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MULTIPLICATIVE STRUCTURES IN MOD q COHOMOLOGY THEORIES II

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This paper is the continuation of part I, Osaka J. Math. 2 (1965), pp. 71–115. §§1–5 are contained in Part I and this part consists of §§6–12. We use all notations and notions defined in Part I.

In §6 we discuss admissible multiplications μ_q in $\tilde{K}(\ ;Z_q)$ and show that they induce multiplications μ_q^* in periodic cohomology $\tilde{K}^*(\ ;Z_q)$; Künneth isomorphism $\tilde{K}^*(X;Z_p)\otimes\tilde{K}^*(Y;Z_p)\cong\tilde{K}^*(X\wedge Y;Z_p)$ holds for any prime p; an important property of mod q K-theory, Propositions 6.3 and 6.4, is discussed. §7 is devoted to the discussion of commutativity criteria of admissible multiplications (Corollary 7.7, Theorems 7.11 and 7.13); we can establish the existence or non-existence of commutative admissible multiplications in $\tilde{K}(\ ;Z_q)$ for all $q>1$. §8 is a preparation mainly for §9. The existence of associative admissible multiplications is proved in §9 ($q=2$) and §10 ($q\neq 2$). In case $q=2$, it is guaranteed whenever $\eta^{**}=0$ (which is required even for the existence proof of admissible μ_2 of Theorem 5.9) (Theorem 9.9). In case $q\neq 2$, it is proved only under some conditions (Theorems 10.6 and 10.7). These are sufficient to prove the associativity for every admissible multiplication of $\tilde{K}(\ ;Z_q)$ (Corollary 10.8). In §11 we discuss Bockstein spectral sequences for general cohomologies and multiplicative structures in them. We see many analogous properties as those of ordinary Bockstein spectral sequences. Whenever the existence of admissible μ_p is guaranteed by Theorem 5.9, then some μ_p induces multiplications m_r in E_r-terms of mod p Bockstein spectral sequences for each prime p. It is noticeable that d_r behaves as a derivation to m_r (Theorem 11.10) even though the compatibility of the reduction $\rho_{sp,p}$ with μ_{sp} and μ_p is generally not proved, from which follows Künneth's isomorphism for each term of Bockstein spectral sequences of periodic K^*-cohomology (Theorem 11.11). §12 is an appendix treating some further properties of the maps $\bar{a}: M_q\to M_r$ of 2.4, not treated there.

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6. Multiplications in mod q K-cohomology.

6.1. The Atiyah-Hirzebruch K-cohomology theory of complex vector bundles has the commutative and associative multiplication μ defined by tensor products [2]. Thus its associated multiplications μ_R and μ_L satisfies $(H_1)-(H_8)$. By Theorem 2.3 and Proposition 2.5 any homomorphism $f: Z_q \to Z_r$ induces a natural map

$$f_* : \tilde{K}^i(\ ; Z_q) \to \tilde{K}^i(\ ; Z_r)$$

for each i, and (H_9) holds.

Since $\tilde{K}^i(S^0) \cong Z$ or 0 according as i is even or odd, we easily see that

$$\tilde{K}^i(S^0) = \rho_q \tilde{K}^i(S^0 ; Z_q) \cong Z_q \quad or \quad 0 \tag{6.1}$$

according as i is even or odd. From (6.1), for $i=-2$, Theorems 2.3, 5.9, Corollaries 3.10 and 3.11 follows

Theorem 6.1. *For every integer $q>1$ there exist just q distinct admissible multiplications in $\tilde{K}(; Z_q)$.*

6.2. Let g be the generator of $\tilde{K}^0(S^2)$, given by the reduced Hopf bundle. Bott's isomorphism

$$\beta : \tilde{K}^i(X) \approx \tilde{K}^{i-2}(X)$$

is given by the formula $\beta = \sigma^{-2}\mu(\ \otimes g)$. Making use of μ_R, we define Bott's isomorphism

$$\beta_q : \tilde{K}^i(X ; Z_q) \approx \tilde{K}^{i-2}(X ; Z_q) \tag{6.2}$$

in $\tilde{K}(\ ; Z_q)$ for each i by

$$\beta_q = \sigma_q^{-2}\mu_R(\ \otimes g)$$

in the same way as β, which is *natural*. $\beta_q = \beta$ as a map: $\tilde{K}^{i+2}(X \wedge M_q) \to \tilde{K}^i(X \wedge M_q)$, hence (6.2) is an isomorphism.

By (H_4)-(H_7) and (H_9) we have the commutativities:

$$\begin{aligned} \sigma_q\beta_q &= \beta_q\sigma_q, & \beta_q\rho_q &= \rho_q\beta \\ \delta\beta_q &= \beta\delta, & \delta_q\beta_q &= \beta_q\delta_q \\ f_*\beta_q &= \beta_r f_* \end{aligned} \tag{6.3}$$

for any homomorphism $f: Z_q \to Z_r$.

β_q gives isomorphisms of exact sequences of mod q $\tilde{K}$-cohomology associated with cofibrations. Consequently we can define periodic Z_2-graded mod q $\tilde{K}^*$-cohomology theory by putting

$$\tilde{K}^*(X; Z_q) = \tilde{K}^0(X; Z_q) \oplus \tilde{K}^1(X; Z_q)$$

and identifying $\tilde{K}^{2i}(X; Z_q)$ with $\tilde{K}^0(X; Z_q)$ and $\tilde{K}^{2i+1}(X; Z_q)$ with $\tilde{K}^1(X; Z_q)$ *via* β_q^i. By (6.3) natural maps σ_q, ρ_q, δ, δ_q and f_* are introduced also in the periodic theory $\tilde{K}^*(\ ; Z_q)$.

Let μ_q be an admissible multiplication in $\tilde{K}(\ ; Z_q)$. By (H_8), (Λ_1), and (Λ_3) we obtain the equalities

(6.4) $$\mu_q(\beta_q \otimes 1) = \mu_q(1 \otimes \beta_q) = \beta_q \mu_q ,$$

which imply that

(6.5) *μ_q induces a multiplication, denoted by μ_q^*, in the periodic cohomology $\tilde{K}^*(\ ; Z_q)$ satisfying also the admissibility conditions* $(\Lambda_1)-(\Lambda_3)$.

By a general argument using an induction on cells and (6.1) we get a Künneth isomorphism:

Theorem 6.2. *If $\tilde{K}^*(X; Z_q)$ or $\tilde{K}^*(Y; Z_q)$ is a Z_q-free module, then μ_q^* induces an isomorphism*

$$\tilde{K}^*(X; Z_q) \otimes \tilde{K}(Y; Z_q) \cong \tilde{K}^*(X \wedge Y; Z_q) .$$

6.3. The following proposition is important for our later discussions.

Proposition 6.3. *Let $\bar{\eta}$ be a generator of* $\{S^2M_2, S^2\}$ *given in* (4.1). *There holds the relation*

$$\bar{\eta}^{**} = \sigma^2 \beta \pi_2^{**}$$

for $\tilde{K}$-theory.

Proof. Let $\bar{\eta}$ be represented by a map f: $S^4M_2 \to S^4$ such that $f(S^4 i) = S^2\eta$, η is the Hopf map. Then the mapping cone of f is

$$L = S^4 \cup_f CS^4M_2 = S^4 \cup e^6 \cup e^7 = S^2P \cup_h e^7 ,$$

where h: $S^6 \to S^2P$ is the attaching map of e^7 and P is the complex projective plane. From the cell structure of L we see easily that

$$H^i(L; Z) \cong \begin{cases} Z & \text{for } i = 0, 4 \\ Z_2 & \text{for } i = 7 \\ 0 & \text{others,} \end{cases}$$

and $Sq^3 | H^4(L; Z)$ is non-trivial. Thus, by discussing the Atiyah-Hirzebruch spectral sequence with $E_2^* = H^*(L; Z)$ and $E_\infty^* = \mathcal{G}K^*(L; Z)$ [2], we see that

$$\tilde{K}^1(L) \cong 0 \quad \text{and} \quad \tilde{K}^0(L) \cong Z .$$

Then, from the exact sequence

$$\tilde{K}^0(S^4) \xrightarrow{\bar{\eta}^*} \tilde{K}^0(S^4M_2) \longrightarrow \tilde{K}^1(L)$$

associated with the cofibration $S^4 \to L \to S^4M_2$, it follows that the above $\bar{\eta}^*$ and, *via* Bott isomorphism, the homomorphism $\bar{\eta}^*: \tilde{K}^0(S^2) \to \tilde{K}^0(S^2M_2)$ are epimorphic. $\pi^*: \tilde{K}^0(S^2) \to \tilde{K}^0(M_2)$ is also epimorphic. Thus $\bar{\eta}^*$ and π^* are both equal to the projection: $Z \to Z_2$. Hence

$$\bar{\eta}^* = \sigma^2\beta\pi^*$$

as maps: $\tilde{K}^0(S^2) \to \tilde{K}^0(S^2M_2)$.

For any W and any $x \in \tilde{K}^i(W \wedge S^2)$, there exists $x' \in \tilde{K}^i(W)$ such that $\mu(x' \otimes g) = x$ by the Bott isomorphism theorem. Then we have

$$\begin{aligned}\bar{\eta}^{**}x &= (1_W \wedge \bar{\eta})^*\mu(x' \otimes g) = \mu(x' \otimes \bar{\eta}^* g)\\ &= \mu(x' \otimes \mu(\pi^* g \otimes g)) = \mu((1_W \wedge \pi^*)\mu(x' \otimes g) \otimes g)\\ &= \mu(\pi^{**}x \otimes g) = \sigma^2\beta\pi^{**}x .\end{aligned}$$

q.e.d.

6. 4. The above proposition can be generalized to the following

Proposition 6. 4. *Let q be even. For any class $\bar{\eta} \in \{S^2M_q, S^2\}$ of* (4. 1) *the following relation*

$$\bar{\eta}^{**} = (q/2) \cdot \sigma^2\beta\pi_q^{**}$$

holds for $\tilde{K}$-theory.

Proof. Denote by $\bar{\eta}_2 \in \{S^2M_2, S^2\}$ an element $\bar{\eta}$ for $q=2$. Let $\gamma \in \{S^2M_q, S^2M_2\}$ be the class of $S^2(\overline{q/2})$, where $\overline{q/2}: M_q \to M_2$ is a map of (2. 5), i.e.,

$$\gamma(S^2 i_q) = S^2 i_2 \quad \text{and} \quad (S^2\pi_2)\gamma = (q/2) \cdot S^2\pi_q .$$

We have

$$\bar{\eta}(S^2 i_q) = \eta = \bar{\eta}_2(S^2 i_2) = \bar{\eta}_2\gamma(S^2 i_q) .$$

Then, from the exact sequence

$$\{S^4, S^2\} \xrightarrow{S^2\pi_q^*} \{S^2M_q, S^2\} \xrightarrow{S^2 i_q^*} \{S^3, S^2\}$$

follows the relation

$$\bar{\eta} \equiv \bar{\eta}_2\gamma \bmod \eta^2(S^2\pi_q) .$$

Since $\eta^{**} = 0$ in $\tilde{K}$, we have

$$\begin{aligned}\bar{\eta}^{**} &= \gamma^{**}\bar{\eta}_2^{**} = \gamma^{**}\sigma^2\beta\pi_2^{**}\\ &= \sigma^2\beta\gamma^{**}\pi_2^{**} = (q/2) \cdot \sigma^2\beta\pi_q^{**} .\end{aligned}$$

q.e.d.

7. (Non-) Commutativity of mod q multiplications.

7.1. In this section we use the following convention: for each $x\in\tilde{h}^i(X;Z_q)$, which is the same as $\tilde{h}^{i+2}(X\wedge M_q)$ by definition, we denote x as $\bar{x}$ when we consider it as an element of $\tilde{h}^{i+2}(X\wedge M_q)$.

Let μ be a commutative and associative multiplication in a cohomology theory $\tilde{h}$ and μ_q an admissible multiplication in $\tilde{h}(\ ;Z_q)$ constructed in §5 (assuming that $\eta^{**}=0$ in case $q\equiv 2\ (\mathrm{mod}\ 4)$ and fixing an element $\bar{\alpha}$ of (4.17)). That is, for $x\in\tilde{h}^i(X;Z_q)$, $y\in\tilde{h}^j(Y;Z_q)$ and $w=\mu_q(x\otimes y)$, we have

$$\bar{w} = \sigma^{-2}\gamma_W(1_W\wedge\alpha)^*(1_X\wedge T'\wedge 1_M)^*\mu(\bar{x}\otimes\bar{y})\,, \tag{7.1}$$

where $W=X\wedge Y$ and $T'=T(Y, M_q)$.

Put

$$w' = \mu_q'(x\otimes y) = (-1)^{ij}T''^*\mu_q(y\otimes x)$$

for $T''=T(X, Y)$. μ_q' is also an admissible multiplication. In fact, by a routine calculation making use of the commutativity of μ and the naturality of γ etc., we see that

$$\bar{w}' = \sigma^{-2}\gamma_W(1_W\wedge(T\alpha))^*(1_X\wedge T'\wedge 1_M)^*\mu(\bar{x}\otimes\bar{y})\,, \tag{7.2}$$

where $T=T(M_q, M_q)$.

Computing the difference $\bar{w}'-\bar{w}$ for $x=y=\kappa_1$ we discuss the (non-) commutativity of μ_q. To do this, we may choose κ_1 suitably.

Lemma 7.1. *We can choose* $\kappa_1\in\tilde{h}^1(M_q;Z_q)$ *satisfying* (3.7) *as*

$$\bar{\kappa}_1 = \beta_0^*(\sigma^3 1)\,. \tag{7.3}$$

Proof. Discussing the integral cohomology map β_0^* by using (4.9′) we see that

$$\beta_0(1_M\wedge i) = \pi\wedge 1_S \quad \textit{and} \quad \beta_0(i\wedge 1_M) = -1_S\wedge\pi\,, \tag{7.3′}$$

where $S=S^1$, which show (3.7) immediately for the element κ_1 defined by (7.3). q.e.d.

In this section we define κ_1 always by (7.3).

Lemma 7.2. *Let q be even. There holds the relation*

$$1_M\wedge\eta = \eta_1+\eta_2$$

in $\{S^2M_q, SM_q\}$ *after choosing* η_3 *suitably in case* $q\equiv 0\ (\mathrm{mod}\ 4)$.

