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Osaka University
CONDITIONS AGAINST RAPID DECREASE OF OSCILLATORY INTEGRALS AND THEIR APPLICATIONS TO INVERSE SCATTERING PROBLEMS

HIDEO SOGA

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Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter $\sigma > 0$:

$$I(\sigma) = \int_{\mathbb{R}^n} e^{-i\varphi(x)} \rho(x; \sigma) dx \quad \text{or} \quad \int e^{i\varphi(x)} \rho(x; \sigma) dx,$$

where $\varphi(x)$ is a real-valued $C^\infty$ function and $\rho(x; \sigma)$ is a $C^\infty$ function with an asymptotic expansion

$$\rho(x; \sigma) \sim \rho_0(x) + \rho_1(x)(i\sigma)^{-1} + \rho_2(x)(i\sigma)^{-2} + \cdots \quad \text{as } \sigma \to \infty.$$

In this paper we study conditions for the integral $I(\sigma)$ not to decrease rapidly as $\sigma \to \infty$, and solve some inverse scattering problems.

As is well known, if stationary points of $\varphi(x)$ are non-degenerate (i.e. $\det (\partial_x^2 \varphi(x)) \neq 0$ when $\partial_x \varphi(x) = 0$), $I(\sigma)$ is expanded asymptotically as $\sigma \to \infty$, and we can know whether $I(\sigma)$ decreases rapidly as $\sigma \to \infty$. Also when the stationary points are degenerate, the asymptotic expansion of $I(\sigma)$ is obtained if $\varphi(x)$ is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of $\varphi(x)$ vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of $I(\sigma)$. In the previous paper [13], the author examined the case that $n=2$ and $\rho_1(x)=0$ ($j \geq 1$): If $\rho_0(x) \geq 0$ on $\mathbb{R}^2$ and $\rho_0(x_0) > 0$ for a degenerate stationary point $x_0$ of $\varphi(x)$, then $(1+|\sigma|)^n I(\sigma) \in L^2(\mathbb{R}^2)$ for some $m < 2^{-1}$ (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of $n \geq 3$.

Let $\text{supp}[\rho(\cdot; \sigma)]$ and $\text{supp}[\rho_j]$ ($j \geq 0$) be contained in a compact set $D$ in $\mathbb{R}^n$. We set

$$E(s) = \{x: \varphi(x) \leq s\} \quad (s \in \mathbb{R}),$$
One of our main results is the following

**Theorem 1.** Let all $\rho_j$ $(j \geq 0)$ be real-valued. Then, for every $m \in \mathbb{R}$ we have

$$\sigma^m I(\sigma) \in L^2(1, \infty)$$

if and only if for every integer $N(\geq 1)$

$$\tilde{g}_N(s) \equiv \tilde{g}_0(s) + \sum_{j=1}^{N} \frac{(t-t)^{j-1}}{(j-1)!} g_j(t) dt \in C^N(\mathbb{R}^d).$$

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

**Theorem 2.** Let all $\rho_j$ $(j \geq 0)$ be real-valued, and let $\rho_0(x) \geq 0$ on $\mathbb{R}^n$. If $\rho_0$ satisfies

$$\rho_0(x) > 0 \text{ on } E(\min \varphi(x)),$$

then for some $m(\in \mathbb{R})$ depending only on the dimension $n$ we have

$$\sigma^m I(\sigma) \in L^2(1, \infty).$$

Theorem 1 implies that decreasingness of $I(\sigma)$ is connected with smoothness of the measure $|E(s)|$. This is seen also from the discussions in Vasil'ev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of $|E(s)|$.

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle $\Omega \subset \mathbb{R}^n$, with $C^\infty$ boundary $\partial \Omega$. Assume that the domain $\Omega = \mathbb{R}^n - \Omega$ is connected, and consider the initial-boundary value problem

$$\begin{cases}
\Box u(t, x) = 0 \quad \text{in } \mathbb{R}^1 \times \Omega \\
u(t, x') = 0 \quad \text{on } \mathbb{R}^1 \times \partial \Omega \\
u(0, x) = f_1(x) \quad \text{on } \Omega \\
\partial_t u(0, x) = f_2(x) \quad \text{on } \Omega.
\end{cases}$$

We denote by $k_-(s, \omega)$, $(k_+(s, \omega)) \in L^2(\mathbb{R}^1 \times S^{n-1})$ the incoming (outgoing) translation representation of the data $(f_1, f_2)$ (cf. Lax and Phillips [6], [7]). The operator $S: k_- \rightarrow k_+$ is called the scattering operator and represented by a distribution kernel $S(s, \theta, \omega)$ called the scattering kernel:
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\[(Sk_{-})(s, \theta) = \iiint S(s-t, \theta, \omega)k_{-}(t, \omega)dtd\omega\]

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of \(O \subset R^3\) (i.e. \(n=3\)) that for any fixed \(\omega \in S^2\)

\[(0.2)\]

\[
\begin{align*}
&\text{(i) } \text{supp } S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)], \\
&\text{(ii) } S(s, -\omega, \omega) \text{ is singular (not } C^\infty) \text{ at } s = -2r(\omega),
\end{align*}
\]

where \(r(\omega) = \min x \cdot \omega\). He reduced proof of the above (ii) to verifying that the integral of the form

\[\int_{R^n} e^{-i\sigma\varphi(x)}\rho(x; \sigma)dx\]

does not decrease rapidly as \(\sigma \to \infty\) (cf. §2 of Majda [8] or §4 of the author [14]). His methods are not applicable to the case of \(n>3\), one of whose reasons is that the stationary points of the phase function \(\varphi(x)\) are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when \(n>3\):

**Theorem 3.** For any fixed \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \pm \theta\), we have

\[
\begin{align*}
&\text{(i) } \text{supp } S(\cdot, \theta, \omega) \subset (-\infty, -r(\omega-\theta)], \\
&\text{(ii) } S(s, \theta, \omega) \text{ is singular at } s = -r(\omega-\theta).
\end{align*}
\]

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

\[
\begin{align*}
\partial_t u(t, x) - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(t, x)) &= 0 \quad \text{in } R^1 \times R^n, \\
u(0, x) &= f_1(x) \quad \text{on } R^n, \\
\partial_t u(0, x) &= f_2(x) \quad \text{on } R^n,
\end{align*}
\]

where \(a_{ij}(x)\) are real-valued \(C^\infty\) functions satisfying

\[
\begin{align*}
a_{ij}(x) &= a_{ji}(x), \quad x \in R^n, \\
a_{ij}(x) &= 0 \quad (i \neq j), \quad a_{ii}(x) = 1 \quad \text{when } |x| \geq r_0, \\
\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j &= \delta |\xi|^2, \quad x \in R^n, \quad \xi \in R^n.
\end{align*}
\]

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of \(n \geq 3\). By means of Theorem 2 we get rid of the restriction to the dimension \(n\).

