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FROBENIUS MANIFOLDS FOR ELLIPTIC ROOT SYSTEMS

IKUO SATAKE

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Abstract

In this paper, we show that the orbit space of the domain by the elliptic Weyl group for an elliptic root system of codimension 1 has a structure of Frobenius manifold. We also give a characterization of this structure of the Frobenius manifold under suitable conditions.

1. Introduction

A Frobenius manifold is a complex manifold with a multiplication and a metric on the holomorphic tangent bundle and with two global vector fields, the unit field and the Euler field satisfying some integrable conditions. The notion of Frobenius manifold was introduced by Dubrovin in order to give a geometric description of the integrable structures in the topological field theory. The study of the Frobenius manifolds is important from the viewpoint of mirror symmetry (cf. [6]).

On a Frobenius manifold, the tensor, called "the intersection form", is defined. It is a holomorphic symmetric tensor on the cotangent bundle of the Frobenius manifold. It appears first in the work of Saito ([14]) on the study of a semiuniversal unfolding of an isolated hypersurface singularity. Dubrovin ([4]) generalized the definition of the intersection form in cases of Frobenius manifolds.

In this paper we discuss the following problem:

PROBLEM. Let (M, I^*, E) be a triple of a complex manifold M, a holomorphic symmetric tensor I^* on the cotangent bundle of M and the vector field E. Establish the structure of a Frobenius manifold on M such that its intersection form coincides with the tensor I^* and its Euler field coincides with the vector field E.

If M is the complex orbit space of a finite irreducible Coxeter group, I^* is the tensor descended from the standard holomorphic metric and E is the Euler field derived from the standard $\mathbb{C}^*$-action, the problem was solved by Saito [13] and Dubrovin [4].

In this paper, we solve the problem for the complex orbit spaces of the elliptic Weyl groups for the elliptic root systems of codimension 1 with the tensors descended from the standard holomorphic metrics and with the vector fields derived from the

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canonical $\mathbb{C}$-action. It is a natural generalization of the complex orbit space of a finite Coxeter group. We also prove the uniqueness (up to $\mathbb{C}^*$ action) of the structure of the Frobenius manifold on the orbit spaces of the elliptic Weyl groups under the condition that the intersection form I^* and the Euler vector field are fixed.

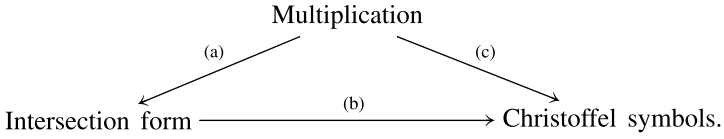
We remark that the complex orbit spaces of the elliptic Weyl groups for the elliptic root systems occur in various contexts in the following subjects: the invariant theory (I.N. Bernšteĭn and O.V. Švarcman [1], [2], E. Looijenga [12], K. Saito [16], K. Wirthmüller [22]), the representation theory of the affine Lie algebras (V.G. Kac and D.H. Peterson [11], P. Slodowy [21]), the theory of the elliptic Lie groups (S. Helmke and P. Slodowy [7]) and the theory of the G-principal bundles over elliptic curves (R. Friedman and J.W. Morgan [5]).

We shall explain the outline of our method of the construction of the Frobenius manifolds for our cases, which is similar to those in cases of the finite Coxeter groups [4].

In [16], the normalized lowest degree vector field e is defined and the flat holomorphic metric J on M was constructed by I^* and e.

For the construction of the multiplication, we need to recall the following facts.

The structure of the multiplication of the Frobenius manifold gives the intersection form (step (a) in the diagram). The Levi–Civita connection is defined for the intersection form and gives the Christoffel symbols (step (b)). Then the structure of the multiplication gives the Christoffel symbols (step (c)).



The explicit description of step (c) is given as follows. For given flat coordinates of the Frobenius manifold, we have the following simple relation between the structure coefficients $C_\gamma^{\alpha\beta}$ of the structure of the multiplication and the Christoffel symbols $\Gamma_\gamma^{\alpha\beta}$ ([4, p. 194, Lemma 3.4]):

$$\Gamma_\gamma^{\alpha\beta} = \left(d^\beta + \frac{1-D}{2}\right) C_\gamma^{\alpha\beta}, \tag{1.1}$$

where D is a degree of the flat metric (see Definition 3.5) of the Frobenius manifold and d^β is a degree of the homogeneous flat coordinate t^β. Thanks to the equation (1.1), we could recover the structure of the multiplication from the Christoffel symbols for some cases. This is a clue to solve our problem.

For the case of the complex orbit space of a finite irreducible Coxeter group, every factor $d^\beta + (1-D)/2$ in (1.1) is non-zero. Thus each $C_\gamma^{\alpha\beta}$ is determined by $\Gamma_\gamma^{\alpha\beta}$. Furthermore it is shown ([4]) that the set of $C_\gamma^{\alpha\beta}$ satisfies the conditions of Frobenius manifold.

For the case of the complex orbit space of the elliptic Weyl group for an elliptic root system of codimension 1, some factors $d^\beta + (1-D)/2$ in (1.1) are zero. However we could determine all $C_\gamma^{\alpha\beta}$'s also for this case by another condition that the normalized lowest degree vector field e must be the unit field. Then we can prove that the set of $C_\gamma^{\alpha\beta}$ satisfies the conditions of Frobenius manifold.

Among the results of this paper, the existence of the structures of the Frobenius manifolds on the complex orbit spaces of the elliptic Weyl groups is already announced in [18]. For the explicit construction of the flat coordinate system, it is done for G_2 case [17], D_4 case [18] and for E_6 case [19]. For the explicit calculation of the structure of the Frobenius manifold, it is done for D_4 case [18] and for G_2 case [3].

This paper is organized as follows.

In Section 2, we review the notions which are necessary in later sections. We recall the definitions in [16] such as the elliptic root systems, the elliptic Weyl groups, the domains, the symmetric tensors on the domains, invariant rings and the Euler operators. A signed marking and an orientation of an elliptic root system are introduced in this paper in order to give a natural definition of the domains.

In Section 3, we give the statements of the main results (Theorem 3.7). This theorem will be proved in §4 and §5. First we introduce the analytic spectrums of the invariant rings of the elliptic Weyl groups. We call these spectrums the (modified) complex orbit spaces of the elliptic Weyl groups. On the complex orbit spaces, the Euler field is defined. Then the main theorem (Theorem 3.7) is stated, where the existence and the uniqueness of the structures of the Frobenius manifolds on the complex orbit spaces of the elliptic Weyl groups for the cases of "codimension 1" (see the before of Theorem 3.7 for the definition) are asserted.

In Section 4, we prove that the complex orbit spaces have a structure of Frobenius manifold (Theorem 3.7 (1)). First we review the construction of the holomorphic metric in [16]. We construct a flat pencil. We define the multiplication on the tangent bundle of the orbit spaces. We show that the Euler fields, the holomorphic metrics, the multiplications and its unit fields constitute the structure of the Frobenius manifold. Here the flat pencil is used.

In Section 5, we prove Theorem 3.7 (2), (3), which asserts that the structure of the Frobenius manifold with the Euler vector field E and the intersection form I^* is unique up to $\mathbb{C}^*$-action.

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2. Invariant rings of the elliptic Weyl groups

The purpose of this section is to review the invariant ring and the Euler operator introduced in [16]. A signed marking and an orientation of an elliptic root system are introduced in this paper in order to give a natural definition of the domains.

2.1. Elliptic root system. In this subsection, we define the elliptic root systems and its orientations.

Let l be a positive integer. Let F be a real vector space of rank $l+2$ with a negative semi-definite or positive semi-definite symmetric bilinear form $I\colon F\times F\to\mathbb{R}$, whose radical $\operatorname{rad} I := \{x\in F \mid I(x,y)=0,\ \forall y\in F\}$ is a vector space of rank 2. For a non-isotropic element $\alpha\in F$ (i.e. $I(\alpha,\alpha)\neq 0$), we put $\alpha^\vee := 2\alpha/I(\alpha,\alpha)\in F$. The reflection w_α with respect to α is defined by

$$w_\alpha(u) := u - I(u,\alpha^\vee)\alpha \quad (\forall u\in F). \tag{2.1}$$

DEFINITION 2.1 ([15, p. 104, Definition 1]). A set R of non-isotropic elements of F is an elliptic root system belonging to (F, I) if it satisfies the axioms 1–4:

1. The additive group generated by R in F, denoted by $Q(R)$, is a full sub-lattice of F. That is, the embedding $Q(R)\subset F$ induces the isomorphism: $Q(R)\otimes_{\mathbb{Z}}\mathbb{R}\simeq F$.
2. $I(\alpha,\beta^\vee)\in\mathbb{Z}$ for $\alpha,\beta\in R$.
3. $w_\alpha(R)=R$ for $\forall\alpha\in R$.
4. If $R=R_1\cup R_2$, with $R_1\perp R_2$, then either R_1 or R_2 is void.

For an elliptic root system R belonging to (F, I), the additive group $\operatorname{rad} I\cap Q(R)$ is isomorphic to $\mathbb{Z}^2$.

DEFINITION 2.2. An elliptic root system R is called oriented if the $\mathbb{R}$-vector space $\operatorname{rad} I$ is oriented. A frame $\{a,b\}$ of $\operatorname{rad} I$ is called admissible if $\operatorname{rad} I\cap Q(R)\simeq\mathbb{Z}a\oplus\mathbb{Z}b$ and it gives the orientation of $\operatorname{rad} I$.

REMARK 2.3. If an elliptic root system (R, F, I) comes from vanishing cycles of a Milnor fiber of a simple elliptic singularity, then $\operatorname{rad} I\cap Q(R)\simeq H_1(E_\infty,\mathbb{Z})$ for an elliptic curve E_∞ at infinity (cf. [16, p. 18]). Then (R, F, I) is canonically oriented by the complex structure of the elliptic curve E_∞.

2.2. Hyperbolic extension and the elliptic Weyl group. In this subsection, we define a signed marking, a hyperbolic extension and the elliptic Weyl group.

DEFINITION 2.4. Let R be an elliptic root system R belonging to (F, I). By a signed marking, we mean a non-zero element a of $\operatorname{rad} I\cap Q(R)$ such that $Q(R)\cap\mathbb{R}a=\mathbb{Z}a$.

Hereafter we fix an oriented elliptic root system with a signed marking (R, a) such that the quotient root system $R/\mathbb{R}a$ ($:=\operatorname{Image}(R\hookrightarrow F\to F/\mathbb{R}a)$) is reduced (i.e. $\alpha, c\alpha\in R/\mathbb{R}a$ implies $c\in\{\pm 1\}$).

Let F^1 be a real vector space of rank $l+3$ and $I^1\colon F^1\times F^1\to\mathbb{R}$ an $\mathbb{R}$-symmetric bilinear form. The pair (F^1, I^1) is called a hyperbolic extension of (F, I) if F^1 contains

F as a linear subspace, $\operatorname{rad} I^1 = \mathbb{R}a$ and $I^1|_F = I$. A hyperbolic extension is unique up to isomorphism. Hereafter we fix a hyperbolic extension (F^1, I^1).

We define a reflection $\tilde{w}_\alpha \in GL(F^1)$ by $\tilde{w}_\alpha(u) := u - I^1(u, \alpha^\vee)\alpha$ for $u \in F^1$.

We define the elliptic Weyl group $\tilde{W}$ (resp. W) by

$$\tilde{W} := \langle \tilde{w}_\alpha \mid \alpha \in R \rangle \quad (\text{resp. } W := \langle w_\alpha \mid \alpha \in R \rangle). \tag{2.2}$$

We have a natural exact sequence:

$$0 \to K_{\mathbb{Z}} \to \tilde{W} \to W \to 1, \tag{2.3}$$

where $\tilde{W} \to W$ is given by the restriction of $\tilde{W}$ on F and $K_{\mathbb{Z}}$ is the kernel of $\tilde{W} \to W$. The group $K_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}$.

2.3. Domain. In this subsection, we define a domain $\tilde{\mathbb{E}}$ for the oriented elliptic root system with the signed marking (R, a) belonging to (F, I) such that $R/\mathbb{R}a$ is reduced.

For (F, I), the set $\{c \in \mathbb{R} \mid$ the bilinear form cI defines a semi-negative even lattice structure on $Q(R)\}$ has the unique element of the smallest absolute value. We denote it by $(I_R^- : I)$.

