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THE GLAUBERMAN CORRESPONDENT
OF A NILPOTENT BLOCK OF A FINITE GROUP

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Abstract
We prove that a nilpotent block of a finite group and its Glauberman correspondent
are basic Morita equivalent.

1. Introduction

Let $A$ and $G$ be finite groups such that $A$ is solvable, $A$ acts on $G$ by automorphisms
and that the orders $|A|$ and $|G|$ are coprime. Let $C = C_G(A)$. Let $(\mathcal{K}, \mathcal{O}, k)$ be a $p$-modular
system such that $\mathcal{K}$ contains the $|G|$-th roots of unity and $k$ is algebraically closed. We
denote by $\text{Irr}(G)$ the set of ordinary irreducible characters of $G$ and by $\text{Irr}(G)^A$ the set
of $A$-invariant elements of $\text{Irr}(G)$. Then there is a natural one-to-one and onto map
$\pi (G, A) : \text{Irr}(G)^A \to \text{Irr}(C)$ which we call the Glauberman character correspondence
([5] and [10], Chapter 13).

Let $B$ be a ($p$-)block of $G$. $B$ is a block algebra of $\mathcal{O}_G$. By [19], Proposition 1 and
Theorem 1, if $B$ is $A$-invariant and a defect group $D$ of $B$ is centralized by $A$, then
any $\chi \in \text{Irr}(B)$, the set of ordinary irreducible characters of $B$, is $A$-invariant and
there exists a block $b$ of $C$ such that $\pi (G, A)(\text{Irr}(B)) = \text{Irr}(b)$ (see [12], Theorem 4.3
for another proof of [19], Proposition 1). Moreover $D$ is a defect group of $b$, and
$\pi (G, A)$ gives an isotypy between $B$ and $b$ in the sense of [3], by [19], Proposition 5.
Then we call $b$ the Glauberman correspondent of $B$. In [8], Theorem 1.1, it is shown
that if $G$ is $p$-solvable, then there is a basic Morita equivalence, in the sense of [15],
Chapter 7, between $B$ and $b$ such that the bijection between $\text{Irr}(B)$ and $\text{Irr}(b)$ induced
by it is $\pi (G, A)_{|\text{Irr}(B)}$. Moreover in [7], Theorem 2 and [17], Theorem 4.9, it is shown
that if the defect group $D$ is normal in $G$, then there is a splendid Morita equivalence
between $B$ and $b$ such that the bijection between $\text{Irr}(B)$ and $\text{Irr}(b)$ induced by it is
$\pi (G, A)_{|\text{Irr}(B)}$ (see also [11], [9] and [16]). In this paper we prove the following.

Theorem 1. With the above notations, let $B$ be an $A$-invariant block of $G$ such
that a defect group $D$ of $B$ is contained in $C$, and let $b$ be the Glauberman corre-
respondent of $B$. Assume that $B$ is nilpotent. Then $b$ is nilpotent and there is a basic
Morita equivalence between $B$ and $b$ such that the bijection $\text{Irr}(B)$ and $\text{Irr}(b)$ induced by it is $\pi(G, A)|_{\text{Irr}(B)}$.

For notations and terminologies in this paper, we follow [18]. A module is a left module unless stated explicitly.

We remark that in [6], it is shown that a block of a finite group and its Isaacs correspondent are basic Morita equivalent.

2. Proof of Theorem 1

Let $G$ be a finite group and $B$ be a block of $G$. We denote by $1_B$ the identity element of $B$. Let $D_i$ be a maximal local pointed group contained in the pointed group $G_{\{1\}}$ on $OG$ and let $i \in \gamma$, then $i$ is called a $D$-source idempotent of $B$, and $Bi$ is called a source algebra of $B$. If $L$ is a left (resp. right) $Bi$-module, then $L$ is regarded as a left (resp. right) $OD$-module via the $O$-algebra homomorphism from $OD$ to $Bi$.

