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On the Continuous Function Defined on a Sphere

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1. Prof. S. Kakutani proposed the following problems¹⁾: Given a bounded convex body in an $(n+1)$ -dimensional Euclidean space R^{n+1} , is it always possible to find a circumscribing cube around it? Or more generally if $f(x)$ is a real-valued continuous function on an n -dimensional sphere S^n with the center o , then is it possible to find $(n+1)$ points x_0, x_1, \dots, x_n on S^n perpendicular to one another (which means that the vectors ox_0, ox_1, \dots, ox_n are perpendicular to one another), such that

$$f(x_0) = f(x_1) = \dots = f(x_n) ? \quad (1)$$

The purpose of this paper is to answer these questions in the affirmative.

2. Let R^{m+1} be an $(m+1)$ -dimensional Euclidean space with the origin o and with the rectangular co-ordinate axes $oe_1, oe_2, \dots, oe_{m+1}$. The cartesian co-ordinates of a point p will be denoted by $(p^1, p^2, \dots, p^{m+1})$ and the distance from o by $\|p\|$. Let us consider the concentric spheres whose common center coincides with o and whose radius runs over the interval $[1, 2]$. We denote the aggregate of these concentric spheres by C .

Now we shall prove the following

LEMMA A'. Let S_0 and S_1 be the set $\{p; \|p\| = 1\}$ and $\{p; \|p\| = 2\}$ respectively. If L is a closed set in C which intersects any continuous curve joining S_0 and S_1 , then L contains $(m+1)$ points q_0, q_1, \dots, q_m such that

$$\|q_0\| = \|q_1\| = \dots = \|q_m\| \quad (2)$$

and oq_0, oq_1, \dots, oq_m are perpendicular to one another.

Proof. If $m=0$, the lemma is evidently true. Let us assume that this lemma is true when the dimension of the space $< m+1$. We can easily find in an ε -neighbourhood²⁾ of L an closed m -dimension manifold L_ε ³⁾ which also intersects any continuous curve joining

¹⁾ S. Kakutani: "Circumscribing cube around convex body". Annals of Math. vol. 43.

²⁾ ε -neighbourhood is the set of points whose distances from L are $< \varepsilon$.

³⁾ If such a L_ε does not exist we can draw a curve from S_0 to S_1 which does not intersect L .

S_0 and S_1 . Clearly there are two points $p(1)$, $p(0)$ in L such that

$$\begin{aligned} \sup_{p \in L_\varepsilon} \|p\| &= \|p(1)\| \\ \inf_{p \in L_\varepsilon} \|p\| &= \|p(0)\|. \end{aligned} \quad (3)$$

Now we join $p(1)$ and $p(0)$ by a curve $p(\tau)$ ($0 \leq \tau \leq 1$) on L_ε . For every $p(\tau)$ we can easily determine a rotation of axes ρ_τ such that

$$\rho_\tau p(\tau) \in oe_{m+1},^4)$$

and such that ρ_τ is a continuous function of τ . Let π be the hyperplane $\{p; p^{m+1} = 0\}$, and let $H^{m-1}(p)$ be the aggregate of q such that

$$\|q\| = \|p\|,$$

and such that oq is perpendicular to op . Then $\rho_\tau H^{m-1}(p(\tau))$ are all in π . Let C' be the intersection of π and C , and let S'_0 and S'_1 be $S_0 \cap \pi$ and $S_1 \cap \pi$ respectively. To a point $y \in C'$ corresponds a point $x \in C$ in such a way that

$$\begin{aligned} x &= \rho_\tau^{-1} \{o y \cap \rho_\tau H^{m-1}(x(t))\} \\ &\equiv \varphi(y), \end{aligned}$$

where $\tau = \|y\| - 1$.

This mapping $\varphi(y)$ is continuous; moreover if $\|y\| = \|z\|$, then $\|\varphi(y)\| = \|\varphi(z)\|$; and if oy is perpendicular to oz , then so is $o\varphi(y)$ to $o\varphi(z)$.

The closed set $\varphi^{-1}(L_\varepsilon)$ intersects any continuous curve γ joining S'_0 and S'_1 in C' because $\varphi(S'_0)$ and (S'_1) are joined by $\varphi(\gamma)$, the former of them being inside or on L_ε , the latter being outside or on L_ε . Therefore by the assumption $\varphi^{-1}(L_\varepsilon)$ contains m points r'_1, r'_2, \dots, r'_m such that

$$\|r'_1\| = \|r'_2\| = \dots = \|r'_m\| = \tau_0 + 1,$$

and $or'_1, or'_2, \dots, or'_m$ are perpendicular to one another. Hence

⁴⁾ $\overrightarrow{oe_{m+1}}$ and \overrightarrow{oy} denote half lines beginning at o .

$$\|p(\tau_0)\| = \|\varphi(r'_1)\| = \dots = \|\varphi(r'_m)\|,$$

and $o p(\tau_0), o \varphi(r'_1), \dots, o \varphi(r'_m)$ are also perpendicular to one another.