Let us review the results of [15]. We set

\[
\lambda_0(x, \xi) = -\left\{\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\right\}^{1/2}.
\]
Denote by \((q^-(t; s, x, \xi), p^-(t; s, x, \xi))\) the solution of the equation
\[
\begin{align*}
\frac{dq^-}{dt} &= -\partial_t\lambda\overline{q}^-, \quad \frac{dp^-}{dt} = \partial_x\lambda\overline{q}^-, \\
q^-|_{t=0} &= x, \quad p^-|_{t=0} = \xi,
\end{align*}
\]
and for \(\omega, \theta \in S^{n-1}\) set
\[
M_\omega(\theta) = \{y: y \cdot \omega = -r_0, \lim_{t \to \infty} p^-(t; -r_0, y, \omega) = \theta\},
\]
\[
s_\omega(\theta) = \sup_{y \in M_\omega(\theta)} \{\lim_{t \to \infty} (q^-(t; -r_0, y, \omega) \cdot \theta - t)\},
\]
\[
\bar{M}_\omega(\theta) = \{y \in M_\omega(\theta): s_\omega(\theta) = \lim_{t \to \infty} (q^-(t; -r_0, y, \omega) \cdot \theta - t)\}.
\]
We assume that for any \(y (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\)
\begin{equation}
\lim_{t \to \infty} |\overline{q}^-(t; -r_0, y, \omega)| = \infty.
\end{equation}
Then singular support of the scattering kernel \(S(\cdot, \theta, \omega)\) for the equation (0.3) is contained in the interval \((-\infty, s_\omega(\theta)]\) (cf. Theorem 2 in the author [15]); furthermore, when \(n=2\), it is proved under some assumptions that \(S(s, \theta, \omega)\) is singular at \(s=s_\omega(\theta)\) (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of \(n>2\):

**Theorem 4.** Assume (0.4) for any \(y (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\). Fix \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), and let the assumption
\begin{equation}
\det[\partial_t q^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_\omega(\theta)
\end{equation}
be satisfied. Then \(S(s, \theta, \omega)\) is singular at \(s=s_\omega(\theta)\).

The assumption (0.5) means that there is no caustic on \(\{(t, x): x=q^-(t; -r_0, y, \omega), -r_0 \leq t < \infty, y \in \bar{M}_\omega(\theta)\}\), namely, the mapping: \((t, y) \to q^-(t; -r_0, y, \omega) (-r_0 \leq t < \infty, y \cdot \omega = -r_0)\) is diffeomorphic on \([-r_0, \infty) \times \bar{M}_\omega(\theta)\). In the previous paper [15] we added the assumption
\[\det[\partial_t p^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_\omega(\theta),\]
but this is not necessary.

1. **Proofs of Theorem 1 and Theorem 2**

We denote by \(H^m(M)\) the Sobolev space of order \(m\) on \(M\), and by \(H_{lsc}(M)\) the space of functions \(g(x)\) satisfying \(\alpha(x)g(x) \in H^m(M)\) for any \(\alpha(x) \in C_0^\infty(M)\) \((C_0^\infty(M)\) is the space of \(C^\infty\) functions on \(M\) with compact support).

**Lemma 1.1.** Let \(\varphi(x)\) be a real-valued \(C^m\) function on \(\mathbb{R}^n\), and let \(\rho(x)\) be a \(C^m\) function on \(\mathbb{R}^n\) with compact support. Then the function
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\[ g(s) = \int_{E(s)} \rho(x) dx \]

(where \( E(s) = \{ x : \varphi(x) \leq s \} \) satisfies

(i) \( g(s) = 0 \) if \( s < \min_{x \in \text{supp}[\rho]} \varphi(x) \),

(ii) \( g(s) \) is constant if \( s > \max_{x \in \text{supp}[\rho]} \varphi(x) \),

(iii) \( g(s) \in H_{10}^m(\mathbb{R}^n) \) for any \( m < 2^{-1} \).

Proof. Set

\[ H(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \]

Then it follows that \( H(s) \in H_{10}^m(\mathbb{R}^n) \) for any \( m < 2^{-1} \), and so \( H(s - \varphi(x)) \) becomes a \( H_{10}^m(\mathbb{R}^n) \)-valued continuous function on \( \mathbb{R}^n \). Therefore, noting that \( g(s) = \int_{\mathbb{R}^n} \rho(x) H(s - \varphi(x)) dx \), we obtain (iii). If \( s < \min_{x \in \text{supp}[\rho]} \varphi(x) \) we have \( E(s) \cap \text{supp}[\rho] = \emptyset \), which proves (i). If \( s > \max_{x \in \text{supp}[\rho]} \varphi(x) \), \( E(s) \) contains \( \text{supp}[\rho] \), which yields (ii). The proof is complete.

Proof of Theorem 1. It follows from (iii) of Lemma 1.1 that the function \( g_j(s) \) defined in (0.1) belongs to \( L^1_{10}(\mathbb{R}^n) \). Therefore we have

\[ \int_{s_0}^s (s-t)^{j-1} g_j(t) dt \in H_{10}^j(\mathbb{R}^n) \quad (j \geq 1), \]

\[ \partial^j_s \int_{s_0}^s (s-t)^{j-1} g_j(t) dt = g_j(s). \]

Hence the function \( \tilde{g}_j(s) (= g_j(s) + \sum_{j=1}^N \int_{s_0}^s (s-t)^{j-1} g_j(t) dt) \) satisfies

(1.1) \[ \partial^j_s \tilde{g}_j(s) = \sum_{j=0}^N \partial^N_j g_j(s). \]

We define \( \tilde{I}(\sigma) \) by

\[ \tilde{I}(\sigma) = \begin{cases} I(\sigma) & \text{for } \sigma > 0, \\ I(-\sigma) & \text{for } \sigma < 0. \end{cases} \]

Then \( \sigma \tilde{I}(\sigma) \in L^2(1, \infty) \) if and only if \( (1 + |\sigma|)^{n} \tilde{I}(\sigma) \in L^2(\mathbb{R}^n) \). Furthermore, since \( \rho_j(x) \) are assumed real-valued, it follows that for any integer \( N(\geq 0) \)

(1.2) \[ \tilde{I}(\sigma) = \sum_{j=0}^N \int_{\mathbb{R}^n} e^{-i \sigma \varphi(x)} \rho_j(x) dx (i \sigma)^{-j} + O(|\sigma|^{-N-1}). \]