Let $G$ and $G'$ be finite groups. If $M$ is an $(OG', OG)$-bimodule, $M$ is considered as an $O(G' \times G)$-module through $(x', x)m = x'mx^{-1}$ for any $x' \in G'$, $x \in G$ and $m \in M$, which we denote by $G' \times G M$. Similarly, if $M$ is an $O(G' \times G)$-module, then $M$ is considered as an $(OG', OG)$-bimodule. Moreover, for $(OG', OG)$-bimodules $M_1$, $M_2$, those are isomorphic if and only if $G' \times G M_1$ and $G' \times G M_2$ are so. Let $D$ and $D'$ be subgroups of $G$ and $G'$ respectively and $N$ be an $(OD', OD)$-bimodule. We have an isomorphism of $O(G' \times G)$-modules

$$N_{D' \times D}^{G' \times G} \cong OG' \otimes_{OD'} N \otimes_{OD} OG$$

mapping $(x', x) \otimes n$ to $x' \otimes n \otimes x^{-1}$ for any $x' \in G'$, $x \in G$ and $n \in N$ where $N_{D' \times D}^{G' \times G}$ is the induced $O(G' \times G)$-module from the $O(D' \times D)$-module $N$.

**Lemma 2.** Let $G$ and $G'$ be finite groups, and let $B$ and $B'$ be blocks of $G$ and $G'$ with defect groups $D$ and $D'$ respectively. Let $i$ (resp. $i'$) be a $D$ (resp. $D'$)-source idempotent of $B$ (resp. $B'$). If $N$ is an $(i'B'i', iBi)$-bimodule, then as $O(G' \times G)$-modules

$$B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB \mid N_{D' \times D}^{G' \times G}$$

where $N_{D' \times D}^{G' \times G}$ is the induced $O(G' \times G)$-module from the $O(D' \times D)$-module $N$.

Proof. By the above it suffices to show that as $(OG', OG)$-bimodules, $B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB$ is a component of $OG' \otimes_{OD'} N \otimes_{OD} OG$. By [13], Theorem 1.2, there exist $c, d \in (OG)^D$ such that $1_B = \sum_{u \in U} ucidu^{-1}$ where $U$ is the set of representatives of $G/D$. Here $(OG)^D$ denotes the set of $D$-fixed elements of $OG$. Similarly there exist $c', d' \in (OG')^{D'}$ such that $1_B = \sum_{u' \in U'} u'c'i'd'u'^{-1}$ where $U'$ is the set of representatives...
of $G'/D'$. We can construct a homomorphism of $(O'G', OG)$-bimodules

$$\varphi: B'i' \otimes_{i'B'_i} N \otimes_{iB_i} iB \to O'G' \otimes_{O'D'} N \otimes_{O'D} OG$$

which satisfies

$$\varphi(a' \otimes n \otimes a) = \sum_{d' \in U'} \sum_{u \in U} a'u'^{-1} \otimes (i'd'u'^{-1}i')n(iuci) \otimes du'^{-1}a$$

for any $a' \in B'i'$, $n \in N$ and $a \in iB$. On the other hand there is a homomorphism of $(O'G', OG)$-bimodules

$$\psi: O'G' \otimes_{O'D'} N \otimes_{O'D} OG \to B'i' \otimes_{i'B'_i} N \otimes_{iB_i} iB$$

which satisfies

$$\psi(x' \otimes n \otimes x) = x'i' \otimes n \otimes ix$$

for any $x' \in O'G'$, $n \in N$ and $x \in OG$. For $a' \otimes n \otimes a \in B'i' \otimes_{i'B'_i} N \otimes_{iB_i} iB$ where $a' \in B'i'$, $n \in N$ and $a \in iB$, we have

$$(\psi \circ \varphi)(a' \otimes n \otimes a) = \sum_{d' \in U'} \sum_{u \in U} a'u'^{-1} \otimes (i'd'u'^{-1}i')n(iuci) \otimes du'^{-1}a$$

$$= \sum_{d' \in U'} \sum_{u \in U} a'i'u'^{-1} \otimes (i'd'u'^{-1}i')n(iuci) \otimes du'^{-1}a$$

$$= a' \otimes i'n_i \otimes a$$

$$= a' \otimes n \otimes a.$$

This implies that $B'i' \otimes_{i'B'_i} N \otimes_{iB_i} iB$ is a component of $(O'G' \otimes_{O'D'} N \otimes_{O'D} OG)$ as $(O'G', OG)$-bimodules. This completes the proof. \qed

