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Osaka University
Introduction

Analysing singularities of distributions, we often examine the following integral with a parameter $\sigma > 0$:

$$I(\sigma) = \int e^{-i\varphi(x)} \rho(x; \sigma) dx \quad \text{(or } \int e^{i\varphi(x)} \rho(x; \sigma) dx\text{)},$$

where $\varphi(x)$ is a real-valued $C^\infty$ function and $\rho(x; \sigma)$ is a $C^\infty$ function with an asymptotic expansion

$$\rho(x; \sigma) \sim \rho_0(x) + \rho_1(x)(i\sigma)^{-1} + \rho_2(x)(i\sigma)^{-2} + \cdots \quad \text{(as } \sigma \to \infty).$$

In this paper we study conditions for the integral $I(\sigma)$ not to decrease rapidly as $\sigma \to \infty$, and solve some inverse scattering problems.

As is well known, if stationary points of $\varphi(x)$ are non-degenerate (i.e. $\det (\partial^2_x \varphi(x)) \neq 0$ when $\partial_x \varphi(x) = 0$), $I(\sigma)$ is expanded asymptotically as $\sigma \to \infty$, and we can know whether $I(\sigma)$ decreases rapidly as $\sigma \to \infty$. Also when the stationary points are degenerate, the asymptotic expansion of $I(\sigma)$ is obtained if $\varphi(x)$ is analytic (cf. Varchenko [16], Duistermaat [1], etc.), and then we can know it through the expansion. But it seems difficult to do so when all derivatives of $\varphi(x)$ vanish at some points, whose case we take into consideration. In our methods we do not employ the asymptotic expansion of $I(\sigma)$. In the previous paper [13], the author examined the case that $n=2$ and $\rho_j(x) = 0$ $(j \geq 1)$: If $\rho_0(x) \geq 0$ on $\mathbb{R}^2$ and $\rho_0(x_0) > 0$ for a degenerate stationary point $x_0$ of $\varphi(x)$, then $(1+|\sigma|)^m I(\sigma) \in L^2(\mathbb{R}^2)$ for some $m < 2^{-1}$ (cf. Theorem 1 of [13]). Improving the methods in [13], whose idea is due to [8], we shall obtain similar results also in the case of $n \geq 3$.

Let $\text{supp} [\rho(\cdot; \sigma)]$ and $\text{supp} [\rho_j] (j \geq 0)$ be contained in a compact set $D$ in $\mathbb{R}^n$. We set

$$E(s) = \{x: \varphi(x) \leq s\} \quad (s \in \mathbb{R}),$$
One of our main results is the following

**Theorem 1.** Let all $\rho_j \ (j \geq 0)$ be real-valued. Then, for every $m \in \mathbb{R}$ we have

$$\sigma^m I(\sigma) \in L^2(1, \infty)$$

if and only if for every integer $N \geq 1$

$$g_N(s) = g_0(s) + \sum_{j=1}^{N} \int_{\mathbb{R}} (t-t)^{j-1} g_j(t) dt \in C^N(\mathbb{R})^r.$$

The following theorem, derived from the above theorem, seems useful to estimate singular points of distributions.

**Theorem 2.** Let all $\rho_j \ (j \geq 0)$ be real-valued, and let $\rho_0(x) \geq 0$ on $\mathbb{R}^n$. If $\rho_0$ satisfies

$$\rho_0(x) > 0 \quad \text{on } E(\min_{x \in D} \varphi(x)), $$

then for some $m(\in \mathbb{R})$ depending only on the dimension $n$ we have

$$\sigma^m I(\sigma) \in L^2(1, \infty).$$

Theorem 1 implies that decreasingness of $I(\sigma)$ is connected with smoothness of the measure $|E(s)|$. This is seen also from the discussions in Vasil’ev [17] or Kaneko [3] (cf. §2 in Chapter I of [3]). Our methods in the proof of Theorem 2 (and in the author [13]) are based on analysis of $|E(s)|$.

In the latter of the present paper we shall consider some inverse scattering problems, and solve them by means of the above results. In §2 we deal with the scattering by a bounded obstacle $\mathcal{O} (\subset \mathbb{R}^n, n \geq 2)$ with a $C^\infty$ boundary $\partial \mathcal{O}$. Assume that the domain $\Omega = \mathbb{R}^n - \mathcal{O}$ is connected, and consider the initial-boundary value problem

$$
\begin{align*}
\Box u(t, x) &= 0 \quad \text{in } \mathbb{R}^1 \times \Omega \quad (\Box = \partial^2_t - \Delta), \\
u(t, x') &= 0 \quad \text{on } \mathbb{R}^1 \times \partial \Omega \quad (\partial \Omega = \partial \mathcal{O}), \\
(u(0, x) = f_1(x)) &\quad \text{on } \Omega, \\
\partial_t u(0, x) = f_2(x) &\quad \text{on } \Omega.
\end{align*}
$$

We denote by $k_-(s, \omega) \ (k_+(s, \omega)) \in L^2(\mathbb{R}^1 \times S^{n-1})$ the incoming (outgoing) translation representation of the data $(f_1, f_2)$ (cf. Lax and Phillips [6], [7]). The operator $S: k_- \rightarrow k_+$ is called the scattering operator and represented by a distribution kernel $S(s, \theta, \omega)$ called the scattering kernel:
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\[(Sk_\cdot)(s, \theta) = \iint S(s-t, \theta, \omega)h_\cdot(t, \omega)dtd\omega\]

(cf. Majda [8] or §1 of the author [14]).

Majda [8] showed in the case of \( O \subset \mathbb{R}^3 \) (i.e. \( n=3 \)) that for any fixed \( \omega \in S^2 \)

\[(0.2)\]

(i) \( \text{supp } S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)] \),

(ii) \( S(s, -\omega, \omega) \) is singular (not \( C^\infty \)) at \( s = -2r(\omega) \),

where \( r(\omega) = \min x \cdot \omega \). He reduced proof of the above (ii) to verifying that the integral of the form

\[ \int_{\mathbb{R}^n} e^{-i\sigma\varphi(x)}\rho(x; \sigma)dx \]

does not decrease rapidly as \( \sigma \to \infty \) (cf. §2 of Majda [8] or §4 of the author [14]). His methods are not applicable to the case of \( n>3 \), one of whose reasons is that the stationary points of the phase function \( \varphi(x) \) are not necessarily non-degenerate.

Using Theorem 2, we can prove that (0.2) is valid also when \( n>3 \):

**Theorem 3.** For any fixed \( \omega \) and \( \theta \in S^{n-1} \) with \( \omega \perp \theta \), we have

(i) \( \text{supp } S(\cdot, \theta, \omega) \subset (-\infty, -r(\omega-\theta)] \),

(ii) \( S(s, \theta, \omega) \) is singular at \( s = -r(\omega-\theta) \).

