

Title	A base point free theorem for log canonical surfaces
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Citation	Osaka Journal of Mathematics. 1999, 36(2), p. 337-341
Version Type	VoR
URL	<a href="https://doi.org/10.18910/4419">https://doi.org/10.18910/4419</a>
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## A BASE POINT FREE THEOREM FOR LOG CANONICAL SURFACES

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(Received August 14, 1997)

### 0. Introduction

Let  $(X, \Delta)$  be a complete, log canonical algebraic surface defined over the field of complex numbers  $\mathbf{C}$ . A nef and big Cartier divisor  $D$  on  $X$  is *nef and log big* on  $(X, \Delta)$  by definition if  $\deg(D|_C) > 0$  for all irreducible components  $C$  of the reduced part  $[\Delta]$  of  $\Delta$ .

We follow the notation and terminology of [5].

In [6] Miles Reid introduced the notion of "log big" and gave the statement as follows:

*Let  $(M, \Gamma)$  be a complete, log canonical algebraic variety over  $\mathbf{C}$  and  $L$  a nef Cartier divisor on  $M$ . Suppose that  $aL - (K_M + \Gamma)$  is nef and log big on  $(M, \Gamma)$  for some  $a \in \mathbf{N}$ . Then the linear system  $|mL|$  is free from base points for every  $m \gg 0$ .*

In this paper we give a proof to this statement in the surface case.

**Main Theorem.** *Let  $H$  be a nef Cartier divisor on  $X$  such that  $aH - (K_X + \Delta)$  is nef and log big on  $(X, \Delta)$  for some  $a \in \mathbf{N}$ . Then the complete linear system  $|mH|$  is free from base points for every sufficiently large integer  $m$ .*

REMARK 1. In the case where  $(X, \Delta)$  is a weakly kawamata log terminal projective surface, we gave a proof to the theorem above in [1].

REMARK 2. Under the assumption that  $aH - (K_X + \Delta)$  is not nef and log big on  $(X, \Delta)$  but nef and big, there exists a counterexample due to Zariski in which the theorem is not valid ([3], remark 3-1-2).

The author would like to thank the referee for pointing out the misprints and errors in the original version of the paper.

### 1. Preliminaries

First we collect some well known results concerning normal surfaces, which will be required for the proof of Main Theorem.

Let  $\mu : V \rightarrow W$  be a birational morphism between complete,  $\mathbf{Q}$ -factorial, normal algebraic surfaces over  $\mathbf{C}$ .

**Lemma 1 (Projection Formula).** *For  $\mathbf{Q}$ -divisors  $D$  on  $V$  and  $G$  on  $W$ ,  $(D, \mu^*G) = (\mu_*D, G)$ .*

**Lemma 2.** *If  $D$  is a nef  $\mathbf{Q}$ -divisor on  $V$ ,  $\mu_*D$  is also nef on  $W$ .*

Proof. For all irreducible curves  $C$  on  $W$ ,  $(\mu_*D, C) = (D, \mu^*C) \geq 0$  from Lemma 1.

**Lemma 3.** *If  $D$  is a big  $\mathbf{Q}$ -divisor on  $V$ ,  $\mu_*D$  is also big on  $W$ .*

Proof. For a Cartier divisor  $A$  on  $V$ ,  $H^0(V, \mathcal{O}_V(A)) \hookrightarrow H^0(W, \mathcal{O}_W(\mu_*A))$ , because  $V$  and  $W$  are normal.

**Lemma 4.** *Let  $A$  be a non  $\mu$ -exceptional prime divisor and  $B$  a nef  $\mathbf{Q}$ -divisor on  $V$ .  $(\mu_*A, \mu_*B) \geq (A, B)$ .*

Proof. From Lemma 1,  $(\mu_*A, \mu_*B) = (\mu^*\mu_*A, B)$ . Here  $\mu^*\mu_*A \geq A$ , because  $A$  is not  $\mu$ -exceptional. Thus  $(\mu^*\mu_*A, B) \geq (A, B)$ .

**Lemma 5.** *Every complete,  $\mathbf{Q}$ -factorial, normal algebraic surface over  $\mathbf{C}$  is projective.*

Proof. Assume that  $\mu$  is a resolution of singularities of  $W$  and  $A$  an ample divisor on  $V$ . Then  $\mu_*A$  is an ample  $\mathbf{Q}$ -divisor on  $W$  from Lemma 3 and 4 and the Nakai-Moishezon criterion.

Next we note a well known result concerning surface singularities, which will be used without mentioning it throughout this paper. For the convenience of the reader we indicate a proof, which relies on the log minimal model program.