Proof of Theorem 1. Firstly we recall the arguments in [19], §3. Let $(D, B_D)$ be an $A$-invariant maximal $B$-Brauer pair. For $Q \subseteq D$, let $(Q, B_Q)$ be a unique $B$-Brauer pair contained in $(D, B_D)$. Then $B_Q$ is $A$-invariant and a defect group of $B_Q$ is centralized by $A$. Let $b_Q$ be the Glauberman correspondent of $B_Q$. Then $b_Q$ is associated with $b$ and $(Q, b_Q) \subseteq (D, b_D)$. In particular $(D, b_D)$ is a maximal $b$-Brauer pair. Moreover, by [19], Proposition 4, the Brauer categories of $B$ and $b$ are equivalent, and $N_G(Q, B_Q) = C_G(Q)N_C(Q, b_Q)$. Therefore $N_C(Q, b_Q)/C_C(Q)$ is a $p$-group for any $Q \subseteq D$, and hence $b$ is a nilpotent block.

Let $U$ be a set of representatives for the conjugacy classes of $D$ and for each $u \in U$, set $B_u = B_{[u]}$ and $b_u = b_{[u]}$. By [2], Theorem 1.2 or [14], 1.7, $\{(u, B_u) \mid u \in U\}$ is a set of representatives for the $G$-conjugacy classes of $B$-Brauer elements and $\{(u, b_u) \mid u \in U\}$ is a set of representatives for the $C$-conjugacy classes of $b$-Brauer elements.
elements. By [2], Theorem 1.2 or [14], 1.9, $B_u$ (resp. $b_u$) has a unique irreducible Brauer character $\varphi^{(a)}$ (resp. $\varphi^{s(u)}$). For $\chi \in \text{Irr}(B)$, we denote by $d(\chi, u, \varphi^{(a)})$ the generalized decomposition number of $\chi$ with respect to $\varphi^{(a)}$. Also we set $\chi^* = \pi(G, A)\chi)$. Since there is an isotypy between $B$ and $b$ given by the Glauberman character correspondence by [19], Proposition 5, for any $\chi \in \text{Irr}(B)$ and $u \in U$

$$d(\chi, u, \varphi^{(a)}) = \epsilon_{\chi} \tilde{\omega}(u) d(\chi^*, u, \varphi^{s(u)})$$

where $\epsilon_{\chi} = \pm 1$ and $\tilde{\omega}(u) = \pm 1$ because $B_u$ and $b_u$ has the same defect.

Let $i$ (resp. $j$) be a $D$-source idempotent of $B$ (resp. $b$). By [13], Theorem 3.5, there is a Morita equivalence between the source algebra $iBi$ of $B$ and the block algebra $B$ realized by the $(B, iBi)$-bimodule $Bi$, and similarly there is a Morita equivalence between $jbj$ and $b$ realized by the $(b, jbj)$-bimodule $bj$. Moreover by [14], 1.6 and 1.8, there exist $O$-simple interior $D$-algebras $S$ and $T$ such that

$$iBi \cong S \otimes_O OD \quad \text{and} \quad jbj \cong T \otimes_O OD$$

as interior $D$-algebras. In fact $S$ and $T$ are primitive Dade $D$-algebras. Suppose that

$$S \cong \text{End}_O(V) \quad \text{and} \quad T \cong \text{End}_O(W)$$

for some $O$-free modules $V$ and $W$. Thus $V$ and $W$ become indecomposable endo-permutation $OD$-modules with vertex $D$. Now there is a Morita equivalence between the group algebra $OD$ and $iBi$ (resp. $jbj$) realized by the $(iBi, OD)$ (resp. $(jbj, OD)$)-bimodule $V \otimes_O OD$ (resp. $W \otimes OD$). Hence we obtain the equivalences $\Psi_b : \text{mod}(OD) \rightarrow \text{mod}(B)$ and $\Psi_b : \text{mod}(OD) \rightarrow \text{mod}(b)$ where $\text{mod}(B)$ denotes the category of finitely generated $B$-modules. Thus $\Psi_b$ is realized by $Bi \otimes_{iBi} (V \otimes_O OD)$, and $\Psi_b$ is realized by $bj \otimes_{bjb} (W \otimes_O OD)$.