In §3 we consider the scattering by inhomogeneity of media expressed by the equation

\[ \begin{aligned}
\partial_t u(t, x) - \sum_{i,j} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(t, x)) &= 0 \quad \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
u(0, x) &= f_1(x) \quad \text{on } \mathbb{R}^n, \\
\partial_t u(0, x) &= f_2(x) \quad \text{on } \mathbb{R}^n,
\end{aligned} \]

where \( a_{ij}(x) \) are real-valued \( C^\infty \) functions satisfying

- \( a_{ij}(x) = a_{ji}(x), x \in \mathbb{R}^n \),
- \( a_{ij}(x) = 0 \) (\( i \neq j \)), \( a_{ii}(x) = 1 \) \quad \text{when } |x| \geq r_0,
- \( \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta |\xi|^2 \), \( x \in \mathbb{R}^n \), \( \xi \in \mathbb{R}^n \).

We can apply the scattering theory of Lax and Phillips [6], [7] to the equation (0.3). For this scattering the author in [15] has obtained the results corresponding to (0.2), but they are not satisfactory in the case of \( n \geq 3 \). By means of Theorem 2 we get rid of the restriction to the dimension \( n \).

Let us review the results of [15]. We set

\[ \lambda_\xi(x, \xi) = -\{ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \}^{1/2}. \]
Denote by \((q^-(t; s, x, \xi), p^-(t; s, x, \xi))\) the solution of the equation

\[
\begin{align*}
\frac{dq^-}{dt} &= -\partial_x \lambda_\omega(q^-, p^-), \\
\frac{dp^-}{dt} &= \partial_t \lambda_\omega(q^-, p^-), \\
q^-|_{t=s} &= x, \\
p^-|_{t=s} &= \xi,
\end{align*}
\]

and for \(\omega, \theta \in S^{n-1}\) set

\[
M_\omega(\theta) = \{y: y \cdot \omega = -r_0, \lim_{t \to \infty} p^-(t; -r_0, y, \omega) = \theta\},
\]

\[
s_\omega(\theta) = \sup_{x \in M_\omega(\theta)} \{\lim_{t \to \infty}(q^-(t; -r_0, y, \omega) \cdot \theta - t)\},
\]

\[
\bar{M}_\omega(\theta) = \{y \in M_\omega(\theta): s_\omega(\theta) = \lim_{t \to \infty}(q^-(t; -r_0, y, \omega) \cdot \theta - t)\}.
\]

We assume that for any \(y (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\)

\[
(0.4) \quad \lim_{t \to \infty} |q^-(t; -r_0, y, \omega)| = \infty.
\]

Then singular support of the scattering kernel \(S(\cdot, \theta, \omega)\) for the equation (0.3) is contained in the interval \((-\infty, s_\omega(\theta)]\) (cf. Theorem 2 in the author [15]); furthermore, when \(n=2\), it is proved under some assumptions that \(S(s, \theta, \omega)\) is singular at \(s=s_\omega(\theta)\) (cf. Theorem 3 in [15]).

We show in §3 that this is valid also in the case of \(n>2\):

**Theorem 4.** Assume (0.4) for any \(y (y \cdot \omega = -r_0)\) and \(\omega \in S^{n-1}\). Fix \(\omega\) and \(\theta \in S^{n-1}\) with \(\omega \neq \theta\), and let the assumption

\[
(0.5) \quad \det[\partial_t q^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_\omega(\theta)
\]

be satisfied. Then \(S(s, \theta, \omega)\) is singular at \(s=s_\omega(\theta)\).

The assumption (0.5) means that there is no caustic on \(\{(t, x): x=q^-(t; -r_0, y, \omega), -r_0 \leq t < \infty, y \in \bar{M}_\omega(\theta)\}\), namely, the mapping: \((t, y) \to q^-(t; -r_0, y, \omega) (\omega = -r_0 \leq t < \infty, y \cdot \omega = -r_0)\) is diffeomorphic on \([-r_0, \infty) \times \bar{M}_\omega(\theta)\). In the previous paper [15] we added the assumption

\[
\det[\partial_t p^-(t; -r_0, y, \omega)] \neq 0 \quad \text{for any } (t, y) \in [-r_0, \infty) \times \bar{M}_\omega(\theta),
\]

but this is not necessary.

1. **Proofs of Theorem 1 and Theorem 2**

We denote by \(H^m(M)\) the Sobolev space of order \(m\) on \(M\), and by \(H^m_{\text{loc}}(M)\) the space of functions \(g(x)\) satisfying \(\alpha(x)g(x) \in H^m(M)\) for any \(\alpha(x) \in C^\infty_0(M)\) \((C^\infty_0(M)\) is the space of \(C^\infty\) functions on \(M\) with compact support).

**Lemma 1.1.** Let \(\varphi(x)\) be a real-valued \(C^\infty\) function on \(\mathbb{R}^n\), and let \(\rho(x)\) be a \(C^\infty\) function on \(\mathbb{R}^n\) with compact support. Then the function
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\[ g(s) = \int_{E(s)} \rho(x) \, dx \]

(whence \( E(s) = \{ x : \varphi(x) \leq s \} \)) satisfies

(i) \( g(s) = 0 \) if \( s < \min_{x \in \text{supp } \rho} \varphi(x) \),

(ii) \( g(s) \) is constant if \( s > \max_{x \in \text{supp } \rho} \varphi(x) \),

(iii) \( g(s) \in H^m_{\text{loc}}(\mathbb{R}^n) \) for any \( m < 2^{-1} \).

Proof. Set

\[ H(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \]

Then it follows that \( H(s) \in H^m_{\text{loc}}(\mathbb{R}^n) \) for any \( m < 2^{-1} \), and so \( H(s - \varphi(x)) \) becomes a \( H^m_{\text{loc}}(\mathbb{R}^n) \)-valued continuous function on \( \mathbb{R}^n \). Therefore, noting that

\[ g(s) = \int_{\mathbb{R}^n} \rho(x) H(s - \varphi(x)) \, dx, \]

we obtain (iii). If \( s < \min_{x \in \text{supp } \rho} \varphi(x) \) we have \( E(s) \cap \text{supp } \rho = \emptyset \), which proves (i). If \( s > \max_{x \in \text{supp } \rho} \varphi(x) \), \( E(s) \) contains \( \text{supp } \rho \), which yields (ii). The proof is complete.

