the definitions of \( M_\omega(\theta) \) and \( s_\omega(\theta) \) we have for any \( y \in U(\bar{y}) \) and \( \tilde{t} \geq t_0 \)

\[ \psi(y; \tilde{t}) \leq q^-(t_0; -r_0, \bar{y}, \omega) \cdot \theta - t_0 p^-(t_0; -r_0, \bar{y}, \omega) \cdot \theta + \tilde{t} p^-(t_0; -r_0, \omega) \cdot \theta + \varepsilon \leq s_\omega(\tilde{t}) + \varepsilon + \tilde{t}. \]

On the other hand, for any neighborhood \( U \) of \( M_\omega(\varepsilon) \) it follows that \( \delta = \inf \{ 1 - p^-(t_0; -r_0, y, \omega) \cdot \theta \} > 0 \), which yields that \( \psi(y; t) \leq (C - \delta t) + t \) for any \( y \in U \) (\( |y| \leq r_1 \)) and \( t \geq t_0 \) (\( C \) is a constant independent of \( y \) and \( t \)). This means that

\[ \psi(y; \tilde{t}) \leq s_\omega(\theta) - 1 + \tilde{t} \]

if \( y \in U \), \( |y| \leq r_1 \) and \( \tilde{t} \) is large enough. Therefore we obtain the lemma.

References


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