Proof. By Theorem 4.1 we may put

$$1_M\wedge\eta = x\cdot\eta_1+y\cdot\eta_2+z\cdot(Si)\eta^2(S^2\pi)$$

for x, y, and $z \in Z_2$, where $z=0$ in case $q \equiv 2 \pmod 4$. Composing $S\pi$ on both sides from the left we see that

$$\eta(S^2\pi) = \pi \wedge \eta = (S\pi)(1_M \wedge \eta) = y \cdot \eta(S^2\pi).$$

Thus $y=1$. Similarly, composing $S^2 i$ from the right, we obtain that $x=1$. In case $q \equiv 0 \pmod 4$, replacing η_3 by $\eta_3+\eta_1^2$ if necessary, we see that η_3, hence η_1 and η_2, can be chosen so that $z=0$. q.e.d.

7.2. First we discuss *the case* $q=2$. Put

(7.4) $$\eta' = (1\wedge i)\eta_1(1\wedge \pi) \quad and \quad \eta'' = (1\wedge i)\eta_2(1\wedge \pi),$$

which belong to $\{M_2\wedge M_2, M_2\wedge M_2\}$. By (4.3),

$$\eta' = (i\wedge i)\bar{\eta}(1\wedge \pi) \quad \text{and} \quad \eta''=(1\wedge i)\tilde{\eta}(\pi\wedge\pi).$$

Proposition 7.3. (i) *$\{M_2\wedge M_2, SM_2\} \cong Z_2+Z_2+Z_2$ with generators $\eta_1(1\wedge \pi)$, $\eta_2(1\wedge\pi)$ and $(Si)\eta\beta_0$.*

(ii) *$\{M_2\wedge M_2, M_2\wedge M_2\} \cong Z_4+Z_4+Z_2$ with generators $1_{M\wedge M}$, $\alpha_0\beta_0$ and η'.*

(iii) *There hold the relations*

$$\eta'(\alpha_0\beta_0) = 2\cdot\alpha_0\beta_0 \quad and \quad (\alpha_0\beta_0)(\alpha_0\beta_0) = (\alpha_0\beta_0)\eta' = \eta'\eta' = 0.$$

Proof. (i) In the exact sequence of (1.7):

$$0 \to \{S^2M_2, SM_2\} \xrightarrow{(1\wedge\pi)^*} \{M_2\wedge M_2, SM_2\} \xrightarrow{(1\wedge i)^*} \mathrm{Tor}(\{SM_2, SM_2\}, Z_2) \to 0,$$

$(1\wedge i)^*((Si)\eta\beta_0)$ generates $\mathrm{Tor}(\{SM_2, SM_2\}; Z_2)$ since $(1\wedge i)^*((Si)\eta\beta_0) = (Si)\eta(S\pi)=2\cdot 1_{SM}$. And $2\cdot(Si)\eta\beta_0=0$. Thus the above sequence splits and (i) follows from Theorem 4.1.

(ii) In the exact sequence

$$0 \to \{M_2\wedge M_2, SM_2\} \xrightarrow{(1\wedge i)_*} \{M_2\wedge M_2, M_2\wedge M_2\} \xrightarrow{(1\wedge\pi)_*} \{M_2\wedge M_2, S^2M_2\} \to 0,$$

((1.7′), the above (i) and (4.10)), $\{M_2\wedge M_2, S^2M_2\}$ is generated by $1\wedge\pi = (1\wedge\pi)_* 1_{M\wedge M}$ and $(S^2 i)\beta_0 = (1\wedge\pi)_*\alpha_0\beta_0$ by (4.10) and (4.9′). Here

$$\begin{aligned} 2\cdot 1_{M\wedge M} &= 1_M\wedge(2\cdot 1_M) = 1_M\wedge(i\eta\pi) \\ &= (1\wedge i)(\eta_1+\eta_2)(1\wedge\pi) \qquad \text{by Lemma 7.2} \\ &= \eta'+\eta'' \neq 0 \end{aligned}$$

by (i) and the above exact sequence, and

$$\begin{aligned} 2\cdot\alpha_0\beta_0 &= (i\wedge i)\eta\beta_0 \qquad \text{by (4.15′)} \\ &= (1\wedge i)_*(Si)\eta\beta_0 \neq 0 \end{aligned}$$

by (i) and the above exact sequence. Thus (ii) follows.

(iii) $\beta_0\alpha_0=p\bar{\beta}\bar{\alpha}\bar{i}_1=p\bar{i}_1=0$ by (4.9) and (4.7). Thus $(\alpha_0\beta_0)(\alpha_0\beta_0)=0$. $\beta_0(i\wedge i)=(S\pi)(Si)=0$ by (4.9′). Thus $(\alpha_0\beta_0)\eta'=0$. $(\pi\wedge\pi)\alpha_0=(S^2\pi)(S^2i)=0$ by (4.9′), hence $\eta''(\alpha_0\beta_0)=0$; $2\cdot\alpha_0\beta_0=(2\cdot 1_{M\wedge M})\alpha_0\beta_0=(\eta'+\eta'')\alpha_0\beta_0=\eta'(\alpha_0\beta_0)$. $\eta'\eta'=(1\wedge i)\bar{\eta}(1\wedge\pi)(1\wedge i)\bar{\eta}(1\wedge\pi)=0$. Hence (iii) is proved.

Theorem 7.4. *For $T=T(M_2, M_2)$ there holds the relation*

$$T\equiv 1_{M\wedge M}+\alpha_0\beta_0+\eta' \qquad \bmod 2\cdot\{M_2\wedge M_2,\ M_2\wedge M_2\}\ .$$

Proof. Set

$$T=x\cdot 1_{M\wedge M}+y\cdot\alpha_0\beta_0+z\cdot\eta', \qquad x, y\in Z_4 \text{ and } z\in Z_2,$$

by Proposition 7.3, (ii). By (4.9′) and (4.11) we have

$$(1\wedge\pi)+(S^2i)\beta_0=(1\wedge\pi)T=x\cdot(1\wedge\pi)+y\cdot(S^2i)\beta_0\,.$$

Thus

$$(*)\qquad x\equiv y\equiv 1\ (\mathrm{mod}\ 2)$$

by (4.10). Next, if $z=0$,

$$1_{M\wedge M}=T^2=x^2\cdot 1_{M\wedge M}+(2xy)\cdot\alpha_0\beta_0$$

by the above setting and Proposition 7.3, (iii), which implies that $xy\equiv 0$ (mod 2) contradicting to (*), i.e.,

$$z=1\,. \qquad \text{q.e.d.}$$

Corollary 7.5. $T\alpha\equiv\alpha+\eta'\alpha \quad \bmod 2\cdot\{N_2,\ M_2\wedge M_2\}$.
Because: $\beta_0\alpha=p\bar{\beta}\bar{\alpha}j=pj=0$ by (4.7).

7.3. New we compute $\bar{w}'-\bar{w}$ of (7.1)-(7.2) for $x=y=\kappa_1$ in case $q=2$. Since $2\cdot\mu(\bar{\kappa}_1\otimes\bar{\kappa}_1)=0$ by Proposition 3.2,

$$\begin{aligned}
\bar{w}'-\bar{w} &= \sigma^{-2}\gamma_W(1_W\wedge\eta'\alpha)^*(1_M\wedge T'\wedge 1_M)^*\mu(\bar{\kappa}_1\otimes\bar{\kappa}_1)\\
&\qquad\text{by Corollary 7.5 for } W=M_2\wedge M_2\\
&= \sigma^{-2}\gamma_W\alpha^{**}(1_M\wedge\pi)^{**}\bar{\eta}^{**}(1_M\wedge T_1\wedge 1_S)^*(1\wedge i\wedge 1\wedge i)^*\mu(\bar{\kappa}_1\otimes\bar{\kappa}_1)\\
&\qquad\text{for } T_1=T(M_2, S^1) \text{ and } S=S^1\\
&= \sigma^{-2}\gamma_W\pi_0^{**}\bar{\eta}^{**}(1_M\wedge T_1\wedge 1_S)^*\mu((S\pi)^*\sigma^3 1\otimes(S\pi)^*\sigma^3 1)\\
&\qquad\text{by (7.3), (7.3′), (4.9′), (4.18′)}\\
&= \sigma^{-2}\bar{\eta}^{**}(\pi\wedge\pi\wedge 1_{S^2})^*\mu(\mu(\sigma^2 1\otimes\sigma^2 1)\otimes\sigma^2 1)\\
&\qquad\text{by Lemma 5.2, (i) and } T(S^2, S^1)\simeq 1\\
&= \mu(\mu(\pi^*\sigma^2 1\otimes\pi^*\sigma^2 1)\otimes\bar{\eta}^* 1)\,.
\end{aligned}$$

Thus, putting

$$\bar{a} = \bar{\eta}^*1, \quad a \in \tilde{h}^{-2}(S^0; Z_2),$$
$$w' - w = \mu_L(\mu(\pi^*\sigma^2 1 \otimes \pi^*\sigma^2 1) \otimes a)$$
$$= \mu_2(\rho_2\mu(\pi^*\sigma^2 1 \otimes \pi^*\sigma^2 1) \otimes a)$$
$$= \mu_2(a \otimes \mu_2(\kappa_2 \otimes \kappa_2)),$$

from which we deduce the following

Theorem 7.6. *Let $\tilde{h}$ be equipped with a commutative and associative multiplication μ and $\eta^{**}=0$ in $\tilde{h}$. For any admissible multiplication μ_2 in $\tilde{h}(\ ;Z_2)$, putting*

$$a = a(\mu_2),$$

there holds the relation

$$\bar{a} = \bar{\eta}^*1.$$

Proof. The above formula and (3.13) imply that the theorem is true for a suitably constructed μ_2. Since $2\cdot b(\mu_2, \mu_2')=0$ for any two admissible multiplications μ_2 and μ_2' by Proposition 3.2, (3.18) implies that the theorem is true for any admissible μ_2. q.e.d.

Since $\bar{\eta}^*1=0$ if and only if $\eta^{**}=0$ in $\tilde{h}$, we have

Corollary 7.7. *Under the assumption of Theorem 7.6, the necessary and sufficient condition for the existence of commutative admissible μ_2 in $\tilde{h}(\ ;Z_2)$ is that $\eta^{**}=0$. When this condition is satisfied, every admissible μ_2 is commutative.*

Because of Proposition 6.3 and $\pi_2^{**}\neq 0$ in $\tilde{K}$, we have

Corollary 7.8. *$\tilde{K}(\ ;Z_2)$ has no commutative admissible μ_2. If we denote the two distinct admissible multiplications in $\tilde{K}(\ ;Z_2)$ by $\wedge$ and $\wedge'$, respectively, then*

$$T^*(y\wedge x) = x\wedge' y = x\wedge y + \beta_2(\delta_2 x\wedge \delta_2 y)$$

for $x\in\tilde{K}^*(X;Z_2)$, $y\in\tilde{K}^*(Y;Z_2)$ and $T=T(X, Y)$.

This corollary means that *the admissible multiplication in $\tilde{K}(\ ;Z_2)$ is essentially unique.*

7.4. Next we discuss the *case* $q\not\equiv 2$ (mod 4).

$N_q = S^2\vee S^2M_q$ and $\bar{N}_q = SM_q\vee S^2M_q$ by (4.6), (i). Let

$$i' : S^2M_q \to N_q, \quad \bar{i}' : S^2M_q \to \bar{N}_q$$

and

$$\pi' : N_q \to S^2, \quad \bar{\pi}' : \bar{N}_q \to SM_q$$

be the inclusions and the maps collapsing S^2M_q, respectively. Besides (5.2) we have the relations

$$\bar{i}' = ji', \quad \bar{\pi}'j = (Si)\pi', \quad \bar{\pi}'\bar{i}' = 0, \tag{7.5}$$
$$p = (S\pi)\bar{\pi}', \quad i_1 = i'(S^2 i) \quad \textit{and} \quad \bar{\pi}'\bar{i}_1 = 0.$$

In the present case we choose γ_0 as

$$\gamma_0 = \pi'^{*}\sigma^2 1.$$

Then γ_W satisfies (5.4′).

Put

$$i'' = \bar{\alpha}\bar{i}' : S^2M_q \to M_q\wedge M_q, \quad \pi'' = \bar{\pi}'\bar{\beta} : M_q\wedge M_q \to SM_q, \tag{7.6}$$

then we obtain the relations

$$\pi'' i'' = 0, \quad \pi''(1_M\wedge i) = 1_{SM} \quad \textit{and} \quad (1_M\wedge\pi)i'' = 1_{S^2M} \tag{7.7}$$

by Lemmas 4.3, (ii), (4.7), (5.2) and (7.5).

By (4.6), (i), we see immediately

Proposition 7.9. *In case* $q\not\equiv 2 \pmod 4$, $(1_M\wedge i)_*$, i''_*, $(1_M\wedge\pi)^*$ *and* π''^* *are monomorphic, and we have the following direct sum decompositions*

(i) $\{W, M_q\wedge M_q\} = (1\wedge i)_*\{W, SM_q\}\oplus i''_*\{W, S^2M_q\}$,

(ii) $\{M_q\wedge M_q, W\} = (1\wedge\pi)^*\{S^2M_q, W\}\oplus\pi''^*\{SM_q, W\}$

for any W, *and in particular*

(iii) $\{M_q\wedge M_q, M_q\wedge M_q\} = (1\wedge i)_*\pi''^*\{SM_q, SM_q\}\oplus i''_*\pi''^*\{SM_q, S^2M_q\}\oplus(1\wedge i)_*(1\wedge\pi)^*\{S^2M_q, SM_q\}\oplus i''_*(1\wedge\pi)^*\{S^2M_q, S^2M_q\}$.