Proof of Theorem 1. It follows from (iii) of Lemma 1.1 that the function \( g_j(s) \) defined in (0.1) belongs to \( L^2_{\text{loc}}(\mathbb{R}^n) \). Therefore we have

\[ \int_{t_0}^t (s - t)^{i-1} g_j(t) \, dt \in H^i_{\text{loc}}(\mathbb{R}^n) \quad (j \geq 1), \]

\[ \partial_t^i \int_{t_0}^t (s - t)^{i-1} g_j(t) \, dt = g_j(s). \]

Hence the function \( \tilde{g}_N(s) = g_0(s) + \sum_{j=1}^{N} \int_{t_0}^t (s - t)^{j-1} g_j(t) \, dt \) satisfies

\[ \partial_t^i \tilde{g}_N(s) = \sum_{j=0}^{N} \partial_t^{N-j} g_j(s). \]

We define \( \bar{I}(\sigma) \) by

\[ \bar{I}(\sigma) = \begin{cases} I(\sigma) & \text{for } \sigma > 0, \\ I(-\sigma) & \text{for } \sigma < 0. \end{cases} \]

Then \( \sigma^N I(\sigma) \in L^2(1, \infty) \) if and only if \( (1 + |\sigma|^N) \bar{I}(\sigma) \in L^2(\mathbb{R}^n) \). Furthermore, since \( \rho_j(x) \) are assumed real-valued, it follows that for any integer \( N(\geq 0) \)

\[ \bar{I}(\sigma) = \sum_{j=0}^{N} \int_{\mathbb{R}^n} e^{-i\sigma \varphi(x)} \rho_j(x) \, dx (i\sigma)^{-j} + 0(|\sigma|^{-N-1}). \]

Here \( 0(|\sigma|^k) \) means that \( |0(|\sigma|^k)| \leq C |\sigma|^k \) \( |\sigma| \geq 1 \) for some constant \( C \) independent of \( \sigma \).
Noting that $\delta(s-\phi(x))$ is a $H^m(\mathbb{R}^1)$-valued continuous function of $x (m < -2^i)$ and equal to $\partial_s H(s-\phi(x))$, we obtain

$$e^{-is\phi(x)} = \int e^{-is\phi(x)}ds = F[\partial_s H(s-\phi(x))](\sigma),$$

where $F$ is the Fourier transformation in $s$ (the above integral is in the sense of distributions). Therefore we can represent the Riemann sum $\int_{\mathbb{R}^n} e^{-is\phi(x)} \rho_j(x)dx$ in the following way:

$$(1.3) \int_{\mathbb{R}^n} e^{-is\phi(x)} \rho_j(x)dx = F[\partial_s \int_{\mathbb{R}^n} H(s-\phi(x))\rho_j(x)dx](\sigma) = F[\partial_s \tilde{g}_j](\sigma).$$

(1.1), (1.2) and (1.3) yield that

$$(1.4) (i\sigma)^{-1} \tilde{I}(\sigma) = F[\partial_s \tilde{g}_j(s)](\sigma) + O(|\sigma|^{-2}).$$

Let $(1 + |\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^1)$ for every $m \in \mathbb{R}$. Then it follows from (1.4) that

$$\partial_s \tilde{g}_j(s) \in H^1(\mathbb{R}^1),$$

which implies

$$\tilde{g}_j(s) \in C^N(\mathbb{R}^1).$$

Conversely, let $\tilde{g}_j(s) \in C^N$ for every non-negative integer $N$. Then we have $\partial_s \tilde{g}_j(s) \in H^1(\mathbb{R}^1)$, which means that $\partial_s \tilde{g}_j(s) \in H^1(\mathbb{R}^1)$ since $\partial_s \tilde{g}_j(s)=0$ for large $|s|$ (cf. (i), (ii) of Lemma 1.1 and (1.1)). Therefore, by (1.4) we obtain $(1 + |\sigma|)^{-1} \tilde{I}(\sigma) \in L^2(\mathbb{R}^1)$ for every integer $N \geq 1$. This shows that

$$(1 + |\sigma|)^N \tilde{I}(\sigma) \in L^2(\mathbb{R}^1) \quad \text{for every } m \in \mathbb{R}.$$ The proof is complete.

Proof of Theorem 2. We can assume without loss of generality that $s_0 = \min_{x \in D} \phi(x) = 0$. Since $\max_{0 \leq t \leq s} |g_j(t)| \leq |E(s)| \max_{x \in D} |\rho_j(x)| \quad (|E(s)| = \int_{E(s)} dx)$, there is a constant $C$ independent of $s$ such that

$$\left| \int_0^s \frac{(s-t)^{j-1}}{(j-1)!} g_j(t)dt \right| \leq C |s|^j |E(s)| \quad (j \geq 1).$$

Therefore we have

$$|\tilde{g}_j(s)| \geq |g_j(s)| - \sum_{j=1}^N \left| \int_0^s \frac{(s-t)^{j-1}}{(j-1)!} g_j(t)dt \right| \geq (\min_{x \in D} \rho_j(x) - C \sum_{j=1}^N |s|^j |E(s)|).$$
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Since \( \min_{x \in \mathbb{R}^d} \rho(x) > 0 \), we obtain \( \min_{x \in \mathbb{R}^d} \rho(x) \geq 2\delta \) for a constant \( \delta > 0 \) independent of \( s \) if \( |s| \) is small enough. Therefore, if \( |s| \) is small enough, it follows that

\[
|\tilde{g}_N(s)| \geq \delta |E(s)|.
\]

Take a point \( x_0 \) satisfying \( \varphi(x_0) = 0 \) \((= \min_{x \in \partial} \varphi(x))\). Then there is a constant \( d (>0) \) such that

\[
E(s) \supset \tilde{E}(s) = \{ x : d|x - x_0| \leq s \},
\]

which yields \( |E(s)| \geq |\tilde{E}(s)| = \delta' s^n \) for \( s \geq 0 \) (the constant \( \delta' \) does not depend on \( s \)). Thus, for any sufficiently small \( s > 0 \) we have

(1.5) \[
|\tilde{g}_N(s)| \geq \delta' s^n.
\]

Now, assume that \( \sigma^m I(\sigma) \in L^2(1, \infty) \) for every \( m \in \mathbb{R} \). Then it follows from Theorem 1 that \( \tilde{g}_N(s) \in C^N \) for any integer \( N \geq 0 \). Take the \( N \) so that \( N \geq n + 1 \). All the derivatives \( g_N(0), \partial_\sigma g_N(0), \ldots, \partial_{(n)} g_N(0) \) vanish because of (i) in Lemma 1.1, and so, by the Taylor expansion, we obtain

\[
|\tilde{g}_N(s)| \leq C |s|^{n+1}.
\]

This is not consistent with (1.5) as \( s \to +0 \). Therefore we have

\[
\sigma^m I(\sigma) \in L^2(1, \infty)
\]

for some constant \( m \in \mathbb{R} \) depending only on \( n \).

2. Proof of Theorem 3

In this section we review some results obtained in Majda [8] and the author [14], and prove Theorem 3.

Let \( v(t, x; \omega) \) be the solution of the equation

(2.1) \[
\begin{align*}
\Box v(t, x) &= 0 & \text{in } \mathbb{R}^4 \times \Omega, \\
v(t, x') &= -2^{-1}(-2\pi i)^{1-n} \delta(t-x\cdot \omega) & \text{on } \mathbb{R}^4 \times \partial \Omega, \\
v(t, x) &= 0 & \text{for } t < r(\omega).
\end{align*}
\]

Then \( v(t, x; \omega) \) is a \( C^\infty \) function of \( x \) and \( \omega \) with the value \( S'(R^4) \).

