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## ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES $\blacktriangledown$

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Throughout this note,  $R$  is a ring with identity,  $J$  the Jacobson radical of  $R$ , and all  $R$ -modules are unitary right  $R$ -modules. Further we let  $e$  be a local idempotent of  $R$  and  $U_R$  a submodule of  $eJ$  such that  $eR/U$  is of finite length. Put  $D:=eRe/eJe$  and  $D(U):=\{x+eJe \in D \mid x \in eRe, xU \leq U\}$ . Then  $D(U)$  is a division subring of  $D$ . In [3]–[6], we have given a relationship between the dimension  $[D: D(U)]_r$  of  $D$  as a *right*  $D(U)$ -vector space and the property  $(**_n)$  of maximal submodules of  $(eR/U)^{(n)}$  defined there. (The dual result had been obtained in [1, Proposition 2.1] from another point of view.)

In this short note, we shall study the dimension  $[D: D(U)]_l$  of  $D$  as a *left*  $D(U)$ -vector space and give it a meaning. Originally our considerations had been restricted to the case of uniform modules of Loewy length 2 over an artinian ring with proofs along the line of Sumioka [8, Lemma 5.3], and later by different proofs we generalized and dualized to get the present form. Hence it should be noted that by dualizing the arguments all the parallel results hold for uniform modules if we assume that  $[D: D(U)]_l < \infty$ .

In what follows, we denote by  $|M|$  and by  $\#I$  the composition length of each  $R$ -module  $M$  and the cardinality of each set  $I$ , respectively. Let  $L$  be a submodule of an  $R$ -module  $M$ . Then we say that  $L$  is a *characteristic* submodule of  $M$  if  $fL \leq L$  for every endomorphism  $f$  of  $M$ . By  $M^{(I)}$  we denote the direct sum of  $\#I$  copies of  $M$  for each  $R$ -module  $M$  and each set  $I$ . We regard  $Me = \text{Hom}_R(eR, M)$  for every  $R$ -module  $M$  by identifying each  $t \in Me$  with the map  $eR \rightarrow M$  defined via  $x \mapsto tx$  for each  $x \in eR$ . So in particular for each  $t \in eRe = \text{End}_R(eR)$ ,  $t^{-1}U$  means the inverse image  $\{x \in eR \mid tx \in U\}$  of  $U$  under  $t$ .

Now the canonical epimorphism  $\pi: eR \rightarrow eR/U$  induces a monomorphism  $\text{End}_R(eR/U) \rightarrow \text{Hom}_R(eR, eR/U)$  by which we regard  $\text{End}_R(eR/U) = \{f \in \text{Hom}_R(eR, eR/U) \mid f(U) = 0\} \leq \text{Hom}_R(eR, eR/U)$ . Consider the epimorphism

$$\delta: \text{Hom}_R(eR, eR/U) \rightarrow \text{Hom}_R(eR, eR/eJ) = eRe/eJe$$

induced from the canonical epimorphism  $eR/Ue \rightarrow R/eJ$  by the exact functor

$\text{Hom}_R(eR, -)$ . Then  $\delta$  induces a ring homomorphism  $\delta_U := \delta|_{\text{End}_R(eR/U)} : \text{End}_R(eR/U) \rightarrow eRe/eJe$  and a ring isomorphism  $\text{Im } \delta_U \cong \text{End}_R(eR/U)/K_U$  by which we identify these rings, where  $K_U := \text{Ker } \delta_U = \{f \in \text{End}_R(eR/U) \mid \text{Im } f \leq eJ/U\}$  is the Jacobson radical of the local ring  $\text{End}_R(eR/U)$ . Hence  $\text{Im } \delta_U$  is a division subring of  $eRe/eJe$ . Putting  $D := eRe/eJe$  and  $D(U) := \text{Im } \delta_U$ , we examine the dimension  $[D : D(U)]_I$  of  $D$  as a left  $D(U)$ -vector space. Note that this definition of  $D(U)$  coincides with that defined in the introduction. Observe that the left module structure of  $\text{Hom}_R(eR, eR/U)$  over  $\text{End}_R(eR/U)$  induces that of  $\overline{\text{Hom}}_R(eR, eR/U) := \text{Hom}_R(eR, eR/U)/K$  over  $\text{End}_R(eR/U)/K_U = D(U)$  where  $K := \text{Ker } \delta = \{f \in \text{Hom}_R(eR, eR/U) \mid \text{Im } f \leq eJ/U\}$ . Then  $\delta$  induces an isomorphism

$$\#) \quad {}_{D(U)}\overline{\text{Hom}}_R(eR, eR/U) \cong {}_{D(U)}D$$

which plays a basic role in our study. By  $\tilde{f}$  we denote the coset of  $f$  in  $\overline{\text{Hom}}_R(eR, eR/U)$  for each  $f \in \text{Hom}_R(eR, eR/U)$ . Now the isomorphism  $\#)$  tells us that  $[D : D(U)]_I = n$  iff there exists a map  $f = (f_i)_{i \in I}^T : eR \rightarrow (eR/U)^{(I)}$  satisfying the following condition  $(\#)$  in case  $\#I = n$  but not in case  $\#I = k$  for any  $k < n$ .

