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Osaka J. Math.
36 (1999), 337-341

A BASE POINT FREE THEOREM FOR LOG CANONICAL SURFACES

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(Received August 14, 1997)

0. Introduction

Let (X,Δ) be a complete, log canonical algebraic surface defined over the field of complex numbers $\mathbf{C}$. A nef and big Cartier divisor D on X is *nef and log big* on (X,Δ) by definition if $\deg(D|_C) > 0$ for all irreducible components C of the reduced part $\lfloor\Delta\rfloor$ of Δ.

We follow the notation and terminology of [5].

In [6] Miles Reid introduced the notion of "log big" and gave the statement as follows:

Let (M,Γ) be a complete, log canonical algebraic variety over $\mathbf{C}$ and L a nef Cartier divisor on M. Suppose that $aL-(K_M+\Gamma)$ is nef and log big on (M,Γ) for some $a \in \mathbf{N}$. Then the linear system $|mL|$ is free from base points for every $m \gg 0$.

In this paper we give a proof to this statement in the surface case.

Main Theorem. *Let H be a nef Cartier divisor on X such that $aH-(K_X+\Delta)$ is nef and log big on (X,Δ) for some $a \in \mathbf{N}$. Then the complete linear system $|mH|$ is free from base points for every sufficiently large integer m.*

Remark 1. In the case where (X,Δ) is a weakly kawamata log terminal projective surface, we gave a proof to the theorem above in [1].

Remark 2. Under the assumption that $aH-(K_X+\Delta)$ is not nef and log big on (X,Δ) but nef and big, there exists a counterexample due to Zariski in which the theorem is not valid ([3], remark 3-1-2).

The author would like to thank the referee for pointing out the misprints and errors in the original version of the paper.

1. Preliminaries

First we collect some well known results concerning normal surfaces, which will be required for the proof of Main Theorem.

Let $\mu : V \to W$ be a birational morphism between complete, $\mathbf{Q}$-factorial, normal algebraic surfaces over $\mathbf{C}$.

Lemma 1 (Projection Formula). *For $\mathbf{Q}$-divisors D on V and G on W, $(D, \mu^* G) = (\mu_* D, G)$.*

Lemma 2. *If D is a nef $\mathbf{Q}$-divisor on V, $\mu_* D$ is also nef on W.*

Proof. For all irreducible curves C on W, $(\mu_* D, C) = (D, \mu^* C) \geq 0$ from Lemma 1.

Lemma 3. *If D is a big $\mathbf{Q}$-divisor on V, $\mu_* D$ is also big on W.*

Proof. For a Cartier divisor A on V, $H^0(V, \mathcal{O}_V(A)) \hookrightarrow H^0(W, \mathcal{O}_W(\mu_* A))$, because V and W are normal.

Lemma 4. *Let A be a non μ-exceptional prime divisor and B a nef $\mathbf{Q}$-divisor on V. $(\mu_* A, \mu_* B) \geq (A, B)$.*

Proof. From Lemma 1, $(\mu_* A, \mu_* B) = (\mu^* \mu_* A, B)$. Here $\mu^* \mu_* A \geq A$, because A is not μ-exceptional. Thus $(\mu^* \mu_* A, B) \geq (A, B)$.

Lemma 5. *Every complete, $\mathbf{Q}$-factorial, normal algebraic surface over $\mathbf{C}$ is projective.*

Proof. Assume that μ is a resolution of singularities of W and A an ample divisor on V. Then $\mu_* A$ is an ample $\mathbf{Q}$-divisor on W from Lemma 3 and 4 and the Nakai-Moishezon criterion.

Next we note a well known result concerning surface singularities, which will be used without mentioning it throughout this paper. For the convenience of the reader we indicate a proof, which relies on the log minimal model program.

Proposition 0. *If (X, Δ) is weakly kawamata log terminal, then X is $\mathbf{Q}$-factorial.*

Proof. Let $f : M \to X$ be a log resolution of (X, Δ) such that $K_M + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + E$ with $E \geq 0$ and Supp(E)=Exc(f), where $F = \Sigma\{F_i;\quad F_i$ is an f-exceptional prime divisor $\}$.