**Proposition 2.1.** \( S(s, \theta, \omega) \) is represented of the form

\[
S(s, \theta, \omega) = \int_{\partial \Omega} \{ \partial_{\sigma} \partial_{\theta} v(x+ \theta - s, x; \omega) - v \cdot \theta \partial_{\sigma} v(x+\theta - s, x; \omega) \} dS_x \quad (\omega \neq \theta),
\]

where \( v \) is the outer unit vector normal to \( \partial \Omega \) (cf. Theorem 1 in Majda [8] and Theorem 2.1 in §2 of the author [14]).

In the above proposition the integral \( \int \cdot dS_x \) is in the sense of the Riemann
integral with the value \( S'(R^1) \). For the proof see Majda [8] and the author [14].

It is seen that the wave front set of \( \delta(t-x\cdot \omega) |_{R^1 \times \partial \Omega} \) is non-glancing in \( \{(t,x)\colon -r(\omega-\theta)-2\eta \leq x\cdot \theta - t \} \cap (R^1 \times \partial \Omega) (\omega \neq \theta) \) if \( \eta > 0 \) is small enough (for description of wave front sets, see Hörmander [2], Kumano-go [5], etc.). Therefore we can construct there the solution \( v(t,x;\omega) \) of (2.1) mod \( C^\infty \) by means of the Fourier integral operators (cf. §9 of Nirenberg [10]), and get information about \( \partial_x v |_{R^1 \times \partial \Omega} \). This is indicated by Majda [8] in the case of \( \theta = -\omega \) (cf. Lemma 2.1 of [8]). We have

**Lemma 2.2.** There exists a first order pseudo-differential operator \( B \) on \( R^1 \times \partial \Omega \) independent of \( t \) such that

1. its symbol \( B(\xi'; \tau, \xi') \) represented near

\[
N(\omega-\theta) = \{ x: x \cdot (\theta - \omega) = r(\omega-\theta) \} \cap \partial \Omega
\]

by local coordinates \((t, \tilde{x}')\), has a homogeneous asymptotic expansion \( \sum_{j=0} B_j(\xi'; \tau, \xi') \) satisfying

\[
(2.2) \quad -iB_0(\xi'; \pm 1, \mp \phi') > 0 \text{ on } N(\omega-\theta) (\phi' \text{ is the tangential component of } \theta \text{ to the plane } \{ x: x \cdot (\omega-\theta) = r(\omega-\theta) \})
\]

\[
(2.3) \quad B_j(\xi'; \tau, \xi') \text{ are purely imaginary-valued for even } j \text{ and real-valued for odd } j,
\]

2. \( \partial_x v |_{R^1 \times \partial \Omega} \) is equal to \( B(v |_{R^1 \times \partial \Omega}) \mod C^\infty \) in \( \{(t,x)\colon -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \cap R^1 \times \partial \Omega \) for some small constant \( \eta > 0 \).

In the above lemma, “a homogeneous asymptotic expansion \( \sum_{j=0} B_j(\xi'; \tau, \xi') \)’’ means that \( B_j(\xi'; \mu \tau, \mu \xi') = \mu^{-j} B_j(\xi'; \tau, \xi') \) for \( \mu \geq 1, |\tau| + |\xi'| \geq 1 \) and that

\[
|B(\xi'; \tau, \xi') - \sum_{j=1}^N B_j(\xi'; \tau, \xi')| \leq C_N |\tau| + |\xi'| + 1)^{-N-1} \text{ for any non-negative integer } N.
\]

For detailed description of pseudo-differential operators on manifolds, see Seeley [11], etc.; (ii) in the lemma states that \( \alpha(t, x') \partial_x v |_{R^1 \times \partial \Omega} - B(v |_{R^1 \times \partial \Omega}) \in C^\infty \) for any \( \alpha(t, x') \in C^\infty(R^1 \times \partial \Omega) \) with \( \text{supp } [\alpha] \subset \{(t,x): -r(\omega-\theta)-\eta \leq x \cdot \theta - t \} \).

**Proof of Lemma 2.2.** Let \( \sum_{i=1}^I \chi_i(x) \) be a partition of unity on a neighborhood of \( N(\omega-\theta) \) satisfying \( \max_{1 \leq i \leq I} |\text{supp } [\chi_i]| \leq \varepsilon_0 \) (\( \varepsilon_0 \) is a sufficiently small positive constant). Then there is a constant \( \varepsilon_1 > 0 \) such that \( \sum_{i=1}^I \chi_i(x) = 1 \) for any \( x \in \partial \Omega \) satisfying \( -r(\omega-\theta)-\varepsilon_1 \leq x \cdot \theta - x \cdot \omega \). Let \( v_i(t,x) \) be the solution of the equation

\[
\begin{cases}
\Box v_i(t,x) = 0 & \text{in } R^1 \times \Omega, \\
v_i(t,x') = \chi_i(x')v(t,x';\omega) & \text{on } R^1 \times \partial \Omega, \\
v_i(t,x) = 0 & \text{for } t < r(\omega).
\end{cases}
\]
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Since \( \text{supp}[v | R^1 \times \Omega] \subset \{(t, x'): x' \cdot \omega = t\} \), \( \sum_{i=1}^{l} v_i(t, x') \) is equal to \( v(t, x'; \omega) \) on \( (R^1 \times \Omega) \cap \{(t, x'): -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t \} \), and so, noting that the propagation speed is less than one, we have

\[
v(t, x; \omega) = \sum_{i=1}^{l} v_i(t, x) \quad \text{in} \quad (R^1 \times \Omega) \cap \{(t, x): -r(\omega - \theta) - \varepsilon_1 \leq x' \cdot \theta - t \}.
\]

We denote by \( \text{WF}[f(t, x)] \) the wave front set of \( f(t, x) \). It is seen that \( \text{WF}[v | R^1 \times \Omega] = \text{WF}[\delta(x' \cdot \omega - t) | R^1 \times \Omega] \) where \( \delta \) is the delta function. Hence, for any \( (t, x', \tau, \xi') \in \text{WF}[v | R^1 \times \Omega] \) the equation \( \tau^2 - |\xi' + \lambda \xi|^2 = 0 \) in \( \lambda \) has real roots, and the null-bicharacteristics associated with \( \partial_t - \Delta \) through \( \text{WF}[v | R^1 \times \Omega] \) are transversal to \( R^1 \times \partial \Omega \) (non-glancing). This implies that \( \text{sing supp}[\partial_v v | R^1 \times \Omega] \subset \text{sing supp}[v | R^1 \times \Omega] \) (cf. Theorem 7 in §9 of (Lax and) Nirenberg [10]), and so it suffices to examine \( v_i(t, x) \) only in a neighborhood \( (t_i - \varepsilon_0, t_i + \varepsilon_0) \times \Omega \) of \( (t_i, x_i) \) \((x_i \in \text{supp}[\chi_i] \cap N(\omega - \theta) \) and \( t_i = x_i \cdot \omega) \).

