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ON ALMOST QF RINGS

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0. Introduction

Let R be ring. A right R-module M is said to be small, if M is small in some right R-module containing M. Clearly, any right R-module containing a non-zero injective module is not small. Harada [5] studied rings satisfying the converse of this fact and rings dual to those. In [12], Oshiro called such rings with some chain conditions right H rings and right co-H rings, respectively, and in [13], he obtained a result that a ring R is a left H ring if and only if R is a right co-H ring. On the other hand, in [7], Harada introduced notions of almost iniective modules and almost projective modules. In [8], he defined right almost QF rings and right almost QF$^{\#}$ rings by using these notions, and proved that those rings coincide with right co-H rings and right H rings, respectively. (But it seems that there is a gap in the proof of [8, Theorem 2], because indecomposable injective modules considered in [7] (so also in [8]) are only ones which are finitely generated.) Thus we have the following result:

Theorem (Harada [8], Oshiro [13]). *For a two-sided artinian ring R, the following conditions are equivalent.*

(1) *R is a right almost* QF *ring.*
(2) *R is a left almost* QF$^{\#}$ ring.
(3) *R is a right* co-H *ring.*
(4) *R is a left* H *ring.*

In this note, we give an elementary proof for the equivalence of these four rings under a slight generalization. The proof fills the gap mentioned above.

Throughout this note we always assume R is a ring with identity and J its Jacobson radical, and unless otherwise stated, "a module" means a unitary right R-module. By M_R (${}_RM$) we stress what M is a right (left) R-module. Let M be a module. Then $L \leq M$ (resp. $L < M$) means L is a submodule of M (resp. $L \leq M$ and $L \neq M$). By Top(M), Soc(M) and $E(M)$, we denote the top, the socle and an injecitve hull of M, respectively. Assume every homomorphism always operates from opposite side of scalar. "Acc" ("dcc") means the ascending (descending) chain condition. When R satisfies acc on annihilator right (left) ideals, we briefly say that R satisfies acc-rann (acc-lann). We denote the set of primitive idempotents of R by pi(R).

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1. Almost injective modules

In [7], Harada defined "almost injective" and "almost projective" for modules over right artinian rings, and clarified the structures of such modules. In this section, we give slight generalizations of these definitions and some results in [7].

Let M and N be modules. M is called almost N-injective, if for any monomorphism $\alpha: L \to N$ and homomorphism $\psi: L \to M$, either the following (AI1) or (AI2) holds:

(AI1) There exists a homomorphism $\varphi: N \to M$ with $\psi = \varphi\alpha$.

(AI2) There exist a non-zero split epimorphism $\eta: N \to K$ and a homomorphism $\theta: M \to K$ with $\theta\psi = \eta\alpha$, where K is some module.

M is called almost injective if M is almost N-injective for any (right R-) module N.

Assume M is almost N-injective for an indecomposable module N. As note in [6], if $\psi: L \to M$ is not monic, then the case (AI2) above dose not occur, because in this case, η would be an automorphism and so $\eta\alpha$ monic.

Let M be a module. M is called completely indecomposable if its endomorphism ring $\mathrm{End}\, M$ is a local ring. A submodule N of M is called a waist in M if either $N \leq X$ or $N \geq X$ is satisfied for any submodule X of M, and M is called uniserial if every submodule of M is a waist in M.

The following lemma is immediate from the definition of almost injectives.

Lemma 1.1. *Let $M = \prod_{i \in I} M_i$ be a direct product of modules M_i ($i \in I$). Then M is almost injective if and only if so is M_i for each $i \in I$.*

Lemma 1.2 (See [7, Theorem 1]). *Let R be a semiperfect ring, and M a completely indecomposable module. Then the following are equivalent.*

(1) *M is almost injective.*

(2) (i) *M is uniform.*

(ii) *Any proper essential extension N of M is projective.*

Moreover, in this case M is a waist in $E(M)$ and $E(M)/M$ is uniserial.

Proof. All the diagrams (D1)$\sim$(D6) below signify commutative ones with exact rows.