( $\#$ ) For each  $g : eR \rightarrow eR/U$ , there exists  $h = (h_i)_{i \in I} : (eR/U)^{(I)} \rightarrow eR$  such that  $g = \sum_{i \in I} \bar{h}_i \tilde{f}_i$ , i.e.  $(g - hf)(eR) \leq eJ/U$ .

For, ( $\#$ ) is equivalent to saying that the set  $\{\tilde{f}_i\}_{i \in I}$  generates  ${}_{D(U)}\overline{\text{Hom}}_R(eR, eR/U)$ . We shall relate this condition with an inner structure of  $eR$  under suitable assumptions.

**Lemma 1.** *Let  $I$  be a set,  $t_i \in eRe$  for each  $i \in I$  and  $t \in eRe$ . Put  $s_i := \pi t_i$  and  $s := \pi t : eR \rightarrow eR/U$ . Consider the following conditions.*

- (a)  $\bar{s} \in \sum_{i \in I} D(U) \bar{s}_i$ .
- (b)  $\bigcap_{i \in I} t_i^{-1} U \leq t^{-1} U$ .

Then

- (1) If  $(eJe)U \leq U$  and  $t_l \in eJe$  for some  $l \in I$ , then (a) implies (b).
- (2) If  $eR/U$  is quasi-injective, then (b) implies that  $s \in \sum_{i \in I} \text{End}_R(eR/U) s_i$ ; whence also implies (a).

**Proof.** (1) By (a),  $s = \sum_{i \in I} r_i s_i + k$  for some  $(r_i)_{i \in I} : (eR/U)^{(I)} \rightarrow eR/U$  and some  $k : eR \rightarrow eJ/U$ . By the projectivity of  $eR$ ,  $k = \pi j$  for some  $j \in eJe$ . So  $t^{-1}U = \text{Ker } s \geq \bigcap_{i \in I} \text{Ker } s_i \cap \text{Ker } k = (\bigcap_{i \in I} t_i^{-1}U) \cap j^{-1}U$ . Here since  $jt_i^{-1} \in eJe$ ,  $(jt_i^{-1})U \leq U$  by assumption. Thus  $t_i^{-1}U \leq j^{-1}U$ . Hence  $t^{-1}U \geq \bigcap_{i \in I} t_i^{-1}U$ .

(2). Put  $s_I := (s_i)_{i \in I}^T : eR \rightarrow (eR/U)^{(I)}$ . It follows from (b) that  $\bigcap_{i \in I} \text{Ker } s_i \leq \text{Ker } s$ . So there is some  $r : \text{Im } s_I \rightarrow eR/U$  which makes the diagram (without the broken arrow)

$$\begin{array}{ccc}
 eR & \xrightarrow{s_I} & \text{Im } s_I \hookrightarrow (eR/U)^{(I)} \\
 s \downarrow & r \swarrow & q \searrow \\
 eR/U & \leftarrow &
 \end{array}$$

commutative. Since  $eR/U$  is quasi-injective,  $eR/U$  is  $(eR/U)^{(I)}$ -injective by Azumaya, Mbuntum and Varadarajan [2, Proposition 1.16 (2)]. Hence we have a homomorphism  $q=(q_i)_{i \in I}: (eR/U)^{(I)} \rightarrow eR/U$  which completes the commutative diagram above. Thus  $s = \sum_{i \in I} q_i s_i \in \sum_{i \in I} \text{End}_R(eR/U) s_i$ .

We put  $U^* = \bigcap_{t \in eRe} t^{-1}U$ . Then as easily seen,  $U^*$  is the largest characteristic submodule of  $eR$  contained in  $U$ .

The following is a direct consequence of Lemma 1.

**Proposition.** *Let  $I$ ,  $t_i$  and  $s_i$  be as in Lemma 1.*

- (1) *Assume that  $(eJe)U \leq U$  and  $t_i \notin eJe$  for all  $i \in I$ . Then*
- (i) *If  $\{\bar{s}_i\}_{i \in I}$  generates  ${}_{D(U)}\overline{\text{Hom}}_R(eR, eR/U)$ , then  $U^* = \bigcap_{i \in I} t_i^{-1}U$ .*
- (ii) *If the intersection  $\bigcap_{i \in I} t_i^{-1}U$  is irredundant, then  $\{\bar{s}_i\}_{i \in I}$  is linearly independent in  ${}_{D(U)}\overline{\text{Hom}}_R(eR, eR/U)$ .*
- (2) *Assume that  $eR/U$  is quasi-injective. Then the converse assertions of (i) and (ii) above hold.*

**Theorem.** *Assume that  $(eJe)U \leq U$  and  $eR/U$  is quasi-injective. Then the following cardinal numbers are equal.*

- (1)  $n := [D:D(U)]_I$ .
- (2)  $k := \min \{ \#K \mid U^* = \bigcap_{i \in K} t_i^{-1}U \text{ for some } t_i \in eRe \}$ .
- (3)  $l := \#I$  for a set  $I$  such that there is a  $t_i \in eRe \setminus eJe$  for each  $i \in I$  and  $U^* = \bigcap_{i \in I} t_i^{-1}U$  is an irredundant intersection.