By Proposition 7.9, (iii), we can put

$$1_{M\wedge M} = (1\wedge i)a_1\pi'' + i''a_2\pi'' + (1\wedge i)a_3(1\wedge\pi) + i''a_4(1\wedge\pi)$$

with $a_1\in\{SM_q, SM_q\}$, $a_2\in\{SM_q, S^2M_q\}$, $a_3\in\{S^2M_q, SM_q\}$ and $a_4\in\{S^2M_q, S^2M_q\}$. Compose $1\wedge\pi$ on both sides from the left, then by (7.7) we get

$$1\wedge\pi = a_2\pi'' + a_4(1\wedge\pi),$$

hence

$$a_2 = 0 \quad a_4 = 1_{S^2M}$$

by Proposition 7.9, (ii), for $W = S^2M_q$. Similarly, composing π'' from the left, we get

$$a_1 = 1_{SM} \quad \text{and} \quad a_3 = 0.$$

Thus we obtain

$$1_{M\wedge M} = i''(1\wedge\pi) + (1\wedge i)\pi''. \tag{7.8}$$

Next, put

$$T = (1\wedge i)b_1\pi'' + i''b_2\pi'' + (1\wedge i)b_3(1\wedge\pi) + i''b_4(1\wedge\pi)$$

with $b_1 \in \{SM_q, SM_q\}$, $b_2 \in \{SM_q, S^2M_q\}$, $b_3 \in \{S^2M_q, SM_q\}$ and $b_4 \in \{S^2M_q, S^2M_q\}$ for $T = T(M_q, M_q)$. Compose $1 \wedge \pi$ on both sides from the left. By (4.11), (i) and (7.7) we get

$$(1_M \wedge \pi) + (S^2 i)\beta_0 = b_2 \pi'' + b_4(1 \wedge \pi).$$

Here

$$(S^2 i)\beta_0 = (S^2 i)p\bar{\beta} = (S^2 i)(S\pi)\pi''$$

by (4.9), (7.5) and (7.6). Thus, by Proposition 7.9, (ii), for $W = S^2 M_q$ we get

$$b_2 = (S^2 i)(S\pi) \quad \text{and} \quad b_4 = 1_{S^2 M}.$$

Similarly, composing $1 \wedge i$ from the right and making use of (4.11), (ii), with a remark that

$$\alpha_0(S\pi) = \bar{\alpha}\bar{i}_1(S\pi) = i''(S^2 i)(S\pi),$$

Proposition 7.9, (i), for $W = SM_q$, implies that

$$b_1 = -1_{SM}.$$

Therefore we see that

$$T = -(1 \wedge i)\pi'' + i''(S^2 i)(S\pi)\pi'' + i''(1 \wedge \pi) + (1 \wedge i)\xi(1 \wedge \pi)$$

with $\xi \in \{S^2 M_q, SM_q\}$, from which follows

$$T^2 = (1 \wedge i)\pi'' + i''(1 \wedge \pi) + i''(S^2 i)(S\pi)\xi(1 \wedge \pi) + (1 \wedge i)\xi(S^2 i)(S\pi)\pi''.$$

Since $T^2 = 1_{M \wedge M}$, by (7.8) and Proposition 7.9, (ii), we obtain

$$(S^2 i)(S\pi)\xi = 0 \quad \text{and} \quad \xi(S^2 i)(S\pi) = 0.$$

These relations, Theorem 4.1 and Corollary 4.2, (i), imply

$$\xi = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \varepsilon_q \cdot (Si)\eta^2(S^2\pi) & \text{if } q \equiv 0 \pmod 4 \end{cases}$$

with $\varepsilon_q \in Z_2$. Thus we obtain

Theorem 7.10. *In case* $q \not\equiv 2 \pmod 4$ *there holds the relation*

$$T = -(1 \wedge i)\pi'' + i''(S^2 i)(S\pi)\pi'' + i''(1 \wedge \pi) + \varepsilon_q \cdot (i \wedge i)\eta^2(\pi \wedge \pi)$$

for $T = T(M_q, M_q)$, *where* $\varepsilon_q \in Z_2$ *in case* $q \equiv 0 \pmod 4$ *and* $\varepsilon_q = 0$ *in case* q *odd.*

7.5. Now we compute $\bar{w}' - \bar{w}$ of (7.1)–(7.2) for $x = y = \kappa_1$ in case $q \not\equiv 2 \pmod 4$. By our choice of γ_0, $\gamma_W = (1_W \wedge i')^*$ by (5.4'). Thus

(7.9) $$\gamma_W(1_W\wedge\alpha)^*(1_W\wedge(1\wedge i)\pi'')^* = 0$$

since $(1\wedge i)\pi''\alpha i'=(1\wedge i)\bar{\pi}'\bar{i}'=0$ by (7.5), (7.6) and (4.18). Also

(7.10) $$i''(S^2 i)(S\pi)\pi''\alpha = 0$$

because of (7.5), (7.6) and (4.18). Making use of (7.8)-(7.10) and Theorem 7.10, a parallel computation to the case of $q=2$ implies

$$\bar{w}'-\bar{w} = \mu(\mu(\pi^*\sigma^2 1\otimes\pi^*\sigma^2 1)\otimes\varepsilon_q\cdot\pi^*\eta^{2*}1)\,.$$

Thus, putting $\bar{a}=\pi^*\eta^{2*}1$, $a\in\tilde{h}^{-2}(S^0;Z_q)$, we have

$$w'-w = \mu_q(\varepsilon_q\cdot a\otimes\mu_q(\kappa_2\otimes\kappa_2))\,,$$

from which by (3.13) we conclude

(7.11) $$a(\mu_q) = \varepsilon_q\cdot a\,,\quad \bar{a} = \pi^*\eta^{2*}1\,.$$

Therefore,

Theorem 7.11. *Let μ be a commutative and associative multiplication in $\tilde{h}$. By a suitable construction of § 5 in case $q\not\equiv 2$ (mod 4) we can obtain a commutative admissible multiplication μ_q if q is odd or if $q\equiv 0$ (mod 4) and $(\eta^2\pi)^{**}=0$.*

This theorem for q odd asserts a slightly different thing from Corollary 3.11, i.e., the unique commutative admissible μ_q can be constructed by the manner of § 5, which is necessary for the proof of the existence of a commutative and associative admissible μ_q later in § 10.

In case $q\equiv 0$ (mod 4) it is an open question whether we can choose α such that $\varepsilon_q=0$ or not. If we can do so, then the existence of a commutative admissible μ_q without any condition follows.

In case $\tilde{K}$-cohomology, $(\eta^2\pi)^{**}=\pi^{**}\eta^{**}\eta^{**}=0$. Thus

Corollary 7.12. *If $q\equiv 0$ (mod 4) there exist just two distinct commutative admissible multiplications μ_q in $\tilde{K}(\,;Z_q)$.*

7.6. In case $q\equiv 2$ (mod 4), from Theorems 7.6, 7.11 and 3.14 follows

Theorem 7.13. *Let $\tilde{h}$ be equipped with a commutative and associative multiplication μ and $\eta^{**}=0$ in $\tilde{h}$. If $q\equiv 2$ (mod 4), a necessary and sufficient condition for the existence of a commutative admissible multiplication in $\tilde{h}(\,;Z_q)$ is that $\bar{\eta}^{**}=0$ in $\tilde{h}$ for $\bar{\eta}\in\{S^2M_2, S^2\}$.*

Corollary 7.14. *If $q\equiv 2$ (mod 4), there exist no commutative admissible multiplications in $\tilde{K}(\,;Z_q)$.*

8. Stable homotopy of some elementary complexes. II.

8.1. Let P be the complex projective plane, i.e., $P=S^2\cup_\eta e^4$. Let $i_P: S^2\to P$ and $\pi_P: P\to S^4$ be the inclusion and the map collapsing S^2. We have a cofibration

$$(8.1) \qquad S^2 \xrightarrow{i_P} P \xrightarrow{\pi_P} S^4 .$$

From Puppe's exact sequence and its dual, (1.5) and (1.5′), associated with (8.1) for X and Y spheres, we obtain

(8.2) *the groups* $\{S^{n+k+3}, S^nP\}$ *and* $\{S^nP, S^{n-k+3}\}$, $k\leqq 3$, *are both isomorphic to the corresponding groups in the following table*:

generators of	$k\leqq -2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$
	0	Z	0	Z	Z_{12}	0
$\{S^{n+k+3}, S^nP\}$		i_P		$\tilde{\zeta}$	$i_P\nu$	
$\{S^nP, S^{n-k+3}\}$		π_P		$\bar{\zeta}$	$\nu\pi_P$	

,

where $\tilde{\xi}$ *and* $\bar{\xi}$ *are defined by*

$$(8.3) \qquad \pi_P\tilde{\xi} = 2\cdot 1_{n+4} \quad and \quad \bar{\xi}i_P = 2\cdot 1_{n+2},\ 1_m\in\{S^m, S^m\} .$$

Theorem 8.1. *The groups* $\{P, S^iP\}$, $0\leq i\leq 2$, *and* $\{SP, P\}$ *are isomorphic to the corresponding groups in the following table*:

		generators
$\{P, S^2P\}$	Z	$(S^2i_P)\pi_P$
$\{P, SP\}$	0	
$\{P, P\}$	$Z\oplus Z$	1_P, $\tilde{\zeta}\pi_P$ (*or* $i_P\bar{\zeta}$)
$\{SP, P\}$	Z_6	$i_P\nu(S\pi_P)$

We have relations

$$(8.4) \qquad i_P\bar{\xi}+\tilde{\xi}\pi_P = 2\cdot 1_P$$

and

$$(8.5) \qquad 1_P\wedge\eta = 3\cdot i_P\nu(S\pi_P)$$

for $\eta\in\{S^1, S^0\}$.

Proof. From (8.2), (1.5) and (1.5′) suitably used, we see easily the results for $\{P, S^iP\}$, $0\leq i\leq 2$. Then, by (8.3), we obtain the relation (8.4).

Now we observe an exact sequence (1.5) associated with (8.1):

(*1) $$\{S^4, P\} \xrightarrow{\eta^*} \{S^5, P\} \xrightarrow{(S\pi_P)^*} \{SP, P\} \xrightarrow{(Si_P)^*} \{S^3, P\}.$$

(8.2) and the fact that every element of η^*-images is at most of order 2 show that

(*2) $\{SP, P\} \cong Z_6$ or Z_{12} with the generator $i_P\nu(S\pi_P)$.

And we can put

(*3) $$1_P \wedge \eta = a\cdot(Si_P)\nu(S^2\pi_P) \in \{S^2P, SP\}$$

for some integer a, $\eta \in \{S^2, S^1\}$ and $\nu \in \{S^6, S^3\}$.

By Theorem 1.1 and (*3),

$$2\cdot 1_{P\wedge M} = 1_P \wedge (2\cdot 1_M) = a\cdot(i_P \wedge i_2)\nu(\pi_P \wedge \pi_2), \qquad M = M_1.$$

If a were even, then $2\cdot 1_{P\wedge M} = 0$ since i_2 is of order 2; then

$$\begin{aligned} P\wedge M_2 \wedge M_2 &= S(P\wedge M_2) \cup_{2\cdot 1_{P\wedge M}} CS(P\wedge M_2) \\ &\simeq S(P\wedge M_2) \vee S^2(P\wedge M_2) \end{aligned}$$

in the stable range, which is but a contradiction because $Sq^4 \neq 0$ in $P\wedge M_2\wedge M_2$ whereas $=0$ in $S(P\wedge M_2)\vee S^2(P\wedge M_2)$. Hence

(*4) a is odd.

Next, by (8.1), (8.2), (*1), and (*3),

$$\begin{aligned} 0 &= (S\pi_P)^*\eta^*\xi = (S\pi_P)^*(\xi\wedge\eta) = (1_P\xi\pi_P)\wedge\eta \\ &= (1_P\wedge\eta)\xi\pi_P = 2a\cdot i_P\nu\pi_P, \end{aligned}$$

which implies *via* (*2) that

$$a = 3 \quad \text{and} \quad \{SP, P\} \cong Z_6. \qquad \text{q.e.d.}$$

8.2. From now on through this section M, i, and π stand for M_2, i_2, and π_2. We shall compute the groups $\{M_2\wedge P, S^iM_2\}$ for $2\le i\le 4$. We have

$$M_2\wedge P = M_2\wedge(S^2\cup_\eta CS^3) = S^2M_2\cup_{1\wedge\eta} CS^3M_2, \qquad 1 = 1_M.$$

From its associated exact sequence (1.5) for $X = S^iM_2$:

$$\begin{aligned} &\{S^3M, S^iM\} \xrightarrow{(1\wedge\eta)^*} \{S^4M, S^iM\} \xrightarrow{(1\wedge\pi_P)^*} \{M\wedge P, S^iM\} \\ &\xrightarrow{(1\wedge i_P)^*} \{S^2M, S^iM\} \xrightarrow{(1\wedge\eta)^*} \{S^3M, S^iM\}, \end{aligned}$$

follows the exact sequence

(8.6) $0 \to \mathrm{Coker}(1\wedge\eta)^*_{i-1} \xrightarrow{(1\wedge\pi_P)^*} \{M\wedge P, S^iM\} \xrightarrow{(1\wedge i_P)^*} \mathrm{Ker}(1\wedge\eta)^*_i \to 0$,

where $(1\wedge\eta)^*_i$ stands for $(1_M\wedge\eta)^*: \{S^{n+2}M, S^{n+i}M\} \to \{S^{n+3}M, S^{n+i}M\}$, $n=0$ or 1. Marking use of the relations in Corollary 4.2, (ii), and the relation of Lemma 7.2 we have

$$(1_M\wedge\eta)^*(i\pi)=2\cdot 1_M, \quad (1_M\wedge\eta)^*1_M = \eta_1+\eta_2$$

and

$$(1_M\wedge\eta)^*\eta_i = \eta_i^2, \qquad i = 1, 2.$$

Then, from Theorem 4.1. follow

(8.7) (i) $\mathrm{Ker}(1\wedge\eta)^*_4=\mathrm{Ker}(1\wedge\eta)^*_3=0$,
(ii) $\mathrm{Ker}(1\wedge\eta)^*_2\cong Z_2$ *generated by* $2\cdot 1_M$,
(iii) $\mathrm{Coker}(1\wedge\eta)^*_3\cong Z_2$ *generated by the class of* 1_M,
(iv) $\mathrm{Coker}(1\wedge\eta)^*_2\cong Z_2$ *generated by the class of* η_1 (or η_2),
(v) $\mathrm{Coker}(1\wedge\eta)^*_1\cong Z_2$ *generated by the class of* $i\nu\pi$.