For $\lambda \in \text{Irr}(D)$, let $L_\lambda$ be an $OD$-lattice with the character $\lambda$. Also set $M_\lambda = Bi \otimes_{iBi} (V \otimes_O L_\lambda)$ and $N_\lambda = bj \otimes_{bjb} (W \otimes_O L_\lambda)$. Then $\lambda \in \text{Irr}(D) \iff \chi_\lambda \in \text{Irr}(b)$ is a bijection induced by the equivalence $\Psi_B$, where $\chi_\lambda$ is the character of $M_\lambda$. By [14], 1.12, we have for any $u \in U$

$$d(\chi_\lambda, u, \varphi^{(a)}) = \omega(u) \lambda(u)$$

where $\omega(u) = \pm 1$. Similarly $\lambda \in \text{Irr}(D) \iff \zeta_\lambda \in \text{Irr}(b)$ is a bijection induced by the equivalence $\Psi_b$, where $\zeta_\lambda$ is the character of $N_\lambda$. We have also for any $u \in U$

$$d(\zeta_\lambda, u, \varphi^{s(u)}) = \omega^s(u) \lambda(u)$$

where $\omega^s(u) = \pm 1$.

From (2) and (3) we have $\chi_v \ast \lambda = \chi_{\lambda \chi}$ and $\zeta_v \ast \lambda = \zeta_{\lambda \chi}$ for a linear character $v$ of $D$ and $\lambda \in \text{Irr}(D)$ where $\chi_v \ast \lambda$ is a Broué-Puig’s generalized character defined in [1]
(see [2], Theorem 1.2). Let \( 1_D \) be the trivial character of \( D \). From (1)–(3), if we set \((\chi_{1_D})^* = \zeta_\eta \) for some \( \eta \in \text{Irr}(D) \), then \( \eta \) is linear. Note that if \( p \) is odd, then \( \eta = 1_D \). Now, since there is an isotypy between \( B \) and \( b \) given by the Glauberian character correspondence, we can see

\[
(\chi_{1_D})^* = (\chi_{1_D} \ast \lambda)^* = (\chi_{1_D})^* \ast \lambda = \zeta_\eta \ast \lambda = \zeta_{\eta \lambda}.
\]

Now since \( \eta \) is linear, the \( \mathcal{O}D \)-bimodule \( L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \) realizes an equivalence of \( \text{mod}(\mathcal{O}D) \), which we denote by \( \Psi \). Let \( \Pi = \Psi_b \Psi^{-1} : \text{mod}(B) \rightarrow \text{mod}(b) \) where \( \Psi^{-1}_B \) is the equivalence from \( \text{mod}(B) \) to \( \text{mod}(\mathcal{O}D) \) realized by the \((\mathcal{O}D, B)\)-bimodule \((V^* \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\text{mod}(B)} iB \) where \( V^* \) is the dual module. Note that \( V^* \) is a right \( S \)-module. Thus \( \Pi \) is realized by the \((b, B)\)-bimodule

\[
\begin{align*}
  b j \otimes b j (W \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (L_\eta \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (V^* \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\text{mod}(B)} iB \ni B.
\end{align*}
\]

Moreover we see that the bijection between \( \text{Irr}(B) \) and \( \text{Irr}(b) \) induced by \( \Pi \) is \( \pi(G, A)_{|_{\text{Irr}(B)}} \) from (4).

Now we have a \((b, B)\)-bimodule isomorphism

\[
\begin{align*}
  b j \otimes b j (W \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (L_\eta \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (V^* \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\text{mod}(B)} iB
  \cong b j \otimes b j (W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*) \otimes_{\text{mod}(B)} iB.
\end{align*}
\]

Here \( W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^* \) is regarded as a \((bj, iBi)\)-bimodule through

\[
(t \otimes d_1)(w \otimes l \otimes a \otimes v^*)(s \otimes d_2) = tw \otimes dl \otimes ad_2 \otimes v^*s
\]

for any \( t \in T, d_1, d_2 \in D, w \in W, l \in L_\eta, a \in \mathcal{O}D, v^* \in V^* \) and \( s \in S \) identifying \( iBi \) (resp. \( jbj \)) with \( S \otimes_{\mathcal{O}} \mathcal{O}D \) (resp. \( T \otimes_{\mathcal{O}} \mathcal{O}D \)). Let \( \Delta D = \{(d, d) \in C \times G \mid d \in D \} \) and let

\[
M = b j \otimes b j (W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*) \otimes_{\text{mod}(B)} iB.
\]

In order to complete the proof, it suffices to show that \( M \) as an \( \mathcal{O}(C \times G) \)-module has \( \Delta D \) as a vertex by [15], Corollary 7.4.