**Proof.** Let  $\{\bar{s}_i\}_{i \in N}$  be a basis of  ${}_{D(U)}\overline{\text{Hom}}_R(eR, eR/U)$  where  $s_i \in \text{Hom}_R(eR, eR/U)$  for each  $i \in N$  and  $n = \#N$ . Since  $eR$  is projective,  $s_i = \pi t_i$  for some  $t_i \in eRe$  for each  $i$ . Noting that  $t_i \notin eJe$  since  $\bar{s}_i \neq 0$  for each  $i$ , we have  $U^* = \bigcap_{i \in N} t_i^{-1}U$  by Proposition (1) (i). Thus  $k \leq n$ . Also by Proposition (2) (i),  $n \leq k$ , i.e.  $n = k$ . So since the intersection  $U^* = \bigcap_{i \in N} t_i^{-1}U$  above is irredundant,  $l$  exists.

Clearly  $k \leq l$  for every  $I$  in (3). It follows immediately from Proposition (1) (ii) that  $l \leq n$  for every  $I$ . Hence  $l = k = n$ .

**Lemma 2.** *Let  $U^* = \bigcap_{i \in I} t_i^{-1}U$  be an irredundant intersection with each  $t_i \in eRe \setminus eJe$ . If there is a characteristic submodule  $T$  of  $eR$  containing  $U$  with  $|T/U| = 1$ , then  $\#I = |T/U^*|$ .*

**Proof.** For each  $i \in I$ , since  $|T/t_i^{-1}U|=1$  and  $\bigcap_{j \neq i} t_j^{-1}U \not\leq t_i^{-1}U$ , we have  $T=t_i^{-1}U + \bigcap_{j \neq i} t_j^{-1}U$ . Hence  $|T/U^*|=|\bigoplus_{i \in I} T/t_i^{-1}U|=\#I$ .

By Lemma 2, Proposition (1) (i) and (2) (ii), we obtain the following.

**Corollary.** *Assume that  $[D: D(U)]_l < \infty$  and that there is a characteristic submodule  $T$  of  $eR$  containing  $U$  with  $|T/U|=1$  (e.g.  $T=eJ$  and  $U$  is a maximal submodule of  $eJ$ ). Then*

- (1) *If  $(eJe)U \leq U$ , then  $|T/U^*| \leq [D: D(U)]_l$ .*
- (2) *If  $eR/U$  is quasi-injective, then  $[D: D(U)]_l \leq |T/U^*|$ .*

**REMARK.** (1) If  $\text{Ker } \delta=0$  (e.g.  $eJe=0$ ), then  ${}_{D(U)}D \cong {}_{\text{End}_R(eR/U)} \text{Hom}_R(eR, eR/U)$ .

(2) Assume that  $R$  is a finite dimensional algebra over a field. If  $U$  is a maximal submodule of  $eJ$  (i.e.  $eR/U$  is a uniserial module of length 2), then  $(eJe)U \leq U$  since  $eJ^2 \leq U \leq eJ$ . Put  $\mathcal{C} := \{U \mid U \text{ is a maximal submodule of } eJ\}$ , and let  $\{eR/U_i\}_{i=1}^n$  be a complete set of representatives of isomorphism classes of the class  $\{eR/U \mid U \in \mathcal{C}\}$ . Then since  $eJ^2 \leq U^* \leq U$  for each  $U \in \mathcal{C}$ , and  $eR/U \cong eR/V$  implies  $U^* = V^*$  for any  $U$  and  $V$  in  $\mathcal{C}$ , we have  $eJ^2 \leq \bigcap_{U \in \mathcal{C}} U^* = \bigcap_{i=1}^n U_i^* \leq \bigcap_{U \in \mathcal{C}} U = (eJ)J$  whence  $eJ^2 = \bigcap_{i=1}^n U_i^*$ . So  $|eJ/eJ^2| \leq |\bigoplus_{i=1}^n eJ/U_i^*| \leq \sum_{i=1}^n [D: D(U_i)]_l$  by Corollary (1). On the other hand, by [3, Lemma 3 and Proposition 6 i)], [5, Condition I] implies  $\sum_{i=1}^n [D: D(U_i)]_l = \sum_{i=1}^n [D: D(U_i)]_r \leq 2$ . Hence in this case [5, Condition I] implies  $|eJ/eJ^2| \leq 2$ , i.e. [5, Condition II]. The dual argument also works in [7] and [9].

(3) By using [2, Proposition 1.16 (1)] in the proof of the dual version of Lemma 1 (2), we see that all the results dual to the above hold if we assume that  $[D: D(U)]_l < \infty$ . The details are left to the reader.

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