(1)$\Rightarrow$(2). (i) Suppose $L_1 \oplus L_2 \leq M$ with $L_i \neq 0$ ($i = 1,2$). Since M is indecomposable, the homomorphism $\psi: L_1 \oplus L_2 \to M$ defined by $\psi(x_1, x_2) = x_1$ ($x_i \in L_i$) can be extended to an endomorphism φ of M as noted above. Then neither φ nor $1_M - \varphi$ is clearly an automorphism. But this is a contradiction since $\mathrm{End}\, M$

is a local ring.

(ii) Let N be a proper essential extension of M. We may put $M=L/K$, $N=P/K$, where P is projective with $K<L<P$. Let $\alpha: L \to P$ and $\psi: L \to L/K$ be canonical maps. Suppose the diagram (D1) below is satisfied with some map $\varphi: P \to L/K$. Then (D1) clearly induces (D2). This diagram implies $\alpha': L/K \to P/K$ is a split monomorphism, which contradicts indecomposability of $N=P/K$ by (i). Thus only the case (AI2) occurs. Hence we have (D3) with a projection $\eta: P \to P_1$ for some direct decomposition $P=P_1 \oplus P_2$ of P ($P_1 \neq 0$) and a map $\theta: L/K \to P_1$. Then $K \leq P_2$ since $\eta(K)=\eta\alpha(K)=\theta\psi(K)=0$. Hence $N=P/K \cong P_1 \oplus(P_2/K)$, so $N \cong P_1$ is projective.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\psi"'] & P \arrow[dl, "\varphi"] \\ & L/K & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & L/K \arrow[r, "\alpha'"] \arrow[d, "1_{L/K}"'] & P/K \arrow[dl, "\varphi'"] \\ & L/K & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\psi"'] & P \arrow[d, "\eta"] \\ & L/K \arrow[r, "\theta"] & P_1 \end{tikzcd}$$

(D1) (D2) (D3)

$(2) \Rightarrow (1)$. First we show M is a waist in $E=E(M)$. Let X be a submodule of E with $X \not\leq M$. If we put $P=M+X$, M is small in P since $M<P\leq E$ and P is indecomposable projective by (ii). Hence $P=X$ so $M<X$, which shows M is a waist in E.

Let $\alpha: L \to N$ and $\psi: L \to M$ be a monomorphism and a homomorphism, respectively. For the inclusion map $v: M \to E$, we have (D4) below with some map $\varphi: N \to E$ since E is injective. Put $U=\varphi(N)$. In the case $M \geq U$ we have (D5) naturally, and in the other case $M<U$ we have (D6) with a non-zero split epimorphism $\varphi'': N \to U$ since U is projective by (ii). Thus M is almost injective. Moreover E/M is uniserial since any essential extension V of M also satisfies (2) and so V is a waist in E.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\psi"'] & N \arrow[ddl, "\varphi"] \\ & M \arrow[d, "v"'] & \\ & E & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\psi"'] & N \arrow[dl, "\varphi'"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\psi"'] & N \arrow[d, "\varphi''"] \\ & M \arrow[r, "v'"'] & U \end{tikzcd}$$

(D4) (D5) (D6)

Corollary 1.3 (See [7, Corollary 1$^{\#}$]). *Let R be a semiprimary ring and M a* *completely indecomposable module. Then M is almost injective if and only if either the following* (1) *or* (2) *holds.*

(1) *M is injective.*

(2) *$M \cong fJ^i$ for some indecomposable injective right ideal fR ($f \in \mathrm{pi}(R)$) and some $i \geq 0$ such that fJ^j is projective for any j with $0 \leq j < i$.*

We call a module M satisfying the above condition (2) i-almost injective. Note in the condition (2), $E(M) \cong fR$ and i is uniquely determined by M.

Proposition 1.4. *Let R be a right artinian ring and M a finitely gererated module. Then M is almost injective if and only if M is almost L-injective for any finitely generated module L.*

Proof. By Lemma 1.1, it suffices to show "if" part in case M is indecomposable. Assume M is indecomposable and almost L-injective for any finitely generated module L. Then from the proof of Lemma 1.2, it follows

(i) M is uniform.

(ii′) Any finitely generated submodule X of $E(M)$ with $M < X$ is projective and so $|X| \leq |R|$ since X is indecomposable projective by (i), where $|X|$ denotes the composition length of X.

This shows $|E(M)| \leq |R| < \infty$, so the condition (2) in Lemma 1.2 is satisfied. Thus M is almost injective.