To analyze \( v_i \) more precisely, we transform \( \Omega \) in \( U_i \) into the half-space \( R^d_+ = \{x = (x', x_0): x_0 > 0\} \). Let the derivative \( \partial_x \) be transformed in \( U_i \) into \( -\partial_x \). For any set \( M \) in \( R^d \) we denote by \( M \) the set transformed by the coordinates \( \tilde{x} \). Let \( -\Delta_x \) be represented by \( \tilde{x} \) of the form \( \tilde{A} = \sum_{|a| \leq 2} a(\tilde{x}) \tilde{\xi}^a \). Here we can assume that the coefficients \( a(\tilde{x}) \) are real-valued \( C^\infty \) functions defined on \( R^d \) and constant out of \( \tilde{U_i} \). Let us examine the solution \( \partial(t, \tilde{x}) \) of the following equation instead of \( v_i(t, x) \):

\[
\begin{cases}
(\partial^2_t + \tilde{A})\partial(t, \tilde{x}) = 0 & \text{in} \quad R^1 \times R^d_+, \\
\partial(t, \tilde{x}) = g(t, \tilde{x}') & \text{on} \quad R^1 \times R^{d-1}, \\
\partial(t, \tilde{x}) = 0 & \text{for} \quad t < t_i - \varepsilon_0,
\end{cases}
\]

where \( g(t, \tilde{x}') \) is contained in a sufficiently small conic neighborhood of \((t_i, x'; \pm 1, \mp \theta') \) \((\theta' \in \text{the component of} \theta \text{transformed by the coordinates} \tilde{x} \text{tangent to the plane} \tilde{x}_0 \text{is near} |(\pm 1, \mp \theta')|^{-1}(\pm 1, \mp \theta') \text{the equation} \)

(2.4)

\[
\tau^2 + \tilde{A}(\tilde{x}, \xi', \xi_0) = 0
\]

(\( \tilde{A}(\tilde{x}, \xi', \xi_0) = \sum_{|a| \leq 2} a(\tilde{x}) \xi^a \)) in \( \xi_0 \) has two real roots. Furthermore, examining the forms of these roots, by the same procedure as in Nirenberg [10] or Kumano-go [5] (see Lemma 1 in §5 of [10] or Appendix II of [5]) we can construct first order pseudo-differential operators \( \xi^j(\tilde{x}; D_t, D_x) \) on \( R^1 \times R^d_0 \) (independent of \( t \)) with homogeneous asymptotic expansions \( \sum_{j=0}^{\infty} \xi^j(\tilde{x}; \tau, \xi') \) such that

(1) \( \xi^j(\tilde{x}; \tau, \xi') \) are real-valued for even \( j \) and purely imaginary-valued for odd \( j \),
(ii) if \(|(\tau, \xi')|^{-1}(\tau, \xi')\) is near \(|(-1, \bar{\theta}')|^{-1}(1, \bar{\theta}')\) or \(|(1, -\bar{\theta}')|^{-1}(1, -\bar{\theta}')\),
\(\xi^\circ(\bar{x}; \tau, \xi')\) are equal to the roots of the equation (2.4), and

\[\xi^\circ(\bar{x}^i; \pm 1, \mp \bar{\theta}') = \mp (1 - |\bar{\theta}'|^2)^{1/2},\]

(iii) all the null-bicharacteristic curves associated with \(D_{\bar{x}_0} - \xi^\circ(\bar{x}; D_t, D_{\bar{x}})\) through \(WF[g(t, \bar{x}')]\) are transversal to the boundary \(\{\bar{x}_0 = 0\}\) and proceed in the direction \(t > 0\) as they leave the boundary,

(iv) if the wave front set of \(u(t, x)\) is near the bicharacteristic curves stated in the above (iii), then we have

\[(D_{\bar{x}_0} - \xi^\circ(\bar{x}; D_t, D_{\bar{x}}))(D_{\bar{x}_0} - \xi^\circ(\bar{x}; D_t, D_{\bar{x}}))u = \zeta(\bar{x})(\partial_t^2 + A)u \mod C^\infty,\]

where \(\zeta(\bar{x})\) is a \(C^\infty\) function on \(\mathbb{R}^n\) satisfying \(\zeta(\bar{x}) < 0\) for every \(\bar{x}\).

(iii) and (iv) imply that \(\vartheta(t, \bar{x}', \bar{x}_0)\) is approximated \(\mod C^\infty\) by the solution \(w(\bar{x}_0; t, \bar{x}')\) of the equation

\[\begin{align*}
(D_{\bar{x}_0} - \xi^\circ(\bar{x}; D_t, D_{\bar{x}}))w &= 0, \\
|w|_{\bar{x}_0 = 0} &= h(t, \bar{x}').
\end{align*}\]

Therefore we have

\[-\partial_{\bar{x}_0}\vartheta |_{\bar{x}_0 = 0} = -i\xi^\circ(\bar{x}', 0; D_t, D_{\bar{x}}) (\vartheta |_{\bar{x}_0 = 0}) \mod C^\infty.\]

Combining this with the above (i) and (ii) yields the lemma. The proof is complete.

Proof of Theorem 3. The solution \(v(t, x; \omega)\) in (2.1) satisfies \(\text{supp}[v |_{t' < \omega}] \subset \{(t, x): x \cdot \omega = t\}\). Therefore, noting that the propagation speed is less than one, we see that \(\text{supp}[v(t, x; \omega)] \subset \{(t, x): x \cdot \omega \leq t\}\), which yields

\[v(x \cdot \theta - s, x; \omega) = 0 \quad \text{if} \quad s > x \cdot (\theta - \omega).\]

Hence, if \(s > \max x \cdot (\theta - \omega) = -r(\omega - \theta) (\omega \neq \theta)\), we obtain \(S(s, \theta, \omega) = 0\) from Proposition 2.1.

Next, let us prove that \(S(s, \theta, \omega)\) is singular at \(s = -r(\omega - \theta)\). Take \(\alpha(s) \in C^\infty(\mathbb{R})\) such that \(0 \leq \alpha \leq 1\) on \(\mathbb{R}\), \(\alpha(s) = 1\) for \(|s| \leq 2^{-1}\) and \(\alpha(s) = 0\) for \(|s| \geq 1\). For any \(\varepsilon > 0\) set

\[\alpha_\varepsilon(s) = \alpha\left(\frac{s + r(\omega - \theta)}{2\varepsilon}\right).\]

Then we have only to prove that \(\alpha_\varepsilon(s)S(s, \theta, \omega)\) is not \(C^\infty\) for any small \(\varepsilon > 0\). Proposition 2.1 yields

\[\alpha_\varepsilon(s)S(s, \theta, \omega) = \int_{\Theta_\Omega} \alpha_\varepsilon(s)(\partial_t^{-2}\partial_\nu v)(x \cdot \theta - s, x; \omega) dS_x,

