Nishimura, T. and Nagamachi, S.
Osaka J. Math.
38 (2001), 667–680

SUPPORT AND KERNEL THEOREM FOR FOURIER HYPERFUNCTIONS

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(Received November 8, 1999)

1. Introduction

A Fourier hyperfunction is defined to be an element of the dual space of the test function space $\mathcal{Q}(\mathbf{D}^n)$. On the other hand a Fourier hyperfunction is considerd to be a sum of boundary values of slowly increasing holomorphic functions $F_j(x+iy)$ $(y \to 0)$ defined on the wedge $\mathbf{R}^n + i\Gamma_j$. This situation is expressed by an isomorphism $\mathcal{Q}(\mathbf{D}^n)' \cong H^n_{\mathbf{D}^n}(V, \tilde{\mathcal{O}})$. It is sometimes useful to express a hyperfunction as a continuous sum (integral) of boundary values of slowly increasing holomorphic functions $F_\omega(x+iy)$ $(y \to 0)$ defined on a tube $\mathbf{R}^n + iB^n_\omega$ where $B^n_\omega = \{y \in \mathbf{R}^n; |y+\omega| < 1\}$ for $\omega \in S^{n-1} = \{y \in \mathbf{R}^n; |y| = 1\}$. In fact, S. Nagamachi and T. Nishimura [7] showed that a slowly increasing holomorphic function $U(z)$ in the tube $\mathbf{R}^n + iB^n$ where $B^n = \{y \in \mathbf{R}^n; |y| < 1\}$ defines a Fourier hyperfunction as a (continuous) sum of boundary values of slowly increasing holomorphic functions $F_\omega(z) = U(z+i\omega)$ by (2.3) and that a Fourier hyperfunction u determines a slowly increasing function $U(z)$ in $\mathbf{R}^n + iB^n$ which reproduces the Fourier hyperfunction u.

This expression was used in [7] to define an analytic wave front set of a Fourier hyperfunction and to prove the edge of the wedge theorem. In this paper, we want to show that this expression can be also applied to prove the existence of the support of a Fourier hyperfunction and to prove the kernel theorem for Fourier hyperfunctions. By this expression we show where the support of a Fourier hyperfunction lies and specify the set which carries the kernel Fourier hyperfuction.

Since a Fourier hyperfunction is a kind of analytic functional, it is not clear that it has a support, the minimum carrier. The proof in this paper is more intuitive, elementary and shorter than the one in [9]. The kernel theorem for Fourier hyperfunctions of the following form

$$\mathcal{B}(\mathcal{Q}(\mathbf{D}^n), \mathcal{Q}(\mathbf{D}^m)) \cong \mathcal{Q}(\mathbf{D}^{n+m})'$$

was known in [6], and S.Y. Chung et al [2] gave another proof for it. Y. Ito [4] proved a kernel theorem of the form

$$L(\mathcal{Q}(K), \mathcal{Q}(L)') \cong \mathcal{Q}(K)' \hat{\otimes} \mathcal{Q}(L)' \cong \mathcal{Q}(K \times L)'$$

for closed subsets $K \subset \mathbf{D}^n$, $L \subset \mathbf{D}^m$, by employing several deep results from [12] and [3]. Recently, E. Brüning and S. Nagamachi [1] gave a simple proof of the kernel theorem of the form

$$\mathcal{B}(\mathcal{Q}(K), \mathcal{Q}(L)) \cong \mathcal{Q}(K \times L)'.$$

They showed

$$\mathcal{Q}(K)\hat{\otimes}\mathcal{Q}(L) \cong \mathcal{Q}(K \times L)$$

and proved the kernel theorem using the theory of nuclear spaces, the abstract kernel theorem ([10]) and the fact that the space $\mathcal{Q}(K)$ is nuclear ([5]). The proof in this paper is constructive and more elementary than the one in [1] in the sense that the theory of nuclear spaces and the abstract kernel theorem are not used.

In Section 2, we review the theory of Fourier hyperfunctions we have developed. In Section 3, we define a set in $\mathbf{D}^n$ for each Fourier hyperfucntions u through the corresponding $F_\omega(z) = U(z+i\omega)$ and show that it is a carrier of u and that it is contained in every carrier of u. As a result we prove the existence of the minimum carrier of a Fourier hyperfucntion u and give its characterization other than the intersection of all carriers of u. In Section 4, using methods summarized or developed in previous sections, we prove a Schwartz type kernel theorem for Fourier hyperfunctions. For a seperately continuous bilinear form B on $\mathcal{Q}(K) \times \mathcal{Q}(L)$ we construct a kernel Fourier hyperfunction F in $\mathbf{D}^n \times \mathbf{D}^m$ ($K \subset \mathbf{D}^n$, $L \subset \mathbf{D}^m$) repesenting B. F is constructed as a continuous sum of boundary values of holomorphic functions in the product of cylinders $\{|\operatorname{Im} z| < 1\} \times \{\operatorname{Im} w| < 1\} \subset \mathbf{C}^n \times \mathbf{C}^m$. Then using methods developed in Section 3 we show that F has a carrier in $K \times L$. Also, in order to specify the set in $\mathbf{D}^{n+m}$ which carries F we introduce a new product set $K \times_{\mathbf{D}^{n+m}} L$ in $\mathbf{D}^{n+m}$.

2. Preliminaries

First, we give a quick review of the theory of Fourier hyperfuctions developed in [9], [8] and [7].

Let $\mathbf{D}^n = \mathbf{R}^n \cup S^{n-1}_\infty$ be the radial compactification of $\mathbf{R}^n$. It is determined so that the homeomorphism $\varphi : B^n = \{x \in \mathbf{R}^n ; |x| < 1\} \to \mathbf{R}^n$, $x \mapsto (\tan \pi|x|/2)x$ is extended to a homeomorphism $\tilde{\varphi} : \bar{B}^n = B^n \cup S^{n-1} \to \mathbf{D}^n = \mathbf{R}^n \cup S^{n-1}_\infty$. Let $\mathbf{Q}^n = \mathbf{D}^n \times i\mathbf{R}^n$ (topologically). $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$ is embedded in $\mathbf{Q}^n$. For $z = (x, iy) \in \mathbf{Q}^n$ we use the notations $\operatorname{Re} z = x$, $\operatorname{Im} z = y$ and $z = x + iy$ even if $x \in \mathbf{D}^n$ is a point at infinty. We note here that a subset K of $\mathbf{Q}^n$ is compact if and only if K is closed in $\mathbf{Q}^n$ and $\sup\{|\operatorname{Im} z| ; z \in K\} < \infty$.