This proposition shows our definition of "almost injective" for finitely generated modules generalizes one in [7].

Almost projective modules are defined as dual to almost injective modules. But for the sake of completeness we give its definition and a proof of a lemma dual to Lemma 1.2.

Let M and N be modules. M is called almost N-projective, if for any epimorphism $\alpha: N \to L$ and homomorphism $\psi: M \to L$, either the following (AP1) or (AP2) holds:

(AP1) There exists a homomorphism $\varphi: M \to N$ such that $\psi = \alpha\varphi$.

(AP2) There exist a non-zero split monomorphism $\eta: K \to N$ and a homomorphism $\theta: K \to M$ such that $\psi\theta = \alpha\eta$, where K is some module.

M is called almost projective if M is almost N-projective for any (right R-) module N. An epimorphism $\pi: N \to M$ is called a small cover of M if $\mathrm{Ker}\,\pi$ is small in N ([12]). We call a module M local if M has the largest proper submodule.

Lemma 1.1′. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of modules M_i ($i \in I$). Then M is almost projective if and only if so is M_i for each $i \in I$.*

Lemma 1.2′ (See [7, Theorem 1]). *Let R be a right perfect ring, and M a completely indecomposable module. Then the following are equivalent.*

(1) *M is almost projective.*

(2) (i) *M is local.*

(ii) *For any proper small cover $N \to M$ of M, N is injective.*

Moreover, in this case, if $\pi: P \to M$ is a projective cover of M with $K = \operatorname{Ker} \pi$, then K is a uniserial waist in P.

Proof. All the diagrams (D1′)~(D6′) below signify commutative ones with exact rows.

(1)$\Rightarrow$(2). (i) Suppose M is not local. Then we have $M = L_1 + L_2$ and $\bar{M} = \bar{L}_1 \oplus \bar{L}_2$ for some modules L_i with $MJ < L_i < M$, where $\bar{M} = M/MJ$ and $\bar{L}_i = L_i/MJ$ ($i = 1,2$). If $\psi: M \to \bar{L}_1 \oplus \bar{L}_2$ is a composition map $M \to (\bar{M} =)\ \bar{L}_1 \oplus \bar{L}_2 \to \bar{L}_1 \to \bar{L}_1 \oplus \bar{L}_2$ of canonical maps and $\alpha: M \to (\bar{M} =)\ \bar{L}_1 \oplus \bar{L}_2$ is a canonical epimorphism, then ψ can be lifted to an endomorphism φ of M with $\psi = \alpha\varphi$. Then neither φ nor $1_M - \varphi$ is clearly an automorphism, which is a contradiction.

(ii) Let $M = N/K$ for modules N and K such that K is a non-zero small submodule of N. Since M is local by (i) and K is small in N, N is also local. Let $\psi: N/K \to E/K$ and $\alpha: E \to E/K$ be canonical maps, where $E = E(N)$. Suppose (D1′) below is satisfied with some map $\varphi: N/K \to E$. Then (D1′) clearly induces (D2′). This diagram implies $\alpha': N \to N/K$ is a split epimorphism, which is a contradiction. Thus we have (D3′) with an injection $\eta: E_1 \to E$ for some direct decomposition $E = E_1 \oplus E_2$ of E ($E_1 \neq 0$) and a map $\theta: E_1 \to N/K$. Then $E_1 \leq N$ since $(E_1 + K)/K = \alpha\eta(E_1) = \psi\theta(E_1) \leq N/K$. Hence $N = (E_1 \oplus E_2) \cap N = E_1 \oplus (E_2 \cap N)$, so $N = E_1$ is injective.

$$\begin{tikzcd} & N/K \arrow[dl, "\varphi"'] \arrow[d, "\psi"] & \\ E \arrow[r, "\alpha"'] & E/K \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} & N/K \arrow[dl, "\varphi'"'] \arrow[d, "1_{N/K}"] & \\ N \arrow[r, "\alpha'"'] & N/K \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} E_1 \arrow[r, "\theta"] \arrow[d, "\eta"'] & N/K \arrow[d, "\psi"] & \\ E \arrow[r, "\alpha"'] & E/K \arrow[r] & 0 \end{tikzcd}$$

(D1′) (D2′) (D3′)

(2)$\Rightarrow$(1). Let $\mu: P \to M$ be a projective cover of M with $\operatorname{Ker} \mu = K$. We may assume $M = P/K$. We show K is a waist in P. Let X be a submodule of P with $K \nleq X$. Since P is local and $K \cap X < K$, by the assumption $P/(K \cap X)$ is indecomposable injective, in particular uniform. Hence $X/(K \cap X) = 0$ so $X < K$, which implies K is a waist in P.