- \int_{\Theta_\Omega} v \cdot \theta \alpha_\varepsilon(s)(\partial_t^{-1} v)(x \cdot \theta - s, x; \omega) dS_x = f_1(s) + f_2(s).\]
Let $F[k(s)](\sigma) = \int e^{i\sigma \varphi} k(s) ds$. As is readily seen, it follows that
\begin{equation}
F[J_\varphi(s)](\sigma) = -2^{-1}(-2\pi i)^{1-n} \sum_{j=0}^{n-1} \frac{\Gamma^{n-1}((i\sigma)^{n-1})}{j!} \int_{\partial \Omega} e^{i\sigma \varphi(\theta-w)} (-v \cdot \theta) \cdot \alpha_{\varphi}^{(j)}(x \cdot (\theta-w)) dS_x \tag{2.5}
\end{equation}
(where $C^{n-1}_j = (n-1)!/(n-1-j)! j!$). Taking the $\varepsilon(>0)$ so that $2\varepsilon \leq \eta$, by Lemma 2.2 we have
\begin{equation}
F[J_\varphi(s)](\sigma) = \int_{R^1 \times \partial \Omega} e^{i\sigma (\varphi - t)} \alpha_{\varphi}(x \cdot \theta - t) \partial_{\theta}^{-2} [B_{\varphi} |_{R^1 \times \partial \Omega}](t, x) dS_t dS_x \tag{2.6}
\end{equation}

Here $'B$ denotes the transposed operator of $B$ (i.e. $\langle 'Bf, g \rangle = \langle f, Bg \rangle$ for any $f$ and $g \in C^\infty_0(R^1 \times \partial \Omega)$). Let us note that the symbol of $'B$ expressed near $\text{supp} \{\alpha_{\varphi}(x \cdot \theta - t)\} \cap (R^1 \times \partial \Omega)$ by the local coordinates $(t, x')$, has a homogeneous asymptotic expansion $\sum_{j=0} \int_{R^1 \times \partial \Omega} e^{i\sigma (\varphi - t)} \alpha_{\varphi}(x \cdot \theta - t) \partial_{\theta}^{-2} [B_{\varphi} |_{R^1 \times \partial \Omega}](t, x) dS_t dS_x$

and purely imaginary valued for even $j$ and that $-i 'B_{\varphi}(x'; \pm 1, \mp \theta') = -i B_{\varphi}(x'; \mp 1, \pm \theta') \equiv 0$ for $\bar{x}' \in N(\omega - \theta)$, which follows from Lemma 2.2. By the methods of stationary phases (cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we can expand $'B[e^{i\sigma (\varphi - t)} \alpha_{\varphi}^{(j)}(x \cdot \theta - t)]$ asymptotically (as $\sigma \to \infty$) in the same way as in Proposition 4.1 of the author [12]. Therefore we obtain the asymptotic expansion
\begin{equation}
\beta_{\varphi}(x) = -i 'B_{\varphi}(x'; -1, \theta') \alpha_{\varphi}(x \cdot (\theta - \omega)) > 0 \quad \text{for } x \in N(\omega - \theta). \tag{2.6}
\end{equation}

Combining (2.5) and (2.6) yields that for any integer $N(>0)$
\begin{equation}
F[\alpha_{\varphi}(s)]S(s, \theta, \omega)(\sigma) = -2^{-1}(-2\pi i)^{1-n} (i\sigma)^{n-1} \int_{R^1 \times \partial \Omega} e^{-i\sigma \varphi(\bar{x}') \cdot (\omega - \theta)} \beta_{\varphi}(x) dS_x \tag{2.6}
\end{equation}

\begin{equation}
\cdot \{ \sum_{j=0}^{n-1} \rho_j(\bar{x}') (i\sigma)^{-j} \} d\bar{x}' + o(\sigma^{-N}). \tag{2.6}
\end{equation}

Here $\bar{x}'$ is the local coordinates on $\partial \Omega$ near $N(\omega - \theta)$ and
\begin{equation}
\rho_j(\bar{x}') = \beta_{\varphi}(x(\bar{x}')) + (-v \cdot \theta) \alpha_{\varphi}^{(j)}(x(\bar{x}) \cdot (\theta - \omega)) \quad (\alpha_{\varphi}^{(j)} = 0, j \geq n). \tag{2.6}
\end{equation}

Noting that $\rho_j(\bar{x}') > 0$ when the phase function $x(\bar{x}) \cdot (\omega - \theta)$ is minimum, and applying Theorem 2, we obtain for some constant $m \in \mathbb{R}$
\begin{equation}
\sigma^m F[\alpha_{\varphi}(s)]S(s, \theta, \omega)(\sigma) \in L^2(1, \infty), \tag{2.6}
\end{equation}
which shows that \( \alpha_t(s)S(s, \theta, \omega) \) is not \( C^\infty \). The proof is complete.

3. **Proof of Theorem 4**

We use the same notations as for the scattering by obstacles in §2. The scattering operator \( S \) for the equation (0.3) is represented as follows (see Theorem 1 and (3.1) of the author [15]):

**Proposition 3.1.** Set

\[
S_0(s, \theta, \omega) = \int_{\mathbb{R}^n} (\partial_t^2 - \Delta) \omega(t - s, x) dt dx,
\]

where \( \omega(t, x) \) is the solution of the equation

\[
\begin{cases}
(\partial_t^2 - A) \omega(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
\omega(-r_0, x) = -2^{-1}(-2\pi i)^{1-n} \delta(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n, \\
\partial_t \omega(-r_0, x) = -2^{-1}(-2\pi i)^{1-n} \delta'(-r_0 - x \cdot \omega) & \text{on } \mathbb{R}^n.
\end{cases}
\]

Then we have

\[
(Sk)(s, \theta) = \int_{\mathbb{R}^n} S_0(s - t, \theta, \omega) k(t, \omega) dt d\omega + (Kk)(s, \theta).
\]

Note that \( S_0(s, \theta, \omega) = S(s, \theta, \omega) \) if \( \omega = \theta \).