Let V be an open set in $\mathbf{Q}^n$. A function f is said to be *holomorphic in V* if f is holomorphic in $V \cap \mathbf{C}^n$. f is called *rapidly decreasing* in V if for any compact set $K \subset V$ there exists $\eta > 0$ such that $f(z)e^{\eta|z|}$ is bounded in $K \cap \mathbf{C}^n$. f is said to be *slowly increasing* in V if for every compact set K in V and every $\eta > 0$ $f(z)e^{-\eta|z|}$

is bounded in $K \cap \mathbf{C}^n$. The space of all slowly increasing holomorphic functions in V is denoted by $\tilde{\mathcal{O}}(V)$. For $\eta > 0$, $\mathcal{Q}_c(V;\eta)$ denotes the space of functions which are holomorphic in $V \cap \mathbf{C}^n$, continuous on the closure of $V \cap \mathbf{C}^n$ in $\mathbf{C}^n$, and satisfy

$$\| f \|_{V,\eta} = \sup_{z \in V \cap \mathbf{C}^n} |f(z)| e^{\eta|z|} < \infty.$$

$\mathcal{Q}_c(V;\eta)$ is a Banach space with the norm $\| \cdot \|_{V,\eta}$.

Let K be a compact set in $\mathbf{D}^n$. Then $\mathcal{Q}(K)$ denotes the inductive limit of $\mathcal{Q}_c(V_m; 1/m)$, where $\{V_m\}_{m\in\mathbf{N}}$ is a fundamental system of neighbourhoods of K. An element of the dual space $\mathcal{Q}(\mathbf{D}^n)'$ is called a *Fourier hyperfunction*. There is a natural injection (embedding) $\mathcal{Q}(K)' \to \mathcal{Q}(\mathbf{D}^n)'$ and if a Fourier hyperfunction u is identified with an element of $\mathcal{Q}(K)'$, we say u is *carried by* K or K is a *carrier of* u.

By $\mathcal{Q}(\mathbf{Q}^n)$ we denote the space of all rapidly decreasing holomorphic functions f in $\mathbf{Q}^n$ (i.e., entire analytic in $\mathbf{C}^n$ and $e^{\eta|z|}f(z)$ are bounded in $|\mathrm{Im}\, z| \le r$ for any $r > 0$ and for some $\eta = \eta(r) > 0$).

The space $\mathcal{Q}(\mathbf{Q}^n)$ is a linear subsapce of $\mathcal{Q}(\mathbf{D}^n)$ and known to be a dense subspace of $\mathcal{Q}(K)$ for any compact subset K of $\mathbf{D}^n$ (Theorem 2.7 of [9]). So, $u \in \mathcal{Q}(\mathbf{D}^n)'$ can be extended to a continuous linear functional on $\mathcal{Q}(K)$ if u is continous on $\mathcal{Q}(\mathbf{Q}^n)$ with respect to the relative topology from $\mathcal{Q}(K)$. This observation gives the following lemma.

Lemma 2.1. *$u \in \mathcal{Q}(\mathbf{D}^n)'$ is carried by K if and only if*
(C): *for any neighbourhood V of K and $\eta > 0$ there exists $C = C_{V,\eta}$ such that*

$$|u(f)| \le C \| f \|_{V,\eta},$$

for every $f \in \mathcal{Q}(\mathbf{Q}^n)$.

(For some f, $\| f \|_{V,\eta}$ may be ∞ and then the above inequality holds trivially. We do not exclude such cases throughout this paper.) If a linear functional u on $\mathcal{Q}(\mathbf{Q}^n)$ satisfies (C), then it extends uniquly to a continuous linear functional on $\mathcal{Q}(K)$. (We identify u with the unique extension.) The minimum carrier of $u \in \mathcal{Q}(\mathbf{D}^n)'$ is called the *support of u*, but its existence is not trivial.

As main tools of the present paper, we summarize here some of the results in [7] on the representation of Fourier hyperfucntions by slowly increasing holomorphic functions in a domain containing the cylinder

$$\Omega = \{ z \in \mathbf{Q}^n \,;\, |\mathrm{Im}\, z| < 1 \} = \mathbf{D}^n + iB^n. \tag{2.1}$$

(N.B. In [7], Ω denoted $\Omega \cap \mathbf{C}^n = \mathbf{R}^n + iB^n$.) Let $I(\xi)$ be the function defined in $\mathbf{R}^n$

by the following integral on the unit sphere S^{n-1}

$$I(\xi) = \int_{|\omega|=1} e^{-\langle \omega, \xi \rangle} d\omega,$$

where $d\omega$ is the surface measure on S^{n-1}. Let $K(z) = K_n(z)$ be a function in $\Omega \cap \mathbf{C}^n$ defined by

$$K(z) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle z, \xi \rangle} / I(\xi)\, d\xi, \quad z \in \Omega \cap \mathbf{C}^n.$$

The integral converges for $z \in \Omega \cap \mathbf{C}^n$ and defines a holomorphic function $K(z)$ in $\Omega \cap \mathbf{C}^n$ (so, in Ω) which is analytically continued to the connected open set

$$\tilde{\Omega} = \{ z \in \mathbf{C}^n \, ; \, \langle z, z \rangle \notin (-\infty, -1] \} (\supset \Omega \cap \mathbf{C}^n).$$

Lemma 2.2 (Lemma 3.1. of [7]). *$K(z)$ is a holomorphic function in*

$$Y = \{ z \in \mathbf{C}^n \, ; \, |\mathrm{Im}\, z|^2 < 1 + |\mathrm{Re}\, z|^2 \},$$

and we have, for some $c > 0$, $K(z) = O(e^{-c|z|})$ as $z \to \infty$ in

$$Z = \left\{ z \in \mathbf{C}^n \, ; \, |\mathrm{Im}\, z| < \frac{|\mathrm{Re}\, z|}{2} \right\}.$$

REMARK 2.3. $K(z)$ is rapidly decreasing in $Y_{\mathbf{Q}^n} = \{ z \in \mathbf{Q}^n \, ; \, |\mathrm{Im}\, z|^2 < 1 + |\mathrm{Re}\, z|^2 \} = Y \cup (\mathbf{Q}^n \setminus \mathbf{C}^n)$ since every compact subset P of $Y_{\mathbf{Q}^n}$ is represented as $P = P_1 \cup P_2$ where $P_1 \cap \mathbf{C}^n \subset Z$ and P_2 is a compact subset of Y.