Theorem 8.2. (i) *$\{M_2\wedge P, S^4M_2\}\cong Z_2$ generated by $1_M\wedge\pi_P$.*
(ii) *$\{M_2\wedge P, S^3M_2\}\cong Z_2$ generated by $\eta_1(1_M\wedge\pi_P)=\eta_2(1_M\wedge\pi_P)$.*
(iii) *$\{M_2\wedge P, S^2M_2\}\cong Z_2\oplus Z_2$ generated by $(S^2i)\nu(\pi\wedge\pi_P)$ and $1_M\wedge\xi$, where $\nu\in\{S^6, S^3\}$ and $\xi\in\{P, S^2\}$.*
(iv) *We have the relation $\eta_1^2(1_M\wedge\pi_P)=\eta_2^2(1_M\wedge\pi_P)=0$.*

Proof. (i) and (ii) follow from (8.6) and (8.7). (iii) follows from (8.3), (8.6), (8.7), and the following:

$$\begin{aligned}2\cdot 1_M\wedge\xi &= (i\eta\pi)\wedge\xi = (S^2i)(\eta\wedge\xi)(\pi\wedge 1_P), \qquad \eta\in\{S^2, S^1\},\\ &= (S^2i)(1_S\wedge\xi)(\eta\wedge 1_P)(\pi\wedge 1_P)\\ &= 3\cdot(S^2i)(1_S\wedge\xi)(1_S\wedge i_P)\nu(\pi\wedge\pi_P) \qquad \text{by (8.5)}\\ &= 6\cdot(S^2i)\nu(\pi\wedge\pi_P)\in 2\cdot(1\wedge\pi_P)^*\{S^4M, S^2M\} = 0.\end{aligned}$$

(iv) follows from

$$(1_M\wedge\eta)^*\eta_i = \eta_i^2, \qquad i = 1, 2. \qquad \text{q.e.d.}$$

Next we compute the group $\{M_2\wedge P, M_2\wedge P\}$.

Theorem 8.3. *$\{M_2\wedge P, M_2\wedge P\} = Z_4\oplus Z_2$ generated by $1_{M\wedge P}$ and $1_M\wedge i_P\xi$. We have a relation*

$$2\cdot 1_{M\wedge P} = (i\wedge i_P)\nu(\pi\wedge\pi_P) = 1_M\wedge i_P\xi+1_M\wedge\xi\pi_P \neq 0.$$

Proof. Let us consider an exact sequence (1.5′):

$$\{M\wedge P, S^3M\} \xrightarrow{(1\wedge\eta)_*} \{M\wedge P, S^2M\} \xrightarrow{(1\wedge i_P)_*} \{M\wedge P, M\wedge P\} \xrightarrow{(1\wedge\pi_P)^*} \{M\wedge P, S^4M\} .$$

$(1\wedge\pi_P)_*$ is epimorphic since $(1\wedge\pi_P)_*1_{M\wedge P}=1\wedge\pi_P$ generates $\{M\wedge P, S^4M\}$ (Theorem 8.2, (i)). $(1\wedge i_P)_*$ is monomorphic since $\{M\wedge P, S^3M\}$ is generated by $\eta_1(1\wedge\pi_P)$ and $(1\wedge\eta)_*\eta_1(1\wedge\pi_P)=\eta_1^2(1\wedge\pi_P)=0$ (Theorem 8.2, (ii) and (iv)). The image group $(1\wedge i_P)_*\{M\wedge P, S^2M\}\cong Z_2\oplus Z_2$ is generated by $1_M\wedge i_P\xi$ and $(i\wedge i_P)\nu(\pi\wedge\pi_P)$ (Theorem 8.2, (iii)). Finally, by Theorem 8.1,

$$2\cdot 1_{M\wedge P} = 3\cdot(i\wedge i_P)\nu(\pi\wedge\pi_P) = (i\wedge i_P)\nu(\pi\wedge\pi_P)$$

since it is of order 2 at most. Thus the theorem is proved.

8.3. We shall discuss some structures of $M_2\wedge N_2$ and $M_2\wedge M_2\wedge M_2$. Let $\pi_1: N_2\to P$ be the map collapsing S^3 such that

(8.8) $$\pi_1 i_0 = i_P \quad and \quad \pi_P\pi_1 = (S^2\pi)\pi_0$$

stably. We have a cofibration

(8.8′) $$S^3 \xrightarrow{i_1} N_2 \xrightarrow{\pi_1} P .$$

Proposition 8.4. *There exists a stable homotopy equivalence*

$$\varepsilon\in\{S^3M_2\vee(M_2\wedge P), M_2\wedge N_2\}$$

such that $(1_M\wedge\pi_1)\varepsilon$ *is stably homotopic to the projection of* $S^3M_2\vee(M_2\wedge P)$ *onto* $M_2\wedge P$.

Proof. By (4.6), (ii), there exists a homotopy equivalence $SN_2\simeq S^4\cup_{2\cdot\pi_P}CP$ such that $S\pi_1$ is equivalent to the map $S^4\cup_{2\cdot\pi_P}CP\to SP$ collapsing S^4 to a point. Denoting N_2 by N,

$$S(M\wedge N) = M\wedge SN\simeq M\wedge(S^4\cup_{2\cdot\pi_P}CP) = S^4M\cup_{1_M\wedge 2\cdot\pi_P}C(M\wedge P) .$$

By Theorem 8.2, (i), $1_M\wedge 2\cdot\pi_P=2\cdot(1_M\wedge\pi_P)\in 2\cdot\{M\wedge P, S^4M\}=0$. Thus, by a general argument (cf., the beginning part of the proof of Lemma 4.3) we can conclude the proposition.

Proposition 8.5. *Choosing an element* $\bar{\alpha}$ *of Lemma* 4.3, *we have a stable homotopy equivalence*

$$\tilde{\alpha}\in\{S^3M_2\vee(M_2\wedge P)\vee S^3M_2, M_2\wedge M_2\wedge M_2\}$$

such that

$$(1_M\wedge\alpha)\varepsilon = \tilde{\alpha}k ,$$

where ε *is the stable map of Proposition* 8.4 *and* $k: S^3M_2 \vee (M_2\wedge P) \to S^3M_2\vee(M_2\wedge P)\vee S^3M_2$ *the inclusion to the first two factors.*

Proof. For $1_M\wedge 2\cdot i_0 \in \{S^2M, M\wedge N\}$ we have

$$\begin{aligned}1_M\wedge 2\cdot i_0 &= 2\cdot 1_M\wedge i_0 = (i\eta\pi)\wedge i_0 = (i\pi)\wedge\eta\wedge i_0\\ &= (i\pi)\wedge 2\cdot i_1 = (2\cdot i\pi)\wedge i_1 = 0\,.\end{aligned}$$

Thus

$$M\wedge \bar{N} = M\wedge(N\cup_{2\cdot i_0} CS^2) = (M\wedge N)\cup_{1_M\wedge 2\cdot i_0} CS^2M$$

is stably homotopy equivalent to $(M\wedge N)\vee S^3M$ preserving the subspace $M\wedge N$, i.e., we have a stable homotopy equivalence $\varepsilon_1\in\{M\wedge\bar{N}, (M\wedge N)\vee S^3M\}$ such that $\varepsilon_1(1_M\wedge j)=k_0$, the inclusion map: $M\wedge N\to(M\wedge N)\vee S^3M$. Put

$$\tilde{\alpha} = (1_M\wedge\bar{\alpha})\varepsilon_1^{-1}(\varepsilon\vee 1_{S^3M})\,,$$

which is a stable homotopy equivalence. And

$$\begin{aligned}\tilde{\alpha}k &= (1_M\wedge\bar{\alpha})\varepsilon_1^{-1}k_0\varepsilon = (1_M\wedge\bar{\alpha})(1_M\wedge j)\varepsilon \quad \text{by Proposition 8.4}\\ &= (1_M\wedge\alpha)\varepsilon\,.\end{aligned}$$

q.e.d.

By Theorems 8.2, 8.3, Propositions 8.4 and 8.5 we have

Theorem 8.6. (i) $\{M_2\wedge P, M_2\wedge N_2\}\approx Z_4\oplus Z_2\oplus Z_2$.
(ii) $\{M_2\wedge P, M_2\wedge M_2\wedge M_2\}\approx Z_4\oplus Z_2\oplus Z_2\oplus Z_2$.

8.4. The following two lemmas will be used in the next section.

Lemma 8.7. *We can choose an element* $p_0\in\{M_2\wedge P, S^2N_2\}$ *satisfying the relations*:

(8.9) $$(S^2\pi_0)p_0 = 1_M\wedge\pi_P,\ p_0(1_M\wedge i_P) = S^2(i_0\pi)$$

and

(8.10) $$(S^2\pi_1)p_0 = (1_P\wedge\pi)T \qquad \textit{for } T = T(M_2, P)\,.$$

Proof. Consider the complex $M\wedge P/S^1\wedge S^2$. It has the same cell structure as S^2N, hence we have a homotopy equivalence $M\wedge P/S^1\wedge S^2 \simeq S^2N$. The map collapsing $S^1\wedge S^2$ followed by the homotopy equivalence gives rise to a map $p_0': M\wedge P\to S^2N$ such that

$$S^3 = S^1\wedge S^2 \xrightarrow{i\wedge i_P} M\wedge P \xrightarrow{p_0'} S^2N$$

is a cofibration. The candidates for p_0 are p_0' and $p_0'+p_0'(1_M\wedge\xi\pi_P)$, $\xi\in\{S^4, P\}$. First we see easily that both candidates satisfy (8.9) because both sides of the equalities induce the same mod 2 homology maps, and $\{M\wedge P, S^4M\}\approx Z_2$ and $\{S^2M, S^2N\}\approx Z_2$ as is easily seen.

Discussing an exact sequence (1.5) for the cofibration $S^1\wedge P \xrightarrow{i\wedge 1_P} M\wedge P \xrightarrow{\pi\wedge 1_P} S^2\wedge P$ and $X=S^2P$ by making use of Theorem 8.1, we see that

$$\{M\wedge P, S^2P\} \cong Z_2\oplus Z_2 \text{ generated by } (1_P\wedge\pi)T \text{ and } (S^2\tilde{\xi})(\pi_P\wedge\pi)T.$$

Thus

$$(\#1) \qquad (S^2\pi_1)p_0' = (1_P\wedge\pi)T$$

or

$$(\#2) \qquad (S^2\pi_1)p_0' = (1_P\wedge\pi)T+(S^2\tilde{\xi})(\pi_P\wedge\pi)T$$

since $(S^2\pi_1)p_0'$ induce a non-zero mod 2 homology map. If (#1) holds, then putting $p=p_0'$ we finish the proof. If (#2) holds, then put

$$p_0 = p_0'+p_0'(1_M\wedge\tilde{\xi}\pi_P).$$

Since

$$\begin{aligned}(1_P\wedge\pi)T(1_M\wedge\tilde{\xi}\pi_P) &= (1_P\wedge\pi)(\tilde{\xi}\pi_P\wedge 1_M)T \\ &= (S^2\tilde{\xi})(\pi_P\wedge\pi)T\end{aligned}$$

and

$$\begin{aligned}(\pi_P\wedge\pi)T(1_M\wedge\tilde{\xi}\pi_P) &= (\pi_P\wedge\pi)(\tilde{\xi}\pi_P\wedge 1_M)T \\ &= 2\cdot(\pi_P\wedge\pi)T = 0,\end{aligned}$$

we see that

$$\begin{aligned}(S^2\pi_1)p_0 &= (S^2\pi_1)p_0'+(S^2\pi_1)p_0'(1_M\wedge\tilde{\xi}\pi_P) \\ &= (1_P\wedge\pi)T+(S^2\tilde{\xi})(\pi_P\wedge\pi)T \\ &\quad +(1_P\wedge\pi)T(1_M\wedge\tilde{\xi}\pi_P)+(S^2\tilde{\xi})(\pi_P\wedge\pi)T(1_M\wedge\tilde{\xi}\pi_P) \\ &= (1_P\wedge\pi)T.\end{aligned}$$

q.e.d.

Lemma 8.8. *For any $\alpha\in\{N_2, M_2\wedge M_2\}$ satisfying $(1_M\wedge\pi)\alpha=\pi_0$ there exists an element $\kappa=\kappa_\alpha\in\{M_2\wedge P, M_2\wedge N_2\}$ such that*

$$(8.11) \qquad (1_M\wedge\pi_1)\kappa = 1_{M\wedge P} \quad \textit{and} \quad (1_M\wedge\pi_0)\kappa = (S^2\alpha)p_0.$$

Proof. Consider the following commutative diagram of exact rows:

$$\begin{array}{ccccc}\{M\wedge P, S^2M\} & \xrightarrow{(1\wedge i_0)_*} & \{M\wedge P, M\wedge N\} & \xrightarrow{(1\wedge\pi_0)_*} & \{M\wedge P, M\wedge S^2M\} \\ \| & & \downarrow{\scriptstyle (1\wedge\pi_1)_*} & & \downarrow{\scriptstyle (1\wedge S^2\pi)_*} \\ \{M\wedge P, S^2M\} & \xrightarrow{(1\wedge i_P)_*} & \{M\wedge P, M\wedge P\} & \xrightarrow{(1\wedge\pi_P)_*} & \{M\wedge P, S^4M\}.\end{array}$$

The group $\{M\wedge P, M\wedge S^2M\}$ has 4 elements because of Theorem 8.2,

(i) and (ii), and (1.7′). By Theorems 8.2, (iii), and 8.6, (i), the groups $\{M\wedge P, S^2M\}$ and $\{M\wedge P, M\wedge N\}$ have 4 and 16 elements, respectively. Hence $(1\wedge\pi_0)_*$ is epimorphic and there exists an element $\kappa'\in\{M\wedge P, M\wedge N\}$ such that

$$(1\wedge\pi_0)_*\kappa' = (S^2\alpha)p_0 .$$

Now

$$\begin{aligned}(1\wedge\pi_P)_*1_{M\wedge P} &= 1\wedge\pi_P = (S^2\pi_0)p_0 && \text{by } (8.9)\\ &= (1\wedge S^2\pi)(S^2\alpha)p_0 && \text{by assumption}\\ &= (1\wedge S^2\pi)_*(1\wedge\pi_0)_*\kappa' \\ &= (1\wedge\pi_P)_*(1\wedge\pi_1)_*\kappa' && \text{by } (8.8).\end{aligned}$$

Thus

$$1_{M\wedge P}-(1\wedge\pi_1)_*\kappa' = (1\wedge i_P)_*x$$

for some element $x\in\{M\wedge P, S^2M\}$. Put

$$\kappa = \kappa'+(1\wedge i_0)_*x ;$$

then

$$\begin{aligned}(1\wedge\pi_1)\kappa &= (1\wedge\pi_1)_*\kappa'+(1\wedge\pi_1)_*(1\wedge i_0)_*x \\ &= (1\wedge\pi_1)_*\kappa'+(1\wedge i_P)_*x && \text{by } (8.8)\\ &= 1_{M\wedge P}\end{aligned}$$

and

$$\begin{aligned}(1\wedge\pi_0)\kappa &= (1\wedge\pi_0)_*\kappa'+(1\wedge\pi_0)_*(1\wedge i_0)_*x \\ &= (S^2\alpha)p_0 .\end{aligned}$$

q.e.d.