To prove Theorem 4, we have only to show that for any small \( \varepsilon(>0) \) there exist a real number \( m \) and a function \( \rho(s) \in C_0^\infty(s_0 - 2\varepsilon, s_0 + 2\varepsilon) \) such that

\[
(1 + |\sigma|)^m F[\rho(s)S(s, \theta, \omega)](\sigma) \in L^2(\mathbb{R}^l).
\]

Let \( \gamma(x) \in C_0^\infty(\mathbb{R}^n) \) with \( \gamma(x) = 1 \) in a neighborhood of \( \bar{M}_\omega(\theta) \), and denote by \( \bar{\omega}(t, x) \) the solution of the equation

\[
\begin{cases}
(\partial_t^2 - A) \bar{\omega}(t, x) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}^n, \\
\bar{\omega}(-r_0, x) = \gamma(x) \omega(-r_0, x) & \text{on } \mathbb{R}^n, \\
\partial_t \bar{\omega}(-r_0, x) = \gamma(x) \partial_t \omega(-r_0, x) & \text{on } \mathbb{R}^n.
\end{cases}
\]

The author [15] showed that if \( \tilde{l} \) is large enough we have for any integer \( N(>0) \)

\[
F[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1} e^{-i\sigma \tilde{l}} \sum_{j=0}^{N-1} (i \sigma)^{n-1-j} \mathcal{F}'[\beta_j(x) \{\bar{\omega}(\tilde{l}, x) \\
+ (i \sigma)^{-1} \partial_t \bar{\omega}(\tilde{l}, x)\}](-\sigma \theta) + O(\sigma^{-N+N_0})
\]

as \( \sigma \to \infty \) (\( N_0 \) is an integer independent of \( N \) (cf. (4.5) in [15]). Here, \( \mathcal{F}' \)

denotes the Fourier transformation in \( x \), and the functions \( \beta_j(x) \in C_0^\infty(\mathbb{R}^n) \) are all real-valued.
We take \( t \) so large as to have (i) and (ii) stated in the following

**Lemma 3.2.** Let \( r_1 \) be an arbitrary constant \(( \geq r_0)\), and set

\[
\psi(x; t) = q^-(t; -r_0, x, \omega) \cdot \theta.
\]

Then, for any \( \varepsilon ( > 0) \) there is a constant \( \bar{t}_0 \) such that for any fixed \( \bar{t} \geq \bar{t}_0 \)

(i) \( \max_{|r| \leq r_1} |\psi(x; \bar{t})| \leq \varepsilon_0(\theta) + \bar{t} + \varepsilon \),

(ii) all points at which \( \psi(x; \bar{t}) \) is maximum \((x \cdot \omega = -r_0, |x| \leq r_1)\), are contained in \( \varepsilon \)-neighborhood \(( M_\omega(\theta))_\varepsilon \) of \( M_\omega(\theta) \) \( ((M) = \{ x : \text{dis}(x, M) < \varepsilon \}) \).

This lemma will be proved later. Choose the \( p(s) \) so that \( p(s) > 0 \) on \( R^1 \) and \( p(s) > 0 \) on \([s_\omega(\theta) - \varepsilon, s_\omega(\theta) + \varepsilon]\). Then it is seen from the form of \( \beta_\omega(x) \) (cf. (4.4) and (4.6) in [15]) and the above lemma that

\[
\beta_\omega(x) \geq 0 \quad \text{on} \quad R^n \quad \text{and} \quad \beta_\omega(q^-(\bar{t}; -r_0, y, \omega)) > 0
\]

for any \( y \in ( M_\omega(\theta))_\varepsilon \) \((y \cdot \omega = -r_0)\).

We take the \( \gamma(x) \) so that \( \gamma(x) \geq 0 \) on \( R^n \), \( \gamma(x) > 0 \) on \(( M_\omega(\theta))_\varepsilon \) and \( \text{supp}[\gamma] \subset ( M_\omega(\theta))_{2\varepsilon} \).

By the same procedure as in Nirenberg [10], Kumano-go [5] (cf. §5 of [10] or Appendix II of [5]), we can construct a symbol \( \lambda(x, \xi) \) with a homogeneous asymptotic expansion \( \sum_{j=0}^{\infty} \lambda_j(x, \xi) \) such that

\[
\lambda_0(x, \xi) = \{ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \} \frac{1}{12} + \partial_i A = ( D_i + \lambda(x, D_2)) ( D_i - \lambda(x, D_2)) \]

modulo a smoothing operator (cf. Corollary 2.5 in the author [15] also). Furthermore we see that \( \lambda_j(x, \xi) \) are real-valued for even \( j \) and purely imaginary valued for odd \( j \) since the coefficients \( a_{ij}(x) \) are all real-valued (recall the construction of \( \xi^\pm (x'; \tau, \xi') \) in §2).

Consider the Cauchy problem

\[
\begin{align*}
\{ ( D_i - \lambda(x, D_2)) u(t, x) &= 0 \quad \text{in} \quad R^1 \times R^n, \\
| u \rangle_{t=0} &= u_0(x) \quad \text{on} \quad R^n,
\end{align*}
\]

and denote by \( E(t) \) the operator: \( u_0 \rightarrow u(t, \cdot) \). Then \( \check{\omega}(\bar{t}, x) \) and \( \partial_t \check{\omega}(\bar{t}, x) \) are represented as follows:

\[
\begin{align*}
\check{\omega}(\bar{t}, x) &= 2^{-1} E(\bar{t} + r_0) (\check{\omega}(-r_0, \cdot) - i \check{\mu} \partial_t \check{\omega}(-r_0, \cdot)) (x) \\
&\quad + 2^{-1} E(-\bar{t} - r_0) (\check{\omega}(-r_0, \cdot) + i \check{\mu} \partial_t \check{\omega}(-r_0, \cdot)) (x), \\
\check{\partial}_t \check{\omega}(\bar{t}, x) &= 2^{-1} E(\bar{t} + r_0) i \check{\lambda} (\check{\omega}(-r_0, \cdot) - i \check{\mu} \partial_t \check{\omega}(-r_0, \cdot)) (x) \\
&\quad + 2^{-1} E(-\bar{t} - r_0) i \check{\lambda} (\check{\omega}(-r_0, \cdot) + i \check{\mu} \partial_t \check{\omega}(-r_0, \cdot)) (x),
\end{align*}
\]

where \( \check{\lambda} \) and \( \check{\mu} \) are pseudo-differential operators whose symbols coincide with
\( \lambda(x, \xi) \) and \( \mu(x, \xi) \) (\( \mu(x, D_x) \)) is the parametrix of \( \lambda(x, D_x) \) respectively in a neighborhood of \( \text{supp}[\gamma(x)] \) and vanish for large \( |x| \). Therefore, noting that
\[
\mathcal{L}^{*}\beta \mathcal{E}(\bar{\xi} - r_o)(\bar{w}(r_o, \cdot) + i\bar{\mu} \partial \bar{w}(r_o, \cdot)) (-\sigma \theta) = 0(\sigma^{-\infty}), \\
\mathcal{L}^{*}\beta \mathcal{E}(\bar{\xi} - r_o) \lambda(\bar{w}(r_o, \cdot) + i\bar{\mu} \partial \bar{w}(r_o, \cdot)) (-\sigma \theta) = 0(\sigma^{-\infty})
\]
as \( \sigma \to \infty \) (cf. §4 of the author [15]), we have
\[
\mathcal{F}[\rho(s)S(s, \theta, \omega)](\sigma) = 2^{-1} e^{-i\pi \gamma} \sum_{j=0}^{\infty} (i\sigma)^{j+1-i} \mathcal{L}^{*}\beta \mathcal{E}(\bar{\xi} - r_o)(1 + |i\sigma - \lambda|)
\]
\[\cdot \left( \bar{w}(r_o, \cdot) - i\bar{\mu} \partial \bar{w}(r_o, \cdot) \right) (-\sigma \theta) + O(\sigma^{-\infty}) \cdot \mathcal{F}[\rho(\sigma)S(\sigma, \theta, \omega)](\sigma) = 0(\sigma^{-\infty})
\]