Proposition 2.4 (Proposition 3.3 (i) of [7]). *Let V be an open set contained in Ω. If $u \in \mathcal{Q}_c(V; \eta)'$ for every $\eta > 0$, then*

$$U(z) = K * u(z) = u(K(z - \cdot))$$

is a slowly increasing holomorphic function in any open set W in $\mathbf{Q}^n$ satisfying

$$W \cap \mathbf{C}^n \subset \{ z \in \mathbf{C}^n \, ; \, |\mathrm{Im}\, z - \mathrm{Im}\, t|^2 < 1 + |\mathrm{Re}\, z - \mathrm{Re}\, t|^2 \ \textit{for any } t \in V \cap \mathbf{C}^n \}. \tag{2.2}$$

In particular, if $u \in \mathcal{Q}(\mathbf{D}^n)'$, $U(z)$ is holomorphic in $|\mathrm{Im}\, z| < 1$ and we have

$$u(\phi) = \lim_{r \to 1-0} \int_{S^{n-1}} d\omega \int_{\mathbf{R}^n} U(x + ir\omega) \phi(x) dx. \tag{2.3}$$

DEFINITION 2.5. For a compact set K in $\mathbf{D}^n$, we define W_K to be the union of all open sets $W \subset \mathbf{Q}^n$ which satisfy (2.2) for some neighbourhood V of K in Ω.

Proposition 2.6. *Let $K \subset \mathbf{D}^n$ be compact.*
(1) *Then,*

$$\Omega \cup \{ z \in \mathbf{Q}^n ; |\operatorname{Im} z| = 1,\ \operatorname{Re} z \notin K \} \subset W_K.$$

(2) *If $u \in \underset{\sim}{\mathcal{Q}}(K)'$ and $U(z) = K * u(z) = u(K(z - \cdot))$, then U is a slowly increasing holomorphic function in W_K.*

Proof. (2) is obvious from Proposition 2.4. We prove (1). By the definition of W_K, obviously $\Omega \subset W_K$. So, assume that $z_0 = x_0 + iy_0 \in \mathbf{Q}^n$ and $|y_0| = 1$, $x_0 \notin K$ and show that $z_0 \in W_K$. We can take an open neighbourhood V_1 of K in $\mathbf{D}^n$ and an open neighbourhood W_1 of x_0 in $\mathbf{D}^n$ such that $V_1 \cap W_1 = \emptyset$ and $\delta \equiv \operatorname{dist}(W_1 \cap \mathbf{R}^n, V_1 \cap \mathbf{R}^n) > 0$. We take $\eta > 0$ small enough so that $4\eta + 4\eta^2 < \delta^2$ and form $V = V_1 \times i\{ y \in \mathbf{R}^n ; |y| < \eta \}$ as an open neighbourhood of K in $\mathbf{Q}^n$ and $W = W_1 \times i\{ y \in \mathbf{R}^n ; |y - y_0| < \eta \}$ as an open neighbourhood of z_0 in $\mathbf{Q}^n$. Now, let $z \in W \cap \mathbf{C}^n$ and $t \in V \cap \mathbf{C}^n$. Then, $|\operatorname{Im} z| < 1 + \eta$, $|\operatorname{Im} t| < \eta$, $|\operatorname{Re} z - \operatorname{Re} t| > \delta$ and, so,

$$|\operatorname{Im} z - \operatorname{Im} t|^2 < (1 + 2\eta)^2 < 1 + \delta^2 < 1 + |\operatorname{Re} z - \operatorname{Re} t|^2$$

Thus, (2.2) holds and $z_0 \in W \subset W_K$. □

Sometimes it becomes necessary to introduce $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ $(n = n_1 + n_2)$ as a compactification of $\mathbf{R}^n$ and then $\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2} (= \mathbf{D}^{n_1} \times \mathbf{D}^{n_2} \times i\mathbf{R}^n \supset \mathbf{C}^n)$. The space $\underset{\sim}{\mathcal{Q}}(K)$ and its dual $\underset{\sim}{\mathcal{Q}}(K)'$ are defined for any compact set K in $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ in the same way as for compact sets in $\mathbf{D}^n$ by exact word-for-word repetition. Takeing $K = \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ we obtain $\underset{\sim}{\mathcal{Q}}(\mathbf{D}^{n_1} \times \mathbf{D}^{n_2})$. Note that the family of sets $V_m = \{ (x, iy) \in \mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2} = \mathbf{D}^{n_1} \times \mathbf{D}^{n_2} \times i\mathbf{R}^n ; |y| < 1/m \}$, $m \in \mathbf{N}$ (resp. $\tilde{V}_m = \{ (x, iy) \in \mathbf{Q}^n ; |y| < 1/m \}$, $m \in \mathbf{N}$) constitutes a fundamental system of neighbourhoods of $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ (resp. $\mathbf{D}^n$) in $\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}$ (resp. $\mathbf{Q}^n$). Since $V_m \cap \mathbf{C}^n = \tilde{V}_m \cap \mathbf{C}^n$, it is obvious that $\underset{\sim}{\mathcal{Q}}(\mathbf{D}^n) = \underset{\sim}{\mathcal{Q}}(\mathbf{D}^{n_1} \times \mathbf{D}^{n_2})$. So, for any compact $K \subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ the space $\underset{\sim}{\mathcal{Q}}(K)'$ is naturally embedded in $\underset{\sim}{\mathcal{Q}}(\mathbf{D}^n)' = \underset{\sim}{\mathcal{Q}}(\mathbf{D}^{n_1} \times \mathbf{D}^{n_2})'$. We say K $(\subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2})$ is a carrier of a Fourier hyperfunction $u \in \underset{\sim}{\mathcal{Q}}(\mathbf{D}^n)'$ if $u \in \underset{\sim}{\mathcal{Q}}(K)'$. If $u \in \underset{\sim}{\mathcal{Q}}(\mathbf{D}^n)$ has a minimal carrier among all such $K \subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$, we call it the *support of u in $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$*.