9. Associativity of mod 2 multiplications.

9.1. Let μ be a commutative and associative multiplication in $\tilde{h}$, and assume that $\eta^{**}=0$ in $\tilde{h}$. Under this assumption the exact sequence of $\tilde{h}$ associated to the cofibration $S^2 \xrightarrow{i_P} P \xrightarrow{\pi_P} S^4$ of coefficients breaks into the following short exact sequences

$$(9.1)\qquad 0\to\tilde{h}^k(W\wedge S^4)\xrightarrow{(1\wedge\pi_P)^*}\tilde{h}^k(W\wedge P)\xrightarrow{(1\wedge i_P)^*}\tilde{h}^k(W\wedge S^2)\to 0$$

for any W and k. In particular, for $W=S^0$ and $k=2$, we can choose an element $\gamma_1\in\tilde{h}^2(P)$ such that

$$(9.2)\qquad i_P^*\gamma_1 = \sigma^2 1 .$$

Put

$$(9.3)\qquad \gamma_0 = \pi_1^*\gamma_1 .$$

Then, because of (8.8), γ_0 defined by (9.3) satisfies (5.3). Hence any multipilcation μ_2 constructed in §5 by making use of the above γ_0 is admissible. We discuss the associativity of such a μ_2.

Choosing an element γ_1 satisfying (9.2), we define a homomorphism

$$\tilde{\gamma} = \tilde{\gamma}_W : \tilde{h}^k(W\wedge P) \to \tilde{h}^k(W\wedge S^4)$$

for any W by

(9.4) $$\tilde{\gamma}_W(x) = (1_W\wedge\pi_P)^{*-1}(x-\mu(\sigma^{-2}(1_W\wedge i_P)^*x\otimes\gamma_1)),$$

$x\in\tilde{h}^k(W\wedge P)$. Since

$$(1_W\wedge i_P)^*\mu(\sigma^{-2}(1_W\wedge i_P)^*x\otimes\gamma_1) = \mu(\sigma^{-2}(1_W\wedge i_P)^*x\otimes\sigma^2 1) = (1_W\wedge i_P)^*x,$$

$x-\mu(\sigma^{-2}(1_W\wedge i_P)^*x\otimes\gamma_1)$ is in the kernel of $(1_W\wedge i_P)^*$. By (9.1) $(1_W\wedge\pi_P)^*$ is monomorphic. Thus $\tilde{\gamma}$ is a well-define homomorphism.

Similarly as in Lemma 5.2, (i) and (ii), we see

Lemma 9.1. (i) *$\tilde{\gamma}_W$ is a left inverse of $(1_W\wedge\pi_P)^*$.*
(ii) *$\tilde{\gamma}$ is natural in the sense that*

$$(S^4 f)^*\tilde{\gamma}_W = \tilde{\gamma}_{W'}(f\wedge 1_P)^*$$

for $f: W'\to W$.

9.2. The following lemmas are crucial in later discussions. We define γ_W by using $\pi_1^*\gamma_1$ as γ_0.

Lemma 9.2. *For the element $p_0\in\{M_2\wedge P, S^2N_2\}$ of Lemma 8.7 there holds the relation*

$$\tilde{\gamma}_{W\wedge M}(1_W\wedge p_0)^* = \sigma^2\gamma_W\sigma^{-2}.$$

Proof. For any $x\in\tilde{h}^k(W\wedge N\wedge S^2)$ we have

$$\begin{aligned}
&(1_{W\wedge M}\wedge\pi_P)^*\sigma^2\gamma_W\sigma^{-2}x\\
&= (1_{W\wedge M}\wedge\pi_P)^*(1_W\wedge S^2\pi_0)^{*-1}(x-\sigma^2\mu(\sigma^{-2}(1_W\wedge i_0)^*\sigma^{-2}x\otimes\gamma_0))\\
&= (1_W\wedge p_0)^*x-(1_W\wedge p_0)^*\sigma^2\mu(\sigma^{-2}(1_W\wedge i_0)^*\sigma^{-2}x\otimes\pi_1^*\gamma_1)\\
&\qquad\qquad \text{by (8.9) and (9.3)}\\
&= (1_W\wedge p_0)^*x-(1_W\wedge(T_1(S^2\pi_1)p_0))^*\mu((1_W\wedge i_0)^*\sigma^{-2}x\otimes\gamma_1)\\
&\qquad\qquad \text{for } T_1=T(P, S^2)\\
&= (1_W\wedge p_0)^*x-(1_W\wedge\pi\wedge 1_P)^*\mu((1_W\wedge i_0)^*\sigma^{-2}x\otimes\gamma_1) \quad \text{by (8.10)}\\
&= (1_W\wedge p_0)^*x-\mu(\sigma^{-2}(1_W\wedge S^2(i_0\pi))^*x\otimes\gamma_1)\\
&= (1_W\wedge p_0)^*x-\mu(\sigma^{-2}(1_{W\wedge M}\wedge i_P)^*(1_W\wedge p_0)^*x\otimes\gamma_1) \quad \text{by (8.9)}\\
&= (1_{W\wedge M}\wedge\pi_P)^*\tilde{\gamma}_{W\wedge M}(1_W\wedge p_0)^*x \quad \text{by (9.4)}.
\end{aligned}$$

Since $(1_{W\wedge M}\wedge\pi_P)^*$ is monomorphic, the proof is complete.

Lemma 9.3. *For* $\alpha\in\{N_2, M_2\wedge M_2\}$ *and* $\kappa=\kappa_\alpha\in\{M_2\wedge P, M_2\wedge N_2\}$ *of Lemma* 8.8 *there holds the relation*

$$\gamma_W(1_W\wedge\alpha)^*\sigma^{-2}\gamma_{W\wedge M}=\sigma^{-2}\tilde{\gamma}_{W\wedge M}(1_W\wedge\kappa)^*.$$

Proof. For any $x\in\tilde{h}^k(W\wedge M\wedge N)$,

$$\begin{aligned}&(1_{W\wedge M}\wedge\pi_0)^*\gamma_{W\wedge M}x\\&\quad=x-\mu(\sigma^{-2}(1_{W\wedge M}\wedge i_0)^*x\otimes\pi_1^*\gamma_1)\\&\quad=x-(1_{W\wedge M}\wedge\pi_1)^*\sigma^{-2}(1_{W\wedge M}\wedge(i_0\wedge 1_P)T_1)^*\mu(x\otimes\gamma_1)\end{aligned}$$

for $T_1=T(P, S^2)$. Then, by (8.11),

$$\begin{aligned}&(1_W\wedge p_0)^*(1_W\wedge S^2\alpha)^*\gamma_{W\wedge M}x\\&\quad=(1_W\wedge\kappa)^*(1_{W\wedge M}\wedge\pi_0)^*\gamma_{W\wedge M}x\\&\quad=(1_W\wedge\kappa)^*x-x',\end{aligned}$$

where $x'=\sigma^{-2}(1_{W\wedge M}\wedge(i_0\wedge 1_P)T_1)^*\mu(x\otimes\gamma_1)$. Here

$$\begin{aligned}&(1_{W\wedge M}\wedge i_P)^*x'\\&\quad=\sigma^{-2}(1_{W\wedge M}\wedge(i_0\wedge 1_P)T_1(S^2i_P))^*\mu(x\otimes\gamma_1)\\&\quad=\sigma^{-2}(1_{W\wedge M}\wedge(i_0\wedge 1_P)(1_S\wedge i_P))^*\mu(x\otimes\gamma_1),\qquad S=S^2,\\&\quad=\sigma^{-2}(1_{W\wedge M}\wedge i_0\wedge i_P)^*\mu(x\otimes\gamma_1)\\&\quad=\sigma^{-2}\mu((1_{W\wedge M}\wedge i_0)^*x\otimes\sigma^2 1)\\&\quad=(1_{W\wedge M}\wedge i_0)^*x.\end{aligned}$$

Hence

$$\begin{aligned}\tilde{\gamma}_{W\wedge M}x'&=(1_{W\wedge M}\wedge\pi_P)^{*-1}(x'-\mu(\sigma^{-2}(1_{W\wedge M}\wedge i_0)^*x\otimes\gamma_1))\\&=(1_{W\wedge M}\wedge\pi_P)^{*-1}(x'-x')=0.\end{aligned}$$

Thus

$$\begin{aligned}\tilde{\gamma}_{W\wedge M}(1_W\wedge\kappa)^*x&=\tilde{\gamma}_{W\wedge M}((1_W\wedge\kappa)^*x-x')\\&=\tilde{\gamma}_{W\wedge M}(1_W\wedge p_0)^*(1_W\wedge S^2\alpha)^*\gamma_{W\wedge M}x\\&=\sigma^2\gamma_W(1_W\wedge\alpha)^*\sigma^{-2}\gamma_{W\wedge M}x\qquad\text{by Lemma 9.2,}\end{aligned}$$

i.e.,

$$\sigma^{-2}\tilde{\gamma}_{W\wedge M}(1_W\wedge\kappa)^*=\gamma_W(1_W\wedge\alpha)^*\sigma^{-2}\gamma_{W\wedge M}.\qquad\text{q.e.d.}$$

9.3. For any element $\xi\in\{M_2\wedge P, M_2\wedge M_2\wedge M_2\}$ we define a *triple product*

(9.5) $\qquad\tau_\xi:\tilde{h}^i(X;Z_2)\otimes\tilde{h}^j(Y;Z_2)\otimes\tilde{h}^k(Z;Z_2)\to\tilde{h}^{i+j+k}(W;Z_2)$

as the composition

(9.5′)
$$\begin{aligned}\tau_\xi = \sigma^{-4}\tilde{\gamma}_{W\wedge M}\xi^{**}U^*\mu(1\otimes\mu):\\ \tilde{h}^i(X;Z_2)\otimes\tilde{h}^j(Y;Z_2)\otimes\tilde{h}^k(Z;Z_2)\\ = \tilde{h}^{i+2}(X\wedge M_2)\otimes\tilde{h}^{j+2}(Y\wedge M_2)\otimes\tilde{h}^{k+2}(Z\wedge M_2)\\ \to \tilde{h}^{i+j+k+6}(X\wedge M_2\wedge Y\wedge M_2\wedge Z\wedge M_2)\\ \to \tilde{h}^{i+j+k+6}(W\wedge M_2\wedge M_2\wedge M_2)\\ \to \tilde{h}^{i+j+k+6}(W\wedge M_2\wedge P)\\ \to \tilde{h}^{i+j+k+6}(W\wedge M_2\wedge S^4)\\ \to \tilde{h}^{i+j+k+2}(W\wedge M_2) = \tilde{h}^{i+j+k}(W;Z_2)\,,\end{aligned}$$

where $W=X\wedge Y\wedge Z$ and $U: W\wedge M_2\wedge M_2\wedge M_2\to X\wedge M_2\wedge Y\wedge M_2\wedge Z\wedge M_2$ is the map given by a permutation of factors as $U(x, y, z, m, m', m'')=(x, m, y, m', z, m'')$. τ_ξ is defined for all (i, j, k) and natural with respect to three variables X, Y and Z by Lemma 9.1.

Denote by $T(2, h)$ the set of all triple products τ_ξ, $\xi\in\{M_2\wedge P, M_2\wedge M_2\wedge M_2\}$. $\tau_\xi=\tau_{\xi'}$ if and only if they are equal as natural transformations for all (i, j, k). Clearly

$$\tau_{\xi+\xi'} = \tau_\xi+\tau_{\xi'}\,.$$

Thus $T(2, h)$ forms an additive group. Define a map

(9.6) $$\tau_2: \{M_2\wedge P, M_2\wedge M_2\wedge M_2\} \to T(2, h)$$

by $\tau_2(\xi)=\tau_\xi$. Then, by definitions,

(9.6′) *τ_2 is an epimorphism of groups.*

Since $\tilde{h}(\,;Z_2)$ is a functor of Z_2-modules by Proposition 3.2, $T(2, h)$ is a Z_2-module. Thus, by Theorem 8.6, (ii), and (9.6′),

(9.7) *$T(2, h)$ is a factor group of $Z_2\oplus Z_2\oplus Z_2\oplus Z_2$.*

9.4. Let $\alpha\in\{N_2, M_2\wedge M_2\}$ be an element of (4.18) and μ_2 be the multiplication in $\tilde{h}(\,;Z_2)$ defined by (5.6) by making use of this α and γ_0 of (9.3). By (4.18′) and Proposition 4.7, (i), both α and $\alpha'=T\alpha$, $T=T(M_2, M_2)$, satisfy the condition of Lemma 8.8. Let $\kappa=\kappa_\alpha$ and $\kappa'=\kappa_{\alpha'}$ be the elements satisfying (8.11) for α and α', respectively.