Here \( 0(|\sigma|^k) \) means that \( |0(|\sigma|^k)| \leq C |\sigma|^k \) for some constant \( C \) independent of \( \sigma \).
Noting that \( \delta(s-\varphi(x)) \) is a \( H^m(\mathbb{R}^1) \)-valued continuous function of \( x \) \((m< -2^i)\) and equal to \( \partial_x H(s-\varphi(x)) \), we obtain
\[
e^{-i\sigma \varphi(x)} = \int e^{-i\sigma \varphi(x)} ds = F(\partial_x H(s-\varphi(x))) (\sigma) ,
\]
where \( F \) is the Fourier transformation in \( s \) (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum \( \int_{\mathbb{R}^n} e^{-i\sigma \varphi(x)} \rho_j(x) dx \) in the following way:
\[
(1.3) \quad \int_{\mathbb{R}^n} e^{-i\sigma \varphi(x)} \rho_j(x) dx = F(\partial_x H(s-\varphi(x))) \rho_j(x) dx (\sigma) = F(\partial_x \tilde{g}_N(s) ) (\sigma) .
\]

(1.1), (1.2) and (1.3) yield that
\[
(1.4) \quad (i\sigma)^{N-1} \tilde{I}(\sigma) = F(\partial_x \tilde{g}_N(s) ) (\sigma) + o(|\sigma|^{-2}).
\]

Let \( (1+|\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^1) \) for every \( m \in \mathbb{R} \). Then it follows from (1.4) that
\[
\partial_x^N \tilde{g}_N(s) \in H^1(\mathbb{R}^1) ,
\]
which implies
\[
\tilde{g}_N(s) \in C^N(\mathbb{R}^1) .
\]

Conversely, let \( \tilde{g}_N(s) \in C^N \) for every non-negative integer \( N \). Then we have \( \partial_x^{N+1} \tilde{g}_N(s) \in H^{-1}(\mathbb{R}^1) \), which means that \( \partial_x^{N+1} \tilde{g}_N(s) \in H^{-1}(\mathbb{R}^1) \) since \( \partial_x^{N+1} \tilde{g}_N(s) = 0 \) for large \( |s| \) (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain \( (1+|\sigma|)^{N-1} \tilde{I}(\sigma) \in L^2(\mathbb{R}^1) \) for every integer \( N(\geq 1) \). This shows that
\[
(1+|\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^1) \quad \text{for every} \quad m \in \mathbb{R} .
\]
The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that \( s_0 = \min_{s \in D} \varphi(x) = 0 \). Since \( \max_{0 \leq t \leq s} |g_j(t)| \leq |E(s)| \max_{x \in D} |\rho_j(x)| \) \(( |E(s)| = \int_{E(s)} dx \) ), there is a constant \( C \) independent of \( s \) such that
\[
\left| \int_0^s (s-t)^{j-1} g_j(t) dt \right| \leq C |s|^j |E(s)| \quad (j \geq 1) .
\]

Therefore we have
\[
|\tilde{g}_N(s) | \geq |g_0(s) | - \sum_{j=1}^N \left| \int_0^s (s-t)^{j-1} g_j(t) dt \right| \\
\geq (\min_{s \in G_j} |\rho_j(x) | - C \sum_{j=1}^N |s|^j |E(s)| ) .
\]
Since \( \min_{x \in \mathbb{R}^n} \rho_0(x) > 0 \), we obtain \( \min_{x \in \mathbb{R}^n} \rho_0(x) \geq 2\delta \) for a constant \( \delta > 0 \) independent of \( s \) if \( |s| \) is small enough. Therefore, if \( |s| \) is small enough, it follows that

\[
|\tilde{g}_N(s)| \geq \delta |E(s)|.
\]

Take a point \( x_0 \) satisfying \( \varphi(x_0) = 0 \) \( = \min_{x \in \Omega} \varphi(x) \). Then there is a constant \( d > 0 \) such that

\[
E(s) \supset \tilde{E}(s) = \{ x : d |x - x_0| \leq s \},
\]

which yields \( |E(s)| \geq |\tilde{E}(s)| = \delta s^n \) for \( s \geq 0 \) (the constant \( \delta' \) does not depend on \( s \)). Thus, for any sufficiently small \( s \geq 0 \) we have

\[
(1.5) \quad |\tilde{g}_N(s)| \geq \delta \delta' s^n.
\]

Now, assume that \( \sigma^m I(\sigma) \in L^2(1, \infty) \) for every \( m \in \mathbb{R} \). Then it follows from Theorem 1 that \( \tilde{g}_N(s) \in C^N \) for any integer \( N \geq 0 \). Take the \( N \) so that \( N \geq n+1 \). All the derivatives \( \partial_1 g_N(0), \partial_2 g_N(0), \ldots, \partial_n g_N(0) \) vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

\[
|\tilde{g}_N(s)| \leq C |s|^{n+1}.
\]

This is not consistent with (1.5) as \( s \to +0 \). Therefore we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

for some constant \( m \in \mathbb{R} \) depending only on \( n \).

2. **Proof of Theorem 3**

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let \( v(t, x; \omega) \) be the solution of the equation

\[
(2.1) \quad \begin{cases}
\Box v(t, x) = 0 & \text{in } \mathbb{R}^d \times \Omega, \\
v(t, x') = -2^{-1}(-2\pi i)^{1-n} \delta(t-x' \cdot \omega) & \text{on } \mathbb{R}^d \times \partial \Omega, \\
v(t, x) = 0 & \text{for } t < r(\omega).
\end{cases}
\]

Then \( v(t, x; \omega) \) is a \( C^m \) function of \( x \) and \( \omega \) with the value \( S'(\mathbb{R}^d) \).

**Proposition 2.1.** \( S(s, \theta, \omega) \) is represented of the form

\[
S(s, \theta, \omega) = \int_{\partial \Omega} \{ \partial_1^{-1} \partial_2 v(x \cdot \theta - s, x; \omega) - v \cdot \theta \partial_1^{-1} v(x \cdot \theta - s, x; \omega) \} \cdot dS_x \quad (\omega \neq \theta),
\]

where \( v \) is the outer unit vector normal to \( \partial \Omega \) (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral \( \int \cdot dS_x \) is in the sense of the Riemann
integral with the value $S'(R^i)$. For the proof see Majda [8] and the author [14].

It is seen that the wave front set of $\delta(t-x\cdot\omega)|_{R^i\times\partial\Omega}$ is non-glancing in $\{(t,x): -r(\omega-\theta)-2\eta \leq x\cdot\theta-t\} \cap (R^i\times\partial\Omega) (\omega \neq \theta)$ if $\eta (>0)$ is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution $\psi(t,x;\omega)$ of (2.1) mod $C^\infty$ by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about $\partial_\nu \psi|_{R^i\times\partial\Omega}$. This is indicated by Majda [8] in the case of $\theta=-\omega$ (cf. Lemma 2.1 of [8]).