Also, in pararell with the definition of $\underset{\sim}{\mathcal{Q}}(\mathbf{Q}^n)$, we define $\underset{\sim}{\mathcal{Q}}(\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2})$ to be the space of all the rapidly decreasing holomorphic functions in $\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}$. Of course "holomorphic functions in $\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}$" means holomorphic functions in $(\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}) \cap \mathbf{C}^n = \mathbf{C}^n$ and thus, $\underset{\sim}{\mathcal{Q}}(\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}) = \underset{\sim}{\mathcal{Q}}(\mathbf{Q}^n)$. Now, the space $\underset{\sim}{\mathcal{Q}}(\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2})$ $(= \underset{\sim}{\mathcal{Q}}(\mathbf{Q}^n))$ is proven to be dense in $\underset{\sim}{\mathcal{Q}}(K)$ for any compact $K \subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ by an argument parallel to the one in the proof of Theorem 2.7 of [9]. Thus, also, Lemma 2.1 holds for any compact $K \subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$.

DEFINITION 2.7. For a compact set K in $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$, we define $\tilde{W}_K$ to be the union of all open sets $W \subset \mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}$ satisfying (2.2) for some neighbourhood V of K in $\{ z \in \mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2} ; |\operatorname{Im} z| < 1 \}$.

We set

$$\Omega_{n_1,n_2} = \{ z \in \mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2} ; |\operatorname{Im} z| < 1 \} = \mathbf{D}^{n_1} \times \mathbf{D}^{n_2} \times i\{ y \in \mathbf{R}^n ; |y| < 1 \}.$$

It is obvious that the following proposition is proven by an argument similar to the one in the proof in Proposition 2.6.

Proposition 2.8. *Let $K \subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ be compact.*

(1) *Then,*

$$\Omega_{n_1,n_2} \cup \{ z \in \mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2} ; |\operatorname{Im} z| = 1, \ \operatorname{Re} z \notin K \} \subset \tilde{W}_K.$$

(2) *If $u \in \mathcal{Q}(K)'$ and $U(z) = K * u(z) = u(K(z - \cdot))$, then U is a slowly increasing holomorphic function in $\tilde{W}_K$.*

3. Existence of support of Fourier hyperfunctions

In this section we take an arbitrary $u \in \mathcal{Q}(\mathbf{D}^n)'$ and fix it. $U(z)$ denotes $U(z) = u(K(z - \cdot))$.

Lemma 3.1. *Let K be a compact set in $\mathbf{D}^n$ and let W be an open set in $\mathbf{Q}^n$ such that*

$$\{ z \in \mathbf{Q}^n ; |\operatorname{Im} z| < 1 \} \cup \{ z \in \mathbf{Q}^n ; |\operatorname{Im} z| = 1, \ \operatorname{Re} z \notin K) \} \subset W. \tag{3.1}$$

Assume that the function U is a slowly increasing holomorphic function in W. Let $\chi(x)$ be a continuous function in $\mathbf{D}^n$ valued in $[0, 1]$ with bounded continuous first derivatives in $\mathbf{R}^n$ which takes the value 1 on K. Let $\Gamma_\omega(\chi, \epsilon)$, $\omega \in S^{n-1}$ be the surface in $\mathbf{C}^n$ defined by

$$\Gamma_\omega(\chi, \epsilon) : x \mapsto x + i(1 - \epsilon\chi(x))\omega, \ \mathbf{R}^n \to \mathbf{C}^n,$$

where $0 < \epsilon < 1/2$. Then, we have, for $\phi \in \mathcal{Q}(\mathbf{Q}^n)$,

$$\begin{aligned} u(\phi) &= \int_{S^{n-1}} d\omega \int_{\Gamma_\omega(\chi,\epsilon)} U(z)\phi(z - i\omega) dz_1 \wedge \cdots \wedge dz_n \\ &= \int_{S^{n-1}} d\omega \int_{\mathbf{R}^n} U(x + i(1 - \epsilon\chi(x))\omega)\phi(x - i\epsilon\chi(x)\omega) J_\omega(x) dx, \end{aligned}$$

where $J_\omega(x) = J_\omega(\chi, \epsilon, x)$ *is the bounded continuous function of* (ω, x) *on* $S^{n-1} \times \mathbf{R}^n$ *given by*

$$(dx_1 - i\epsilon\omega_1 d\chi(x)) \wedge \cdots \wedge (dx_n - i\epsilon\omega_n d\chi(x)) = J_\omega(x) dx_1 \wedge \cdots \wedge dx_n. \tag{3.2}$$

Proof. Fix $0 < \epsilon < 1/2$. The following equation holds for $0 < r < 1$ by Stokes' theorem since $U(z)\phi(z - ir\omega)$ is a rapidly decreasing holomorphic function in $\{ z \in \mathbf{Q}^n ; |\operatorname{Im} z| < 1 \}$:

$$\int_{S^{n-1}} d\omega \int_{\mathbf{R}^n} U(x + ir\omega)\phi(x)\, dx = \int_{S^{n-1}} d\omega \int_{\mathbf{R}^n} U(x + i(r - \epsilon)\omega)\phi(x - i\epsilon\omega)\, dx.$$

Note that the right-hand side is a continuous function of r in $0 < r < 1 + \epsilon$. Taking the limit as $r \to 1 - 0$, we get, by (2.3),

$$u(\phi) = \int_{S^{n-1}} d\omega \int_{\Gamma_{\omega,\epsilon}} U(z)\phi(z - i\omega) dz_1 \wedge \cdots \wedge dz_n,$$

where $\Gamma_{\omega,\epsilon}$ is a surface defined by

$$\Gamma_{\omega,\epsilon} : x \mapsto x + i(1 - \epsilon)\omega, \ \ \mathbf{R}^n \to \mathbf{C}^n.$$