Let $\alpha: N \to L$ and $\psi: M \to L$ be an epimorphism and a homomophism, respectively. Then we have (D4′) below with some map $\varphi: P \to N$. Put $U = \operatorname{Ker} \varphi$. In the case $K \leq U$ (i.e. $\operatorname{Ker} \mu \leq \operatorname{Ker} \varphi$) we have (D5′), and in the other case $K > U$ we have (D6′) with a non-zero split monomorphism $\varphi'' : P/U \to N$ since P/U is injective by the (ii). Thus M is almost projective. If V is a submodule of K, P/V also satisfies (2) and so V is a waist in P. This shows K is uniserial.

$$\begin{tikzcd} & P \arrow[d, "\mu"] \arrow[ddl, "\varphi"'] & \\ & M \arrow[d, "\psi"] & \\ N \arrow[r, "\alpha"'] & L \arrow[r] & 0 \end{tikzcd}$$

(D4′)

$$\begin{tikzcd} & M \arrow[d, "\psi"] \arrow[dl, "\varphi'"'] & \\ N \arrow[r, "\alpha"'] & L \arrow[r] & 0 \end{tikzcd}$$

(D5′)

$$\begin{tikzcd} P/U \arrow[r, "\mu'"] \arrow[d, "\varphi''"'] & M \arrow[d, "\psi"] & \\ N \arrow[r, "\alpha"'] & L \arrow[r] & 0 \end{tikzcd}$$

(D6′)

Corollary 1.3′ (See [7, Corollary 1]). *Let R be a semiprimary ring and M a completely indecomposable module. Then M is almost projective if and only if either the following* (1) *or* (2) *holds.*

(1) *M is projective.*

(2) *$M \cong fR/\mathrm{Soc}_i(fR)$ for some indecomposable injective right ideal fR ($f \in \mathrm{pi}(R)$) and some $i \geq 0$ such that $fR/\mathrm{Soc}_j(fR)$ is injective for any j with $0 \leq j < i$.*

We call a module M satisfying the above condition (2) *i-almost projective.* Note in the condition (2), the natural map $fR \to M$ is a projective cover of M and i is uniquely determined by M.

2. Equivalence of right almost QF rings and left almost co-QF rings

Lemma 2.1 (See Fuller [3, Lemmas 2.1 and 2.2]). *Let R be a semiprimary ring and e an idempotent of R, and put ${}_RE = E(\mathrm{Top}({}_RRe))$ and $H = eRe$. Then the following hold.*

(1) *$r_E(K)/r_E(I) \cong \mathrm{Hom}_H(eI/eK, eE)$ (as left R-modules) for any two-sided ideals K and I of R with $K \leq I$, where $r_E(K) = \{x \in E \mid Kx = 0\}$.*

(2) *$eE \cong E(\mathrm{Top}({}_HH))$ (as left H-modules).*

Lemma 2.2. *Let R be a semiprimary ring and $e \in \mathrm{pi}(R)$, and put ${}_RE = E(\mathrm{Top}({}_RRe))$. Then for any non-negative integer i, the following hold.*

(1) *eJ_R^i is projective if and only if ${}_RE/\mathrm{Soc}_i(E)$ is injecitive.*

(2) *$\mathrm{Top}(eJ_R^i)g \neq 0$ if and only if $g\,\mathrm{Soc}({}_RE/\mathrm{Soc}_i(E)) \neq 0$, where $g \in \mathrm{pi}(R)$.*

Proof. (1) Put $H = eRe$. By Lemma 2.1 (1), for any two-sided ideal I of R and any left R-module M we have ${}_RE/r_E(I) \cong {}_R\mathrm{Hom}_H(eI, eE)$, so $\mathrm{Hom}_R(M, E/r_E(I)) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_H(eI, eE)) \cong \mathrm{Hom}_H(eI \otimes_R M, eE))$ (as abelian groups). Since by

Lemma 2.1 (2) ${}_HeE$ is a injective cogenerator, eI_R is projective if and only if ${}_RE/r_E(I)$ is injective. This implies (1) by taking J^i as I in it.