We have

**Lemma 2.2.** There exists a first order pseudo-differential operator $B$ on $R^i\times\partial\Omega$ independent of $t$ such that

(i) its symbol $B(\bar{x}';\tau,\xi')$ represented near

$$N(\omega-\theta) = \{x: x-(\theta-\omega) = r(\omega-\theta)\} \cap \partial\Omega$$

by local coordinates $(t, \bar{x}')$, has a homogeneous asymptotic expansion $\sum_{j=0} B_j(\bar{x}';\tau,\xi')$ satisfying

$$-iB_0(\bar{x}';\pm 1, \mp \theta') > 0 \text{ on } N(\omega-\theta) (\theta' \text{ is the tangential component of } \theta \text{ to the plane } \{x: x-(\omega-\theta) = r(\omega-\theta)\})$$

$$B_j(\bar{x}';\tau,\xi') \text{ are purely imaginary-valued for even } j \text{ and real-valued for odd } j,$$

(ii) $\partial_\nu \psi|_{R^i\times\partial\Omega}$ is equal to $B(\psi|_{R^i\times\partial\Omega})$ mod $C^\infty$ in $\{(t,x): -r(\omega-\theta)-\eta \leq x\cdot\theta-t\} \cap R^i\times\partial\Omega$ for some small constant $\eta > 0$.

In the above lemma, “a homogeneous asymptotic expansion $\sum_{j=0} B_j(\bar{x}';\tau,\xi')$” means that $B_j(\bar{x}';\mu\tau,\mu\xi') = \mu^{-j} B_j(\bar{x}';\tau,\xi')$ for $\mu \geq 1$, $|\mu\tau| + |\mu\xi'| \geq 1$ and that $|B(\bar{x}';\tau,\xi') - \sum_{j=1}^N B_j(\bar{x}';\tau,\xi')| \leq C_N (|\tau| + |\xi'| + 1)^{-N-1}$ for any non-negative integer $N$ (for detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.); (ii) in the lemma states that $\alpha(t,x')(\partial_\nu \psi|_{R^i\times\partial\Omega} - B(\psi|_{R^i\times\partial\Omega})) \in C^\infty$ for any $\alpha(t,x') \in C^\infty(R^i\times\partial\Omega)$ with $\text{supp} [\alpha] \subset \{(t,x): -r(\omega-\theta)-\eta \leq x\cdot\theta-t\}$.

**Proof of Lemma 2.2.** Let $\sum_{i=1}^l \chi_i(x)$ be a partition of unity on a neighborhood of $N(\omega-\theta)$ satisfying $\max_{1 \leq i \leq l} |\text{supp} [\chi_i]| \leq \epsilon_0$ ($\epsilon_0$ is a sufficiently small positive constant). Then there is a constant $\epsilon_1 > 0$ such that $\sum_{i=1}^l \chi_i(x) = 1$ for any $x \in \partial\Omega$ satisfying $-r(\omega-\theta)-\epsilon_1 \leq x\cdot\theta-x\cdot\omega$. Let $\psi_i(t,x)$ be the solution of the equation

$$(\square \psi_i(t,x) = 0 \quad \text{in } R^i\times\Omega, \psi_i(t,x') = \chi_i(x') \psi(t,x';\omega) \quad \text{on } R^i\times\partial\Omega, \psi_i(t,x) = 0 \quad \text{for } t < r(\omega).$$
RAPID DECREASE OF OSCILLATORY INTEGRALS

Since \( \text{supp}[v |_{\mathbb{R}^1 \times \partial \Omega}] \subset \{(t, x') : x' \cdot \omega = t\} \), \( \sum_{i=1}^I v_i(t, x') \) is equal to \( v(t, x'; \omega) \) on \((\mathbb{R}^1 \times \partial \Omega) \cap \{(t, x') : -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t\} \), and so, noting that the propagation speed is less than one, we have

\[
v(t, x; \omega) = \sum_{i=1}^I v_i(t, x) \quad \text{in} \quad (\mathbb{R}^1 \times \Omega) \cap \{(t, x) : -r(\omega - \theta) - \varepsilon_1 \leq x \cdot \theta - t\}.
\]

We denote by \( \text{WF}[f(t, x)] \) the wave front set of \( f(t, x) \). It is seen that \( \text{WF}[v |_{\mathbb{R}^1 \times \partial \Omega}] = \text{WF}[\delta(x' \cdot \omega - t) |_{\mathbb{R}^1 \times \partial \Omega}] = \{(t, x') : (t, x') \in \mathbb{R}^1 \times \partial \Omega, x' \cdot \omega - t = 0, \xi' = -\tau(\omega - (\omega \cdot \nu) \nu), \tau \neq 0 \} \) (\( \nu \) is the outer unit normal to \( \partial \Omega \)). Hence, for any \( (t, x'; \tau, \xi') \in \text{WF}[v_i |_{\mathbb{R}^1 \times \partial \Omega}] \) the equation \( \tau^2 - |\xi' + \lambda \nu|^2 = 0 \) in \( \lambda \) has real roots, and the null-bicharacteristics associated with \( \partial_t - \Delta \) through \( \text{WF}[v_i |_{\mathbb{R}^1 \times \partial \Omega}] \) are transversal to \( \mathbb{R}^1 \times \partial \Omega \) (non-glancing). This implies that \( \text{sing supp}[\partial_\nu v_i |_{\mathbb{R}^1 \times \partial \Omega}] \subset \text{sing supp}[v_i |_{\mathbb{R}^1 \times \partial \Omega}] \) (cf. Theorem 7 in §9 of (Lax and) Nirenberg [10]), and so it suffices to examine \( v_i(t, x) \) only in a neighborhood \((t_i - \varepsilon_0, t_i + \varepsilon_0) \times \bar{U}_i \) of \((t_i, x_i)\) (\( x' \in \text{supp}[X_\nu] \cap N(\omega - \theta) \) and \( t_i = x_i \cdot \omega \)).

To analyze \( v_i \) more precisely, we transform \( \Omega \) in \( \mathbb{R}_+^n \) into the half-space \( \mathbb{R}^+_n = \{x = (x', x_0) : x_0 > 0\} \). Let the derivative \( \partial_\nu \) be transformed in \( U_i \) into \( -\partial_{x_0} \). For any set \( M \) in \( \mathbb{R}_+^n \) we denote by \( M' \) the set transformed by the coordinates \( \bar{x} \). Let \( -\Delta_x \) be represented by \( \bar{x} \) of the form \( \bar{A} = \sum_{|\alpha| \leq 2} a_\alpha(\bar{x}) \bar{x}_\alpha^2 \). Here we can assume that the coefficients \( a_\alpha(\bar{x}) \) are real-valued \( C^\infty \) functions defined on \( \mathbb{R}_+^n \) and constant out of \( \bar{U}_i \). Let us examine the solution \( \psi(t, \bar{x}) \) of the following equation instead of \( v_i(t, x) \):