Take $b \in \operatorname{rad} I \cap Q(R)$ such that $\{a, b\}$ gives an admissible frame. Then we could choose an isomorphism $\rho\colon \mathbb{Z} \simeq K_{\mathbb{Z}}$ and $\tilde{\lambda} \in F^1 \setminus F$ such that

$$(\rho(n))(\tilde{\lambda}) = \tilde{\lambda} + na \ (n \in \mathbb{Z}), \quad (I_R^- : I) I^1(\tilde{\lambda}, b) > 0. \tag{2.4}$$

By the condition (2.4), $\tilde{\lambda}$ is unique up to adding an element of F, and such an isomorphism ρ is unique.

REMARK 2.5. In the paper [16], the signature of the Hermitian form (3.5.5) is incorrect. The correct form is

$$H(z, w) = \frac{m_{\max}}{t(R)(l_{\max} + 1)\operatorname{Im}(\tau)} I_{R^\vee}(z, \bar{w}).$$

Thus the line bundle L in [16, §3.5] becomes ample relative to $\mathbb{H}$ which is not the desired one (see [16, §3.6]).

In order to obtain the desired line bundle, the semi-positive bilinear forms I, I_R, $I_{R^\vee}$ in [16] should be changed to the semi-negative ones. This is the reason of the signature of the definition of $\tilde{\lambda}$ in (2.4).

We define two domains:

$$\tilde{\mathbb{E}} := \{x \in \operatorname{Hom}_{\mathbb{R}}(F^1, \mathbb{C}) \mid \langle a, x \rangle = 1, \ \operatorname{Im}\langle b, x \rangle > 0\}, \tag{2.5}$$

$$\mathbb{H} := \{x \in \operatorname{Hom}_{\mathbb{R}}(\operatorname{rad} I, \mathbb{C}) \mid \langle a, x \rangle = 1, \ \operatorname{Im}\langle b, x \rangle > 0\}, \tag{2.6}$$

where $\langle\ ,\ \rangle$ is the natural pairing $F^1_{\mathbb{C}} \times (F^1_{\mathbb{C}})^* \to \mathbb{C}$ and $F^1_{\mathbb{C}} := F^1 \otimes_{\mathbb{R}} \mathbb{C}$. We have a natural projection

$$\pi : \tilde{\mathbb{E}} \to \mathbb{H}. \tag{2.7}$$

For a root $\alpha \in R$, we define the reflection hyperplane of $\tilde{\mathbb{E}}$ by

$$H_\alpha := \{x \in \tilde{\mathbb{E}} \mid \langle \alpha, x\rangle = 0\}. \tag{2.8}$$

We define a left action of $\tilde{W}$ on $\tilde{\mathbb{E}}$. For $g \in \tilde{W}$ and $x \in \tilde{\mathbb{E}}$, we define $g \cdot x \in \tilde{\mathbb{E}}$ by

$$\langle \eta, g \cdot x\rangle := \langle g^{-1}(\eta), x\rangle \tag{2.9}$$

for $\eta \in F^1_{\mathbb{C}}$.

For a complex manifold M, we denote by $\mathcal{O}_M$ (resp. Ω^1_M, Θ_M) the sheaf of holomorphic functions (resp. holomorphic 1-forms, holomorphic vector fields).

We define a vector field E' on $\tilde{\mathbb{E}}$ by the conditions

$$E'x = 0\ (\forall x \in F), \quad E'\tilde{\lambda} = \frac{1}{2\pi\sqrt{-1}}. \tag{2.10}$$

The vector field E' is uniquely determined by the condition (2.4). The reason of the normalization of E' will be explained after the definition of the sheaf $\boldsymbol{\mathcal{S}}^W_k$ in (2.15).

We define a $\mathbb{C}$-symmetric bilinear form I^* on $\Omega^1_{\tilde{\mathbb{E}}}$. Since we have a canonical isomorphism $T^*_p\tilde{\mathbb{E}} \simeq \mathbb{C} \otimes_{\mathbb{R}} (F^1/\mathbb{R}a)$ for $p \in \tilde{\mathbb{E}}$, we have an $\mathcal{O}_{\tilde{\mathbb{E}}}$-bilinear form

$$I^* : \Omega^1_{\tilde{\mathbb{E}}} \times \Omega^1_{\tilde{\mathbb{E}}} \to \mathcal{O}_{\tilde{\mathbb{E}}} \tag{2.11}$$

induced from $I^1 : F^1/\mathbb{R}a \times F^1/\mathbb{R}a \to \mathbb{R}$. We remark that $Lie_{E'}\, I^* = 0$, where Lie is the Lie derivative.

By the condition $\mathrm{Im}\langle b, x\rangle > 0$ in (2.5) and (2.6), the action of $\tilde{W}$ on $\tilde{\mathbb{E}}$ (resp. $\tilde{\mathbb{E}} \setminus \bigcup_{\alpha\in R} H_\alpha$) is properly discontinuous (resp. properly discontinuous and fixed point free) (cf. [16]), thus the orbit space $\tilde{\mathbb{E}}/\tilde{W}$ (resp. $\big(\tilde{\mathbb{E}} \setminus \bigcup_{\alpha\in R} H_\alpha\big)\big/\tilde{W}$) has a structure of analytic space (resp. complex manifold). We have the following commutative diagram of analytic spaces:

$$\begin{array}{ccc} \left(\tilde{\mathbb{E}} \setminus \bigcup\limits_{\alpha\in R} H_\alpha\right)\Big/\tilde{W} & \xrightarrow{i_1} & \tilde{E}/\tilde{W} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{H} & = & \mathbb{H}. \end{array} \tag{2.12}$$

Since $\bigcup_{\alpha\in R} H_\alpha \subset \tilde{\mathbb{E}}$ is locally finite, $\tilde{\mathbb{E}} \setminus \bigcup_{\alpha\in R} H_\alpha \subset \tilde{\mathbb{E}}$ is open dense. Thus the morphism i_1 is an open immersion and its image is open dense.

Since the tensors E' and I^* on $\tilde{\mathbb{E}}$ are $\tilde{W}$-invariant, these tensors descend to the space $\left(\tilde{\mathbb{E}} \setminus \bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}$:

$$E' \colon \Omega^1_{\left(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}} \to \mathcal{O}_{\left(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}}, \tag{2.13}$$

$$I^* \colon \Omega^1_{\left(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}} \times \Omega^1_{\left(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}} \to \mathcal{O}_{\left(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}}. \tag{2.14}$$

2.4. Invariant rings of the elliptic Weyl groups. In order to extend the domain of the definition of the tensors (2.13) and (2.14), we introduce the invariant ring $\mathcal{S}^W$ of the elliptic Weyl group, $\mathcal{S}^W$ modules $\Omega^1_{\mathcal{S}^W}$, $\mathrm{Der}_{\mathcal{S}^W}$ in this subsection and formulate the tensors E' and I^* by these $\mathcal{S}^W$ modules in the next subsection.

We define an $\mathcal{O}_{\mathbb{H}}$-module $\mathcal{S}^W_k$ of $\tilde{W}$-invariant functions parametrized by $k \in \mathbb{C}$ as the subsheaf of $\pi_*\mathcal{O}_{\tilde{\mathbb{E}}}$ by

$$\mathcal{S}^W_k(U) := \{ f \in \pi_*\mathcal{O}_{\tilde{\mathbb{E}}}(U) \mid f(g\cdot x) = f(x) \quad (\forall g \in \tilde{W},\ \forall x \in \pi^{-1}(U)),\ E'f = kf \} \tag{2.15}$$

for an open set $U \subset \mathbb{H}$. By the normalization of E' in (2.10), $k \in \mathbb{Z}$.

We define the $\mathcal{O}_{\mathbb{H}}$-graded algebra $\mathcal{S}^W$ by

$$\mathcal{S}^W := \bigoplus_{k\in\mathbb{Z}} \mathcal{S}^W_k. \tag{2.16}$$

We have injective homomorphisms

$$\mathcal{S}^W \to \pi_*\mathcal{O}_{\tilde{\mathbb{E}}/\tilde{W}} \to \pi_*\mathcal{O}_{\left(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha\right)/\tilde{W}}. \tag{2.17}$$

Theorem 2.6 ([1], [2], [5], [11], [12], [22]). *The $\mathcal{O}_{\mathbb{H}}$-graded algebra $\mathcal{S}^W$ is an $\mathcal{O}_{\mathbb{H}}$-free algebra, i.e.*

$$\mathcal{S}^W = \mathcal{O}_{\mathbb{H}}[s^1, \dots, s^{n-1}] \tag{2.18}$$

for $s^j \in \mathcal{S}^W_{c^j}(\mathbb{H})$ with $c^1 \geq c^2 \geq \cdots \geq c^{n-1} > 0$ and $n := l + 2$. We remark that j of s^j is a suffix.

We introduce the n-th invariant $s^n \in \mathcal{S}^W_0(\mathbb{H})$. Since an element of $\mathrm{rad}\, I$ naturally gives an element of $\mathcal{S}^W_0(\mathbb{H})$, we define $s^n = b$, where $b \in \mathrm{rad}\, I$ is introduced in Section 2.3. We put $c^n = 0$. Then $s^j \in \mathcal{S}^W_{c^j}(\mathbb{H})$ for $j = 1, \dots, n$.

We define two $\mathcal{S}^W$-modules $\Omega^1_{\mathcal{S}^W}$ and $\mathrm{Der}_{\mathcal{S}^W}$.

For an open set $U \subset \mathbb{H}$, we put

$$\Omega^1_{\mathcal{S}^W}(U) := \Omega^1_{\mathcal{S}^W(U)/\mathbb{C}}, \tag{2.19}$$

where R.H.S. is the module of relative differential forms of a $\mathbb{C}$-algebra $\boldsymbol{\mathcal{S}}^W(U)$. Since $\boldsymbol{\mathcal{S}}^W$ is an $\mathcal{O}_{\mathbb{H}}$-free algebra, $\Omega^1_{\boldsymbol{\mathcal{S}}^W}$ defines a sheaf. $\Omega^1_{\boldsymbol{\mathcal{S}}^W}$ has a structure of an $\boldsymbol{\mathcal{S}}^W$-module. We remark that since a local section of $\Omega^1_{\boldsymbol{\mathcal{S}}^W}$ determines a local section $\pi_*\Omega^1_{\tilde{\mathbb{E}}}$ and $\pi_*\Omega^1_{\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha}$ which is $\tilde{W}$-invariant, there exists a natural lifting map $\Omega^1_{\boldsymbol{\mathcal{S}}^W} \to \pi_*\Omega^1_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}}$.

We put

$$\mathrm{Der}_{\boldsymbol{\mathcal{S}}^W} := \underline{\mathrm{Hom}}_{\boldsymbol{\mathcal{S}}^W}(\Omega^1_{\boldsymbol{\mathcal{S}}^W}, \boldsymbol{\mathcal{S}}^W). \tag{2.20}$$

By the generators of $\boldsymbol{\mathcal{S}}^W$, we have

$$\Omega^1_{\boldsymbol{\mathcal{S}}^W} = \bigoplus_{i=1}^n \boldsymbol{\mathcal{S}}^W ds^i, \quad \mathrm{Der}_{\boldsymbol{\mathcal{S}}^W} = \bigoplus_{i=1}^n \boldsymbol{\mathcal{S}}^W \frac{\partial}{\partial s^i}. \tag{2.21}$$

2.5. Euler operator and bilinear form on the invariant ring. We define the Euler operator E as a vector field on $\tilde{\mathbb{E}}$ defined by

$$E := \frac{1}{c^1} E'. \tag{2.22}$$

As in the case of E' in (2.13), E defines a morphism

$$E\colon \Omega^1_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}} \to \mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}}. \tag{2.23}$$

By [16], we have the $\boldsymbol{\mathcal{S}}^W$-homomorphism and $\boldsymbol{\mathcal{S}}^W$-symmetric bilinear form:

$$E\colon \Omega^1_{\boldsymbol{\mathcal{S}}^W} \to \boldsymbol{\mathcal{S}}^W, \tag{2.24}$$

$$I^*\colon \Omega^1_{\boldsymbol{\mathcal{S}}^W} \times \Omega^1_{\boldsymbol{\mathcal{S}}^W} \to \boldsymbol{\mathcal{S}}^W \tag{2.25}$$

with the following commutative diagrams:

(2.26)
$$\begin{array}{ccc} \pi_*\Omega^1_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}} & \xrightarrow{E} & \pi_*\mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}} \\ \uparrow & & \uparrow \\ \Omega^1_{\boldsymbol{\mathcal{S}}^W} & \xrightarrow{E} & \boldsymbol{\mathcal{S}}^W, \end{array}$$

(2.27)
$$\begin{array}{ccccc} I^*\colon \pi_*\Omega^1_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}} & \times & \pi_*\Omega^1_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}} & \longrightarrow & \pi_*\mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}} \\ \uparrow & & \uparrow & & \uparrow \\ I^*\colon \Omega^1_{\boldsymbol{\mathcal{S}}^W} & \times & \Omega^1_{\boldsymbol{\mathcal{S}}^W} & \longrightarrow & \boldsymbol{\mathcal{S}}^W, \end{array}$$

where the upper line of (2.26) is induced by (2.23) and the upper line of (2.27) is induced by (2.14). The morphisms (2.24) and (2.25) are uniquely characterized by the diagrams (2.26) and (2.27) respectively because $\boldsymbol{\mathcal{S}}^W \to \pi_*\mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}}$ is injective.