(2) By taking $I=J^i$, $K=J^{i+1}$ in Lemma 2.1 (1), we have ${}_R\mathrm{Soc}_{i+1}(E)/\mathrm{Soc}_i(E) \cong {}_R\mathrm{Hom}_H(eJ^i/eJ^{i+1},eE)$. This shows (2) since ${}_HeE$ is a cogenerator.

Proposition 2.3. *Let R be a semiprimary ring which satisfies acc-rann and g a primitive idempotent of R. Then gR_R is i-almost injective if and only if ${}_RE(\mathrm{Top}(Rg))$ is i-almost projective.*

Proof. If eR_R (resp. ${}_RRf$) is injective, there exists an injective module ${}_RRf$ (resp. eR_R) such that $eR\cong E(\mathrm{Top}(fR))$ and $Rf\cong E(\mathrm{Top}(Re))$ by [1], where $e,f\in \mathrm{pi}(R)$. Hence let eR and Rf be such injective modules. By Lemma 2.2 (1) for any $j\geq 0$, it holds that eJ_R^j is projective if and only if ${}_RRf/\mathrm{Soc}_j(Rf)$ is injective. Moreover by Lemma 2.2 (2), $eJ_R^i\cong gR_R$ if and only if ${}_RRf/\mathrm{Soc}_i(Rf) \cong {}_RE(\mathrm{Top}(Rg))$. Therefore the assertion follows from Corollaries 1.3 and 1.3′.

Let R be a semiprimary ring. Then R is called a right almost QF ring if R satisfies acc-rann and every indecomposable projective right R-module is almost injective, and dually R is called a right almost co-QF ring if R satisfies dcc-rann (or equivalently acc-lann) and every indecomposable injective right R-module is almost projective. (Harada [8] called the rings dual to right almost QF rings right almost QF$^{\#}$ rings.)

The following theorem is an immediate consequence of Proposition 2.3.

Theorem 2.4. *Let R be a ring. The following conditions are equivalent.*

(1) *R is a right almost* QF *ring.*

(2) *R is a left almost* co-QF *ring.*

Let M_R be a module and $H=\mathrm{End}\,M$. Then we can consider M as an H-R-bimodule ${}_HM_R$. Put $\mathrm{Ar}(H,M)=\{r_M(X)\mid X\subseteq H\}$ and $\mathrm{Ar}(M,R)=\{r_R(Y)\mid Y \subseteq M\}$, where $r_M(X)=\{m\in M\mid Xm=0\}$ and $r_R(Y)=\{a\in R\mid Ya=0\}$ ([2]).

We recall a module M quasi-injective if for any submodule N of M, any homomorphism $N\rightarrow M$ can be extended to some endomorphism of M. By [9, Theorem 1.1], M is quasi-injective if and only if $SM=M$, where $S=\mathrm{End}\,E(M)$. Hence in case fR is injective ($f\in \mathrm{pi}(R)$), fK is quasi-injective for any two-sided ideal K of R.

Lemma 2.5 (Harada-Ishii [4, Theorem 1]). *Let M_R be quasi-injective and $H=\mathrm{End}\,M$. If the lattice* $\mathrm{Ar}(H,M)$ *satisfies* acc, *then H is semiprimary.*

Let R be a ring satisfying acc-rann and fR an injective right ideal; $f\in \mathrm{pi}(R)$. Then $\mathrm{End} fK$ is semiprimary for any two-sided ideal K of R since fRf

is semiprimary by Lemma 2.5 and there exists a surjective ring homomorphism $fRf \to \mathrm{End} fK$.