Owing to the way χ is taken, we have $1 - \epsilon\chi(x) < 1$ for $x \in K$ and so, $\Gamma_\omega(\chi, \epsilon)$ and $\Gamma_{\omega,\epsilon}$ are both relatively compact subsets of W. So, using Stokes' theorem again, we get the lemma. □

Lemma 3.2. *If* K *is a carrier of* u*, then*

$$\int_{S^{n-1}} U(x + i\omega) d\omega = 0, \ \ x \in \mathbf{R}^n \setminus K.$$

Proof. Without loss of generality, we have only to show the above equality for $x = 0$ assuming that $0 \notin K$. We define a function $E_t(z) \in \mathcal{Q}(\mathbf{Q}^n)$, $t > 0$ by

$$E_t(z) = \left(\frac{1}{\sqrt{\pi t}}\right)^n \exp\left\{\frac{-\langle z, z\rangle}{t}\right\}.$$

Let V be a neighbourhood of K such that $x + iy \in \overline{V}$ means $|x| > 2\delta > \delta > |y|$ (δ is a constant). Then for a small $\eta > 0$ we have $\| E_t \|_{V,\eta} \to 0$ as $t \to +0$, and so, $u(E_t) \to 0$. Take χ and ϵ of Lemma 3.1, so that χ takes the value 0 on a closed ball $\{|x| \leq 2\epsilon\}$ disjoint with K. Since for $|x| \geq 2\epsilon$, $|E_t(x - i\epsilon\chi(x)\omega)| = (1/\sqrt{\pi t})^n e^{-|x|^2/(2t)} e^{-|x|^2/(2t) + \epsilon^2\chi(x)^2/t} \leq (1/\sqrt{\pi t})^n e^{-|x|^2/(2t)} e^{-\epsilon^2/t}$ and U is slowly increasing in W_K, we have

$$u(E_t) = \int_{S^{n-1}} d\omega \int_{|x| \leq 2\epsilon} U(x + i\omega) E_t(x) dx$$

$$
\begin{aligned}
&+\int_{S^{n-1}} d\omega \int_{|x|\geq 2\epsilon} U(x+i(1-\epsilon\chi(x))\omega)E_t(x-i\epsilon\chi(x)\omega)J_\omega(x)dx \\
&\to \int_{S^{n-1}} U(i\omega)\,d\omega \qquad \text{as } t\to +0.
\end{aligned}
$$

Thus, we have shown the equality. □

Theorem 3.3. *Let X_0 be the open subset of $\mathbf{D}^n$ consisting of all $x\in\mathbf{D}^n$ satisfying the following conditions*:

(i) *$U(z)$ is a slowly increasing holomorphic function in a neighbourhood W_x of $\{\,z\in \mathbf{Q}^n\,;\,\mathrm{Re}\,z=x,\ |\,\mathrm{Im}\,z|\leq 1\,\}$ in $\mathbf{Q}^n$.*

(ii) *x has a neighbourhood T in $\mathbf{D}^n$ such that $\int_{S^{n-1}}U(x'+i\omega)d\omega=0$ for every $x'\in T\cap\mathbf{R}^n$.*

Then $K_0=\mathbf{D}^n\setminus X_0$ is a support of u.

Proof. By Proposition 2.6 and Lemma 3.2, for any carrier K of u, we have $\mathbf{D}^n\setminus K\subset X_0$ and so, $K_0\subset K$. Thus we have only to show that K_0 is a carrier of u. We take W to be the union of $\{\,z\in\mathbf{Q}^n\,;\,|\,\mathrm{Im}\,z|<1\,\}$ and all W_x in condition (i) for $x\in X_0$. Then, obviously W safisfies the condition of Lemma 3.1, for $K=K_0$ and U is a slowly increasing holomorphic function in W. Let V be an open neighbourhood of K_0 relatively compact in $\mathbf{Q}^n$. Take $\chi,\ \epsilon$ of the lemma for $K=K_0$ so that they satisfy the conditions described there and

$$
\{z\in\mathbf{C}^n;\,|\,\mathrm{Im}\,z|\leq\epsilon\chi(\mathrm{Re}\,z),\ \ \mathrm{Re}\,z\in\mathrm{supp}\,\chi\}\subset V. \tag{3.3}
$$

Then for $\phi\in\mathcal{Q}(\mathbf{Q}^n)$, we have by Lemma 3.1,

$$
u(\phi)=\int_{S^{n-1}}d\omega\int_{\mathbf{R}^n}U(x+i(1-\epsilon\chi(x))\omega)\phi(x-i\epsilon\chi(x)\omega)J_\omega(x)dx.
$$

Note that for $x\in\mathbf{R}^n\setminus\overline{V}\subset\mathbf{R}^n\setminus K_0=X_0\cap\mathbf{R}^n$ we have $\int_{S^{n-1}}U(x-i\omega)d\omega=0$ by (ii). Thus,

$$
u(\phi)=\int_{S^{n-1}\times(\mathbf{D}^n\cap\overline{V})}U(x+i(1-\epsilon\chi(x))\omega)\phi(x-i\epsilon\chi(x)\omega)J_\omega(x)dxd\omega. \tag{3.4}
$$

This equation shows that K_0 is a carrier of u. In fact, the surfaces $\Gamma_\omega(\chi,\epsilon),\ \omega\in S^{n-1}$ all lie in the compact set $\{x+iy\in\mathbf{Q}^n;|y|\leq 1-\epsilon\chi(x)\}\subset W_{K_0}$. So, for each $\eta>0$, there exists C_η such that $|U(x+i(1-\epsilon\chi(x))\omega)|\leq C_\eta e^{\eta|x|/2}, x\in\mathbf{R}^n$. Also, for $x\in\mathbf{R}^n\cap\overline{V}$, by (3.3), $x-i\epsilon\chi(x)\omega\in\overline{V}$ and so, $|\phi(x-i\epsilon\chi(x)\omega)|\leq\|\phi\|_{V,\eta}e^{-\eta|x|}$. Applying these estimates to the integrand of (3.4) we have

$$
|u(\phi)|\ \leq\ C_{1,\eta}\|\phi\|_{V,\eta},
$$

where

$$C_{1,\eta} = C_\eta \times \sup_{x\in\mathbf{D}^n,\omega\in S^{n-1}} |J_\omega(x)| \times \int_{\mathbf{D}^n} e^{-\eta|x|/2}dx.$$

Since this holds for any relatively compact open neighbourhood V in $\mathbf{Q}^n$ of K_0 and any $\eta > 0$, we conclude that K_0 is a carrier of u. □

By an argument parallel to the above one, we can prove the following theorem.