3. Results

In this section, we first define the (modified) complex orbit space $\tilde{\mathbb{E}}//\tilde{W}$ of the action of the elliptic Weyl group $\tilde{W}$ on the domain $\tilde{\mathbb{E}}$, where $\tilde{\mathbb{E}}$ and $\tilde{W}$ are defined in Section 2. Then we assert that $\tilde{\mathbb{E}}//\tilde{W}$ has a structure of Frobenius manifold under some suitable condition.

3.1. The orbit space of the elliptic Weyl group. In this subsection, we define the (modified) orbit space of the elliptic Weyl group action and study the tensors on the orbit space.

Let $\tilde{\mathbb{E}}$, $\mathbb{H}$ be the domains and $\tilde{W}$ be the elliptic Weyl group defined in Section 2. Let (*An*) and (*Set*) be categories of analytic spaces and sets, respectively. Let $((An)/\mathbb{H})^\circ$ be the dual category of the category of $\mathbb{H}$-objects. Since the $\mathcal{O}_{\mathbb{H}}$-algebra $\boldsymbol{\mathcal{S}}^W$ is of finite presentation (Theorem 2.6), the analytic space Specan $\boldsymbol{\mathcal{S}}^W$ could be defined by [10]. We define the (modified) orbit space $\tilde{\mathbb{E}}//\tilde{W}$ by

$$\tilde{\mathbb{E}}//\tilde{W} := \text{Specan}\, \boldsymbol{\mathcal{S}}^W. \tag{3.1}$$

We denote the structure morphism $\tilde{\mathbb{E}}//\tilde{W} \to \mathbb{H}$ also by π. The space $\tilde{\mathbb{E}}//\tilde{W}$ is isomorphic to $\mathbb{H} \times \mathbb{C}^{n-1}$ by Theorem 2.6.

By definition of Specan, there exists a natural isomorphism:

$$\text{Hom}_{(An)/\mathbb{H}}(X, \tilde{\mathbb{E}}//\tilde{W}) \simeq \text{Hom}_{\mathcal{O}_X}(f^*\boldsymbol{\mathcal{S}}^W, \mathcal{O}_X)$$

for an object $f: X \to \mathbb{H}$ of the category $(An)/\mathbb{H}$. Since there exists a canonical isomorphism: $\text{Hom}_{\mathcal{O}_X}(f^*\boldsymbol{\mathcal{S}}^W, \mathcal{O}_X) \simeq \text{Hom}_{\mathcal{O}_{\mathbb{H}}}(\boldsymbol{\mathcal{S}}^W, f_*\mathcal{O}_X)$, we have

$$\text{Hom}_{(An)/\mathbb{H}}(X, \tilde{\mathbb{E}}//\tilde{W}) \simeq \text{Hom}_{\mathcal{O}_{\mathbb{H}}}(\boldsymbol{\mathcal{S}}^W, f_*\mathcal{O}_X). \tag{3.2}$$

We define a ringed space $(\mathbb{H}, \boldsymbol{\mathcal{S}}^W)$ by the space $\mathbb{H}$ with the sheaf $\boldsymbol{\mathcal{S}}^W$. We define a morphism of the category of ringed spaces:

$$\varphi: (\tilde{\mathbb{E}}//\tilde{W}, \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}) \to (\mathbb{H}, \boldsymbol{\mathcal{S}}^W) \tag{3.3}$$

by the mapping $\pi: \tilde{\mathbb{E}}//\tilde{W} \to \mathbb{H}$ and the morphism

$$\phi: \boldsymbol{\mathcal{S}}^W \to \pi_*\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}} \tag{3.4}$$

which corresponds to the identity element of $\text{Hom}_{(An)/\mathbb{H}}(\tilde{\mathbb{E}}//\tilde{W}, \tilde{\mathbb{E}}//\tilde{W})$ by (3.2).

Proposition 3.1. *We have the canonical isomorphism*:

$$\varphi^*\Omega^1_{\boldsymbol{\mathcal{S}}^W} \simeq \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}}. \tag{3.5}$$

Proof. We define the ringed space $(\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}, \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}})$ as follows: As a set, $\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}} = \tilde{\mathbb{E}}//\tilde{W}$. A topology on $\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}$ is introduced so that $\{(U, f) \subset \tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}} \mid U \subset \mathbb{H}\colon \text{open}, \ f \in \Gamma(U, \boldsymbol{\mathcal{S}}^W)\}$ becomes an open basis, where $(U, f) := \{x \in \tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}} \mid \pi(x) \in U, \ f(x) \neq 0\}$. We define the sheaf $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}}$ associated with the presheaf $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}}((U, f)) := \Gamma(U, \boldsymbol{\mathcal{S}}^W)_f$ for an open set (U, f).

The morphism $\varphi\colon (\tilde{\mathbb{E}}//\tilde{W}, \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}) \to (\mathbb{H}, \boldsymbol{\mathcal{S}}^W)$ factors as the composite of the morphisms:

$$(\tilde{\mathbb{E}}//\tilde{W}, \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}) \xrightarrow{\varphi_1} (\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}, \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}}) \xrightarrow{\varphi_2} (\mathbb{H}, \boldsymbol{\mathcal{S}}^W). \tag{3.6}$$

We define a sheaf $\Omega^1_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}}$ on $\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}$ as a sheafification of the presheaf

$$(U, f) \mapsto \Omega^1_{\boldsymbol{\mathcal{S}}^W}(U)_f. \tag{3.7}$$

Then we have a natural isomorphism

$$\varphi_2^* \Omega^1_{\boldsymbol{\mathcal{S}}^W} \simeq \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}} \tag{3.8}$$

by a discussion of an affine morphism in scheme theory. Also we have a natural isomorphism

$$\varphi_1^* \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}} \simeq \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \tag{3.9}$$

because an $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}^{\mathrm{alg}}}$-locally free basis of algebraic 1-forms is regarded as an $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-locally free basis of analytic 1-forms. □

We define the $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-homomorphism and $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-symmetric bilinear form

$$E\colon \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \to \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}, \tag{3.10}$$

$$I^*_{\tilde{\mathbb{E}}//\tilde{W}}\colon \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \times \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \to \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}, \tag{3.11}$$

by taking the pull-back of (2.24) and (2.25) by φ.

We shall see the relation between (3.10) (resp. (3.11)) on $\tilde{\mathbb{E}}//\tilde{W}$ and (2.23) (resp. (2.14)) on $\big(\tilde{\mathbb{E}} \setminus \bigcup_{\alpha \in R} H_\alpha\big)/\tilde{W}$.

By (3.2), a natural inclusion $\boldsymbol{\mathcal{S}}^W \hookrightarrow \pi_* \mathcal{O}_{\tilde{\mathbb{E}}/\tilde{W}}$ corresponds to the mapping

$$i_2\colon \tilde{\mathbb{E}}/\tilde{W} \to \tilde{\mathbb{E}}//\tilde{W}. \tag{3.12}$$

Proposition 3.2. *The morphism i_2 is an open immersion and its image is open dense.*

Proof. By [16], the group $\tilde{W}$ decomposes into

$$0 \to \tilde{H} \to \tilde{W} \to W_f \to 1,$$

where $\tilde{H}$ is a Heisenberg group and W_f is a finite Weyl group. Then we have the following diagrams:

$$\begin{array}{ccc}
\tilde{\mathbb{E}}/\tilde{H} & \longrightarrow & \tilde{\mathbb{E}}//\tilde{H} \\
\downarrow & & \downarrow \\
(\tilde{\mathbb{E}}/\tilde{H})/W_f & \longrightarrow & (\tilde{\mathbb{E}}//\tilde{H})/W_f \\
\wr\downarrow & & \wr\downarrow \\
\tilde{\mathbb{E}}/\tilde{W} & \xrightarrow{i_2} & \tilde{\mathbb{E}}//\tilde{W},
\end{array}$$

where $\tilde{\mathbb{E}}//\tilde{H} := \operatorname{Specan}\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{S}} := \bigoplus_{k\in\mathbb{Z}} \boldsymbol{\mathcal{S}}_k$ and $\boldsymbol{\mathcal{S}}_k$ is a sheaf defined by $\boldsymbol{\mathcal{S}}_k(U) := \{ f \in \pi_* \mathcal{O}_{\tilde{\mathbb{E}}}(U) \mid f(g \cdot x) = f(x)\ (\forall g \in \tilde{H},\ \forall x \in \pi^{-1}(U)),\ E' f = k f \}$ for an open subset $U \subset \mathbb{H}$. By the geometric description of $\tilde{\mathbb{E}}/\tilde{H} \to \tilde{\mathbb{E}}//\tilde{H}$ in [16], $\tilde{\mathbb{E}}/\tilde{H} \to \tilde{\mathbb{E}}//\tilde{H}$ is an open immersion and its image is open dense. We remark that we have the relations $\tilde{\mathbb{E}}/\tilde{H} = L^*$ and $\tilde{\mathbb{E}}//\tilde{H} = \mathbb{L}$ for L^*, $\mathbb{L}$ in [16]. Then the morphism $(\tilde{\mathbb{E}}/\tilde{H})/W_f \to (\tilde{\mathbb{E}}//\tilde{H})/W_f$ is an open immersion and its image is open dense. Thus we have the result. □

The composite mapping $\big(\tilde{\mathbb{E}} \setminus \bigcup_{\alpha\in R} H_\alpha\big)/\tilde{W} \xrightarrow{i_1} \tilde{\mathbb{E}}/\tilde{W} \xrightarrow{i_2} \tilde{\mathbb{E}}//\tilde{W}$ is also an open immersion and its image is open dense.

We have the following commutative diagram of ringed spaces:

(3.13)

$$\begin{array}{ccccc}
\left(\left(\tilde{\mathbb{E}} \setminus \bigcup_{\alpha\in R} H_\alpha\right)\Big/ \tilde{W},\ \mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}}\right) & \xrightarrow{i_1} & (\tilde{\mathbb{E}}/\tilde{W},\ \mathcal{O}_{\tilde{\mathbb{E}}/\tilde{W}}) & \xrightarrow{i_2} & (\tilde{\mathbb{E}}//\tilde{W},\ \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}) \\
\pi\downarrow & & \pi\downarrow & & \varphi\downarrow \\
(\mathbb{H},\ \mathcal{O}_{\mathbb{H}}) & = & (\mathbb{H},\ \mathcal{O}_{\mathbb{H}}) & \leftarrow & (\mathbb{H},\ \boldsymbol{\mathcal{S}}^W).
\end{array}$$

Proposition 3.3. *The $\mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}}$-homomorphism* (2.23) (*resp. the $\mathcal{O}_{(\tilde{\mathbb{E}}\setminus\bigcup_{\alpha\in R} H_\alpha)/\tilde{W}}$-symmetric bilinear form* (2.14)) *is uniquely extended to the $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-homomorphism $\Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \to \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$ (resp. $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-symmetric bilinear form $\Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \times \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \to \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$) and coincides with* (3.10) (*resp.* (3.11)).

Proof. Since the image of the open immersion $i_2 \circ i_1$ is open dense, we should only prove that the pull-back of (3.10) (resp. (3.11)) by $i_2 \circ i_1$ coincides with (2.23) (resp. (2.14)). The former is the pull-back of (2.24) (resp. (2.25)) by $\varphi \circ i_2 \circ i_1$. The

latter could be written as $E\colon (\varphi\circ i_2\circ i_1)^*\Omega^1_{\boldsymbol{\mathcal{S}}^W}\to(\varphi\circ i_2\circ i_1)^*\boldsymbol{\mathcal{S}}^W$ (resp. $I^*\colon(\varphi\circ i_2\circ i_1)^*\Omega^1_{\boldsymbol{\mathcal{S}}^W}\times(\varphi\circ i_2\circ i_1)^*\Omega^1_{\boldsymbol{\mathcal{S}}^W}\to(\varphi\circ i_2\circ i_1)^*\boldsymbol{\mathcal{S}}^W$).