\[
\left\{ \begin{array}{ll}
(\partial_t^2 + \bar{A})\psi(t, \bar{x}) = 0 & \text{in} \quad \mathbb{R}^1 \times \mathbb{R}^+_n, \\
\psi(t, \bar{x}^0) = g(t, \bar{x}^0) & \text{on} \quad \mathbb{R}^1 \times \mathbb{R}^+_n, \\
\psi(t, \bar{x}) = 0 & \text{for} \quad t < t_i - \varepsilon_0,
\end{array} \right.
\]

where \( g(t, \bar{x}^0) = -2^{-1}(-2\pi i)^{1-n}\delta(x(\bar{x}^0) \cdot \omega - t)\chi_i(x(\bar{x}^0)) \). Note that \( \text{WF}[g(t, \bar{x}^0)] \) is contained in a sufficiently small conic neighborhood of \((t_i, x^0, \pm 1, \mp \theta) \) (\( \bar{\theta} \) is the component of \( \theta \) (transformed by the coordinates \( \bar{x} \)) tangent to the plane \( x_0 = 0 \)), and that if \(|(\tau, \xi')|^{-1}(\tau, \xi') \) is near \(|(\pm 1, \mp \theta')|^{-1}(\pm 1, \mp \theta') \) the equation

\[
\tau^2 + A_0(\bar{x}; \xi', \xi_0) = 0
\]

\( (\bar{A}_0(\bar{x}, \xi) = \sum_{|\alpha| = 2} a_\alpha(\bar{x}) \bar{x}_\alpha^2 \) in \( \xi_0 \) has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators \( \bar{x}^\pm(\bar{x}; D_t, D_\bar{x}) \) on \( \mathbb{R}^1 \times \mathbb{R}^+_n \) (independent of \( t \)) with homogeneous asymptotic expansions \( \sum_{j=0}^\infty \bar{x}^j(\bar{x}; \tau, \xi') \) such that

\[
(1) \quad \bar{x}^j(\bar{x}; \tau, \xi') \text{ are real-valued for even } j \text{ and purely imaginary-valued for odd } j,
\]
(ii) if \( |(\tau, \xi')|^{-1}(\tau, \xi') \) is near \( |(-1, \theta')|^{-1}(1, \theta') \) or \( |(1, \theta')|^{-1}(1, -\theta') \), 
\( \xi_0^\pm(\mathbf{x}; \tau, \xi') \) are equal to the roots of the equation (2.4), and

\[
\xi_0^\pm(\mathbf{x}; \tau, \xi') = \pm(1 - |\theta'|^2)^{1/2},
\]

(iii) all the null-bicharacteristic curves associated with \( D_{\xi_0^\pm}(\mathbf{x}; D_t, D_{\xi'}) \) through \( \text{WF}[g(t, \mathbf{x}')] \) are transversal to the boundary \( \{\mathbf{x}_0 = 0\} \) and proceed in the direction \( t > 0 \) as they leave the boundary,

(iv) if the wave front set of \( u(t, \mathbf{x}) \) is near the bicharacteristic curves stated in the above (iii), then we have

\[
(D_{\xi_0^\pm}(\mathbf{x}; D_t, D_{\xi'})) (D_{\xi_0^\pm}(\mathbf{x}; D_t, D_{\xi'})) u = \xi(\mathbf{x}) (\partial_t^2 + \Lambda) u \mod C^\infty,
\]

where \( \xi(\mathbf{x}) \) is a \( C^\infty \) function on \( \mathbb{R}^n \) satisfying \( \xi(\mathbf{x}) < 0 \) for every \( \mathbf{x} \).

(iii) and (iv) imply that \( \theta(t, \mathbf{x}', \mathbf{x}_0) \) is approximated \( \mod C^\infty \) by the solution \( \omega(\mathbf{x}_0; t, \mathbf{x}') \) of the equation

\[
\begin{align*}
\{ (D_{\xi_0^\pm}(\mathbf{x}; D_t, D_{\xi'})) \omega = 0, & \quad \mathbf{x}_0 > 0, \\
\omega |_{\mathbf{z}_0 = 0} = h(t, \mathbf{x}'). &
\end{align*}
\]

Therefore we have

\[
-\partial_{\mathbf{z}_0} \theta |_{\mathbf{z}_0 = 0} = -i\xi^+(\mathbf{x}', 0; D_t, D_{\xi'})(\theta |_{\mathbf{z}_0 = 0}) \mod C^\infty.
\]

Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution \( v(t, \mathbf{x}; \omega) \) in (2.1) satisfies \( \text{supp} [v |_{\mathbb{R}^n \times \partial \Omega}] \subset \{(t, \mathbf{x}): x \cdot \omega = t\} \). Therefore, noting that the propagation speed is less than one, we see that \( \text{supp} [v(t, \mathbf{x}; \omega)] \subset \{(t, \mathbf{x}): x \cdot \omega \leq t\} \), which yields

\[
v(x \cdot \theta - s, \mathbf{x}; \omega) = 0 \quad \text{if} \quad s > x \cdot (\theta - \omega).
\]

Hence, if \( s > \max x \cdot (\theta - \omega) = -r(\omega - \theta) (\omega \not= \theta) \), we obtain \( S(s, \theta, \omega) = 0 \) from Proposition 2.1.

Next, let us prove that \( S(s, \theta, \omega) \) is singular at \( s = -r(\omega - \theta) \). Take \( \alpha(s) \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \alpha \leq 1 \) on \( \mathbb{R} \), \( \alpha(s) = 1 \) for \( |s| \leq 2^{-1} \) and \( \alpha(s) = 0 \) for \( |s| \geq 1 \). For any \( \varepsilon > 0 \) set

\[
\alpha_\varepsilon(s) = \alpha \left( \frac{s + r(\omega - \theta)}{2\varepsilon} \right).
\]

Then we have only to prove that \( \alpha_\varepsilon(s) S(s, \theta, \omega) \) is not \( C^\infty \) for any small \( \varepsilon > 0 \). Proposition 2.1 yields

\[
\alpha_\varepsilon(s) S(s, \theta, \omega) = \int_{\partial \Omega} \alpha_\varepsilon(s) \partial_\nu v (x \cdot \theta - s, \mathbf{x}; \omega) dS_x - \int_{\partial \Omega} v \cdot \partial_\nu \alpha_\varepsilon(s) (\partial_\nu^{-1} v) (x \cdot \theta - s, \mathbf{x}; \omega) dS_x \equiv f_1(s) + f_2(s).
\]
Let $F[\mathcal{L}(s)](\sigma) = \int e^{i\sigma \mathcal{L}(s)} ds$. As is readily seen, it follows that

$$F[J_2(s)](\sigma) = -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} C_j^{-1}(i\sigma)^{n-1-j} \int_{\mathfrak{M}} e^{i\mathcal{L}((\theta-w))(-\nu \cdot \theta)} \cdot \alpha_j^{(j)}(x \cdot (\theta-w)) ds_x$$