Theorem 3.4. *Let X_0 be the open subset of $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ $(n = n_1 + n_2)$ consisting of all $x \in \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ satisfying the following conditions*:
(i) *$U(z)$ is a slowly increasing holomorphic function in a neighbourhood W_x of $\{ z \in \mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}\,;\, \mathrm{Re}\, z = x,\ |\mathrm{Im}\, z| \le 1 \}$ in $\mathbf{Q}^{n_1} \times \mathbf{Q}^{n_2}$.*
(ii) *x has a neighbourhood T in $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ such that $\int_{S^n} U(x' + i\omega)d\omega = 0$ for every $x' \in T \cap \mathbf{R}^n$.*
Then $K_0 = (\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}) \setminus X_0$ is a support of u.

Proof. Lemmas similar to Lemma 3.1 and Lemma 3.2 are proven for compact sets $K \subset \mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ and $\tilde{W}_K$. Then, from these and Proposition 2.8, obviously follows the theorem. □

4. The kernel theorem for Fourier hyperfunctions

Let B be a separately continuous bilinear form on $\mathcal{Q}(L_1) \times \mathcal{Q}(L_2)$, where L_1 and L_2 are compact subsets of $\mathbf{D}^n$ and $\mathbf{D}^m$ respectively. We note that for any fixed $z \in W_{L_1}$ the function $K_n(z - \cdot)$ belongs to $\mathcal{Q}(L_1)$ and $\phi \mapsto B(K_n(z - \cdot), \phi)$ belogns to $\mathcal{Q}(L_2)'$ and in the same way, $\psi \mapsto B(\psi, K_m(w - \cdot))$ belogns to $\mathcal{Q}(L_1)'$. So, it is easy to see that the function

$$U_{n,m}(z, w) = B(K_n(z - \cdot), K_m(w - \cdot)). \tag{4.1}$$

is defined in $(W_{L_1} \times W_{L_2}) \cap \mathbf{C}^{n+m}$ and separately holomorphic by Proposition 2.6 and consequently a holomorphic function by Hartogs' theorem.

We note that by Lemma 3.2,

$$\int_{S^{n-1}} U_{n,m}(x + i\omega_1, w)d\omega_1 = 0, \quad \int_{S^{m-1}} U_{n,m}(z, y + i\omega_2)d\omega_2 = 0 \tag{4.2}$$

for $x \in \mathbf{R}^n \setminus L_1,\ y \in \mathbf{R}^m \setminus L_2$ and $z \in W_{L_1},\ w \in W_{L_2}$.

Proposition 4.1. *$U_{n,m}(z, w)$ of* (4.1) *is a slowly increasing holomorphic function in $W_{L_1} \times W_{L_2}$, that is, for any compact subset K of $W_{L_1} \times W_{L_2} \subset \mathbf{Q}^n \times \mathbf{Q}^m$, and any $\eta > 0$, the function $U_{n,m}(z, w)e^{-\eta(|z|+|w|)}$ is bounded in $K \cap \mathbf{C}^{n+m}$.*

Proof. Let $(z_0, w_0) \in W_{L_1} \times W_{L_2}$. Since $z_0 = x_0 + iy_0 \in W_{L_1}$, we can take a compact neighbourhood P of z_0 in $\mathbf{Q}^n$ and a neighourhood V_1 in $\mathbf{Q}^n$ of L_1 such that $\{ z - \zeta\,;\, z \in P \cap \mathbf{C}^n,\ \zeta \in \overline{V_1} \cap \mathbf{C}^n \}$ is a relatively compact subset of $Y_{\mathbf{Q}^n}$ of Remark 2.3. By the remark, it follows that

$$\begin{aligned} \| K_n(z - \cdot) \|_{V_1,\eta} &= \sup_{\zeta \in \overline{V_1} \cap \mathbf{C}^n} |K(z-\zeta)| e^{\eta|\zeta|} \\ &\leq \sup_{\zeta \in \overline{V_1} \cap \mathbf{C}^n} |K(z-\zeta)| e^{\eta|\zeta - z|} e^{\eta|z|} \quad \leq \quad C_\eta e^{\eta|z|}, \end{aligned}$$

for some small $\eta > 0$ and a constant $C_\eta > 0$. Similarly we can take a compact neighbourhood Q in $\mathbf{Q}^m$ of w_0 and a neighbourhood V_2 in $\mathbf{Q}^m$ of L_2 so that $\| K_m(w - \cdot) \|_{V_2,\eta} \leq C'_\eta e^{\eta|w|}$, $w \in Q \cap \mathbf{C}^m$ for small $\eta > 0$. On the other hand, since $\mathcal{Q}_c(V_1;\eta)$ and $\mathcal{Q}_c(V_2;\eta)$ are Banach spaces, the seperately continuous bilinear form B restricted on their product space is also jointly continuous (e.g., Corollary of Theorem III.9 of [11]), i.e., there exists $C = C_{V_1,V_2,\eta} > 0$ such that

$$|B(\phi_1, \phi_2)| \leq C \| \phi_1 \|_{V_1,\eta} \| \phi_2 \|_{V_2,\eta},$$

for $\phi_j \in \mathcal{Q}_c(V_j;\eta)$ $(j = 1, 2)$. Thus, combining these inequalities, we have, for $\eta > 0$,

$$|U_{m,n}(z, w)| = |B(K_n(z - \cdot), K_m(w - \cdot))| \leq C C'_\eta C_\eta e^{\eta(|z|+|w|)},$$

for $(z, w) \in (P \times Q) \cap \mathbf{C}^{n+m}$. $\square$

Proposition 4.2. *$U_{n,m}(z, w)$ of (4.1) defines a Fourier hyperfunction $F \in \underset{\sim}{\mathcal{Q}}(\mathbf{D}^{n+m})'$ through*

$$F(\phi) = \lim_{r \to 1-0} \int_{S^{n-1} \times S^{m-1}} d\omega_1 d\omega_2 \int_{\mathbf{R}^n \times \mathbf{R}^m} U_{n,m}(x + ir\omega_1, y + ir\omega_2) \phi(x, y) dx dy$$

for $\phi \in \underset{\sim}{\mathcal{Q}}(\mathbf{D}^{n+m})$.