Let

\[
X = W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} V^*.
\]

We regard \( X \) as an \( \mathcal{O}(\Delta D) \)-module by the following action.

\[
(d, d)(w \otimes l \otimes v^*) = dw \otimes dl \otimes v^*d^{-1}
\]

where \( d \in D, w \in W, l \in L_\eta \) and \( v^* \in V^* \). We show \( jMi \) and \( X_{\Delta D}^{D \times D} \) are isomorphic as \( \mathcal{O}(D \times D) \)-modules. (Note that \( jMi \) is a \((bj, iBi)\)-bimodule, and hence this is an
\(\mathcal{O}D\)-bimodule.) Here \(jMi\) is identified with \(W \otimes_{\mathcal{O}} \Lambda_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*\). Now we have an \(\mathcal{O}\)-linear map

\[
f: X^{D \times D}_{\Delta D} \to jMi
\]

defined by

\[
f((d_1, d_2) \otimes (w \otimes l \otimes v^*)) = d_1w \otimes d_1l \otimes d_1d_2^{-1} \otimes v^*d_2^{-1}
\]

for any \(d_1, d_2 \in D, w \in W, l \in \Lambda_\eta\) and \(v^* \in V^*\). Then \(f\) is an \(\mathcal{O}(D \times D)\)-homomorphism. On the other hand we have an \(\mathcal{O}\)-linear map

\[
g: jMi \to X^{D \times D}_{\Delta D}
\]

defined by

\[
g(w \otimes l \otimes d \otimes v^*) = (d, 1) \otimes (d^{-1}w \otimes d^{-1}l \otimes v^*)
\]

for any \(w \in W, l \in \Lambda_\eta, d \in D\) and \(v^* \in V^*\). We see that \(g\) is an \(\mathcal{O}\)-isomorphism with the inverse \(f\). Thus \(jMi\) and \(X^{D \times D}_{\Delta D}\) are isomorphic as \(\mathcal{O}(D \times D)\)-modules.

By Lemma 2, \(M\) is a component of the induced module \((jMi)^{C \times G}_{D \times D}\) because \(M \simeq \mathcal{O}C \otimes_{iBi} jMi \otimes_{iBi} \mathcal{O}G\). Hence by the above \(M\) is a component of \(X^{C \times G}_{\Delta D}\). Hence \(M\) is \(\Delta D\)-projective. Now \(jMi\) is a component of \(M\) as \(\mathcal{O}(D \times D)\)-modules. Since \(jMi \simeq X^{D \times D}_{\Delta D}\), \(X\) is a component of \(jMi\) as \(\mathcal{O}(\Delta D)\)-modules, and hence \(X\) is a component of \(M\) as \(\mathcal{O}(\Delta D)\)-modules. Since \(p \nmid \text{rank}_\mathcal{O} X\) from [14], 1.6 and 1.8 or [18], Corollary 28.11, an indecomposable component of \(X\) has \(\Delta D\) as a vertex, and hence an indecomposable component of \(M\) as an \(\mathcal{O}(\Delta D)\)-module has \(\Delta D\) as a vertex. This implies that \(\Delta D\) is a vertex of \(M\). Hence \(B\) and \(b\) are basic Morita equivalent. This completes the proof.

Here we give a direct proof of the fact that \(M\) has an endo-permutation \(\mathcal{O}(\Delta D)\)-module as a source (see [15], Corollary 7.4). Since \(W, \Lambda_\eta\) and \(V\) are endo-permutation \(\mathcal{O}D\)-modules, \(X\) is an endo-permutation \(\mathcal{O}(\Delta D)\)-module, in fact \(X\) is capped in the sense of [4]. Now, since \(M \mid X^{C \times G}_{\Delta D}\), there exists an indecomposable component \(Y\) of \(X\) such that \(M \mid Y^{C \times G}_{\Delta D}\). Then \(Y\) is an endo-permutation \(\mathcal{O}(\Delta D)\)-module and it is a source module of \(M\). We note that \(Y = \text{cap}(X)\).

\[
\square
\]

References


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