Lemma 2.6 (See Johnson-Wong [9, Corollary 2.2]). *Let M_R be a quasi-injective and $H = \mathrm{End}\, M$. If the lattice* $\mathrm{Ar}(M,R)$ *satisfies* dcc *and* acc, *then ${}_H M$ has a finite composition length.*

REMARK 1. Let M_R is a quasi-injective module and $\bar{R} = R / r_R(M)$, and assume $\mathrm{Ar}(M,R)$ satisfies dcc. Since $M_{\bar{R}}$ is faithful and $\mathrm{Ar}(M,\bar{R})$ is isomorphic to $\mathrm{Ar}(M,R)$ (as lattices), $\bar{R}_{\bar{R}}$ is embedded in a finite direct sum $M^{(n)}$ of copies of $M_{\bar{R}}$. Then M is injective as a right $\bar{R}$-module (see the proof of [2, Corollary 5.6A]), and so $\mathrm{Ar}(M,R) \cong \mathrm{Ar}(M,\bar{R})$ satisfies acc by [11, Theorem 1.4]. Therefore in the above lemma, the condition "acc" is superfluous.

REMARK 2. Let E be an injective module. By the torsion theory, it is well-known all lengths of maximal chains of right ideals in $\mathrm{Ar}(E,R)$ are same if there exist such chains (see e.g. [16, Lemma 1]). In case R is semiprimary and $\mathrm{Ar}(E,R)$ satisfies acc, $\mathrm{Ar}(E,R)$ has a maximal chain since for any right ideal I, $r_R l_M(I) / I$ is torsion (i.e. $\mathrm{Hom}_R(r_R l_M(I) / I, E) = 0$). This shows that if R is semiprimary and $\mathrm{Ar}(E,R)$ satisfies acc, then $\mathrm{Ar}(E,R)$ satisfies dcc.

We call an idempotent e of R local if eRe is a local ring. It is well-known that any completely indecomposable projective module is local, and any uniform quasi-injective modules is completely indecomposable. Hence if P is a uniform quasi-injective projective module, then P is a local projective module and $P \cong eR$ for some local idempotent e of R.

Proposition 2.7 (Cf. [13, Proposition 3.2]). *Let R be a ring satisfying* acc-rann. *If for any primitive idempotent e, eR is uniform and any essential extension of eR is projective, then R is left artinian.*

Proof. By the assumption, there exists a complete set of orthogonal primitive idempotents $e_1, e_2, \cdots, e_n$ of R. If $e'R$ is local projective ($e' \in \mathrm{pi}(R)$), we have $e'R \cong e_i R$ for some i; $1 \le i \le n$ since there exists an epimorphism $e_i R \to e'R$. Therefore there is only a finite number of isomorphic classes of local projective. Let $e \in \mathrm{pi}(R)$. Then $E(eR) \cong fR$ for some local idempotent f. Let I be a right ideal with $R \cong fI \le fR$. We claim $eR \cong fI = fJ^k$ for some $k \ge 0$. If $fI < fR$, then $fI \le fJ$ since fR is local. By the assumption fJ is uniform quasi-injective and projective because of $eR \cong fI \le fJ < fR = E(fI)$. Hence fJ is local projective. Thus in the case $fI < fJ$, $fI \le fJ^2$ holds and fJ^2 is local projective by the same argument as above. Suppose there exists a chain of local projective right ideals fJ^i containing fI with infinite length: $fR > fJ > fJ^2 > \cdots > fI$. Then as mentioned above, it is impossible that

$(fJ^i \mid i \geq 0)$ consists of the member of non-isomorphic each other. Therefore $fJ^j \cong gR$, $gR \cong gJ^m$ for some $j,m \geq 1$ and some local idempotent g. Then we have $gR \cong (gJ^m)^t \leq (gJ)^t$ for any $t \geq 1$. But $gRg \cong \mathrm{End} fJ^j$ is semiprimary and so $(gJ)^s = 0$ for some $s \geq 1$, which is a contradiction. Thus as required, $eR \cong fI = fJ^k$ is satisfied for some k. Hence eR is quasi-injective and eRe is semiprimary, and in particular R is semiprimary. Put $E = E(R_R)$. Since E is faithful projective by the assumption, $\mathrm{Ar}(E,R) = \mathrm{Ar}(R,R)$ satisfies acc. But R is semiprimary, so $\mathrm{Ar}(R,R) = \mathrm{Ar}(E,R)$ satisfies dcc (see Remark 2). Hence $\mathrm{Ar}(eR,R)$ satisfies dcc and acc. If follows from Lemma 2.6 that eR is an artinian left eRe-module for every $e \in \mathrm{pi}(R)$, and so R is a left artinian ring.