Then we have the result by applying the following lemma to (2.26) and (2.27) using the fact that $(\varphi\circ i_2\circ i_1)_*=\pi_*$. □

Lemma 3.4. *Let $f\colon(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F},\mathcal{G}$ be $\mathcal{O}_Y$-modules. If we have $\alpha\colon f^*\mathcal{F}\to f^*\mathcal{G}$, $\beta\colon\mathcal{F}\to\mathcal{G}$ and a commutative diagram*:

(3.14)
$$\begin{array}{ccc} f_*f^*\mathcal{F} & \xrightarrow{f_*\alpha} & f_*f^*\mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{F} & \xrightarrow{\beta} & \mathcal{G} \end{array}$$

for the natural morphisms $\mathcal{F}\to f_*f^*\mathcal{F}$ and $\mathcal{G}\to f_*f^*\mathcal{G}$, then we have $\alpha=f^*\beta$.

Proof. By the naturality of $f^*f_*\to id.$, we have the commutative diagram:

(3.15)
$$\begin{array}{ccc} f^*\mathcal{F} & \xrightarrow{\alpha} & f^*\mathcal{G} \\ \uparrow & & \uparrow \\ f^*f_*f^*\mathcal{F} & \xrightarrow{f^*f_*\alpha} & f^*f_*f^*\mathcal{G} \\ \uparrow & & \uparrow \\ f^*\mathcal{F} & \xrightarrow{f^*\beta} & f^*\mathcal{G}. \end{array}$$

Since the composite morphism $f^*\mathcal{F}\to f^*f_*f^*\mathcal{F}\to f^*\mathcal{F}$ is the identity morphism, we have the result. □

3.2. Frobenius manifold. In this section, we give the main theorem which asserts that the space $\tilde{\mathbb{E}}//\tilde{W}$ admits a structure of Frobenius manifold and it is unique up to $\mathbb{C}^*$ action under some suitable condition.

We first remind the definition of Frobenius manifold and its intersection form.

DEFINITION 3.5 ([8, p. 146, Definition 9.1]). A Frobenius manifold is a tuple $(M,\circ,e,E,J)$ where M is a complex manifold of dimension ≥ 1 with holomorphic metric J and multiplication $\circ$ on the tangent bundle, e is a global unit field and E is another global vector field, subject to the following conditions:

1. the metric is invariant under the multiplication, i.e., $J(X\circ Y,Z)=J(X,Y\circ Z)$ for local sections $X,Y,Z\in\Theta_M$,

2. (potentiality) the $(3,1)$-tensor $\nabla\circ$ is symmetric (here, ∇ is the Levi–Civita connection of the metric), i.e., $\nabla_X(Y\circ Z)-Y\circ\nabla_X(Z)-\nabla_Y(X\circ Z)+X\circ\nabla_Y(Z)-[X,Y]\circ Z=0$, for local sections $X,Y,Z\in\Theta_M$,

3. the metric J is flat,
4. e is a unit field and it is flat, i.e. $\nabla e = 0$,
5. the Euler field E satisfies $Lie_E(\circ) = 1 \cdot \circ$ and $Lie_E(J) = D \cdot J$ for some $D \in \mathbb{C}$.

DEFINITION 3.6 ([4, p. 191]). For a Frobenius manifold $(M, \circ, e, E, J)$, we define an intersection form $h^*\colon \Omega^1_M \times \Omega^1_M \to \mathcal{O}_M$ by

$$h^*(\omega_1, \omega_2) = J(E, J^*(\omega_1) \circ J^*(\omega_2)) \tag{3.16}$$

for local 1-forms $\omega, \omega' \in \Omega^1_M$, where $J^*\colon \Omega^1_M \to \Theta_M$ is the isomorphism induced by J.

For the oriented elliptic root system with the signed marking (R, a) such that $R/\mathbb{R}a$ is reduced, the condition $c^1 > c^2$ is called "codimension 1" in [16], where $\{c^1, c^2\}$ is a part of degrees of generators of $\mathcal{O}_{\mathbb{H}}$-algebra $\boldsymbol{\mathcal{S}}^W$ in Theorem 2.6.

Theorem 3.7. *If the oriented elliptic root system with the signed marking* (R, a) *such that* $R/\mathbb{R}a$ *is reduced satisfies the condition of codimension* 1, *then we have the following results.*
(1) $\tilde{\mathbb{E}}//\tilde{W}$ *has a structure of Frobenius manifold* $(\tilde{\mathbb{E}}//\tilde{W}, \circ, e, E, J)$ *with the following conditions*:

1. E *is the Euler field defined in* (3.10).
2. $I^*_{\tilde{\mathbb{E}}//\tilde{W}}$ *gives the intersection form of a Frobenius manifold* $(\tilde{\mathbb{E}}//\tilde{W}, \circ, e, E, J)$.

(2) *For* $c \in \mathbb{C}^*$, $(\tilde{\mathbb{E}}//\tilde{W}, c^{-1}\circ, ce, E, c^{-1}J)$ *is also a Frobenius manifold satisfying conditions of* (1).
(3) *Let* $(\tilde{\mathbb{E}}//\tilde{W}, \circ', e', E', J')$ *be a Frobenius manifold which satisfies conditions of* (1). *Then there exists* $c \in \mathbb{C}^*$ *such that* $(\tilde{\mathbb{E}}//\tilde{W}, \circ', e', E', J') = (\tilde{\mathbb{E}}//\tilde{W}, c^{-1}\circ, ce, E, c^{-1}J)$.

REMARK 3.8. In the definition of Frobenius manifold, the homogeneity $Lie_E(J) = D \cdot J$ for some $D \in \mathbb{C}$ is assumed. By the equation (3.16), D must be 1 because $Lie_E\, I^*_{\tilde{\mathbb{E}}//\tilde{W}} = 0$ and $Lie_E(\circ) = \circ$.

REMARK 3.9. The existence of the holomorphic metric J is already shown in [16]. The existence of the multiplication is already announced in [18] in the form of the existence of the potential.

4. Construction of the structure of the Frobenius manifold

In this section, we give a proof of Theorem 3.7 (1), that is, the existence of the structure of the Frobenius manifold on $\tilde{\mathbb{E}}//\tilde{W}$. In Section 4.1, we review a construction [16] of a flat metric on $\tilde{\mathbb{E}}//\tilde{W}$ (Proposition 4.4) and flat coordinates. In Section 4.2, we recall the notion of a flat pencil. In Section 4.3, we construct a multiplication on the tangent space of $\tilde{\mathbb{E}}//\tilde{W}$. In Section 4.4, we construct a potential of the multiplication.

In Section 4.5, we show the properties of the multiplication. In Section 4.6, we show that these constructions give a structure of the Frobenius manifold.

Hereafter we shall calculate tensors by using indices. In that case, we use Einstein's summation convention, that is, if an upper index of one tensor and a lower of the other tensor coincide, then we take summation for the same letter.

4.1. A construction of a flat metric and flat coordinates. Let $\tilde{\mathbb{E}}//\tilde{W}$ be the orbit space of the elliptic Weyl group action defined in (3.1). Hereafter we assume that (R, a) is codimension 1.

We prepare the relation between $\mathcal{S}^W$-modules and $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-modules.

We first define a notion of degree. For $f \in \mathcal{S}^W$ and $d \in \mathbb{Q}$, if $Ef = df$, then we call d *the degree of* f. For $f \in \mathcal{S}^W_k$, the degree of f is k/c^1. Especially the degree of s^i in Theorem 2.6 is $d^i := c^i/c^1$ $(i = 1, \ldots, n)$, i.e.

$$Es^i = d^i s^i \ (i = 1, \ldots, n), \quad 1 = d^1 > d^2 \geq \cdots \geq d^{n-1} > d^n = 0. \tag{4.1}$$

A degree is defined also for local sections of $\Omega^1_{\mathcal{S}^W}$ and $\mathrm{Der}_{\mathcal{S}^W}$. They have $\mathcal{S}^W$-free homogeneous generators by (2.21).

We have morphisms:

$$\mathcal{S}^W \to \varphi_* \varphi^* \mathcal{S}^W \simeq \varphi_* \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}, \tag{4.2}$$

$$\Omega^1_{\mathcal{S}^W} \to \varphi_* \varphi^* \Omega^1_{\mathcal{S}^W} \simeq \varphi_* \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}}, \tag{4.3}$$

$$\mathrm{Der}_{\mathcal{S}^W} \to \varphi_* \varphi^* \mathrm{Der}_{\mathcal{S}^W} \simeq \varphi_* \Theta_{\tilde{\mathbb{E}}//\tilde{W}}, \tag{4.4}$$

where (4.2) is $\phi\colon \mathcal{S}^W \to \pi_* \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$ defined in (3.4). The morphisms (4.3) and (4.4) are defined for a morphism $\varphi\colon (\tilde{\mathbb{E}}//\tilde{W}, \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}) \to (\mathbb{H}, \mathcal{S}^W)$.

Proposition 4.1. *The morphisms* (4.2), (4.3) *and* (4.4) *are injective. A homogeneous local section with respect to the Euler operator* $E \in \Theta_{\tilde{\mathbb{E}}//\tilde{W}}(\tilde{\mathbb{E}}//\tilde{W})$ *of* $\varphi_* \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$ (*resp.* $\varphi_* \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}}$, $\varphi_* \Theta_{\tilde{\mathbb{E}}//\tilde{W}}$) *is an image of* $\mathcal{S}^W$ (*resp.* $\Omega^1_{\mathcal{S}^W}$, $\mathrm{Der}_{\mathcal{S}^W}$).

Proof. The morphism φ decomposes into $\varphi = \varphi_2 \circ \varphi_1$ as in the proof of Proposition 3.5. For a $\mathcal{S}^W$-module $\mathcal{M}$, we have $\mathcal{M} \simeq \varphi_{2*} \varphi_2^* \mathcal{M}$ because φ_2 is an analogue of affine morphism of scheme theory. Also $\varphi_2^* \mathcal{M} \to \varphi_{1*} \varphi_1^* (\varphi_2^* \mathcal{M})$ is injective because φ_1 is faithfully flat by [20]. Thus we obtain the injectivity of $\mathcal{M} \to \varphi_* \varphi^* \mathcal{M}$.

By semi-positivity of the degrees of $s^1, \ldots, s^n$, a homogeneous section of $\varphi_* \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$ is an image of (4.2). Since $\Omega^1_{\mathcal{S}^W}$ (resp. $\mathrm{Der}_{\mathcal{S}^W}$) is a $\mathcal{S}^W$-free with homogeneous gener-

ator $ds^1, \dots, ds^n$ (resp. $\partial/\partial s^1, \dots, \partial/\partial s^n$), the morphism (4.3) (resp. (4.4)) is written as

$$\bigoplus_{i=1}^{n} \boldsymbol{\mathcal{S}}^W ds^i \to \bigoplus_{i=1}^{n} \varphi_* \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}} ds^i, \tag{4.5}$$

$$(\text{resp. } \bigoplus_{i=1}^{n} \boldsymbol{\mathcal{S}}^W \partial/\partial s^i \to \bigoplus_{i=1}^{n} \varphi_* \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}} \partial/\partial s^i). \tag{4.6}$$

Then the assertion is obvious. □

We prepare the notations. We put $S_k^W := \boldsymbol{\mathcal{S}}_k^W(\mathbb{H})$, $S^W := \boldsymbol{\mathcal{S}}^W(\mathbb{H})$. Then S^W is an $\mathcal{O}_{\mathbb{H}}(\mathbb{H})$-free algebra:

$$S^W = \mathcal{O}_{\mathbb{H}}(\mathbb{H})[s^1, \dots, s^{n-1}]. \tag{4.7}$$

We define S^W-modules:

$$\mathrm{Der}_{S^W} := \mathrm{Der}_{\boldsymbol{\mathcal{S}}^W}(\mathbb{H}), \tag{4.8}$$

$$\Omega^1_{S^W} := \Omega^1_{\boldsymbol{\mathcal{S}}^W}(\mathbb{H}). \tag{4.9}$$

We put

$$\mathrm{Der}_{S^W}^{\mathrm{lowest}} := \{\delta \in \Theta_{\tilde{\mathbb{E}}//\tilde{W}}(\tilde{\mathbb{E}}//\tilde{W}) \mid [E, \delta] = -\delta,\ \delta \text{ is non-singular}\}, \tag{4.10}$$

$$\Omega_\delta := \left\{\omega \in \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}} \;\middle|\; Lie_\delta\, \omega = 0\right\} \quad \text{for} \quad \delta \in \mathrm{Der}_{S^W}^{\mathrm{lowest}}, \tag{4.11}$$

$$V := \left\{\delta \in \mathrm{Der}_{S^W}^{\mathrm{lowest}} \;\middle|\; \delta^2 I^*_{\tilde{\mathbb{E}}//\tilde{W}}(\omega, \omega') = 0,\ \forall \omega, \omega' \in \Omega_\delta\right\}. \tag{4.12}$$

Using generators $s^1, \dots, s^n$ in Theorem 2.6, we have $\mathrm{Der}_{S^W}^{\mathrm{lowest}} = \mathcal{O}^*(\mathbb{H})\,\partial/\partial s^1$ by Proposition 4.1.