Lemma 9.4. *$\mu_2(1\otimes\mu_2)=\tau_\xi$ and $\mu_2(\mu_2\otimes 1)=\tau_\zeta$ for $\xi=(1_M\wedge\alpha)\kappa$ and $\zeta=T'(1_M\wedge\alpha)\kappa'$, $T'=T(M_2, M_2\wedge M_2)$.*

Proof. By definition

$$\mu_2(1\otimes\mu_2) = \sigma^{-2}\gamma\alpha^{**}(1_X\wedge T_2\wedge 1_M)^*\mu(1_{X\wedge M}\otimes\sigma^{-2}\gamma\alpha^{**}(1_Y\wedge T_1\wedge 1_M)^*\mu)$$
for $T_1=T(Z, M_2)$ and $T_2=T(Y\wedge Z, M_2)$
$$= \sigma^{-2}\gamma\alpha^{**}\sigma^{-2}(1_X\wedge T_2\wedge S^2 1_M)^*\gamma\alpha^{**}(1_{X\wedge M\wedge Y}\wedge T_2\wedge 1_M)^*\mu(1\otimes\mu)$$
by Lemma 5.2, (iv)
$$= \sigma^{-2}\gamma_W\alpha^{**}\sigma^{-2}\gamma_{W\wedge M}\alpha^{**}U^*\mu(1\otimes\mu)$$
for $W=X\wedge Y\wedge Z$ and $U=$ the map of (9.5′)
$$= \sigma^{-4}\tilde{\gamma}_{W\wedge M}(1_W\wedge\kappa)^*\alpha^{**}U^*\mu(1\otimes\mu) \quad \text{by Lemma 9.3}$$
$$= \tau_\xi.$$

Next, using the commutativity and the associativity of μ, we have

$$\mu(\mu_2\otimes 1)(x\otimes y\otimes z) = T_3^*\mu(z\otimes\mu_2(x\otimes y)) \quad \text{for } T_3=T(X\wedge Y\wedge M, Z\wedge M)$$
$$= T_3^*\sigma^{-2}\gamma\alpha^{**}(1_{Z\wedge M\wedge X}\wedge T_4\wedge 1_M)^*\mu(z\otimes\mu(x\otimes y))$$
for $T_4=T(Y, M)$
$$= T_3^*\sigma^{-2}\gamma\alpha^{**}(1_{Z\wedge M\wedge X}\wedge T_4\wedge 1_M)^*T_5^*\mu(\mu(x\otimes y)\otimes z)$$
for $T_5=T(Z\wedge M, X\wedge M\wedge Y\wedge M)$
$$= T_3^*\sigma^{-2}(T_6\wedge S^2 1_M)^*\gamma_{W\wedge M}(T'(1_M\wedge\alpha))^{**}U^*\mu(\mu\otimes 1)(x\otimes y\otimes z)$$
for $T_6=T(Z\wedge M, X\wedge Y)$
$$= (1_{X\wedge Y}\wedge T_7)^*\sigma^{-2}\gamma_{W\wedge M}(T'(1_M\wedge\alpha))^{**}U^*\mu(1\otimes\mu)(x\otimes y\otimes z)$$
for $T_7=T(M, Z\wedge M)$.

Thus,

$$\mu_2(\mu_2\otimes 1) = \sigma^{-2}\gamma_W\alpha^{**}(1_{X\wedge Y}\wedge T_1\wedge 1_M)^*\mu(\mu_2\otimes 1)$$
$$= \sigma^{-2}\gamma_W\alpha^{**}T^{**}\sigma^{-2}\gamma_{W\wedge M}(T'(1_M\wedge\alpha))^{**}U^*\mu(1\otimes\mu)$$
$$= \sigma^{-2}\gamma_W\alpha'^{**}\sigma^{-2}\gamma_{W\wedge M}(1_W\wedge T'(1_M\wedge\alpha))^*U^*\mu(1\otimes\mu)$$
$$= \sigma^{-4}\tilde{\gamma}_{W\wedge M}(1_W\wedge\kappa')^*(1_W\wedge T'(1_M\wedge\alpha))^*U^*\mu(1\otimes\mu)$$
by Lemma 9.3
$$= \tau_\zeta.$$
q.e.d.

Let μ_2' denote the admissible multiplication defined by

$$\mu_2'(x\otimes y) = T''^*\mu_2(y\otimes x)$$

for $x\in\tilde{h}^i(X;Z_2)$, $y\in\tilde{h}^j(Y;Z_2)$ and $T''=T(X, Y)$. Then, by (7.2) and the similar computations as above, we obtain

Lemma 9.5. *$\mu_2(1\otimes\mu_2')=\tau_{\xi'}$, $\mu_2'(1\otimes\mu_2)=\tau_{\xi''}$, $\mu_2(\mu_2'\otimes 1)=\tau_{\zeta'}$ and $\mu_2'(\mu_2\otimes 1)=\tau_{\zeta''}$ for $\xi'=(1_M\wedge\alpha')\kappa$, $\xi''=(1_M\wedge\alpha)\kappa'$, $\zeta'=T'(1_M\wedge\alpha')\kappa'$ and $\zeta''=T'(1_M\wedge\alpha)\kappa$.*

9.5. Here we discuss first the case of $h=K$.

Lemma 9.6. *In case* $h=K$, $\beta_2\mu_2(\mu_2\otimes 1)(1\otimes\delta_2\otimes\delta_2)$, $\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes 1\otimes\delta_2)$ *and* $\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes\delta_2\otimes 1)$ *are triple products.*

Proof. By Corollary 7.8 and (Λ_3) for μ_2 we have

$$\begin{aligned}\beta_2\mu_2(\mu_2\otimes 1)(1\otimes\delta_2\otimes\delta_2) &= \beta_2\mu_2(1\otimes\mu_2)(1\otimes\delta_2\otimes\delta_2)\\ &= \mu_2(1\otimes\mu_2')+\mu_2(1\otimes\mu_2)\,,\\ \beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes\delta_2\otimes 1) &= \mu_2(\mu_2'\otimes 1)+\mu_2(\mu_2\otimes 1)\,,\end{aligned}$$

and

$$\begin{aligned}&\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes 1\otimes\delta_2)\\ &\quad= \mu_2'(1\otimes\mu_2)+\mu_2(1\otimes\mu_2)+\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes\delta_2\otimes 1)\\ &\quad= \mu_2'(1\otimes\mu_2)+\mu_2(1\otimes\mu_2)+\mu_2(\mu_2'\otimes 1)+\mu_2(\mu_2\otimes 1)\,.\end{aligned}$$

Thus, as sums of triple products, they are triple products. q.e.d.

Lemma 9.7. $\mu_2(\mu_2\otimes 1)$, $\beta_2\mu_2(\mu_2\otimes 1)(1\otimes\delta_2\otimes\delta_2)$, $\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes 1\otimes\delta_2)$ *and* $\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes\delta_2\otimes 1)$ *form a base of* $T(2, K)$.

By Proposition 3.4, $\mu_2(\mu_2(\kappa_i\otimes\kappa_j)\otimes\kappa_k)$, $1\leqq i, j, k\leqq 2$, form a base of $\tilde{K}^*(M_2\wedge M_2\wedge M_2; Z_2)$. Applying the four triple products on $\kappa_2\otimes\kappa_1\otimes\kappa_1$, $\kappa_1\otimes\kappa_2\otimes\kappa_1$ and $\kappa_1\otimes\kappa_1\otimes\kappa_2$ we see their linear independence. Then, by (9.7), we conclude the lemma and see also that

(9.8) $$T(2, K)\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$

and

(9.8′) *in case* $h\cong K$, $\tau_2^{-1}(0) = 2\cdot\{M_2\wedge P, M_2\wedge M_2\wedge M_2\}$.

Lemma 9.8. $\mu_2(\mu_2\otimes 1)=\mu_2(1\otimes\mu_2)$ *in case* $h=K$.

Proof. By the above lemma we can express $\mu_2(1\otimes\mu_2)$ as a linear combination of $\mu_2(\mu_2\otimes 1)$, $\beta_2\mu_2(\mu_2\otimes 1)(1\otimes\delta_2\otimes\delta_2)$, $\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes 1\otimes\delta_2)$ and $\beta_2\mu_2(\mu_2\otimes 1)(\delta_2\otimes\delta_2\otimes 1)$. By (Λ_3) $\mu_2(1\otimes\mu_2)$ and $\mu_2(\mu_2\otimes 1)$ have the same values on $\kappa_2\otimes\kappa_1\otimes\kappa_1$, $\kappa_1\otimes\kappa_2\otimes\kappa_1$, and $\kappa_1\otimes\kappa_1\otimes\kappa_2$, which means theat

$$\mu_2(1\otimes\mu_2) = \mu_2(\mu_2\otimes 1)$$

by the argument in the proof of Lemma 9.7. q.e.d.

By (9.8′), Lemmas 9.4 and 9.8 we see that

$$\xi-\zeta\in 2\cdot\{M_2\wedge P, M_2\wedge M_2\wedge M_2\}\,,$$

that is,

$$\tau_\xi = \tau_\zeta$$

for any h satisfying $\eta^{**}=0$. Hence we obtain

Theorem 9. 9. *Let μ be a commutative and associative multiplication in $\tilde{h}$ and $\eta^{**}=0$ in $\tilde{h}$. There exists an associative admissible multiplication μ_2 in $\tilde{h}(\ ;Z_2)$.*

By Corollary 7. 7 and Theorem 9. 9 we have

Corollary 9. 10. *Under the assumption of Theorem 9. 9 and $\bar{\eta}^{**}=0$, there exists a commutative and associative admissible μ_2 in $\tilde{h}(\ ;Z_2)$.*

10. Associativity of mod q multiplications ($q \neq 2$).

10. 1. Let $\tilde{h}$ be given with a commutative and associative multiplication μ. For each element $\xi \in \{S^4M_q, M_q\wedge M_q\wedge M_q\}$ we define a *triple product*

(10. 1) $$\tau_\xi : \tilde{h}^i(X;Z_q)\otimes\tilde{h}^j(Y;Z_q)\otimes\tilde{h}^k(Z;Z_q) \to \tilde{h}^{i+j+k}(W;Z_q)$$

as the composition

(10. 1′) $$\begin{aligned}\tau_\xi &= \sigma^{-4}\xi^{**}U^*\mu(1\otimes\mu): \\ &\quad \tilde{h}^i(X;Z_q)\otimes\tilde{h}^j(Y;Z_q)\otimes\tilde{h}^k(Z;Z_q) \\ &= \tilde{h}^{i+2}(X\wedge M_q)\otimes\tilde{h}^{j+2}(Y\wedge M_q)\otimes\tilde{h}^{k+2}(Z\wedge M_q) \\ &\to \tilde{h}^{i+j+k+6}(X\wedge M_q\wedge Y\wedge M_q\wedge Z\wedge M_q) \\ &\to \tilde{h}^{i+j+k+6}(W\wedge M_q\wedge M_q\wedge M_q) \\ &\to \tilde{h}^{i+j+k+6}(W\wedge S^4M_q) \\ &\to \tilde{h}^{i+j+k+2}(W\wedge M_q) = \tilde{h}^{i+j+k}(W;Z_q),\end{aligned}$$

where $W=X\wedge Y\wedge Z$ and $U: W\wedge M_q\wedge M_q\wedge M_q \to X\wedge M_q\wedge Y\wedge M_q\wedge Z\wedge M_q$ is the similar map as the corresponding one in (9. 5′). τ_ξ is defined for all (i, j, k) and natural with respect to three variables X, Y and Z.

Similarly to 9. 3 we denote by $T(q, h)$ the set of all triple products τ_ξ, $\xi\in\{S^4M_q, M_q\wedge M_q\wedge M_q\}$. By an easy relation

$$\tau_{\xi+\zeta} = \tau_\xi+\tau_\zeta$$

$T(q, h)$ forms an additive group, and the map

(10. 2) $$\tau_q : \{S^4M_q, M_q\wedge M_q\wedge M_q\} \to T(q, h)$$

defined by $\tau_q(\xi)=\tau_\xi$ satisfies

(10. 2′) τ_q *is an epimorphism of groups.*

10. 2. We shall discuss the case $q\not\equiv 2$ (mod 4). Choosing an $\bar{\alpha}$ of (4. 17) and using notations of (7. 6), by Proposition 7. 9, (i), we have a direct sum decomposition

$$\begin{aligned}(10.3)\quad &\{S^4M_q, M_q\wedge M_q\wedge M_q\}\\ &= (1\wedge i\wedge i)_*\{S^4M_q, S^2M_q\}\oplus(i''\wedge i)_*\{S^4M_q, S^3M_q\}\\ &\oplus(1\wedge i''(S^2i))_*\{S^4M_q, S^3M_q\}\oplus((1\wedge i'')S^2i'')_*\{S^4M_q, S^4M_q\}\\ &\cong\begin{cases} Z_{(q,24)}\oplus Z_q & \textit{if } q \textit{ is odd}\\ (Z_2\oplus Z_2\oplus Z_2\oplus Z_{(q,24)})\oplus(Z_2\oplus Z_2\oplus Z_2)\oplus(Z_2\oplus Z_2\oplus Z_2) & \\ \quad\oplus(Z_q\oplus Z_2) & \textit{if } q\equiv 0 \pmod 4\end{cases}\end{aligned}$$

by Theorem 4.1, where $(1\wedge i\wedge i)_*$, $(i''\wedge i)_*$, $(1\wedge i''(S^2i))_*$ and $((1\wedge i'')S^2i'')_*$ are all monomorphic.

Here we make an assumption that

(10.4) $(\eta\pi_q)^{**}=(\nu\pi_q)^{**}=(1_M\wedge\eta)^{**}=0$ *in* $\tilde{h}$, *where* η *and* ν *are Hopf maps of* 1*-stem and* 3*-stem respectively, and* $M=M_q$.

Under this assumption $(i\eta\pi)^{**}=(i\nu\pi)^{**}=(i\eta^2\pi)^{**}=0$, $(\eta_1+\eta_2)^{**}=(1_M\wedge\eta)^{**}=0$ by Lemma 7.2, $\eta_1^{2**}=(i\eta\pi\eta_3)^{**}=0$ and $\eta_2^{2**}=(\eta_3 i\eta\pi)^{**}=0$ by Corollary 4.2, (i). Thus, defining A as a subgroup of $\{S^4M_q, M_q\wedge M_q\wedge M_q\}$ generated by $(1\wedge i\wedge i)_*(i\eta\pi)$ in case q odd, or by $\{(1\wedge i\wedge i)_*$-images of of η_1^2, η_2^2 and $i\nu\pi$, $(i''\wedge i)^*$- and $(1\wedge i''(S^2i))_*$-images of $\eta_1+\eta_2$ and $i\eta^2\pi$, $((1\wedge i'')S^2i'')_*(i\eta\pi)\}$, we see that τ_q factors through the projection: $\{S^4M_q, M_q\wedge M_q\wedge M_q\}\to\{S^4M_q, M_q\wedge M_q\wedge M_q\}/A=B$, say, and induces an epimorphism

$$(10.5)\qquad \tau_q': B\to T(q, h).$$

Now, by (10.3) and the definition of A we have

$$(10.5')\qquad B\cong\begin{cases} Z_q & \textit{if } q \textit{ is odd}\\ Z_2\oplus Z_2\oplus Z_2\oplus Z_q & \textit{if } q\equiv 0 \pmod 4).\end{cases}$$

Putting $\gamma_0=\pi'^*(\sigma^2 1)$, define an admissble μ_q by (5.6′). Then it satisfies (5.6″). Remarking that $\alpha i'=i''$ by (7.5)-(7.6), routine calculations give a proof of the following

Lemma 10.1. $\mu_q(1\otimes\mu_q)=\tau_\xi$ *and* $\mu_q(\mu_q\otimes 1)=\tau_\zeta$ *for* $\xi=(1_M\wedge i'')S^2i''$ *and* $\zeta=(i''\wedge 1_M)(1_M\wedge T)S^2i''$, *where* $M=M_q$ *and* $T=T(M_q, S^2)$.