The assumption \( (0.5) \) implies that if \( \text{WF}[u_0] \) is contained in a conic neighborhood of \( \tilde{M}_\omega(\theta) \times \{ -\omega \} \) (\( \text{WF}(\bar{w}(r_o, \cdot) - i\bar{\mu} \partial \bar{w}(r_o, \cdot) \) is contained there) \( E(\bar{\xi} + r_o)u_0 \) is represented by the Fourier integral operator:
\[
E(\bar{\xi} + r_o)u_0(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\varphi(q+y, z, \xi)} a(\bar{\xi} + r_o, x, \xi) \tilde{u}_0(\xi) d\xi \mod C^\infty
\]
(cf. the proof of Theorem 2.6 in the author [15]). Moreover note that \( \mathcal{F}[\delta^{(\eta)}(-r_o - x \cdot \omega)] (B_\eta) = (-i\pi)^{n} \delta^\eta (\gamma') \), where \( B = (b_1, \ldots, b_n) \) is an orthogonal matrix with \( b_1 = \omega \). Then, introducing change of the variables \( x = \bar{q}(\bar{\xi} - r_o, y, \omega) \) near \( x = \bar{q}(\bar{\xi} - r_o, \tilde{M}_\omega(\theta), \omega) \) (\( y = (y_0, y') \) is orthogonal coordinates with \( y_0 = x \cdot \omega \)), we obtain
\[
\mathcal{F}[\beta \mathcal{E}(\bar{\xi} + r_o)(1 + |i\sigma - \lambda|) (\bar{w}(r_o, \cdot) - i\bar{\mu} \partial \bar{w}(r_o, \cdot)) (-\sigma \theta) = 0(\sigma^{-\infty})
\]
\[\mathcal{F}[\beta \mathcal{E}(\bar{\xi} + r_o)(1 + |i\sigma - \lambda|) (\bar{w}(r_o, \cdot) - i\bar{\mu} \partial \bar{w}(r_o, \cdot)) (-\sigma \theta) = 0(\sigma^{-\infty})
\]

(\( \gamma(x) \in C_0^\infty(\mathbb{R}^n) \), \( \gamma(x) = 1 \) on a neighborhood of \( q^- \) (\( \text{supp}[\gamma] \)), and \( \tau \) is a positive constant independent of \( \sigma \). The function \( \Phi(y_0, \tau) = g^-(y_0, y') \cdot \theta - \tau(y_0 + r_o) \) has the stationary point \( (y_0, \tau) = (r_0, p^- - r_o, y') \cdot \theta \), at which its Hesse matrix equals \( \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \). Expanding \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\varphi(x, y, \tau)} \beta \gamma dy \) (as \( \sigma \to \infty \)) by the methods of stationary phases (e.g., cf. §3.2 of Hörmander [2], §4 of Matsumura [9], etc.), we have the asymptotic expansion
\[
(3.2) \quad \mathcal{F}[\rho(s)S(s, \theta, \omega)](\sigma) = e^{-i\pi \gamma} \sum_{j=0}^{\infty} e^{i\varphi(x, y, \tau)} (r_0, x, \omega) \cdot \mathcal{F}[\rho_j(x) i\sigma^{\gamma-1}] dx + 0(\sigma^{-N+N_0})
\]
\( (N_0 \) is an integer independent of \( N = 1, 2, \ldots) \). Here \( \rho_j \) are \( C^\infty \) functions with
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supp[ρ] ⊂ supp[γ] and all real-valued, which follows from the fact that the symbol $a(τ, x, ξ)$ has a homogeneous asymptotic expansion $\sum_{k=0}^{\infty} a_k(τ, x, ξ)$ such that $a_k(τ, x, ξ)$ are real-valued for even $k$ and purely imaginary valued for odd $k$; furthermore $ρ_0$ is of the form

$$ρ_0(τ) = γ(τ)β(τ)(q^-(τ; -τ_0, x, ξ))α_k(τ) + r_0 q^- (τ; -r_0, x, ξ), -ω) |det \frac{∂q^-}{∂y}|.$$ 

Combining this with (3.1) and (ii) of Lemma 3.2, we see that $ρ_0(x) ≥ 0$ on $R^n$ and $ρ_0(x) > 0$ for any $x$ at which the function

$$φ(x) = -q^-(τ; -τ_0, x, ξ) \cdot θ (x \cdot ξ = -τ_0)$$

is minimum. Thus, applying Theorem 2 to (3.2), we obtain

$$σ^m Fr(σ) ∈ L^2(1, ∞)$$

for some constant $m ∈ R$, which proves Theorem 4.

Proof of Lemma 3.2. We denote by $y$ the variables on $R^{n-1} = \{x: x \cdot ξ = -τ_0\}$. It follows from (0.4) that for a large constant $τ_0$ independent of $τ$, $y$ and $ω$

$$q^- (τ; -τ_0, y, ξ) = q^- (τ_0; -τ_0, y, ξ) + (τ - τ_0)p^- (τ_0; -τ_0, y, ξ), \ t ≥ τ_0, y ∈ R^{n-1}.$$ 

Fix $y ∈ M_u(ε)$ arbitrarily and take a neighborhood $U(y)$ of $y$ such that

$$|q^- (τ_0; -τ_0, y, ξ) - q^- (τ_0; -r_0, y, ξ)| ≤ ε/2 \quad \text{for any } y ∈ U(y),$$

$$|τ_0 (p^- (τ_0; -τ_0, y, ξ) - p^- (τ_0; -r_0, y, ξ))| ≤ ε/2 \quad \text{for any } y ∈ U(y).$$

Then, in view of the definitions of $M_u(ε)$ and $s_u(ε)$ we have for any $y ∈ U(y)$ and $τ ≥ τ_0$

$$ψ(y; τ) ≤ q^- (τ_0; -τ_0, y, ξ) \cdot θ - τ_0 p^- (τ_0; -τ_0, y, ξ) \cdot θ + τ p^- (τ_0; -τ_0, y, ξ) \cdot θ + ε \leq s_u(ε) + ε + τ.$$ 

On the other hand, for any neighborhood $U$ of $M_u(ε)$ it follows that $δ = \inf_{τ \geq τ_0} \{1 - (p^- (τ_0; -τ_0, y, ξ) \cdot θ) ≥ 0$, which yields that $ψ(y; t) ≤ (C - δ t) + t$ for any $y ∈ U$ (\( |y| ≤ r_1 \)) and $t ≥ τ_0$ ($C$ is a constant independent of $y$ and $t$). This means that

$$ψ(y; τ) ≤ s_u(ε) - 1 + τ$$

if $y ∈ U$, $|y| ≤ r_1$ and $τ$ is large enough. Therefore we obtain the lemma.

References


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