Let us put $p(\tau_0) = q_0[\varepsilon], \varphi(r'_1) = q_1[\varepsilon], \dots, \varphi(r'_m) = q_m[\varepsilon]$. It is easy to take a sequence ε_n converging to zero so that $q_0[\varepsilon_n], q_1[\varepsilon_n], \dots, q_m[\varepsilon_n]$ may converge to limit points q_0, q_1, \dots, q_m respectively. Clearly every $q_i \in L$,

$$\|q_0\| = \|q_1\| = \dots = \|q_m\|,$$

and $o q_i$'s are perpendicular to one another. Thus the lemma is proved.

3. A point P of a cylindrical space (I, S^m) which is a topological product of the interval $I = [0, 1]$ and the m -dimensional sphere S^m , is represented by a pair of co-ordinates (t, s) for $t \in I$ and $s \in S^m$, both $t = t(P)$ and $s = s(P)$ being continuous functions of P . Then we have another lemma which will be obtained without difficulty from the Lemma A'.

LEMMA A. Let S_0^m and S_1^m be the set $\{P; t(P) = 0\}$ and $S_1\{P; t(P) = 1\}$ respectively. If L is a closed set on (I, S^m) which intersects any continuous curve that joins S_0 and S_1 , then L contains $(m+1)$ points Q_0, Q_1, \dots, Q_m such that

$$t(Q_0) = t(Q_1) = \dots = t(Q_m) \tag{2}$$

and such that $s(Q_i)$'s ($0 \leq i \leq m$) are perpendicular to one another.

4. For a real-valued function $f(x)$ on S^n , there exist two points $x(1)$ and $x(0)$ with

$$\begin{aligned} \sup_{t \in S^n} f(x) &= f(x(1)) \\ \inf_{t \in S^n} f(x) &= f(x(0)). \end{aligned} \tag{4}$$

We join $x(0)$ and $x(1)$ by a curve $x(t)$ ($0 \leq t \leq 1$) on S^n . We may consider the S^n as the unit sphere in an $(n+1)$ -dimensional space R^{n+1} with the origin o . For a point p in this R^{n+1} the co-ordinates p^i 's and $\|p\|$ are similarly defined as in § 2. π denotes the hyperplane $\{p; p^{n+1} = 0\}$, and e_i ($0 \leq i \leq n$) denotes the point whose i -th co-ordinate is equal to 1 and other co-ordinates are zero.

We take again the rotations of axes ρ_i such that

$$\rho_i x(t) = e_{n+1},$$

ρ_i being continuous. Then $\rho_i H^{n-1}(x(t))$ are all contained in π . Let

S^{n-1} be $S^n \cap \pi$. Let us consider the topological product (I, S^{n-1}) of I and S^{n-1} , whose point P is represented by $t \in I$ and $u \in S^{n-1}$. We define S_0^{n-1} and S_1^{n-1} in a similar way as in § 4. Put

$$\begin{aligned} \rho_t^{-1} u(P) &= \Psi(P), \\ F(P) &= f(x(t)) - f(\Psi(P)), \end{aligned} \tag{5}$$

and let the set of zero points of $F(P)$ be K . Then any curve which is drawn from S_0^{n-1} to S_1^{n-1} intersects K because $F(P) \leq 0$ for $P \in S_0^{n-1}$ and $F(P) \geq 0$ for $P \in S_1^{n-1}$; therefore K contains n points P_1, P_2, \dots, P_n such that

$$t(P_1) = t(P_2) = \dots = t(P_n) = t_0,$$

and such that $u(P_i)$'s are perpendicular to one another. On the other hand

$$f(x(t_0)) = f(\Psi(P_1)) = \dots = f(\Psi(P_n)),$$

and $x(t_0), \Psi(P_1), \dots, \Psi(P_n)$ are clearly perpendicular to one another. Thus we have the theorem:

THEOREM. *For a continuous function $f(x)$ on S^n , there exist $(n+1)$ points x_0, x_1, \dots, x_n perpendicular to one another on S^n such that*

$$f(x_0) = f(x_1) = \dots = f(x_n).$$

From the above theorem we can obtain by the same argument as Kakutani¹⁾ the following

THEOREM. *For a bounded convex body in an $(n+1)$ -dimensional Euclidean space there exists a circumscribing cube around it.*

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