**Proposition 0.** *If  $(X, \Delta)$  is weakly kawamata log terminal, then  $X$  is  $\mathbf{Q}$ -factorial.*

Proof. Let  $f : M \rightarrow X$  be a log resolution of  $(X, \Delta)$  such that  $K_M + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + E$  with  $E \geq 0$  and  $\text{Supp}(E) = \text{Exc}(f)$ , where  $F = \sum\{F_i; F_i \text{ is an } f\text{-exceptional prime divisor}\}$ .

Apply the relative log minimal model program to  $f : (M, f_*^{-1}\Delta + F) \rightarrow X$ .

We end up with a  $\mathbf{Q}$ -factorial weakly kawamata log terminal surface (over  $X$ )  $g : (Y, g_*^{-1}\Delta + (F)_Y) \rightarrow X$  such that  $K_Y + g_*^{-1}\Delta + (F)_Y$  is  $g$ -nef.

Because  $(E)_Y$  is a  $g$ -exceptional  $g$ -nef divisor,  $(E)_Y = 0$ . Thus  $\text{Exc}(g) = \emptyset$ . Therefore  $g$  is an isomorphism from Zariski's Main Theorem.

Lastly we mention variations of results by Kawamata and Keel-Matsuki-McKernan.

**Proposition 1.** *Suppose that  $(X, \Delta)$  is a weakly kawamata log terminal projective surface. Let  $R$  be a  $(K_X + \Delta)$ -extremal ray. Then there exists a rational curve  $C \in R$  such that  $-(K_X + \Delta).C \leq 4$ .*

Proof. For some  $r \geq 1$ ,  $r(K_X + \Delta)$  is Cartier. Let  $R$  be a  $(K_X + \Delta)$ -extremal ray. Put  $s := \min\{(\Delta, E); E \text{ is an irreducible component of } \Delta\}$ . For  $0 < \varepsilon \ll \frac{1}{(|s|+1)^r}$ ,  $K_X + (1-\varepsilon)\Delta$  is kawamata log terminal and  $R$  is a  $(K_X + (1-\varepsilon)\Delta)$ -extremal ray.

Thus, from [2], there exists a rational curve  $C \in R$  such that  $-(K_X + (1 - \varepsilon)\Delta).C \leq 4$ .

If  $\Delta$  does not include  $C$ , then  $(\Delta, C) \geq 0$ . Hence  $-(K_X + \Delta).C \leq -(K_X + (1 - \varepsilon)\Delta).C \leq 4$ .

If  $\Delta$  includes  $C$ , then  $s \leq (\Delta, C)$ . Hence  $-(K_X + \Delta).C = -(K_X + (1 - \varepsilon)\Delta).C - \varepsilon(\Delta, C) \leq 4 + (-s)\varepsilon$ . By the choice of  $\varepsilon$ ,  $-(K_X + \Delta).C \leq 4$ .

**Proposition 2.** *Suppose that  $(X, \Delta)$  is a weakly kawamata log terminal projective surface.  $D$  is a nef  $\mathbf{Q}$ -divisor, but  $K_X + \Delta$  is not nef. Set  $\lambda := \sup\{\lambda \in \mathbf{Q}; D + \lambda(K_X + \Delta) \text{ is nef}\}$ .*

*Then  $\lambda$  is a rational number and moreover there is a  $(K_X + \Delta)$ -extremal ray  $R$  such that  $(D + \lambda(K_X + \Delta)).R = 0$ .*

Proof. From Proposition 1 and [4], the proof of 2.1, the assertion follows.

### 2. Proof of the main theorem

From a result by Shokurov ([5], 17.10) (cf. [7], 9.1) and Lemma 5, we may find a weakly kawamata log terminal projective surface  $(Y, S + B)$  and a birational morphism  $g : Y \rightarrow X$  such that  $K_Y + S + B = g^*(K_X + \Delta)$ , where  $S$  is the reduced part of  $S + B$ . We note that  $g^*(aH - (K_X + \Delta))$  is nef and log big on  $(Y, g_*^{-1}[\Delta] + B)$ . In the case where  $S = g_*^{-1}[\Delta]$ , [1] implies the assertion. Thus we may assume that  $S - g_*^{-1}[\Delta] > 0$ .