(where $C_j^{-1} = (n-1)!/(n-1-j)!j!$). Taking the $\varepsilon(>0)$ so that $2\varepsilon \leq \eta$, by Lemma 2.2 we have

$$F[J_1(s)](\sigma) = \int_{\mathcal{M}^1} e^{i\mathcal{L}((\theta-w))\alpha_j(x \cdot (\theta-w))\mathfrak{M}} B_0(s, x) dS_x$$

$$= -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-2} C_j^{-2} \int_{\mathfrak{M}^1} e^{i\mathcal{L}((\theta-w))\alpha_j^{(j)}(x \cdot (\theta-w))} |_{x'=\omega} ds_x$$

Here $\mathfrak{M}$ denotes the transposed operator of $B$ (i.e. $\langle kB, g \rangle = \langle f, Bg \rangle$ for any $f$ and $g \in C^\infty_\mathfrak{M}(\mathbb{R}^1 \times \partial \Omega)$). Let us note that the symbol of $\mathfrak{M}$ expressed near supp $[\alpha_\mathfrak{M}(x \cdot \theta-t)] \cap (\mathbb{R}^1 \times \partial \Omega)$ by the local coordinates $(t, x')$, has a homogeneous asymptotic expansion $\sum_{j=0} \mathfrak{M}^j(x'; \tau, \xi')$ such that $\mathfrak{M}^j(x'; \tau, \xi')$ are real-valued for odd $j$ and purely imaginary valued for even $j$ and that $-i\mathfrak{M}^j(x'; \pm 1, \mp \theta') = -iB_0(x'; \mp 1, \pm \theta') \equiv 0$ for $x' \in \mathcal{N}(\omega-\omega)$, which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand $\mathfrak{M}^j(x'; x \cdot (\theta-w))$ asymptotically (as $\sigma \to \infty$) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion

$$F[J_1(s)](\sigma) \sim -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-2} (i\sigma)^{n-1-j} \int_{\mathfrak{M}^1} e^{i\mathcal{L}((\theta-w))\alpha_j(x \cdot (\theta-w))} ds_x$$

as $\sigma \to \infty$.

Combining (2.5) and (2.6) yields that for any integer $N(>0)$

$$F[\mathcal{L}(s)]S(s, \theta, \omega)(\sigma) = -2^{-1}(-2\pi i)^{1-n} (i\sigma)^{n-1} \int_{\mathfrak{M}^1} e^{-i\mathcal{L}(s')(\omega-w)} \cdot \{\sum_{j=0}^{n-1} \rho_j(x')(i\sigma)^{-j} \} d\mathfrak{M}' + O(\sigma^{-N}).$$

Here $x'$ is the local coordinates on $\partial \Omega$ near $\mathcal{N}(\omega-\theta)$ and

$$\rho_j(x') = \beta_j(x(x')) + (-\nu \cdot \theta)x_j^{(j)}(x(x') \cdot (\theta-w)) \quad (x_j^{(j)} = 0, j \geq n).$$

Noting that $\rho_0(x') > 0$ when the phase function $x(x') \cdot (\omega-\theta)$ is minimum, and applying Theorem 2, we obtain for some constant $m \in \mathbb{R}$

$$\sigma^m F[\mathcal{L}(s)]S(s, \theta, \omega)(\sigma) \in L^2(1, \infty),$$
which shows that \( \alpha(s)S(s, \theta, \omega) \) is not \( C^\infty \). The proof is complete.

3. Proof of Theorem 4

We use the same notations as for the scattering by obstacles in §2. The scattering operator \( S \) for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

**Proposition 3.1.** Set

\[
S_0(s, \theta, \omega) = \int_{\mathbb{R}^n} (\partial_t^2 - \Box) w(t, \theta - s, x) dx,
\]

\[Kk = F^{-1}[(\text{sgn } \sigma)^{n-1}(Fk)(\sigma)] ,\]

where \( w(t, x) \) is the solution of the equation

\[
\begin{cases}
(\partial_t^2 - \Box) w(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
\partial_t w(-t_0, x) = -2^i(-2\pi i)^{1-n}\delta(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n, \\
\partial_x w(-t_0, x) = -2^i(-2\pi i)^{1-n}\delta(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n.
\end{cases}
\]

Then we have

\[
(Sk)(s, \theta) = \int S_0(s - t, \theta, \omega)k(t, \omega)dtd\omega + (Kk)(s, \theta) .
\]

Note that \( S_0(s, \theta, \omega) = S(s, \theta, \omega) \) if \( \omega \neq \theta \).

To prove Theorem 4, we have only to show that for any small \( \varepsilon(>0) \) there exist a real number \( m \) and a function \( \rho(s) \in C_0^\infty(\mathbb{R}_\sigma) - 2\varepsilon > 0 \) such that

\[ (1 + |\sigma|^\alpha) F[\rho(s)S(s, \theta, \omega)](\sigma) \not\in L^2(\mathbb{R}^n) . \]

Let \( \gamma(x) \in C_0^\infty(\mathbb{R}^n) \) with \( \gamma(x) = 1 \) in a neighborhood of \( \bar{M}_\omega(\theta) \), and denote by \( \tilde{w}(t, x) \) the solution of the equation

\[
\begin{cases}
(\partial_t^2 - \Box) \tilde{w}(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
\tilde{w}(-t_0, x) = \gamma(x)w(-t_0, x) & \text{on } \mathbb{R}^n, \\
\partial_x \tilde{w}(-t_0, x) = \gamma(x)\partial_x w(-t_0, x) & \text{on } \mathbb{R}^n.
\end{cases}
\]

The author [15] showed that if \( \bar{t} \) is large enough we have for any integer \( N(>0) \)

\[ F[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1}e^{-i\sigma \bar{t}} \sum_{j=0}^{N-1} (i\sigma)^{n-1-j}\mathcal{F}[\beta_j(x) \{w(\bar{t}, x \} \} - (i\sigma)^{-1}\partial_x \tilde{w}(\bar{t}, x)](-\sigma\theta) + O(\sigma^{-N+N_0}) \]

as \( \sigma \rightarrow \infty \) (\( N_0 \) is an integer independent of \( N \)) (cf. (4.5) in [15]). Here, \( \mathcal{F} \) denotes the Fourier transformation in \( x \), and the functions \( \beta_j(x) \in C_0^\infty(\mathbb{R}^n) \) are all real-valued.
We take \( \tilde{t} \) so large as to have (i) and (ii) stated in the following

**Lemma 3.2.** Let \( r_1 \) be an arbitrary constant \((\geq r_0)\), and set

\[
\psi(x; t) = q^-(t; -r_0, x, \omega) \cdot \theta.
\]

Then, for any \( \varepsilon(>0) \) there is a constant \( \tilde{t}_0 \) such that for any fixed \( \tilde{t} \geq \tilde{t}_0 \)

(i) \[ \max_{|x| \leq r_1 \atop x \cdot \omega = -r_0} |\psi(x; \tilde{t})| \leq l_\omega(\theta) + \tilde{t} + \varepsilon, \]

(ii) all points at which \( \psi(x; \tilde{t}) \) is maximum \((x \cdot \omega = -r_0, |x| \leq r_1)\), are contained in \( \varepsilon \)-neighborhood \((M_\omega(\theta))_\varepsilon \) of \( M_\omega(\theta) \) \(((M)_\varepsilon \{x: \text{dis}(x, M) < \varepsilon\})\).