Apply the relative log minimal model program to $f : (M, f_*^{-1}\Delta + F) \to X$.

We end up with a $\mathbf{Q}$-factorial weakly kawamata log terminal surface (over X) $g : (Y, g_*^{-1}\Delta + (F)_Y) \to X$ such that $K_Y + g_*^{-1}\Delta + (F)_Y$ is g-nef.

Because $(E)_Y$ is a g-exceptional g-nef divisor, $(E)_Y = 0$. Thus Exc(g)= $\emptyset$. Therefore g is an isomorphism from Zariski's Main Theorem.

Lastly we mention variations of results by Kawamata and Keel-Matsuki-McKernan.

Proposition 1. *Suppose that (X, Δ) is a weakly kawamata log terminal projective surface. Let R be a $(K_X + \Delta)$-extremal ray. Then there exists a rational curve $C \in R$ such that $(-(K_X + \Delta), C) \leq 4$.*

Proof. For some $r \geq 1$, $r(K_X + \Delta)$ is Cartier. Let R be a $(K_X + \Delta)$-extremal ray. Put $s := \min\{(\Delta, E);\quad E$ is an irreducible component of $\Delta\}$. For $0 < \varepsilon \ll \frac{1}{(|s|+1)^r}$, $K_X + (1-\varepsilon)\Delta$ is kawamata log terminal and R is a $(K_X + (1-\varepsilon)\Delta)$-extremal ray.

Thus, from [2], there exists a rational curve $C \in R$ such that $-(K_X + (1 - \varepsilon)\Delta).C \leq 4$.

If Δ does not include C, then $(\Delta, C) \geq 0$. Hence $-(K_X + \Delta).C \leq -(K_X + (1 - \varepsilon)\Delta).C \leq 4$.

If Δ includes C, then $s \leq (\Delta, C)$. Hence $-(K_X + \Delta).C = -(K_X + (1 - \varepsilon)\Delta).C - \varepsilon(\Delta, C) \leq 4 + (-s)\varepsilon$. By the choice of ε, $-(K_X + \Delta).C \leq 4$.

Proposition 2. *Suppose that (X, Δ) is a weakly kawamata log terminal projective surface. D is a nef* **Q***-divisor, but $K_X + \Delta$ is not nef. Set $\lambda := \sup\{\lambda \in \mathbf{Q};\quad D + \lambda(K_X + \Delta)\quad is\quad nef\}$.*

Then λ is a rational number and moreover there is a $(K_X + \Delta)$-extremal ray R such that $(D + \lambda(K_X + \Delta)).R = 0$.

Proof. From Proposition 1 and [4], the proof of 2.1, the assertion follows.

2. Proof of the main theorem

From a result by Shokurov ([5], 17.10) (cf. [7], 9.1) and Lemma 5, we may find a weakly kawamata log terminal projective surface $(Y, S + B)$ and a birational morphism $g : Y \to X$ such that $K_Y + S + B = g^*(K_X + \Delta)$, where S is the reduced part of $S + B$. We note that $g^*(aH - (K_X + \Delta))$ is nef and log big on $(Y, g_*^{-1}\lfloor\Delta\rfloor + B)$. In the case where $S = g_*^{-1}\lfloor\Delta\rfloor$, [1] implies the assertion. Thus we may assume that $S - g_*^{-1}\lfloor\Delta\rfloor > 0$.