We consider the following exact sequence for  $m \in \mathbf{N}$ :

$$0 \rightarrow \mathcal{O}_Y(mg^*H - (S - g_*^{-1}[\Delta])) \rightarrow \mathcal{O}_Y(mg^*H) \rightarrow \mathcal{O}_{S - g_*^{-1}[\Delta]}(mg^*H) \rightarrow 0$$

Here  $mg^*H - (S - g_*^{-1}[\Delta]) - (K_Y + g_*^{-1}[\Delta] + B) = mg^*H - (K_Y + S + B) = g^*(mH - (K_X + \Delta)) = g^*(m - a)H + g^*(aH - (K_X + \Delta))$  is nef and log big on

$(Y, g_*^{-1}[\Delta] + B)$  for  $m \geq a$ . Thus from [1],

$$H^1(Y, \mathcal{O}_Y(mg^*H - (S - g_*^{-1}[\Delta]))) = 0$$

Therefore the homomorphism

$$H^0(Y, \mathcal{O}_Y(mg^*H)) \rightarrow H^0(S - g_*^{-1}[\Delta], \mathcal{O}_{S - g_*^{-1}[\Delta]}(mg^*H))$$

is surjective. Because  $\dim g(S - g_*^{-1}[\Delta]) = 0$ ,

$$(***) \quad Bs|mg^*H| \cap (S - g_*^{-1}[\Delta]) = \emptyset$$

Now run  $(K_Y + g_*^{-1}[\Delta] + B)$ -Minimal Model Program with extremal rays that are  $g^*H$ -trivial (cf. [3], lemma 3-2-5 and [4]).

We have three cases:

Case (A). *We obtain the morphism  $p : Y \rightarrow Z$  such that  $g^*H = p^*(p_*g^*H)$ ,  $p_*g^*H$  is Cartier,  $(Z, p_*(g_*^{-1}[\Delta] + B))$  is a weakly kawamata log terminal projective surface and  $K_Z + p_*(g_*^{-1}[\Delta] + B)$  gives a non-negative function on  $\{C \in \overline{NE}(Z); (p_*g^*H, C) = 0\}$ .*

We put  $\lambda := \sup\{\lambda \in \mathbf{Q}; p_*g^*H + \lambda(K_Z + p_*(g_*^{-1}[\Delta] + B)) \text{ is nef}\}$ .

If  $K_Z + p_*(g_*^{-1}[\Delta] + B)$  is nef, then  $\lambda = \infty$ . If  $K_Z + p_*(g_*^{-1}[\Delta] + B)$  is not nef and  $\lambda = 0$ , then there exists a  $(K_Z + p_*(g_*^{-1}[\Delta] + B))$ -extremal ray  $R$  such that  $(p_*g^*H, R) = 0$  from Proposition 2, but this is a contradiction ! Thus  $\lambda > 0$ .

We note that  $m(p_*g^*H) - p_*(S - g_*^{-1}[\Delta]) = p_*(g^*(mH) - (S - g_*^{-1}[\Delta])) = K_Z + p_*(g_*^{-1}[\Delta] + B) + p_*(g^*(mH) - (K_Y + S + B)) = K_Z + p_*(g_*^{-1}[\Delta] + B) + p_*(g^*(mH - (K_X + \Delta))) = K_Z + p_*(g_*^{-1}[\Delta] + B) + p_*(g^*(aH - (K_X + \Delta))) + p_*(g^*(m - a)H)$  is nef for  $m$  such that  $m - a > \frac{1}{\lambda}$ . Here  $p_*(g^*(aH - (K_X + \Delta)))$  is nef and log big on  $(Z, p_*(g_*^{-1}[\Delta] + B))$  from Lemmas 2,3 and 4.

Thus  $|m(r(p_*g^*H) - t(p_*(S - g_*^{-1}[\Delta])))|$  is base point free for  $m \gg 0$  from [1], where  $t$  is a positive natural number such that  $t(p_*(S - g_*^{-1}[\Delta]))$  is a Cartier divisor and  $r$  is a sufficiently large prime number.

Therefore  $Bs|m(p_*g^*H)| \subseteq p_*(S - g_*^{-1}[\Delta])$  for every sufficiently large integer  $m$  (cf. [3], the proof of theorem 3-1-1). Noting the fact that  $g^*H = p^*(p_*g^*H)$ , we come to the conclusion that  $Bs|m(p_*g^*H)| = \emptyset$  from (\*\*\*) .

Case (B). *We obtain the morphism  $p : Y \rightarrow Z$ , where  $Z$  is a smooth curve and  $g^*H \sim p^*P$  for some divisor  $P$  on  $Z$ .*

If  $\deg(P) > 0$ , then  $|mP|$  is base point free for  $m \gg 0$ . Thus  $|mH|$  is base point free.

If  $\deg(P) = 0$ , then  $g^*H$  is numerically trivial. From (\*\*\*) ,  $|g^*(mH)| \neq \emptyset$  for  $m \geq a$ . Thus  $mH$  is linearly trivial.

Case (C). *We obtain the morphism  $p : Y \rightarrow Z$ , where  $Z$  is a point and  $g^*H$  is linearly trivial.*

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