In case $q\equiv 0 \pmod 4$, define elements

$$\tilde{i}'\in\{S^2M_q, N_q\}\quad\text{and}\quad\tilde{\pi}'\in\{N_q, SM_q\}$$

by

$$\tilde{i}'=i'+i_0\bar{\eta}\quad\text{and}\quad\tilde{\pi}'=\pi'+\bar{\eta}\pi_0.$$

Then, the obvious relation $1=i_0\pi'+i'\pi_0$ (which was used in the proof of (5.4′)) and $2\cdot\bar{\eta}=0$ (in the present case) imply the relation

$$1 = i_0\tilde{\pi}' + \tilde{i}'\pi_0 .$$

We define γ'_W as γ_W of (5.4) making use of $\tilde{\pi}'^*(\sigma^2 1)$ as γ_0; the above relation implies the formula

$$\gamma'_W = (1_W \wedge \tilde{i}')^*$$

as in the proof of (5.4′). Define an admissible μ'_q by (5.6′) making use of the α and the γ'_W, then we have

Lemma 10.2. $\mu_q(1\otimes\mu'_q)=\tau_\xi$, $\mu'_q(1\otimes\mu_q)=\tau_{\xi''}$ *and* $\mu_q(\mu'_q\otimes 1)=\tau_{\zeta'}$ *for* $\xi'=(1_M\wedge\alpha\tilde{i}')S^2 i''$, $\xi''=(1_M\wedge i'')S^2(\alpha\tilde{i}')$ *and* $\zeta'=(\alpha\tilde{i}'\wedge i'')(1_M\wedge T)S^2 i''$, $T=T(M_q, S^2)$.

The proof is routine likely as Lemma 10.1.

10.3. We shall discuss in a parallel way to 9.5, i.e., first the case $h=K$. In this case the assumption (10.4) is satisfied by Theorem 2.3.

Lemma 10.3. *In case $h=K$ and $q\equiv 0$* (mod 4),

$(q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(1\otimes\delta_q\otimes\delta_q)$, $(q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(\delta_q\otimes 1\otimes\delta_q)$
and $(q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(\delta_q\otimes\delta_q\otimes 1)$

are triple products.

Proof. By definitions

$$\begin{aligned}
\mu'_q-\mu_q &= \sigma^{-2}(\alpha i_0\bar{\eta})^{**}(1_X\wedge T\wedge 1_M)\mu && \text{for } T=T(Y, M_q)\\
&= \sigma^{-2}\bar{\eta}^{**}(i\wedge i)^{**}(1_X\wedge T\wedge 1_M)\mu && \text{by (4.18')}\\
&= (q/2)\cdot\beta\pi^{**}(i\wedge i)^{**}(1_X\wedge T\wedge 1_M)\mu && \text{by Proposition 6.4}\\
&= (q/2)\cdot\beta_q\mu_q(\delta_q\otimes\delta_q).
\end{aligned}$$

Then, as differences of triple products, the following are also triple products:

$$\begin{aligned}
\mu_q(1\otimes\mu'_q)-\mu_q(1\otimes\mu_q) &= \mu_q(1\otimes(\mu'_q-\mu_q))\\
&= (q/2)\cdot\mu_q(1\otimes\beta_q\mu_q(\delta_q\otimes\delta_q))\\
&= (q/2)\cdot\beta_q\mu_q(1\otimes\mu_q)(1\otimes\delta_q\otimes\delta_q)\\
&= (q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(1\otimes\delta_q\otimes\delta_q) \quad \text{by } (\Lambda_3),\\
\mu_q(\mu'_q\otimes 1)-\mu_q(\mu_q\otimes 1) &= (q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(\delta_q\otimes\delta_q\otimes 1),\\
\mu'_q(1\otimes\mu_q)-\mu_q(1\otimes\mu_q) &= (q/2)\cdot\beta_q\mu_q(\delta_q\otimes\delta_q\mu_q)\\
&= (q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(\delta_q\otimes\delta_q\otimes 1+\delta_q\otimes 1\otimes\delta_q).
\end{aligned}$$

q.e.d.

Lemma 10.4. *In case $q\equiv 0$* (mod 4),

$\mu_q(\mu_q\otimes 1)$, $(q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(1\otimes\delta_q\otimes\delta_q)$, $(q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(\delta_q\otimes 1\otimes\delta_q)$
and $(q/2)\cdot\beta_q\mu_q(\mu_q\otimes 1)(\delta_q\otimes\delta_q\otimes 1)$

generate $T(q, K)$.

The linear independence of the four triple products (with $\mu_q(\mu_q\otimes 1)$ replaced by $(q/2)\cdot\mu_q(\mu_q\otimes 1)$) over Z_2, can be seen in the same way as in the proof of Lemma 9.7. Since $\mu_q(\mu_q\otimes 1)$ is exactly of order q as is easily seen, the proof is completed by (10.5′). We can also conclude that

$$\tau_q'^{-1}(0) \cong 0 \tag{10.6}$$

by (10.5′) in case $h=K$ and $q\not\equiv 2 \pmod 4$.

Lemma 10.5. *In case $h=K$ and $q\not\equiv 2 \pmod 4$*

$$\mu_q(\mu_q\otimes 1) = \mu_q(1\otimes\mu_q).$$

Proof. A similar discussion as the proof of Lemma 9.8 shows that

$$\mu_q(1\otimes\mu_q) = s\cdot\mu_q(\mu_q\otimes 1)$$

for $s\not\equiv 0 \pmod q$. Then (Λ_3) implies that

$$s\equiv 1 \pmod q. \qquad \text{q.e.d.}$$

10.4. Lemma 10.5 and (10.6) imply that

$$\xi\equiv\zeta \mod A,$$

i.e.,

$$\tau_\xi = \tau_\zeta$$

for any h satisfying (10.4) for the case $q\not\equiv 2 \pmod 4$. Thus μ_q is associative under the assumption (10.4). Since we can choose the μ_q used in §7 as the same one used here, we obtain

Theorem 10.6. *Let $\tilde{h}$ be given with a commutative and associative multiplication. In case $q\not\equiv 2 \pmod 4$, if $\tilde{h}$ satisfies $(\eta\pi_q)^{**}=(\nu\pi_q)^{**}=(1_M\wedge\eta)^{**}=0$, $M=M_q$, then there exists a commutative and associative admissible multiplication in $\tilde{h}(\ ;Z_q)$, which is unique in case q odd.*

The conditions of Theorem 10.6 are satisfied always if q is prime to 2 and 3, or if q is odd and $(\nu\pi_q)^{**}=0$.

In case $q\equiv 2 \pmod 4$, from Theorems 3.14, 9.9 and 10.6 follows

Theorem 10.7. *Let $\tilde{h}$ be equipped with a commutative and associative multiplication μ and $q\equiv 2 \pmod 4$. If $\tilde{h}$ satisfies $\eta^{**}=\nu^{**}=0$ (or $\eta^{**}=\nu^{**}=\bar{\eta}^{**}=0$), then there exists an associative (or a commutative and associative) admissible multiplication in $\tilde{h}(\ ;Z_q)$.*

By Corollary 3.13, (6.1), Theorems 9.9, 10.6 and 10.7 we have

Corollary 10.8. *For any integer $q>1$ every admissible multiplication is associative.*

In case q odd, $\widetilde{KO}(\ ; Z_q)$ has a unique admissible μ_q (since $\widetilde{KO}^{-2}(S^0 ; Z_q) \cong 0$) and satisfies the assumption of Theorem 10. 6. Hence

Corollary 10. 9. *For q odd, the unique admissible multiplication in $\widetilde{KO}(\ ; Z_q)$ is commutative and associative.*

11. Multiplications in Bockstein spectral sequences.

11. 1. Let h be a cohomology theory and p a prime. Define a (mono-graded) exact couple [11]

$$\tilde{C}_1(X ; Z_p) = \{\tilde{D}_1^*(X ; Z_p),\ \tilde{E}_1^*(X ; Z_p),\ i_1,\ j_1,\ k_1\}$$

by putting

$$\tilde{D}_1^*(X ; Z_p) = \sum_i \tilde{D}_1^i(X ; Z_p), \quad \tilde{E}_1^*(X ; Z_p) = \sum_i \tilde{E}(X ; Z_p),$$
$$\tilde{D}_1^i(X ; Z_p) = \tilde{h}^i(X), \quad \tilde{E}_1^i(X ; Z_p) = \tilde{h}^i(X ; Z_p),$$
$$i_1 = p,\ j_1 = \rho_p \quad \text{and} \quad k_1 = \delta_{p,0},$$

where p is a map sending every element to its p times. The exactness of (2. 3) shows that $\tilde{C}_1(X ; Z_p)$ is an exact couple. From the successive derived couples

$$\tilde{C}_r(X ; Z_p) = (\tilde{D}_r^*(X ; Z_p),\ \tilde{E}_r^*(X ; Z_p),\ i_r,\ j_r,\ k_r\}, \qquad r \geq 1,$$

we obtain a (mono-graded) spectral sequence

$$\{\tilde{E}_r^*(X ; Z_p) = \sum_i \tilde{E}_r^i(X ; Z_p),\quad r \geq 1\} \quad \text{with} \quad d_r = j_r k_r,$$

which is called the *mod p Bockstein spectral sequence* of X for $\tilde{h}$.

The naturality is clear. Replacing X by X^+ we obtain the mod p Bockstein spectral sequence $\{E_r^*(X ; Z_p),\ r \geq 1\}$ of X for h. By the definition of derived couples we see that

$$\tilde{D}_r^*(X ; Z_p) = p^{r-1} \cdot \tilde{h}^*(X) \qquad \text{for} \quad r \geq 1.$$

When h is of finite type, i.e., h^i (a point) is finitely generated for each i, then $\tilde{h}^*(X)$ is of finite type for any X (finite CW-complexes); thus every Bockstein spectral sequences for $\tilde{h}$ converge to an $\tilde{E}_\infty^*$-term.

11. 2. The Bockstein spectral sequences for $\tilde{h}$ have many analogous properties to the ordinary Bockstein spectral sequences. The proofs are also similar to the ordinary cases if we use (2. 3), (2. 7) and (2. 10) (instead of the choice of canonical basis as is often seen in literature). So we give only a sketch of them and the proofs are left to readers.

Proposition 11.1. *$\rho_{p^r,p}:\tilde{h}^i(X;Z_{p^r})\to\tilde{h}^i(X;Z_p)$ and $\rho_p:\tilde{h}^i(X)\to\tilde{h}^i(X;Z_p)$ induce homomorphisms*

$$l_r:\tilde{h}^i(X;Z_{p^r})\to\tilde{E}^i_r(X;Z_p)$$

and

$$q_r:\tilde{h}^i(X)\to\tilde{E}^i_r(X;Z_p)\,,$$

and the following diagram

$$\begin{array}{ccccc}
\tilde{h}^{i-1}(X;Z_{p^r}) & \xrightarrow{\delta_{p^r,0}} & \tilde{h}^i(X) & \xrightarrow{\rho_{p^r}} & \tilde{h}^i(X;Z_{p^r}) \\
\downarrow l_r & & \downarrow p^{r-1} & \searrow q_r & \downarrow l_r \\
\tilde{E}^{i-1}_r(X;Z_p) & \xrightarrow{k_r} & \tilde{D}^i_r(X;Z_p) & \xrightarrow{j_r} & \tilde{E}^i_r(X;Z_p)
\end{array}$$

is commutative for all i and $r\geq 1$, where p^{r-1} is a map sending $x\in\tilde{h}^i(X)$ to $p^{r-1}\cdot x\in\tilde{D}^i_r(X;Z_p)$.

The proposition is clear for $r=1$ by putting $l_1=id$ and $q_1=\rho_p$. By an induction on r, we can prove this proposition.

Proposition 11.2. (i) *$q_r^{-1}(0)=\delta_{p^{r-1},0}\tilde{h}^{i-1}(X;Z_{p^{r-1}})+p\cdot\tilde{h}^i(X)$.*

(ii) *$l_r^{-1}(0)=\delta_{p^{r-1},p^r}\tilde{h}^{i-1}(X;Z_{p^{r-1}})+p_*\tilde{h}^i(X;Z_{p^{r-1}})$ for $p_*:\tilde{h}^i(X;Z_{p^{r-1}})\to\tilde{h}^i(X;Z_{p^r})$ of (2.6).*

(iii) *l_r is epimorphic.*

(iv) *$l_r\rho_{p^{r+1},p^r}\tilde{h}^i(X;Z_{p^{r+1}})=d_r^{-1}(0)$ in $\tilde{E}^i_r(X;Z_p)$.*

In the above proposition we regard $\tilde{h}^*(X;Z_{p^0})=\{0\}$. Properties (i)–(iv) of the above proposition can be proved by a simultaneous induction on r.