We consider the following exact sequence for $m \in \mathbf{N}$:

$$0 \to \mathcal{O}_Y(mg^*H - (S - g_*^{-1}\lfloor\Delta\rfloor)) \to \mathcal{O}_Y(mg^*H) \to \mathcal{O}_{S - g_*^{-1}\lfloor\Delta\rfloor}(mg^*H) \to 0$$

Here $mg^*H - (S - g_*^{-1}\lfloor\Delta\rfloor) - (K_Y + g_*^{-1}\lfloor\Delta\rfloor + B) = mg^*H - (K_Y + S + B) = g^*(mH - (K_X + \Delta)) = g^*(m - a)H + g^*(aH - (K_X + \Delta))$ is nef and log big on

$(Y, g_*^{-1}\lfloor\Delta\rfloor + B)$ for $m \geq a$. Thus from [1],

$$H^1(Y, \mathcal{O}_Y(mg^*H - (S - g_*^{-1}\lfloor\Delta\rfloor))) = 0$$

Therefore the homomorphism

$$H^0(Y, \mathcal{O}_Y(mg^*H)) \to H^0(S - g_*^{-1}\lfloor\Delta\rfloor, \mathcal{O}_{S-g_*^{-1}\lfloor\Delta\rfloor}(mg^*H))$$

is surjective. Because dim $g(S - g_*^{-1}\lfloor\Delta\rfloor) = 0$,

(***) $$Bs|mg^*H| \cap (S - g_*^{-1}\lfloor\Delta\rfloor) = \emptyset$$

Now run $(K_Y + g_*^{-1}\lfloor\Delta\rfloor + B)$-Minimal Model Program with extremal rays that are g^*H-trivial (cf. [3], lemma 3-2-5 and [4]).

We have three cases:

Case (A). *We obtain the morphism $p : Y \to Z$ such that $g^*H = p^*(p_*g^*H)$, p_*g^*H is Cartier, $(Z, p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$ is a weakly kawamata log terminal projective surface and $K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B)$ gives a non-negative function on $\{C \in \overline{NE}(Z); (p_*g^*H, C) = 0\}$.*

We put $\lambda := \sup\{\lambda \in \mathbf{Q};\quad p_*g^*H + \lambda(K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$ is nef $\}$.

If $K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B)$ is nef, then $\lambda = \infty$. If $K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B)$ is not nef and $\lambda = 0$, then there exists a $(K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$-extremal ray R such that $(p_*g^*H, R) = 0$ from Proposition 2, but this is a contradiction ! Thus $\lambda > 0$.

We note that $m(p_*g^*H) - p_*(S - g_*^{-1}\lfloor\Delta\rfloor) = p_*(g^*(mH) - (S - g_*^{-1}\lfloor\Delta\rfloor)) = K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B) + p_*(g^*(mH) - (K_Y + S + B)) = K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B) + p_*(g^*(mH - (K_X + \Delta))) = K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B) + p_*(g^*(aH - (K_X + \Delta))) + p_*(g^*(m-a)H)$ is nef for m such that $m - a > \frac{1}{\lambda}$. Here $p_*(g^*(aH - (K_X + \Delta)))$ is nef and log big on $(Z, p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$ from Lemmas 2,3 and 4.

Thus $|m(r(p_*g^*H) - t(p_*(S - g_*^{-1}\lfloor\Delta\rfloor)))|$ is base point free for $m \gg 0$ from [1], where t is a positive natural number such that $t(p_*(S - g_*^{-1}\lfloor\Delta\rfloor))$ is a Cartier divisor and r is a sufficiently large prime number.

Therefore $Bs|m(p_*g^*H)| \subseteq p_*(S - g_*^{-1}\lfloor\Delta\rfloor)$ for every sufficiently large integer m (cf. [3], the proof of theorem 3-1-1). Noting the fact that $g^*H = p^*(p_*g^*H)$, we come to the conclusion that $Bs|m(p_*g^*H)| = \emptyset$ from (***).

Case (B). *We obtain the morphism $p : Y \to Z$, where Z is a smooth curve and $g^*H \sim p^*P$ for some divisor P on Z.*

If $\deg(P) > 0$, then $|mP|$ is base point free for $m \gg 0$. Thus $|mH|$ is base point free.

If $\deg(P) = 0$, then g^*H is numerically trivial. From (***), $|g^*(mH)| \neq \emptyset$ for $m \geq a$. Thus mH is linearly trivial.

Case (C). *We obtain the morphism $p : Y \to Z$, where Z is a point and g^*H is linearly trivial.*

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