Proposition 4.2 ([16]). *V is non-empty and for any $\delta \in V$, we have*

$$V = \mathbb{C}^* \delta. \tag{4.13}$$

REMARK 4.3. We remind the problem in the beginning of the induction. For the construction of the structure of the Frobenius manifold for (M, I^*), one of the important point is to construct the unit field e.

For the case of the complex orbit space of the finite irreducible Coxeter group, e is characterized up to $\mathbb{C}^*$ multiplication as a lowest degree vector field. So we could construct the unit field e by this characterization.

For the case of the complex orbit space of the elliptic Weyl group for the elliptic root system of codimension 1, we could not characterize e only by its degree. But as we will see in Proposition 5.1, e must be an element of V. The above proposition

shows that the condition $e \in V$ characterize e up to $\mathbb{C}^*$ multiplication. This characterization is a key point of our construction.

However in general cases (e.g. the complex orbit space of the reflection group for the root system with indefinite inner product) whose reflection group invariants contains negative degree ones, we do not know any characterization of e. This is one of the difficulties of the generalization of our construction.

The following proposition gives a flat metric on $\tilde{\mathbb{E}}//\tilde{W}$.

Proposition 4.4 ([16]). *Take an arbitrary element $\hat{e}$ of V. Then there exists a unique non-degenerate $\boldsymbol{\mathcal{S}}^W$-symmetric bilinear form*

$$\hat{J}\colon \mathrm{Der}_{\boldsymbol{\mathcal{S}}^W} \times \mathrm{Der}_{\boldsymbol{\mathcal{S}}^W} \to \boldsymbol{\mathcal{S}}^W \tag{4.14}$$

such that the $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-symmetric bilinear form:

$$\hat{J}\colon \Theta_{\tilde{\mathbb{E}}//\tilde{W}} \times \Theta_{\tilde{\mathbb{E}}//\tilde{W}} \to \mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}, \tag{4.15}$$

obtained by the pull-back of (4.14) *by φ in* (3.3) *satisfies the condition*

$$\hat{J}^*(\omega_1, \omega_2) = \hat{e} I^*_{\tilde{\mathbb{E}}//\tilde{W}}(\omega_1, \omega_2) \tag{4.16}$$

for the dual metric of (4.15) *and $\omega_1, \omega_2 \in \Omega_{\hat{e}}$. The $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-symmetric bilinear form* (4.15) *is homogeneous of degree* 1, *i.e. $Lie_E(\hat{J}) = \hat{J}$. Furthermore, the Levi–Civita connection $\nabla^{\hat{J}}$ for $\hat{J}$ is flat and $\nabla^{\hat{J}}\hat{e} = 0$.*

We introduce flat coordinates. Since $\tilde{\mathbb{E}}//\tilde{W}$ is simply-connected, we could take functions whose differential are flat with respect to $\hat{J}^*$. In Lemma 4.5, we show that they generate the ring $\boldsymbol{\mathcal{S}}^W$, thus they give global coordinates for $\tilde{\mathbb{E}}//\tilde{W}$.

Lemma 4.5. (1) *There exist holomorphic functions $t^1, \ldots, t^n \in S^W$ such that*
(i) $\{dt^1, \ldots, dt^n\}$ *gives a $\mathbb{C}$-basis of flat sections of $\Omega^1_{\tilde{\mathbb{E}}//\tilde{W}}$ with respect to $\hat{J}^*$ on $\tilde{\mathbb{E}}//\tilde{W}$.*
(ii) $t^1, \ldots, t^n$ *are homogeneous elements of S^W with degree d^i (i.e. $Et^i = d^i t^i$), where d^i is defined in* (4.1).
(iii) $t^n = s^n$, *where s^n is defined after* Theorem 2.6.
(iv) $\hat{e} = \partial/\partial t_1$.
(2) *For $t^1, \ldots, t^n$, we have the following results*:
(i) $\boldsymbol{\mathcal{S}}^W = \mathcal{O}_{\mathbb{H}}[t^1, \ldots, t^{n-1}]$.
(ii) $t^1, \ldots, t^n$ *give global coordinates on $\tilde{\mathbb{E}}//\tilde{W}$.*
(iii) $\Omega^1_{S^W} = \bigoplus_{\alpha=1}^n S^W\, dt^\alpha$. *We remark that we use Greek letter for the suffix.*

(iv) *We prepare elements* $\partial/\partial t^\alpha \in \mathrm{Der}_{S^W}$ *by* $(\partial/\partial t^\alpha)t^\beta = \delta_\alpha^\beta$. *Then we have* $\mathrm{Der}_{S^W} = \bigoplus_{\alpha=1}^n S^W\, \partial/\partial t_\alpha$.

Proof. Since $\tilde{\mathbb{E}}//\tilde{W}$ is simply-connected, the space

$$H_1 := \left\{\omega \in \Gamma\left(\tilde{\mathbb{E}}//\tilde{W},\, \Omega^1_{\tilde{\mathbb{E}}//\tilde{W}}\right) \;\middle|\; \nabla^{\hat{J}}\omega = 0\right\} \tag{4.17}$$

is n-dimensional. We see that any element of H_1 is closed because $\nabla^{\hat{J}}$ is torsion-free. Since $Lie_E\, \hat{J} = \hat{J}$, a tensor $\nabla^{\hat{J}} E$ is flat ([8, p. 147]). Then E acts on H_1. Thus H_1 is identified with

$$H_2 := \left\{\omega \in \Gamma\left(\mathbb{H},\, \Omega^1_{\boldsymbol{\mathcal{S}}^W}\right) \;\middle|\; \nabla^{\hat{J}}\omega = 0\right\}, \tag{4.18}$$

by (4.3). Since φ is faithfully flat, the sequence

$$0 \to \mathbb{C} \to \boldsymbol{\mathcal{S}}^W \to \Omega^1_{\boldsymbol{\mathcal{S}}^W} \to \Omega^2_{\boldsymbol{\mathcal{S}}^W} \to \cdots \tag{4.19}$$

is exact. Thus we have an exact sequence

$$0 \to \mathbb{C} \to S^W \to \Omega^1_{S^W} \to \Omega^2_{S^W} \to \cdots \tag{4.20}$$

because each homogeneous part of each graded module of (4.19) is coherent and the domain $\mathbb{H}$ is Stein. Then we could take $t^1, \ldots, t^n \in S^W$ satisfying (1) (i). We could take $t^1, \ldots, t^n \in S^W$ so that $t^1, \ldots, t^n$ are homogeneous of degree $\deg t^1 \geq \cdots \geq \deg t^n$. Since the Jacobian $\partial(t^1, \ldots, t^n)/\partial(s^1, \ldots, s^n)$ is not 0, degrees of t^i must be d^i. For a proof of (1) (iii) (iv), see [16].

We prove (2). For a proof of (2) (i), we first list up the set of degree of s^α.

Put $\{d^1, \ldots, d^n\} = \{p^1, \ldots, p^m\}$ such that $1 = p^1 > p^2 > \cdots > p^m = 0$. We put $Q^i = \{\alpha \mid d^\alpha = p^i\}$.

We show $s^\alpha \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})[t^1, \ldots, t^{n-1}]$ for $\alpha \in Q^i$ by induction on i, that is, we show it in the order of $i = m$, $i = m-1$, $i = m-2, \ldots$ inductively.

If $i = m$, then $Q^m = \{n\}$ and we have $s^n \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})$, thus the assertion is proved for this case.

If $i = m-1$, then for $\alpha \in Q^{m-1}$, we have $t^\alpha = \sum_{\beta \in Q^{m-1}} f_{\alpha\beta} s^\beta$ with $f_{\alpha\beta} \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})$. The matrix $(f_{\alpha\beta})$ of size $\#Q^{m-1}$ is invertible because the Jacobian $(\partial t^\alpha/\partial s^\beta)$ of size n is upper-triangular and invertible. Thus $s^\beta \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})[t^1, \ldots, t^{n-1}]$ for $\beta \in Q^{m-1}$.

We assume that $s^\alpha \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})[t^1, \ldots, t^{n-1}]$ for $\alpha \in Q^{i+1}$ $(1 \leq i \leq m-2)$.

Then by the parallel discussion as above, we could show that s^α $(\alpha \in Q^i)$ is a linear combination of t^α $(\alpha \in Q^i)$ modulo $\mathcal{O}_{\mathbb{H}}(\mathbb{H})$-coefficient polynomials s^γ with $\deg s^\gamma < p_i$. By the assumption of induction, we have $s^\alpha \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})[t^1, \ldots, t^{n-1}]$ for $\alpha \in Q^i$. Thus we have (2) (i).

(2) (ii), (2) (iii), (2) (iv) are direct consequences of (2) (i). □

We call these elements $t^1, \ldots, t^n \in S^W$ with the properties of Lemma 4.5 (1) *the flat coordinates*.

4.2. Flat pencil. The purpose of Section 4.2 is to recall the notion of a flat pencil. We obtain special properties of the Christoffel symbols with respect to flat coordinates by the technique of a flat pencil by the parallel discussion of [4]. They are summarized in Proposition 4.8. They will be used to construct a multiplication in Section 4.3. and its potential in Section 4.4.

First we introduce the rational extensions both of a symmetric S^W-bilinear form and its Levi–Civita connection. Let $K(S^W)$ be the quotient field of the integral domain S^W. We define $\Omega^1_{K(S^W)}$ and $\mathrm{Der}_{K(S^W)}$ by

$$\Omega^1_{K(S^W)} := K(S^W) \otimes_{S^W} \Omega^1_{S^W}, \quad \mathrm{Der}_{K(S^W)} := K(S^W) \otimes_{S^W} \mathrm{Der}_{S^W} . \tag{4.21}$$

Let $g^*\colon \Omega^1_{S^W} \times \Omega^1_{S^W} \to S^W$ be a symmetric S^W-bilinear form with $0 \neq \det g^*(ds^\alpha, ds^\beta) \in S^W$. It induces the $K(S^W)$-linear extension of g^*:

$$g^*\colon \Omega^1_{K(S^W)} \times \Omega^1_{K(S^W)} \to K(S^W), \tag{4.22}$$

which is non-degenerate because $\det g^*(ds^\alpha, ds^\beta)$ is a unit in $K(S^W)$. The Levi–Civita connection and its dual:

$$\nabla^{g^*}\colon \mathrm{Der}_{K(S^W)} \times \mathrm{Der}_{K(S^W)} \to \mathrm{Der}_{K(S^W)}, \tag{4.23}$$

$$\nabla^{g^*}\colon \mathrm{Der}_{K(S^W)} \times \Omega^1_{K(S^W)} \to \Omega^1_{K(S^W)} \tag{4.24}$$

are defined and characterized by the metric condition $\nabla^{g^*} g^* = 0$ and torsion free condition $\nabla^{g^*}_\delta \delta' - \nabla^{g^*}_{\delta'} \delta = [\delta, \delta']$ for $\delta, \delta' \in \mathrm{Der}_{K(S^W)}$. We call the $K(S^W)$-bilinear form g^* *flat* if the curvature of ∇^{g^*} vanishes, i.e.

$$\nabla^{g^*}_\delta \nabla^{g^*}_{\delta'} - \nabla^{g^*}_{\delta'} \nabla^{g^*}_\delta = \nabla^{g^*}_{[\delta, \delta']} \tag{4.25}$$

for any $\delta, \delta' \in \mathrm{Der}_{K(S^W)}$.

We shall come back to our situation. We remind that

$$I^*\colon \Omega^1_{S^W} \times \Omega^1_{S^W} \to S^W \tag{4.26}$$

is defined as a global section of (2.25).

The $K(S^W)$-linear extension of I^* is non-degenerate and flat because $I^*_{\tilde{\mathbb{E}}//\tilde{W}}$ is non-degenerate and flat on the open dense subset $\big(\tilde{\mathbb{E}} \setminus \bigcup_{\alpha \in R} H_\alpha\big)/\tilde{W} \subset \tilde{\mathbb{E}}//\tilde{W}$ by Proposition 3.3.