3. Equivalence of right almost QF rings and right co-H rings

From [15] and [10], we recall definitions of cosmall modules and small modules. A module M is said to be cosmall if there exists a short exact sequence $0 \to L \to N \to M \to 0$ of modules such that L is an essential submodule of N ([15]), and dually M small if there exists a module N containing M as a small submodule ([10]). Moreover, M is said to be non-cosmall (resp. non-small) if M is not cosmall (resp. not small).

The following lemmas are seen easily from the definition of cosmall modules.

Lemma 3.1 (Rayar [15]). *For a module M, the following are equivalent.*

(1) *M is cosmall.*

(2) *If $M \cong P/K$ with a projective module P and its submodule K, then K is an essential submodule of P.*

Lemma 3.2. *For modules M and M_i $(i \in I)$, the following holds.*

(1) (Harada [5, Lemma 3.1]) *If M is cosmall, then for any short exact sequence $0 \to L \to M \to N \to 0$, L and N are cosmall.*

(2) *If M_i is cosmall for each $i \in I$, then $\bigoplus_{i \in I} M_i$ is cosmall.*

(i.e. The class of cosmall modules is closed under taking submodules, factor modules and direct sums.)

For a module M, we denote its singular submodule by $Z(M)$: that is, $Z(M) = \{m \in M \mid r_R(m)$ is an essential submodule of $R_R\}$.

The following corollary is an immediate consequence of Lemmas 3.1 and 3.2.

Corollary 3.3 (Rayar [15, Proposition 2.4]). *For a module M, the following hold.*

(1) *$Z(M) = \{m \in M \mid mR$ is cosmall$\}$.*

(2) *M is cosmall if and only if $M = Z(M)$.*

Lemma 3.1′ (Leonard [10, Theorem 1]). *For a module M, the following are equivalent.*

(1) *M is small.*

(2) *M is small in $E(M)$.*

(3) *M is small in E for any injective module E with $M \leq E$.*

Lemma 3.2′. *For modules M and M_i $(i \in I)$, the following hold.*

(1) (Harada [5, Lemma 1.1]) *If M is small, then for any short exact sequence $0 \to L \to M \to N \to 0$, L and N are small.*

(2) *Assume R is a right perfect ring. If M_i is small for each $i \in I$, then $\bigoplus_{i \in I} M_i$ is small.*

Put $Z^*(M) = \{m \in M \mid mR$ is small$\}$ for a module M ([5]).

Corollary 3.3′ (Harada [5]). *Let R be a right perfect ring. Then for a module M, M is small if and only if $M = Z^*(M)$.*

We consider the following conditions (H) and (H′) for a ring R, which were introduced by Harada [5].

(H) If M is a non-small module, there exists an monomorphism $E \to M$ for some non-zero injective module E.

(H′) If M is a non-cosmall module, there exists an epimorphism $M \to P$ for some non-zero projective module P.

In [12] Oshiro called a right artinian ring satisfying (H) a right H ring, and called a ring satisfying (H′) together with acc-rann a right co-H ring, and in [13] he showed left H rings and right co-H ring are equivalent.

On the other hand, Harada [8] showed, for any artinian rings, equivalences of right almost QF rings and right co-H rings, and of right almost co-QF rings and right H rings.

In this note we define a right H ring and a right co-H ring as following:

A ring R is called a right H ring (resp. a right co-H ring) if R is a semiprimary ring satisfying the condition (H) and dcc-rann (resp. the condition (H′) and acc-rann). (These definitions are slightly different from the original ones by Oshiro [12] or [13], but they are equivalent as seen from [13, Proposition 3.2] and Theorem 3.6 below).

Proposition 3.4. *Let R be a ring with a complete set of orthogonal primitive idempotents. Then for a ring R, the following statements are equivalent.*

(1) *For any primitive idempotent e, eR is uniform, and any essential extension of eR is projective.*

(2) *R satisfies the condition* (H′).