Proof. First we prove the existence of the limit. Assume $\phi \in \mathcal{Q}_c(V_\epsilon;\eta)$ for $V_\epsilon = \{ (z, w) \in \mathbf{Q}^{n+m}\,;\, |\mathrm{Im}(z, w)| \leq 2\epsilon \}$, $\epsilon < 1/2$. Note that by Stokes' formula,

$$\begin{aligned} &\int_{S^{n-1} \times S^{m-1}} d\omega_1 d\omega_2 \int_{\mathbf{R}^n \times \mathbf{R}^m} U_{n,m}(x + ir\omega_1, y + ir\omega_2) \phi(x, y) dx dy \\ &= \int_{S^{n-1} \times S^{m-1}} d\omega_1 d\omega_2 \int_{\mathbf{R}^n \times \mathbf{R}^m} U_{n,m}(x + i(r-\epsilon)\omega_1, y + i(r-\epsilon)\omega_2) \\ &\quad \phi(x - i\epsilon\omega_1, y - i\epsilon\omega_2) dx dy \end{aligned}$$

holds for $0 < r < 1$. Since U is slowly increasing in $W_{L_1} \times W_{L_2}$, the existence of the integral is obvious and the right-hand side has values and is continuous in $0 < r <$

$1+\epsilon$. So, the limit as $r \to 1-0$ exists and equals to

$$F(\phi) = \int_{S^{n-1}\times S^{m-1}} d\omega_1 d\omega_2 \int_{\mathbf{R}^n\times\mathbf{R}^m} U_{n,m}(x+i(1-\epsilon)\omega_1, y+i(1-\epsilon)\omega_2) \phi(x-i\epsilon\omega_1, y-i\epsilon\omega_2)dxdy. \tag{4.3}$$

From the above expression and Proposition 4.1, continuity of F in $\underset{\sim}{Q}_c(V_\epsilon, \eta)$ follows easily and so, $F \in \underset{\sim}{Q}(\mathbf{D}^{n+m})'$. □

DEFINITION 4.3. For compact sets $L_j \subset \mathbf{D}^{n_j}$ $(j = 1, 2)$, we define a compact set $L_1 \times_{\mathbf{D}^{n_1+n_2}} L_2$ in $\mathbf{D}^{n_1+n_2}$ as

$$L_1 \times_{\mathbf{D}^{n_1+n_2}} L_2 = \cap\{\, \overline{(V_1 \times V_2) \cap \mathbf{R}^{n_1+n_2}}\,;\ V_j \text{ is a neighbourhood of } L_j \text{ in } \mathbf{D}^{n_j}\,\},$$

where the closures are taken in $\mathbf{D}^{n_1+n_2}$.

Lemma 4.4. *Let L_j $(j = 1, 2)$ be compact sets in $\mathbf{D}^{n_j}$ and V be a neighbourhood of $L_1 \times_{\mathbf{D}^{n_1+n_2}} L_2$ in $\mathbf{Q}^{n_1+n_2}$. Then there exist neighbourhoods V_j of L_j in $\mathbf{Q}^{n_j}$ such that*

$$(V_1 \times V_2) \cap \mathbf{C}^{n_1+n_2} \subset V.$$

Proof. This lemma follows from the fact that $L_1 \times_{\mathbf{D}^{n_1+n_2}} L_2$ is also written as $L_1 \times_{\mathbf{D}^{n_1+n_2}} L_2 = \cap\{\, \overline{(V_1 \times V_2) \cap \mathbf{C}^{n_1+n_2}}\,;\ V_j$ is a neighbourhood of L_j in $\mathbf{Q}^{n_j}\,\}$ where the closures are taken in $\mathbf{Q}^{n_1+n_2}$. □

Theorem 4.5 (Kernel Theorem). *Let L_1 and L_2 be compact sets in $\mathbf{D}^n$ and $\mathbf{D}^m$ respectively, and B a separately continuous bilinear form on $\underset{\sim}{Q}(L_1) \times \underset{\sim}{Q}(L_2)$. Then, there exists a Fourier hyperfunction F whose support in $\mathbf{D}^{n_1} \times \mathbf{D}^{n_2}$ is contained in $L_1 \times L_2$ safisfying*

$$B(\phi_1, \phi_2) = F(\phi_1 \otimes \phi_2),\ \ \phi_1 \in \underset{\sim}{Q}(L_1),\ \ \phi_2 \in \underset{\sim}{Q}(L_2). \tag{4.4}$$

The suppot of F in $\mathbf{D}^{n+m}$ is contained in $L_1 \times_{\mathbf{D}^{n+m}} L_2$.

Proof. We proceed using only test functions in $\underset{\sim}{Q}(\mathbf{Q}^l), (l = n, m, n+m)$ because these are dense in each $\underset{\sim}{Q}(K)$. Then, (4.3) holds for all $\phi \in \underset{\sim}{Q}(\mathbf{Q}^{n+m})$ and $0 < \epsilon < 1/2$. Take $\phi_1 \in \underset{\sim}{Q}(\mathbf{Q}^n)$ and $\phi_2 \in \underset{\sim}{Q}(\mathbf{Q}^m)$ and set $\phi = \phi_1 \otimes \phi_2$ in the above equation. Then we have, by (4.3),

$$F(\phi_1 \otimes \phi_2) = \int_{S^{n-1}} d\omega_1 \int_{\mathbf{R}^n} \left\{ \int_{S^{m-1}} d\omega_2 \int_{\mathbf{R}^m} \cdots dy \right\} \phi_1(x - i\epsilon\omega_1)dx.$$

Note that in the above $\left\{ \int_{S^{m-1}} d\omega_2 \int_{\mathbf{R}^m} \cdots dy \right\} = B(K_n(x + i(1-\epsilon)\omega_1 - \cdot), \phi_2) = U_{\phi_2}(x + i(1-\epsilon)\omega_1)$ where $U_{\phi_2}(z)$ is a holomorphic function in the cylinder $\{|\operatorname{Im} z| < 1\}$ which is the expression of the Fourier hyperfunction $B(\cdot, \phi_2)$. So, $F(\phi_1 \otimes \phi_2) = B(\phi_1, \phi_2)$ holds. This is sufficient for (4.4), if the statement on the support is shown.