This lemma will be proved later. Choose the \( p(s) \) so that \( p(s) > 0 \) on \( \mathbb{R}^1 \) and \( p(s) > 0 \) on \([s_\omega(\theta) - \varepsilon, s_\omega(\theta) + \varepsilon]\). Then it is seen from the form of \( \beta_0(x) \) (cf. (4.4) and (4.6) in [15]) and the above lemma that

\[
(3.1) \quad \beta_0(x) \geq 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \beta_0(q^-(\tilde{t}; -r_0, y, \omega)) > 0 \quad \text{for any} \quad y \in (M_\omega(\theta))_\varepsilon \quad (y \cdot \omega = -r_0).
\]

We take the \( \gamma(x) \) so that \( \gamma(x) \geq 0 \) on \( \mathbb{R}^n \), \( \gamma(x) > 0 \) on \( (M_\omega(\theta))_\varepsilon \) and \( \text{supp}[\gamma] \subset (M_\omega(\theta))_{2\varepsilon} \).

By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. §5 of [10] or Appendix II of [5]), we can construct a symbol \( \lambda(x, \xi) \) with a homogeneous asymptotic expansion \( \sum_{j=0}^{\infty} \lambda_j(x, \xi) \) such that

\[
\lambda_0(x, \xi) = \left\{ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \right\}^{1/2},
\]

\[-\partial_q + A = (D_t + \lambda(x, D_x))(D_t - \lambda(x, D_x)) \quad \text{modulo a smoothing operator} \]

(cf. Corollary 2.5 in the author [15] also). Furthermore we see that \( \lambda_j(x, \xi) \) are real-valued for even \( j \) and purely imaginary valued for odd \( j \) since the coefficients \( a_{ij}(x) \) are all real-valued (recall the construction of \( \xi \pm (\xi'; \tau, \xi') \) in §2). Consider the Cauchy problem

\[
\begin{cases}
(D_t - \lambda(x, D_x))u(t, x) = 0 & \text{in} \quad \mathbb{R}^1 \times \mathbb{R}^n, \\
|u|_{t=0} = u_0(x) & \text{on} \quad \mathbb{R}^n,
\end{cases}
\]

and denote by \( E(t) \) the operator: \( u_0 \rightarrow u(t, \cdot) \). Then \( \tilde{w}(\tilde{t}, x) \) and \( \partial_t \tilde{w}(\tilde{t}, x) \) are represented as follows:

\[
\begin{align*}
\tilde{w}(\tilde{t}, x) &= 2^{-1}E(\tilde{t} + r_0)(\tilde{w}(-r_0, \cdot) - i\hat{\nu}\partial_t \tilde{w}(-r_0, \cdot))(x) \\
&\quad + 2^{-1}E(\tilde{t} - r_0)(\tilde{w}(-r_0, \cdot) + i\hat{\nu}\partial_t \tilde{w}(-r_0, \cdot))(x), \\
\partial_t \tilde{w}(\tilde{t}, x) &= 2^{-1}E(\tilde{t} + r_0)i\lambda(\tilde{w}(-r_0, \cdot) - i\hat{\nu}\partial_t \tilde{w}(-r_0, \cdot))(x) \\
&\quad + 2^{-1}E(\tilde{t} - r_0)i\hat{\lambda}(\tilde{w}(-r_0, \cdot) + i\hat{\nu}\partial_t \tilde{w}(-r_0, \cdot))(x),
\end{align*}
\]

where \( \hat{\lambda} \) and \( \hat{\nu} \) are pseudo-differential operators whose symbols coincide with
\( \lambda(x, \xi) \) and \( \mu(x, \xi) \) (\( \mu(x, D_x) \)) is the parametrix of \( \lambda(x, D_x) \) respectively in a neighborhood of \( \text{supp}[\gamma(x)] \) and vanish for large \(|x|\). Therefore, noting that

\[
\mathcal{F}'[\beta_j \mathcal{E}(-\tilde{t} - r_0) (\tilde{\omega}(x_0, \cdot) + i\tilde{\mu} \partial_x \tilde{\omega}(x_0, \cdot))] (-\sigma \theta) = 0(\sigma^{-\infty}),
\]

\[
\mathcal{F}'[\beta_j \mathcal{E}(-\tilde{t} - r_0) \lambda(\tilde{\omega}(x_0, \cdot) + i\tilde{\mu} \partial_x \tilde{\omega}(x_0, \cdot))] (-\sigma \theta) = 0(\sigma^{-\infty})
\]

as \( \sigma \to \infty \) (cf. §4 of the author [15]), we have

\[
\mathcal{F}[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1} e^{-i\sigma} \sum_{j=0}^{X-1} (i\sigma)^{x-1-j} \mathcal{F}'[2^{-1} \beta_j \mathcal{E}(\tilde{t} + r_0) (1 + \sigma^{-1} \tilde{\lambda})]
\]

\[
\cdot (\tilde{\omega}(x_0, 0) - i\tilde{\mu} \partial_x \tilde{\omega}(x_0, 0)) (-\sigma \theta) + O(\sigma^{-N+N_0}).
\]

The assumption (0.5) implies that if \( \text{WF}[\mu_0] \) is contained in a conic neighborhood of \( \tilde{M}_\omega(\theta) \times \{0\} \) \( \text{WF}[\omega(x_0, \cdot) - i\tilde{\mu} \partial_x \omega(x_0, \cdot)] \) is contained there \( \mathcal{E}(\tilde{t} + r_0) \mu_0 \) is represented by the Fourier integral operator:

\[
E(\tilde{t} + r_0) \omega(x) = (2\pi)^{-1} \int e^{i\tilde{t} \cdot (x - x_0, \xi)} a(\tilde{t} + r_0, x, \xi) \omega(\xi) d\xi \mod C^\infty
\]