Proof. (1)$\Rightarrow$(2). Let M be a non-cosmall module. Then $M \neq Z(M)$ by Corollary 3.3. Hence $mg \in M$ and mgR is non-cosmall for some $m \in M$ and $g \in \mathrm{pi}(R)$. Since we have an exact sequence $0 \to r_{gR}(m) \to gR \to mgR \to 0$, $r_{gR}(m)$ is not essential in gR. But gR is uniform, which implies $r_{gR}(m)=0$, and consequently $mgR \cong gR$. The inclusion map $mgR \to E(mgR)$ is extended to some homomorphism $\varphi: M \to E(mgR)$. Then $\mathrm{Im}\,\varphi$ is a non-zero projective module because of $gR \cong mgR \leq \mathrm{Im}\,\varphi \leq E(mgR)$. Thus R satisfies (H′).

(2)$\Rightarrow$(1). Let $e \in \mathrm{pi}(R)$. Suppose eR is not uniform. Then we have $L_1 \oplus L_2 \leq eR$ for some modules L_i with $0 \neq L_i < eR$ $(i=1,2)$. Since eR/L_1 is non-cosmall by Lemma 3.1, there exists an epimorphism $eR/L_1 \to P$ for some non-zero projective module P. Hence by this epimorphism we have an exact sequence $0 \to K \to eR \to P \to 0$ for some module K with $eR > K \geq L_1 > 0$. But since P is projective, the sequence splits, which contradicts indecomposability of eR. Thus eR is uniform. On the other hand every non-zero projective module is non-cosmall by Lemma 3.1. Hence if N is an essential extension of eR, N is uniform and non-cosmall by Lemma 3.2 (1). Therefore N is projective by the assumption.

REMARK 3. Let R be a semiperfect ring. Then it follows from Lemmas 1.1 and 1.2 and Proposition 3.4 that R_R is almost inhjective if and only if R satisfies the condition (H′).

Theorem 3.5 (See [8, Theorem 1]). *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is a right almost* QF *ring.*

(2) *R is a right* co-H *ring.*

Moreover, in this case R is left artinian.

Proof. This follows from Lemma 1.2, and Propositions 3.4 and 2.7.

Theorem 3.5′ (See [8, Theorem 2]). *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is a right almost* co-QF *ring.*

(2) *R is a right* H *ring.*

Proof. (1)$\Rightarrow$(2). By Theorem 2.4, Lemma 1.2 and Proposition 2.7, R is right artinian. Let M be a non-small module. Then by Corollary 3.3′ $M \neq Z^*(M)$. Hence $mg \in M$ and mgR is non-small for some $m \in M$ and $g \in \mathrm{pi}(R)$. If we put $E=E(mgR)$, then mgR is not small in E and so $mgR \not\leq EJ$. Since mgR has a finite composition length, there is a decomposition $E=E_1 \oplus \cdots \oplus E_n$ of E into some indecomposable injective modules E_i $(1 \leq i \leq n)$. By Lemma 1.2′ E_i is local for each i. Thus for some j, the canonical map $\pi_j: mgR \to E_j$ is an epimorphism. Since E_j is injective and almost projective and mgR is local, mgR is injective by Lemma

1.2′. This shows R is a right H ring.

$(2)\Rightarrow(1)$. Let E be an indecomposable injective module. We claim E is local. Suppose E is not local. Then $E/EJ=(L_1/EJ)\oplus(L_2/EJ)$ for some modules L_i with $EJ<L_i<E$ $(i=1,2)$. L_1 is clearly uniform. Since $E=L_1+L_2$, L_1 is non-small by Lemma 3.1′. Hence by the assumption L_1 is injective which contradicts indecomposability of E. Thus E is local. Let $\pi: N\rightarrow E$ be a small cover of E. Since E is local and non-small, so is N. Therefore N is injective, which shows R is a right almost co-QF ring by Lemma 1.2′.

Thus we have the following theorem as an immediate consequence of Theorems 3.5 and 3.5′ (Note that Oshiro [13,14] showed that any left H ring is two-sided artinian).

Theorem 3.6 (See Harada [8, Theorems 1 and 2] and Oshiro [13]). *For a ring R, the following conditions are equivalent.*

(1) *R is a right almost* QF *ring.*

(2) *R is a left almost* co-QF *ring.*

(3) *R is a right* co-H *ring.*

(4) *R is a left* H *ring.*

Moreover, in this case R is left artinian.

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