Taking a global section on $\mathbb{H}$ of the dual tensor $\hat{J}^*$ of $\hat{J}$ in (4.14), we have

$$\hat{J}^*\colon \Omega^1_{S^W} \times \Omega^1_{S^W} \to S^W. \tag{4.27}$$

The $K(S^W)$-linear extension of $\hat{J}^*$ is non-degenerate and flat by Proposition 4.4.

We denote the Levi–Civita connections for $K(S^W)$-linear extensions I^* and $\hat{J}^*$ by ∇^{I^*} and $\nabla^{\hat{J}^*}$ respectively.

Hereafter we use the flat coordinates $t^1, \dots, t^n \in S^W$ introduced in Lemma 4.5 (1).

We fix some notations. We simply denote $\partial/\partial t^\alpha$ by ∂_α. Thus $\hat{e} = \partial_1$. We put

$$\eta^{\alpha\beta} := \hat{J}^*(dt^\alpha, dt^\beta) \in \mathbb{C}. \tag{4.28}$$

We have $\det(\eta^{\alpha\beta}) \neq 0$ because the set $\{dt^1, \dots, dt^n\}$ is an S^W-free basis of $\Omega^1_{S^W}$, and $\hat{J}$ is non-degenerate. The complex numbers $\eta_{\alpha\beta}$ are determined by the property

$$\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^\gamma_\alpha, \tag{4.29}$$

where we take summation for the same letter.

We put

$$g^{\alpha\beta} := I^*(dt^\alpha, dt^\beta) \in S^W. \tag{4.30}$$

We put

$$\Gamma^{\alpha\beta}_\gamma := I^*(dt^\alpha, \nabla^{I^*}_\gamma dt^\beta) \in K(S^W), \quad \text{where} \quad \nabla^{I^*}_\gamma := \nabla^{I^*}_{\partial_\gamma}. \tag{4.31}$$

Proposition 4.6. *Let $t^1, \dots, t^n$ be the flat coordinates defined as above.*

(1) *$\Gamma^{\alpha\beta}_\gamma$ is an element of S^W.*

(2) *$g^{\alpha\beta}$ and $\Gamma^{\alpha\beta}_\gamma$ satisfy*

$$\partial_1^2(g^{\alpha\beta}) = 0, \quad \partial_1^2(\Gamma^{\alpha\beta}_\gamma) = 0. \tag{4.32}$$

(3) *$\det(\partial_1 g^{\alpha\beta})$ is a unit in S^W.*

Proof. (1) is a direct consequence of the results of [16]. We only give the outline. By [16, p. 43, (6.7)], $\nabla^{I^*}_\gamma dt^\beta$ becomes a logarithmic form in the sense of [16]. Meanwhile $I^*(\omega, \omega')$ is an element of S^W for $\omega \in \Omega^1_{S^W}$ and a logarithmic form ω' by [16, p. 38, (5.5.1)]. Thus we obtain the assertion of (1).

For a proof of (2), we first check the degrees of $(\partial_1)^2 g^{\alpha\beta}$ and $(\partial_1)^2\Gamma^{\alpha\beta}_\gamma$. We have

$$\deg(\partial_1)^2 g^{\alpha\beta} = d^\alpha + d^\beta - 2 \leq 0, \quad \deg(\partial_1)^2\Gamma^{\alpha\beta}_\gamma = d^\alpha + d^\beta - d^\gamma - 2 \leq 0. \tag{4.33}$$

Their degrees are 0 only when $\alpha = \beta = 1$, $\gamma = n$. In this case, $(\partial_1)^2 g^{11} = (\hat{e})^2 g^{11} = \hat{e}\eta^{11} = 0$. We show $(\partial_1)^2\Gamma^{11}_n = 0$. Since

$$\Gamma^{11}_n = I^*\big(dt^1, \nabla^{I^*}_n dt^1\big) = \frac{1}{2}\,\partial_n I^*(dt^1, dt^1) = \frac{1}{2}\,\partial_n g^{11},$$

it follows that $(\partial_1)^2\Gamma_n^{11} = (1/2)\,\partial_n(\partial_1)^2 g^{11} = 0$. In the case where degrees are negative, then $(\partial_1)^2 g^{\alpha\beta} = (\partial_1)^2\Gamma_\gamma^{\alpha\beta} = 0$.

For a proof of (3), we remind that $\hat{J}^*$ is non-degenerate. Thus $\det J^*(dt^\alpha, dt^\beta) = \det(\partial_1 g^{\alpha\beta})$ is a unit in S^W. □

We show that I^* and $\hat{J}^*$ give a flat pencil in the sense of [4, p. 194, Definition 3.1].

Proposition 4.7. *$K(S^W)$-linear extensions*

$$I^*\colon \Omega^1_{K(S^W)} \times \Omega^1_{K(S^W)} \to K(S^W), \tag{4.34}$$

$$\hat{J}^*\colon \Omega^1_{K(S^W)} \times \Omega^1_{K(S^W)} \to K(S^W) \tag{4.35}$$

form a flat pencil [4, p. 194 (3.35)]. *Namely, if we put $I^*_\lambda := I^* + \lambda\hat{J}^*$ for any $\lambda \in \mathbb{C}$, we have the following.*

(1) *I^*_λ is non-degenerate and flat.*

(2) *Let ∇^λ be the Levi–Civita connection for I^*_λ. Then the equality*

$$I^*_\lambda\big(\omega_1, \nabla^\lambda_\delta\omega_2\big) = I^*\big(\omega_1, \nabla^{I^*}_\delta\omega_2\big) + \lambda\hat{J}^*\big(\omega_1, \nabla^{\hat{J}^*}_\delta\omega_2\big) \tag{4.36}$$

holds for $\omega_1, \omega_2 \in \Omega^1_{S^W}$, $\delta \in \mathrm{Der}_{S^W}$.

Proof. A proof is completely parallel to Lemma D.1 in [4, p. 227].

We assert that for any $(r, s) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, the tensor $rg^{\alpha\beta} + s\,\partial_1 g^{\alpha\beta}$ is non-degenerate, flat and its Christoffel symbol $\Gamma^{\alpha\beta}_{(r,s)\gamma}$ equals $r\Gamma^{\alpha\beta}_\gamma + s\,\partial_1\Gamma^{\alpha\beta}_\gamma$.

We show that the proposition follows from this assertion. We obtain (1) by $(r, s) = (1, \lambda)$ because $\partial_1 g^{\alpha\beta} = \eta^{\alpha\beta}$. If $(r, s) = (0, 1)$, then we see that $\partial_1\Gamma^{\alpha\beta}_\gamma$ is a Christoffel symbol of $\partial_1 g^{\alpha\beta} = \eta^{\alpha\beta}$. Thus we obtain (2) by $(r, s) = (1, \lambda)$.

We show the assertion. Using the flat coordinates, we regard $g^{\alpha\beta}$ and $\Gamma^{\alpha\beta}_\gamma$ as functions on flat coordinates, i.e.

$$g^{\alpha\beta}(t^1, \ldots, t^n), \quad \Gamma^{\alpha\beta}_\gamma(t^1, \ldots, t^n). \tag{4.37}$$

We assume that $r \neq 0$. Then $rg^{\alpha\beta}(t^1 + s/r, t^2, \ldots, t^n)$ is non-degenerate, flat and its Christoffel symbol is $r\Gamma^{\alpha\beta}_\gamma(t^1 + s/r, t^2, \ldots, t^n)$. Since $g^{\alpha\beta}(t^1, \ldots, t^n)$, $\Gamma^{\alpha\beta}_\gamma(t^1, \ldots, t^n)$ are polynomial functions of degree 1 with respect to t^1 by (4.32), we have

$$rg^{\alpha\beta}\left(t^1 + \frac{s}{r}, t^2, \ldots, t^n\right) = rg^{\alpha\beta}(t^1, \ldots, t^n) + s\,\partial_1 g^{\alpha\beta}(t^1, \ldots, t^n), \tag{4.38}$$

$$r\Gamma^{\alpha\beta}_\gamma\left(t^1 + \frac{s}{r}, t^2, \ldots, t^n\right) = r\Gamma^{\alpha\beta}_\gamma(t^1, \ldots, t^n) + s\,\partial_1\Gamma^{\alpha\beta}_\gamma(t^1, \ldots, t^n). \tag{4.39}$$

Thus we proved the assertion for the case $r \neq 0$.

For the case of $r = 0$, $s\,\partial_1 g^{\alpha\beta} = s\eta^{\alpha\beta}$ is non-degenerate and flat because $s \neq 0$. On Christoffel symbol, we see that $\Gamma^{\alpha\beta}_{(r,s)\gamma} - [r\Gamma^{\alpha\beta}_{\gamma} + s\,\partial_1\Gamma^{\alpha\beta}_{\gamma}]$ is a rational function with respect to (r, s). Since it is 0 on the domain $r \neq 0$, we see that it is 0 for any $(r, s) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Thus we proved the assertion. □

The following is a direct consequence of Proposition 4.7 (cf. [4, p. 226, (D.1a), (D.2)]) and (4.20).

Proposition 4.8. (1) *There exists a homogeneous element $f^{\beta} \in S^W$ satisfying the following relations*

$$\Gamma^{\alpha\beta}_{\gamma} = \eta^{\alpha\epsilon}\,\partial_\epsilon\partial_\gamma f^{\beta} \quad (\alpha, \gamma = 1, \ldots, n). \tag{4.40}$$

(2) *We have*

$$\Gamma^{\alpha\beta}_{\gamma}\Gamma^{\gamma\delta}_{\mu} = \Gamma^{\alpha\delta}_{\gamma}\Gamma^{\gamma\beta}_{\mu} \quad (\alpha, \beta, \delta, \mu = 1, \ldots, n). \tag{4.41}$$

We use the following results in Section 4.4 and 4.5.

Lemma 4.9. *We have*

$$g^{n\alpha} = \eta^{1n} d^{\alpha} t^{\alpha}, \tag{4.42}$$

$$\Gamma^{\alpha n}_{\beta} = 0, \tag{4.43}$$

$$\Gamma^{n\alpha}_{\beta} = \eta^{1n} d^{\alpha}\delta^{\alpha}_{\beta}. \tag{4.44}$$

Proof. For (4.42), we should prove

$$I^*(dt^n) = \eta^{1n} E. \tag{4.45}$$

We define the S^W-isomorphism

$$\mathrm{Der}_{S^W} \xrightarrow{\sim} \Omega^1_{S^W}, \quad \delta \mapsto \hat{J}(\delta, \cdot) \tag{4.46}$$

induced by $\hat{J} : \mathrm{Der}_{S^W} \times \mathrm{Der}_{S^W} \to S^W$ and denote it also by $\hat{J}$. By [16, p. 51, (9.8), Assertion (iii)] and [16, p. 52, (9.9), Corollary], we have

$$I^*(\hat{J}(\hat{e})) = E. \tag{4.47}$$

We remark that the Euler field E in [16, p. 38, (5.4.3)] corresponds to our operator E'. Then by $dt^n = \eta^{kn}\hat{J}(\partial/\partial t^k) = \eta^{1n}\hat{J}(\partial/\partial t^1) = \eta^{1n}\hat{J}(\hat{e})$, we have the result.

For (4.43), we should prove

$$\nabla^{I^*}_{\beta}\, dt^n = 0 \tag{4.48}$$

because $\Gamma^{\alpha n}_{\beta} = I^*\big(dt^{\alpha}, \nabla^{I^*}_{\beta} dt^n\big)$ by definition. Since we have

$$\nabla^{I^*}_{\beta} dt^n = \nabla^{I^*}_{\beta}(I^*)^{-1}(\eta^{1n}E) = \eta^{1n}(I^*)^{-1}(\nabla^{I^*}_{\beta}E) = 0$$

by (4.45) and $\nabla^{I^*}_{\beta}E = 0$ (cf. [16, p. 43, (6.6)]), we have the result.