Let V_j $(j = 1, 2)$ be arbitrary neighbourhoods of L_j relatively compact in $\mathbf{Q}^n$ and $\mathbf{Q}^m$ respectively. Take $\chi = \chi_j$, $\epsilon > 0$ which satisfy the conditions in Lemma 3.1 for $K = L_j$ so that

$$\{ z \in \mathbf{C}^n ; |\mathrm{Im}\, z| \leq \epsilon\chi_1(\mathrm{Re}\, z), \ \mathrm{Re}\, z \in \mathrm{supp}\, \chi_1 \} \subset V_1.$$

and similarly for χ_2 concerning V_2. Then we construct, as in Lemma 3.1, the surfaces $\Gamma_{\omega_j}(\chi_j, \epsilon)$ in W_{L_j}. Applying Stokes' theorem to (4.4), we have

$$\begin{aligned} F(\phi) &= \int_{S^{n-1}\times S^{m-1}} d\omega_2 d\omega_2 \int_{\Gamma_{\omega_1}(\chi_1,\epsilon)\times\Gamma_{\omega_2}(\chi_2,\epsilon)} U(z, w)\phi(z - i\omega_1, w - i\omega_2) dz_1 \wedge \cdots \wedge dw_m \\ &= \int_{S^{n-1}\times S^{m-1}} d\omega_1 d\omega_2 \int_{\mathbf{R}^{n+m}} U(x + i(1 - \epsilon\chi_1(x))\omega_1, y + i(1 - \epsilon\chi_2(y))\omega_2) \\ &\qquad \phi(x - i\epsilon\chi_1(x)\omega_1, y - i\epsilon\chi_2(y))\omega_2) J_{1,\omega_1}(x) J_{2,\omega_2}(y) dxdy, \end{aligned} \tag{4.5}$$

where $U = U_{n,m}$ of (4.1) and $J_{1,\omega_1}(x) = J_{1,\omega_1}(\chi_1, \epsilon, x)$, $J_{2,\omega_2}(y) = J_{2,\omega_2}(\chi_2, \epsilon, y)$ are bounded continuous functions of (ω_1, x), (ω_2, y) defined in $S^{n-1} \times \mathbf{R}^n$ and $S^{m-1} \times \mathbf{R}^m$ respectively by equations like (3.2). In (4.5) we can limit the range of integration in $dxdy$ to $\mathbf{R}^{n+m} \cap (\overline{V_1} \times \overline{V_2})$ by (4.2) since for $(x, y) \notin \overline{V}_1 \times \overline{V}_2$, either $\chi_1(x) = 0$ or $\chi_2(y) = 0$. Thus,

$$\begin{aligned} F(\phi) = \int_{S^{n-1}\times S^{m-1}} d\omega_1 d\omega_2 \int_{\mathbf{R}^{n+m}\cap(\overline{V}_1\times\overline{V}_2)} & U(x + i(1 - \epsilon\chi_1(x))\omega_1, y + i(1 - \epsilon\chi_2(y))\omega_2) \\ & \phi(x - i\epsilon\chi_1(x)\omega_1, y - i\epsilon\chi_2(y))\omega_2) J_{1,\omega_1}(x) J_{2,\omega_2}(y) dxdy. \end{aligned} \tag{4.6}$$

By this equation and an argument parallel to the one in the proof of Theorem 3.3, we can show that the support of F is contained in $L_1 \times L_2$ as follows. Note that since $\chi_j \not\equiv 0$ on L_j, all $\Gamma_{\omega_1}(\chi_1, \epsilon) \times \Gamma_{\omega_2}(\chi_2, \epsilon)$ are contained in a compact subset of $W_{L_1} \times W_{L_2}$ where U is slowly increasing. Thus, for each $\eta > 0$ there exists $C_\eta > 0$ such that

$$|U_{n,m}(x + i(1 - \epsilon\chi_1(x))\omega_1, y + i(1 - i\epsilon\chi_2(y))\omega_2)| \leq C_\eta e^{\eta(|x|+|y|)/2}.$$

Also, we have, for $(x, y) \in \overline{V}_1 \times \overline{V}_2$,

$$|\phi(x - i\epsilon\chi_1(x)\omega_1, y - i\epsilon\chi_2(y))\omega_2)| \leq \| \phi \|_{V_1\times V_2,\eta}\, e^{-\eta(|x|+|y|)}.$$

Then, using the above estimates, we have, for any $\eta > 0$ and for some constant $C_{0,\eta}$,

$$|F(\phi)| \leq C_{0,\eta} \| \phi \|_{V_1\times V_2,\eta}, \ \ \phi \in \mathcal{Q}(\mathbf{Q}^{n+m}).$$

The constant $C_{0,\eta}$ is given by

$$C_{0,\eta} = C_\eta \sup_{\omega_1 \in S^{n-1}, x \in \mathbf{R}^n} |J_{1,\omega_1}(x)| \times \sup_{\omega_2 \in S^{m-1}, y \in \mathbf{R}^m} |J_{2,\omega_2}(y)| \times \int_{\mathbf{R}^{n+m}} e^{-\eta(|x|+|y|)/2}.$$

This indicates $L_1 \times L_2$ is the carrier of F in $\mathbf{D}^n \times \mathbf{D}^m$ and so contains the support of F in $\mathbf{D}^n \times \mathbf{D}^m$.

For any relatively compact neighbourhood V of $L_1 \times_{\mathbf{D}^{n+m}} L_2$ in $\mathbf{Q}^{n+m}$ we can take V_j such that $(V_1 \times V_2) \cap \mathbf{C}^{n+m} \subset V$ by Lemma 4.4. So, $L_1 \times_{\mathbf{D}^{n+m}} L_2$ is a carrier of F (in $\mathbf{D}^{n+m}$). $\square$

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