(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that \( \mathcal{F}'[\mathcal{B}(\omega)] \) \( (-r_0 - x \cdot \omega) \) \( \mathcal{B}(\eta) = (-i\eta) e^{\eta} \delta(\eta) \) (\( \eta = (\eta_1, \eta'_1) \)), where \( \mathcal{B} = (b_1, \ldots, b_6) \) is an orthogonal matrix with \( b_i = \omega \). Then, introducing change of the variables

\[
x = q_0(\tilde{t}; -r_0, y, \omega)(=q(y)) \text{ near } x = \tilde{q}(\tilde{t}; -r_0, \tilde{M}_\omega(\theta), \omega)(y = (y_0, y')) \text{ is orthogonal coordinates with } y_0 = x \cdot \omega, \text{ we obtain}
\]

\[
\mathcal{F}'[2^{-1} \beta_j \mathcal{E}(\tilde{t} + r_0) (1 + \sigma^{-1} \tilde{\lambda}) (\tilde{\omega}(x_0, \cdot) - i\tilde{\mu} \partial_x \tilde{\omega}(x_0, \cdot))] (-\sigma \theta)
\]

\[
= \int e^{i\sigma \tilde{\gamma}(\eta)} \beta_j(x) \int_0^{\tilde{\tau}} e^{i\sigma \tilde{t}(x_0, \tilde{\tau}, \omega)} a(\tilde{t} + r_0, x, -\tau \omega) e^{-i\tau \omega d \tau} dx + 0(\sigma^{-\infty})
\]

\[
= \int_{\mathbb{R}^{n-1}} dy \int_{-\infty}^{\infty} dy_0 \int_0^{\tilde{\tau}} \sigma d(\tau) e^{i\sigma \tilde{t}(y_0, \tilde{\tau}, \omega)} \tilde{\beta}_j(q(y)) \gamma(y)
\]

\[
\cdot a(\tilde{t} + r_0, q(y), -\sigma \tau \omega) |\det \frac{\partial q}{\partial y}| + O(\sigma^{-\infty}) \quad (\text{as } \sigma \to \infty)
\]

(\( \tilde{\gamma}(x) \in C^\infty(\mathbb{R}^{n}), \tilde{\gamma}(x) = 1 \) on a neighborhood of \( q^- \) (\( \text{supp}[\gamma] \)), and \( \tilde{\tau} \) is a positive constant independent of \( \sigma \)). The function \( \Phi(y_0, \tau) = q^- (y_0, y') \cdot \theta - \tau (y_0 + r_0) \) has the stationary point \( (y_0, \tau) = (-r_0, p^- (-r_0, y') \cdot \theta) \), at which its Hesse matrix equals

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

Expanding \( \int_{-\infty}^{\tilde{\tau}} \int_{\mathbb{R}^{X-1}} e^{i\sigma \tilde{t}(y_0, \tau)} \beta_j \gamma \cdot dy_0 d\tau \) (as \( \sigma \to \infty \)) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion

\[
(3.2) \quad \mathcal{F}[\rho(s)S(s, \theta, \omega)](\sigma) = e^{-i\sigma \tilde{\gamma}(\delta \sigma)^{x-1}} \int_{x_0} e^{i\sigma \tilde{t}(\tilde{t}; -r_0, \tilde{\tau}, \omega) \cdot \delta}
\]

\[
\cdot \{\sum_{j=0}^{X-1} \rho_j(x) (i\sigma)^{-j}\} dx + 0(\sigma^{-N+N_0})
\]

(\( N_0 \) is an integer independent of \( N = 1, 2, \cdots \)). Here \( \rho_j \) are \( C^\infty \) functions with
supp[ρ_2] ⊂ supp[γ] and all real-valued, which follows from the fact that the symbol \( a(\tilde{t}, x, \xi) \) has a homogeneous asymptotic expansion \( \sum_{k=0}^{\infty} a_k(\tilde{t}, x, \xi) \) such that \( a_k(\tilde{t}, x, \xi) \) are real-valued for even \( k \) and purely imaginary valued for odd \( k \); furthermore \( \rho_0 \) is of the form

\[
\rho_0(y) = \gamma(y)\beta(y)q^{-}(\tilde{t}; -r_0, y, \omega)a_0(\tilde{t} + r_0) q^{-}(\tilde{t}; -r_0, y, \omega), -\omega) |\det \frac{\partial q^{-}}{\partial y}|.
\]

Combining this with (3.1) and (ii) of Lemma 3.2, we see that \( \rho_0(x) \geq 0 \) on \( R^n \) and \( \rho_0(x) > 0 \) for any \( x \) at which the function

\[
\varphi(x) = q^{-}(\tilde{t}; -r_0, x, \omega) \cdot \theta \quad (x \cdot \omega = -r_0)
\]

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

\[
\sigma^m F[\rho S](\sigma) \in L^2(1, \infty)
\]

for some constant \( m \in R \), which proves Theorem 4.

Proof of Lemma 3.2. We denote by \( y \) the variables on \( R^{n-1} = \{x: x \cdot \omega = -r_0\} \). It follows from (0.4) that for a large constant \( t_0 \) independent of \( t, y \) and \( \omega \)

\[
q^{-}(t; -r_0, y, \omega) = q^{-}(t_0; -r_0, y, \omega) + (t - t_0)p^{-}(t_0; -r_0, y, \omega), \quad t \geq t_0, y \in R^{n-1}.
\]

Fix \( y \in M_{\omega}(\theta) \) arbitrarily and take a neighborhood \( U(y) \) of \( y \) such that

\[
|q^{-}(t_0; -r_0, y, \omega) - q^{-}(t_0; -r_0, y, \omega)| < \varepsilon/2 \quad \text{for any } y \in U(y),
\]

\[
|t_0\{p^{-}(t_0; -r_0, y, \omega) - p^{-}(t_0; -r_0, y, \omega)\}| < \varepsilon/2 \quad \text{for any } y \in U(y).
\]

Then, in view of the definitions of \( M_{\omega}(\theta) \) and \( s_{\omega}(\theta) \) we have for any \( y \in U(y) \) and \( \tilde{t} \geq t_0 \)

\[
\varphi(y; \tilde{t}) \leq q^{-}(t_0; -r_0, y, \omega) \cdot \theta - t_0p^{-}(t_0; -r_0, y, \omega) \cdot \theta + \tilde{t}p^{-}(t_0; -r_0, y, \omega) \cdot \theta + \varepsilon
\]

\[
\leq s_{\omega}(\tilde{t}) + \varepsilon + \tilde{t}.
\]

On the other hand, for any neighborhood \( U \) of \( M_{\omega}(\varepsilon) \) it follows that \( \delta = \inf \{1 - p^{-}(t_0; -r_0, y, \omega) \cdot \theta \} > 0 \), which yields that \( \varphi(y; t) \leq (C - \delta t) + t \) for any \( y \in U \) (\( |y| \leq r_1 \)) and \( t \geq t_0 \) (\( C \) is a constant independent of \( y \) and \( t \)). This means that

\[
\varphi(y; \tilde{t}) \leq s_{\omega}(\theta) - 1 + \tilde{t}
\]

if \( y \in U, |y| \leq r_1 \) and \( \tilde{t} \) is large enough. Therefore we obtain the lemma.

References


Faculty of Education
Ibaraki University
Mito, Ibaraki 310
Japan