For (4.44), we have

$$\partial_{\beta} g^{n\alpha} = I^*\big(\nabla^{I^*}_{\beta} dt^n, dt^{\alpha}\big) + I^*\big(dt^n, \nabla^{I^*}_{\beta} dt^{\alpha}\big) = \Gamma^{\alpha n}_{\beta} + \Gamma^{n\alpha}_{\beta}. \tag{4.49}$$

By (4.42) and (4.43), we have (4.44). □

4.3. A construction of a multiplication. The purpose of this subsection is to define a multiplication. The following proposition gives a motivation of Definition 4.11

Proposition 4.10. *Let $t^1, \ldots, t^n$ be the flat coordinates defined right after* Lemma 4.5. *We assume that there exists a multiplication $\circ$ on the tangent bundle of $\tilde{\mathbb{E}}//\tilde{W}$ such that $(\tilde{\mathbb{E}}//\tilde{W}, \circ, \hat{e}, E, \hat{J})$ becomes a Frobenius manifold whose intersection form is $I^*_{\tilde{\mathbb{E}}//\tilde{W}}$. We put the structure coefficients $C^{\alpha\beta}_{\gamma}$ $(\alpha, \beta, \gamma = 1, \ldots, n)$ with respect to the $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-free basis $\hat{J}^*(dt^1), \ldots, \hat{J}^*(dt^n)$ by the equations*:

$$\hat{J}^*(dt^{\alpha}) \circ \hat{J}^*(dt^{\beta}) = C^{\alpha\beta}_{\gamma}\hat{J}^*(dt^{\gamma}).$$

Then we have

$$\Gamma^{\alpha\beta}_{\gamma} = d^{\beta}C^{\alpha\beta}_{\gamma}, \tag{4.50}$$

$$C^{\alpha n}_{\gamma} = \eta^{1n}\delta^{\alpha}_{\gamma}. \tag{4.51}$$

By the equations (4.50), (4.51) *and the fact that $d^{\beta} \neq 0$ if $\beta \neq n$, we see that the structure of the multiplication is unique if it exists.*

Proof. By the uniqueness of the Levi–Civita connection with respect to the tensor $I^*_{\tilde{\mathbb{E}}//\tilde{W}}$ and the discussion of [4, p. 194, Lemma 3.4], we have (4.50). Also by the equation $\hat{J}^*(dt^n) = \eta^{1n}\partial_1 = \eta^{1n}\hat{e}$, we have (4.51). □

DEFINITION 4.11. We define the multiplication $\hat{\circ}$ by the equations:

$$\hat{J}^*(dt^{\alpha}) \,\hat{\circ}\, \hat{J}^*(dt^{\beta}) := \hat{C}^{\alpha\beta}_{\gamma}\hat{J}^*(dt^{\gamma}) \tag{4.52}$$

where

$$\hat{C}_{\gamma}^{\alpha\beta} := \begin{cases} \dfrac{1}{d^{\beta}}\Gamma_{\gamma}^{\alpha\beta}, & \text{if} \quad \beta \neq n, \\ \eta^{1n}\delta_{\gamma}^{\alpha}, & \text{if} \quad \beta = n. \end{cases} \tag{4.53}$$

This definition does not depend on the choice of the flat coordinates $t^1, \ldots, t^n$.

By $\mathcal{S}^W$-linear extension, we have

$$\hat{\circ}\colon \mathrm{Der}_{\mathcal{S}^W} \times \mathrm{Der}_{\mathcal{S}^W} \to \mathrm{Der}_{\mathcal{S}^W}. \tag{4.54}$$

By taking a pull-back of (4.54) by φ in (3.3), we define a multiplication

$$\hat{\circ}\colon \Theta_{\tilde{\mathbb{E}}//\tilde{W}} \times \Theta_{\tilde{\mathbb{E}}//\tilde{W}} \to \Theta_{\tilde{\mathbb{E}}//\tilde{W}}. \tag{4.55}$$

We shall show that $(\tilde{\mathbb{E}}//\tilde{W}, \hat{\circ}, \hat{e}, E, \hat{J})$ becomes a Frobenius manifold in the following subsections.

4.4. Existence of a potential. The purpose of this subsection is to show the existence of a potential for the multiplication defined in Section 4.3. We give it in Proposition 4.12 adding to the ambiguity of a potential.

We explain the idea of the construction of a potential F. We construct a potential of the multiplication by a technique of a flat pencil which is similar to the finite Coxeter group case [4]. But our multiplication $\hat{\circ}$ is defined in a case by case manner (cf. (4.53)). Thus we need to check the compatibility conditions also in a case by case manner.

Let $t^1, \ldots, t^n$ be the flat coordinates defined right after Lemma 4.5.

Proposition 4.12. (1) *There exists $F \in S^W$ of degree* 2 *such that*

$$\hat{J}(X \hat{\circ} Y, Z) = XYZF \tag{4.56}$$

for flat vector fields X, Y, Z on $\tilde{\mathbb{E}}//\tilde{W}$ with respect to $\hat{J}$. Such F is unique up to adding $c(t^1)^2$ for some $c \in \mathbb{C}$.

(2) *For any $F \in S^W$ satisfying* (4.56)*, we have*

$$I^*_{\tilde{\mathbb{E}}//\tilde{W}}(\omega, \omega') = E\,\hat{J}^*(\omega)\hat{J}^*(\omega')F \tag{4.57}$$

for flat 1*-forms ω, ω' on $\tilde{\mathbb{E}}//\tilde{W}$ with respect to $\hat{J}$. Conversely any degree* 2 *element $F \in S^W$ satisfying* (4.57) *satisfies* (4.56).

Proof. The assertions are all linear with respect to flat 1-forms ω, ω' and a flat vector field X, Y, Z. Then we should only prove the following assertions (a), (b), (c):

(a) There exists $F \in S^W$ of degree 2 such that

$$\hat{C}_\gamma^{\alpha\beta} = \eta^{\alpha\epsilon}\eta^{\beta\mu}\partial_\epsilon\partial_\mu\partial_\gamma F \quad (\alpha, \beta, \gamma = 1, \ldots, n), \tag{4.58}$$

$$g^{\beta\gamma} = E\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_\epsilon\partial_\mu F \quad (\beta, \gamma = 1, \ldots, n). \tag{4.59}$$

(b) An element $F \in S^W$ satisfying (4.58) is unique up to adding $c(t^1)^2$ for some $c \in \mathbb{C}$.
(c) An element $F \in S^W$ satisfying (4.59) is unique up to adding $c(t^1)^2$ for some $c \in \mathbb{C}$.

Here we used notations such as $g^{\alpha\beta}$ etc. defined after Lemma 4.5.

We prove (a) in five steps.

As the first step, by Proposition 4.8 (1), we could take a homogeneous element $f^\gamma \in S^W$ satisfying the following relations

$$\Gamma_\sigma^{\alpha\gamma} = \eta^{\alpha\epsilon}\partial_\epsilon\partial_\sigma f^\gamma \quad (\alpha, \sigma = 1, \ldots, n). \tag{4.60}$$

We put

$$F^\gamma = \begin{cases} \dfrac{f^\gamma}{d^\gamma}, & \text{if} \quad \gamma \neq n, \\ \dfrac{1}{2}\eta^{1n}\eta_{\alpha\beta}t^\alpha t^\beta, & \text{if} \quad \gamma = n. \end{cases} \tag{4.61}$$

We show that F^γ satisfies

$$\hat{C}_\sigma^{\alpha\gamma} = \eta^{\alpha\epsilon}\partial_\epsilon\partial_\sigma F^\gamma \quad (\alpha, \sigma = 1, \ldots, n). \tag{4.62}$$

If $\gamma = n$, it is O.K. by definition of $C_\sigma^{\alpha n}$. If $\gamma \neq n$, then it is O.K. by (4.60) and (4.61). $F^\gamma \in S^W$ is homogeneous of degree $1 + d^\gamma = 2 - (1 - d^\gamma)$.

As the second step, we shall check the equation:

$$g^{\beta\gamma} = (d^\beta + d^\gamma)\eta^{\beta\epsilon}\partial_\epsilon F^\gamma. \tag{4.63}$$

If $\gamma \neq n$ in (4.63), then we should only prove the equation

$$d^\gamma g^{\beta\gamma} = (d^\beta + d^\gamma)\eta^{\beta\epsilon}\partial_\epsilon(d^\gamma F^\gamma) \tag{4.64}$$

because $d^\gamma \neq 0$. We use the torsion freeness of ∇^{I^*} (cf. [4, p. 193, (3.27)]):

$$g^{\alpha\sigma}\Gamma_\sigma^{\beta\gamma} = g^{\beta\sigma}\Gamma_\sigma^{\alpha\gamma}. \tag{4.65}$$

We take $\beta = n$. Then L.H.S. of (4.65) becomes

$$g^{\alpha\sigma}\Gamma_\sigma^{n\gamma} = g^{\alpha\sigma}(\eta^{1n}d^\gamma\delta_\sigma^\gamma) = \eta^{1n}d^\gamma g^{\alpha\gamma} \tag{4.66}$$

by (4.44). R.H.S. of (4.65) becomes

$$\begin{aligned} g^{n\sigma}\Gamma_{\sigma}^{\alpha\gamma} &= (\eta^{1n} d^{\sigma} t^{\sigma})(\eta^{\alpha\epsilon}\partial_{\epsilon}\partial_{\sigma} f^{\gamma}) \quad \text{by (4.42) and (4.60)} \\ &= \eta^{1n}(d^{\sigma} t^{\sigma}\partial_{\sigma})(\eta^{\alpha\epsilon}\partial_{\epsilon} f^{\gamma}) \\ &= \eta^{1n}(d^{\alpha} + d^{\gamma})(\eta^{\alpha\epsilon}\partial_{\epsilon} f^{\gamma}) \quad \text{by deg } \eta^{\alpha\epsilon}\partial_{\epsilon} = -(1 - d^{\alpha}). \end{aligned} \tag{4.67}$$

Then by $\eta^{1n} \neq 0$, we have (4.64).

If $\gamma = n$ in (4.63), then we should prove the equation

$$g^{\beta n} = (d^{\beta} + d^{n})\eta^{\beta\epsilon}\partial_{\epsilon} F^{n}. \tag{4.68}$$

L.H.S. is $\eta^{1n} d^{\beta} t^{\beta}$ by (4.42). R.H.S. is $d^{\beta}\eta^{\beta\epsilon}\partial_{\epsilon} F^{n}$ by $d^{n} = 0$. Then (4.68) is a consequence of the definition of F^{n}.

As the third step, we show that there exists a homogeneous element $F \in S^{W}$ of degree $2d^{1} = 2$ such that it satisfies the following equation:

$$F^{\beta} = \eta^{\beta\mu}\partial_{\mu} F. \tag{4.69}$$

By (4.20), we should only prove the integrability conditions

$$\eta^{\beta\epsilon}\partial_{\epsilon} F^{\gamma} = \eta^{\gamma\epsilon}\partial_{\epsilon} F^{\beta} \tag{4.70}$$

for $\beta \neq \gamma$. Since $\beta \neq \gamma$, we have $d^{\beta} + d^{\gamma} \neq 0$. Then by (4.63), the assertion (4.70) reduces to the property of metric $g^{\beta\gamma} = g^{\gamma\beta}$.

As the fourth step, we have (4.59) because $E\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_{\epsilon}\partial_{\mu} F = (d^{\beta} + d^{\gamma})\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_{\epsilon}\partial_{\mu} F$ for $EF = 2F$.

As the fifth step, we have (4.58) because of (4.62) and (4.69).

Thus we finished a proof of the part (a).

We prove the part (b). Let F_1 and F_2 be degree 2 elements of S^{W} satisfying the condition (4.58). Then $F_3 := F_1 - F_2$ satisfies $0 = \eta^{\alpha\epsilon}\eta^{\beta\mu}\partial_{\epsilon}\partial_{\mu}\partial_{\gamma} F_3$. Thus F_3 is a polynomial of $t^{1}, \ldots, t^{n}$ of degree less than or equal to 2. But by the degree condition, F_3 must be constant times $(t^{1})^{2}$. Thus we see the ambiguity of F satisfying the condition (4.58).

We prove the part (c). Let F_4 and F_5 be degree 2 elements of S^{W} satisfying the condition (4.59). Then $F_6 := F_4 - F_5$ satisfies $0 = E\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_{\epsilon}\partial_{\mu} F_6 = (d^{\beta} + d^{\gamma})\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_{\epsilon}\partial_{\mu} F_6$, where the last equality comes from the degree condition. Thus we have

$$\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_{\epsilon}\partial_{\mu} F_6 = \begin{cases} 0, & \text{if } (\beta, \gamma) \neq (n, n), \\ f, & \text{if } (\beta, \gamma) = (n, n) \end{cases} \tag{4.71}$$

for some element $f \in S^{W}$ of degree 0. Thus $F_6 = (1/2) f (t^{1})^{2} + g$, where $g \in S^{W}$ is a linear combination of $t^{1}, \ldots, t^{n}$ plus constant. But by the degree condition, g must be 0. Applying the equation (4.71) for the case of $(\beta, \gamma) = (1, n)$, we see that f must be a constant. Thus we see the ambiguity of F satisfying the condition (4.59). □

4.5. Property of the multiplication. The purpose of this subsection is to show the properties of the multiplication.

Proposition 4.13. *For vector fields X, Y, Z on $\tilde{\mathbb{E}}//\tilde{W}$, we have*

(1) $X \hat{\circ} Y = Y \hat{\circ} X$.

(2) $\hat{e} \hat{\circ} X = X$.

(3) $(X \hat{\circ} Y) \hat{\circ} Z = X \hat{\circ} (Y \hat{\circ} Z)$.

Proof. Let $t^1, \ldots, t^n$ be the flat coordinates defined right after Lemma 4.5.

For (1), it is a direct consequence of Proposition 4.12.

For (2), we need to show

$$\hat{J}^*(dt^\alpha) \hat{\circ} \hat{e} = \hat{J}^*(dt^\alpha) \quad (\alpha = 1, \ldots, n). \tag{4.72}$$

By definition, we have

$$\hat{J}^*(dt^\alpha) \hat{\circ} \hat{J}^*(dt^n) = \hat{C}_\beta^{\alpha n} \hat{J}^*(dt^\beta) = \eta^{1n} \delta_\beta^\alpha \hat{J}^*(dt^\beta) = \eta^{1n} \hat{J}^*(dt^\alpha).$$

Since $\hat{J}^*(dt^n) = \eta^{1n} \hat{e}$ and $\eta^{1n} \neq 0$, we obtain (4.72).

For (3), we need to show

$$\hat{C}_\gamma^{\alpha\beta} \hat{C}_\mu^{\gamma\delta} = \hat{C}_\gamma^{\alpha\delta} \hat{C}_\mu^{\gamma\beta} \quad (\alpha, \beta, \delta, \mu = 1, \ldots, n). \tag{4.73}$$

We show (4.73). We have

$$\Gamma_\gamma^{\alpha\beta} \Gamma_\mu^{\gamma\delta} = \Gamma_\gamma^{\alpha\delta} \Gamma_\mu^{\gamma\beta} \quad (\alpha, \beta, \delta, \mu = 1, \ldots, n) \tag{4.74}$$

by Proposition 4.8 (2).

We show

$$\Gamma_\gamma^{\alpha\beta} = d^\beta \hat{C}_\gamma^{\alpha\beta}. \tag{4.75}$$

If $\beta \neq 0$, it is O.K. by (4.53). If $\beta = 0$, it is O.K. because both hands are 0 by $d^n = 0$ and (4.43).

By (4.75), we have

$$d^\beta d^\delta \hat{C}_\gamma^{\alpha\beta} \hat{C}_\mu^{\gamma\delta} = d^\beta d^\delta \hat{C}_\gamma^{\alpha\delta} \hat{C}_\mu^{\gamma\beta} \quad (\alpha, \beta, \delta, \mu = 1, \ldots, n). \tag{4.76}$$

Therefore we obtain (4.73) for the case $d^\beta d^\delta \neq 0$.

For the case $d^\beta d^\delta = 0$, the index β or δ must be n. Then we have $\hat{C}_\gamma^{\alpha n} = \eta^{1n} \delta_\gamma^\alpha$ by definition. Then the assertion (4.73) is apparent. □

4.6. Construction of the structure of the Frobenius manifold. The purpose of this subsection is to construct a structure of the Frobenius manifold.

Proposition 4.14. *The tuple* $(\tilde{\mathbb{E}}//\tilde{W}, \hat{\circ}, \hat{e}, E, \hat{J})$ *is a Frobenius manifold satisfying the conditions of* Theorem 3.7 (1).

Proof. We shall check the properties of Frobenius manifold.

We check $\hat{J}(X \hat{\circ} Y, Z) = \hat{J}(X, Y \hat{\circ} Z)$ for local fields X, Y, Z. We may assume that X, Y, Z are flat. Then $\hat{J}(X \hat{\circ} Y, Z) = XYZF$. Also we have $\hat{J}(X, Y \hat{\circ} Z) = \hat{J}(Y \hat{\circ} Z, X)YZXF$. Since $XYZF = YZXF$, we have $\hat{J}(X \hat{\circ} Y, Z) = \hat{J}(X, Y \hat{\circ} Z)$.

We check that the $(3,1)$-tensor $\hat{\nabla}\hat{\circ}$ is symmetric. We should only prove $\hat{\nabla}_X(Y \hat{\circ} Z) = \hat{\nabla}_Y(X \hat{\circ} Z)$ for flat vector fields X, Y, Z. We prove $\hat{J}(\hat{\nabla}_X(Y \hat{\circ} Z), W) = \hat{J}(\hat{\nabla}_Y(X \hat{\circ} Z), W)$ for a flat vector field W. Since $\hat{J}(\hat{\nabla}_X(Y \hat{\circ} Z), W) = X\hat{J}(Y \hat{\circ} Z, W) = XYZWF$ and $\hat{J}(\hat{\nabla}_Y(X \hat{\circ} Z), W) = Y\hat{J}(X \hat{\circ} Z, W) = YXZWF$, we have the result.

The flatness of $\hat{J}$ and the property $\hat{\nabla}\hat{e} = 0$ are asserted in Proposition 4.4.

Homogeneity conditions $Lie_E(\hat{\circ}) = 1 \cdot \hat{\circ}$ and $Lie_E(\hat{J}) = \hat{J}$ (i.e. $D = 1$) are consequences of $Lie_E F = 2$, $Lie_E \hat{e} = [E, \hat{e}] = -\hat{e}$ and $Lie_E I^*_{\tilde{\mathbb{E}}//\tilde{W}} = 0$.

We prove $I^*_{\tilde{\mathbb{E}}//\tilde{W}}(\omega, \omega') = \hat{J}(E, \hat{J}^*(\omega) \hat{\circ} \hat{J}^*(\omega'))$ for local 1-forms ω, ω'. We may assume that ω, ω' are flat. By Proposition 4.12 (2), we have

$$\hat{J}(E, \hat{J}^*(\omega) \hat{\circ} \hat{J}^*(\omega')) = E\hat{J}^*(\omega)\hat{J}^*(\omega')F = I^*_{\tilde{\mathbb{E}}//\tilde{W}}(\omega, \omega'). \qquad \square$$

5. Uniqueness of the structure of the Frobenius manifold

In this section, we give a proof of Theorem 3.7 (2) (3), that is, the uniqueness of the structure of the Frobenius manifold on $\tilde{\mathbb{E}}//\tilde{W}$.

Theorem 3.7 (2) is trivial. Theorem 3.7 (3) reduces to Proposition 5.2. We prepare the following proposition.

Proposition 5.1. *Let* $(M, \circ, e, E, J)$ *be a Frobenius manifold with intersection form* I^*. *Put*

$$\mathcal{F} := \{\omega \in \Omega^1_M \mid Lie_e\, \omega = 0\}, \tag{5.1}$$

$$T := \{f \in \mathcal{O}_M \mid e(f) = 0\}, \tag{5.2}$$

$$\Omega^{1\nabla}_M := \{\omega \in \Omega^1_M \mid \nabla\omega = 0\}, \tag{5.3}$$

where ∇ *is the Levi–Civita connection for* J. *Then*

(1) $\mathcal{F} \supset \Omega^{1\nabla}_M$ *and it induces* $\mathcal{F} \simeq T \otimes_{\mathbb{C}} \Omega^{1\nabla}_M$.

(2) e *is non-singular and* $[E, e] = -e$.

(3) $e^2 I^*(\omega, \omega') = 0$ *for local sections* $\omega, \omega' \in \mathcal{F}$.

(4) $eI^*(\omega, \omega') = J^*(\omega, \omega')$ *for local sections* $\omega, \omega' \in \mathcal{F}$.

Proof. We show (1). First we show that $\mathcal{F} \supset \Omega_M^{1\nabla}$. We take a local flat 1-form $\eta \in \Omega_M^{1\nabla}$. For a local flat vector field Y, we have $(Lie_e\,\eta)(Y) = e(\eta(Y)) - \eta([e, Y]) = 0$ because $[e, Y] = \nabla_e Y - \nabla_Y e = 0$. This gives $Lie_e\,\eta = 0$. Thus $\eta \in \mathcal{F}$. We see easily that the isomorphism $\Omega_M^1 \simeq \mathcal{O}_M \otimes_{\mathbb{C}} \Omega_M^{1\nabla}$ induces $\mathcal{F} \simeq T \otimes_{\mathbb{C}} \Omega_M^{1\nabla}$.

We show (2). Since e is flat, e is non-singular or 0. If $e = 0$, then any vector field X must be 0 because $X = X \circ e = X \circ 0 = 0$, which is a contradiction. Thus e is non-singular. Also we have $[E, e] = -e$ because the Lie derivative of $e \circ e = e$ by E gives $Lie_E(e) = -e$ since $Lie_E(\circ) = 1 \cdot \circ$.

We show (3) and (4). We first remark that the local existence of $f \in \mathcal{O}_M$ such that

$$J(X, Y \circ Z) = XYZf \tag{5.4}$$

for local flat fields X, Y, Z is well-known (cf. [8, p. 147]).

Then for local flat 1-forms ω, ω', we have

$$\begin{aligned} eI^*(\omega, \omega') &= eJ(E, J^*(\omega) \circ J^*(\omega')) \\ &= eE\,J^*(\omega)J^*(\omega')f \\ &= (e + Ee)J^*(\omega)J^*(\omega')f \\ &= J^*(\omega, \omega') + E\,J^*(\omega, \omega') \\ &= J^*(\omega, \omega') \end{aligned}$$

because $J^*(\omega, \omega')$ is a constant for flat 1-forms ω, ω'. Then we have $e^2 I^*(\omega, \omega') = eJ^*(\omega, \omega') = 0$.

By the result of (1), it is sufficient to show (3) and (4) only for ω, ω' flat 1-forms, because (3) and (4) are linear over the ring T. Thus we have the result. □

Proposition 5.2. *Let* $(\tilde{\mathbb{E}}//\tilde{W}, \circ, e, E, J)$ *be any Frobenius manifold which satisfies the conditions of* Theorem 3.7 (1). *Let* $(\tilde{\mathbb{E}}//\tilde{W}, \hat{\circ}, \hat{e}, E, \hat{J})$ *be a Frobenius manifold constructed in* Proposition 4.14. *Then there exists* $c \in \mathbb{C}^*$ *such that*

$$(\tilde{\mathbb{E}}//\tilde{W}, c^{-1} \circ, ce, E, c^{-1}J) = (\tilde{\mathbb{E}}//\tilde{W}, \hat{\circ}, \hat{e}, E, \hat{J}). \tag{5.5}$$

Proof. By Proposition 5.1 (2), we have $e \in \mathrm{Der}_{S^W}^{\mathrm{lowest}}$. By Proposition 5.1 (3), $e \in V$, where V is defined in (4.12). By (4.13), we have $e = c^{-1}\hat{e}$ for some $c \in \mathbb{C}^*$. We have $\Omega_e = \Omega_{\hat{e}}$, where Ω_e is defined in (4.11) for $e \in \mathrm{Der}_{S^W}^{\mathrm{lowest}}$.

By Proposition 5.1 (4), $J^*(\omega, \omega') = c^{-1}\hat{J}^*(\omega, \omega')$ for $\omega, \omega' \in \Omega_e = \Omega_{\hat{e}}$. Since Ω_e contains an $\mathcal{O}_{\tilde{\mathbb{E}}//\tilde{W}}$-free basis of $\Omega^1_{\tilde{\mathbb{E}}//\tilde{W}}$ by Proposition 5.1 (1), we have $J^* = c^{-1}\hat{J}^*$. Thus we have $J = c\hat{J}$.

By Theorem 3.7 (2),

$$(\tilde{\mathbb{E}}//\tilde{W}, \circ', e', E, J') := (\tilde{\mathbb{E}}//\tilde{W}, c^{-1} \circ, ce, E, c^{-1}J) \tag{5.6}$$

is also a Frobenius manifold satisfying the conditions of Theorem 3.7 (1). We need to prove $(\tilde{\mathbb{E}}//\tilde{W}, \circ', e', E, J') = (\tilde{\mathbb{E}}//\tilde{W}, \hat{\circ}, \hat{e}, E, \hat{J})$. We already have $e' = \hat{e},\ J' = \hat{J}$.

Since these structures of the Frobenius manifold have the common intersection form $I^*_{\tilde{\mathbb{E}}//\tilde{W}}$, the structure of the multiplication of the Frobenius manifold is uniquely determined by the data of the unit vector e, the Euler field E and the flat metric J by Proposition 4.10. Therefore we have the result. □

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Department of Mathematics
Graduate School of Science
Osaka University
1-1 Machikaneyama-cho, Toyonaka
Osaka 560–0043
Japan
e-mail: satake@math.sci.osaka-u.ac.jp