Corollary 11.3. *If h is of finite type, then the homomorphism*

$$q_\infty:\ \tilde{h}^i(X)\to\tilde{E}^i_\infty(X;Z_p)$$

is induced by ρ_p for each i and epimorphic, and

$$q_\infty^{-1}(0)=\sum_{s\geq 1}\delta_{p^s,0}\tilde{h}^{i-1}(X;Z_{p^s})+p\cdot\tilde{h}^i(X)\,.$$

For any abelian group G we identify $\mathrm{Tor}(G,Z_q)$ with the subgroup of G consisting of all $x\in G$ such that $q\cdot x=0$. Then, by the exactness of (2.3)

$$\delta_{p^r,0}\tilde{h}^{i-1}(X;Z_{p^r})=\mathrm{Tor}\,(\tilde{h}^i(X),Z_{p^r})\,.$$

Thus

(11.1) $$q_r^{-1}(0)=\mathrm{Tor}\,(\tilde{h}^i(X),Z_{p^{r-1}})+p\cdot\tilde{h}^i(X)$$

and

(11.2) $$l_r^{-1}(0) = \rho_r \operatorname{Tor}(\tilde{h}^i(X), Z_{p^{r-1}}) + p_* \tilde{h}^i(X; Z_{p^{r-1}})$$

for all i. If h is of finite type, then

$$\begin{aligned} q_\infty^{-1}(0) &= \sum_{s\geq 1} \operatorname{Tor}(\tilde{h}^i(X), Z_{p^s}) + p\cdot\tilde{h}^i(X) \\ &= p\text{-tors } \tilde{h}^i(X) + p\cdot\tilde{h}^i(X) \\ &= \text{tors } \tilde{h}^i(X) + p\cdot\tilde{h}^i(X), \end{aligned}$$

where tors G (or p-tors G) denotes the subgroup of G consisting of all torsion elements (or of all torsion elements of p-primary order). Thus we have

Theorem 11.4. *If h is of finite type, then*

$$\tilde{E}_\infty^i(X; Z_p) \cong (\tilde{h}^i(X)/\text{tors } \tilde{h}^i(X)) \otimes Z_p$$

for all i.

l_r, q_r and $\delta_{p^r,0}$ induce the corresponding maps in the following diagram (denoted with primes):

$$\begin{array}{ccc} \tilde{h}^{i-1}(X; Z'_{p^r})/l_r^{-1}(0) & \xrightarrow{\delta'_{p^r,0}} & \tilde{h}^i(X)/q_r^{-1}(0) \\ \downarrow l'_r & & \downarrow q'_r \\ \tilde{E}_r^{i-1}(X; Z_p) & \xrightarrow{d_r} & \tilde{E}_r^i(X; Z_p), \end{array}$$

which is commutative by Proposition 11.1. l'_r is isomorphic and q'_r is monomorphic by Proposition 11.2. Thus

$$\begin{aligned} d_r\tilde{E}_r^{i-1}(X; Z_p) &\cong \delta'_{p^r,0}(\tilde{h}^{i-1}(X; Z_{p^r})/l_r^{-1}(0)) \\ &\cong (\operatorname{Tor}(\tilde{h}^i(X), Z_{p^r}) + p\cdot\tilde{h}^i(X))/(\operatorname{Tor}(\tilde{h}^i(X), Z_{p^{r-1}}) + p\cdot\tilde{h}^i(X)) \\ &\cong \operatorname{Tor}(\tilde{h}^i(X), Z_{p^r}))/(\operatorname{Tor}(\tilde{h}^i(X), Z_{p^{r-1}}) + p\cdot\operatorname{Tor}(\tilde{h}^i(X), Z_{p^{r+1}})), \end{aligned}$$

by which we obtain

Theorem 11.5. *$d_r\tilde{E}_r^{i-1}(X; Z_p)$ is a Z_p-module for every i and $r\geq 1$. When h is of finite type, then* $\dim d_r\tilde{E}_r^{i-1}(X; Z_p)$ *is equal to the number of direct summands isomorphic to Z_{p^r} in a direct sum decomposition of* tors $\tilde{h}^i(X)$ *into cyclic groups of primary orders.*

By Theorems 11.4 and 11.5 we see that, in case h is of finite type, if we know Bockstein spectral sequences for all prime p then we can determine $\tilde{h}^*(X)$ additively.

11.3. Let p be a prime and $q=p^r$, $r\geq 1$. Denote $\pi_0: N_q\to S^2M_q$ by $\pi_{0,q}$ and, in case $q\neq 2$, $i': S^2M_q\to N_q$ and $\pi': N_q\to S^2$ by i'_q and π'_q respectively. Choosing a sequence $\{\bar{\alpha}_q\}$, $\bar{\alpha}_q\in\{\bar{N}_q, M_q\wedge M_q\}$ of (4.17) we get a sequence $\{\alpha_q\}$, $\alpha_q=\bar{\alpha}_q \bar{i} \in \{N_q, M_q\wedge M_q\}$, of elements of (4.18).

Let $\tilde{h}$ be given with a commutative and associative multiplication μ, and assume that $\eta^{**}=0$ in $\tilde{h}$ when $p=2$. Put $\gamma_0=\gamma_{0,q}=\pi_q'^{*}(\sigma^2 1)$ in case $q\neq 2$ (cf., (5.4')), and $\gamma_0=\gamma_{0,2}=\pi_1^*\gamma_1$ in case $q=2$ after choosing $\gamma_1\in\tilde{h}^2(P)$ such that $i_P^*\gamma_1=\sigma^2 1$ (cf., 9.1). Making use of α_q and $\gamma_{0,q}$ chosen above, we define an admissible multiplication μ_q for each q by (5.6'). μ_q is associative (or commutative and associative) under suitable conditions for $\tilde{h}$ (Theorems 9.9, 9.10, 10.6 and 10.7).

μ_p defines a multiplication in $\tilde{E}_1^*(\ ;Z_p)$. Since $d_1=\delta_p$ is a derivation by (Λ_2), the term $\tilde{E}_2^*(\ ;Z_p)$ has a multiplication induced from μ_p. Our next task is to prove that μ_p induces multiplications into successive terms $\tilde{E}_r^*(\ ;Z_p)$, $r\geq 3$, so that $\{\tilde{E}_r^*(\ ;Z_p),\ r\geq 1\}$ becomes a functor of spectral sequences with a multiplicative structure.

11.4. Consider the following diagram (in the stable range) for $r\geq 2$:

$$\begin{array}{ccccccc}
S^2M_q & \xrightarrow{i_q'} & N_q & \xrightarrow{\alpha_q} & M_q\wedge M_q & \xrightarrow{1\wedge\pi_q} & S^2M_q \\
\uparrow S^2\bar{1} & & & & \uparrow \bar{1}\wedge\bar{1} & & \uparrow S^2\bar{1} \\
S^2M_p & \xleftarrow{\pi_{0,p}} & N_p & \xrightarrow{\alpha_p} & M_p\wedge M_p & \xrightarrow{1\wedge\pi_p} & S^2M_p\,.
\end{array}$$

By an easy calculation making use of (2.5) we see that the right square is commutative, i.e.,

$$(1\wedge\pi_q)(\bar{1}\wedge\bar{1}) = (S^2\bar{1})(1\wedge\pi_p)\,. \tag{11.3}$$

The left square is generally not commutative. Nevertheless,

$$\begin{aligned}
&(1\wedge\pi_q)_*\{\alpha_q i_q'(S^2\bar{1})\pi_{0,p}-(\bar{1}\wedge\bar{1})\alpha_p\} \\
&\quad = \pi_{0,q} i_q'(S^2\bar{1})\pi_{0,p}-(S^2\bar{1})(1\wedge\pi_p)\alpha_p \qquad \text{by (4.18')} \\
&\quad = (S^2\bar{1})\pi_{0,p}-(S^2\bar{1})\pi_{0,p} = 0 \qquad \text{by (4.18') and (5.2)},
\end{aligned}$$

that is, there exists an element

$$b_q\in\{N_p,\ SM_q\}$$

such that

$$(1\wedge i_q)_* b_q = \alpha_q i_q'(S^2\bar{1})\pi_{0,p}-(\bar{1}\wedge\bar{1})\alpha_p\,. \tag{11.4}$$

We have

$$\begin{aligned}
\rho_{q,p}\mu_q &= \rho_{q,p}\sigma^{-2}(\alpha_q i_q')^{**}(1\wedge T_q\wedge 1)^*\mu \\
&= \sigma^{-2}(\alpha_q i_q'(S^2\bar{1}))^{**}(1\wedge T_q\wedge 1)^*\mu \\
&= \sigma^{-2}\gamma_W(\alpha_q i_q'(S^2\bar{1})\pi_{0,p})^{**}(1\wedge T_q\wedge 1)^*\mu
\end{aligned}$$

and

$$\mu_p(\rho_{q,p}\otimes\rho_{q,p}) = \sigma^{-2}\gamma_W\alpha_p^{**}(1\wedge T_p\wedge 1)^*\mu(\bar{1}^{**}\otimes\bar{1}^{**})$$
$$= \sigma^{-2}\gamma_W((\bar{1}\wedge\bar{1})\alpha_p)^{**}(1\wedge T_q\wedge 1)^*\mu\,,$$

where $T_q=T(Y, M_q)$, $W=X\wedge Y$ and $\gamma_W: \tilde{h}^k(W\wedge N_p)\to\tilde{h}^k(W\wedge S^2M_p)$. Thus by (11.4) we have

(11.5) $$\rho_{q,p}\mu_p-\mu_p(\rho_{q,p}\otimes\rho_{q,p}) = \sigma^{-2}\gamma_W((1\wedge i_q)b_q)^{**}(1\wedge T_q\wedge 1)^*\mu\,.$$

11.5. Here we shall discuss the groups $\{N_p, SM_q\}$ for $r\geq 2$.

The case $p=2$. By a similar discussion as the proof of Theorem 4.1 we see that

(11.6) $$\{SM_2, SM_q\} \approx Z_2\oplus Z_2$$

with generators $S\bar{1}$ *and* $(Si_q)\eta(S\pi_2)$, *and*

(11.7) $$\{S^2M_2, SM_q\} \approx Z_4\oplus Z_2$$

with generators $(Si_q)\bar{\eta}$ (of order 4) *and* $(S\bar{1})\tilde{\eta}(S^2\pi_2)$ (of order 2), *where* $\bar{\eta}\in\{S^2M_2, S^2\}$ *and* $\tilde{\eta}\in\{S^4, SM_2\}$.

Consider the following exact sequence (1.5) associated to the cofibration $S^2\to N_2\to S^2M_2$:

$$\{S^3, SM_q\}\xrightarrow{(\eta(S^2\pi))^*}\{S^2M_2, SM_q\}\xrightarrow{\pi_0^*}\{N_2, SM_q\}$$
$$\xrightarrow{i_0^*}\{S^2, SM_q\}\xrightarrow{(\eta(S\pi))^*}\{SM_2, SM_q\}\,.$$

$\{S^3, SM_q\}\approx Z_2$ generated by $(Si_q)\eta$ (by (4.2)), and

$$(\eta(S^2\pi_2))^*(Si_q)\eta = (Si_q)\eta^2(S^2\pi_2)$$
$$= 2\cdot(Si_q)\bar{\eta} \qquad \text{by (4.2')};$$

thus by (11.7) we see

(11.8) $$\pi_0^*\{S^2M_2, SM_q\} \approx Z_2\oplus Z_2$$

with generators $(Si_q)\bar{\eta}\pi_{0,2}$ *and* $(S\bar{1})\tilde{\eta}(S^2\pi_2)\pi_{0,2}$.

Next, $\{S^2, SM_q\}\approx Z_q$ generated by Si_q ((4.2)) and $(\eta(S\pi_2))^*(Si_q)=(Si_q)\eta(S\pi_2)$ is non-zero and of order 2 by (11.6); hence

(11.9) $$i_0^*\{N_2, SM_q\} \approx Z_{q/2}$$

generated by $2\cdot Si_q$.

Lemma 11.6. $\{N_2, SM_q\}\approx Z_2\oplus Z_q$ *with generators* $(Si_q)\bar{\eta}\pi_{0,2}$ (*of order* 2) *and* $(Si_q)\bar{\zeta}\pi_1$ (*of order* q) *for* $q=2^r$, $r\geq 2$, *where* $\bar{\zeta}\in\{P, S^2\}$ (cf., (8.3)).

Proof. First look at the group $\{P, SM_2\}$. We compute this group in two ways. Discussing an exact sequence (1.5′) for $S^1 \to M_2 \to S^2$, we see by (8.2) that

$$\{P, SM_2\} \cong Z_2 \quad \text{with a generator} \quad (Si_2)\xi .$$

On the other hand, discussing an exact sequence (1.5) for $S^2 \to P \to S^4$ we see by (4.2)-(4.2′) that

$$\{P, SM_2\} \cong Z_2 \quad \text{with generator} \quad \tilde{\eta}\pi_P .$$

Thus we see that

$$\tilde{\eta}\pi_P = (Si_2)\xi .$$

Then

$$(q/2)\cdot(Si_q)\xi\pi_1 = (S\bar{1})(Si_2)\xi\pi_1 \qquad \text{by (2.5)}$$
$$= (S\bar{1})\tilde{\eta}\pi_P\pi_1 = (S\bar{1})\tilde{\eta}(S^2\pi_2)\pi_{0,2} \qquad \text{by (8.8)}.$$

And

$$i_0^*((Si_q)\xi\pi_1) = (Si_q)\xi i_P \qquad \text{by (8.8)}$$
$$= 2\cdot Si_q \qquad \text{by (8.3)}.$$

These, combined with (11.8) and (11.9), prove the lemma.

The case p odd prime. Since $N_p = S^2 \vee S^2M_p$, we have the direct sum decomposition

$$\{N_p, SM_q\} \cong \pi_p'^*\{S^2, SM_q\} \oplus \pi_{0,p}^*\{S^2M_p, SM_q\} ,$$

where $\pi_p'^*$ and $\pi_{0,p}^*$ are monomorphic. Here, $\{S^2, SM_q\} \cong Z_q$ generated by Si_q ((4.2)), and $\{S^2M_p, SM_q\} \cong 0$ as is easily seen from an exact sequence (1.5). Thus we obtain

(11.10) $\{N_p, SM_q\} \cong Z_q$ *generated by* $(Si_q)\pi_p'$ *for p an odd prime and* $q = p^r$, $r \geq 2$.

11.6. Now we shall discuss the deviation

$$\rho_{q,p}\mu_q - \mu_p(\rho_{q,p} \otimes \rho_{q,p}) , \qquad q = p^r \text{ and } r \geq 2 .$$

The case $p=2$. First we prove

Lemma 11.7. *There exists a relation*

$$\pi_2 \wedge \xi = (S^2\tilde{\eta})(1_M \wedge \pi_P)$$

for $\xi \in \{P, S^2\}$ *and* $\tilde{\eta} \in \{S^2M_2, S^2\}$.

Proof. Discussing an exact sequence (1.5) for $M_2 \wedge S^2 \to M_2 \wedge P \to M_2S^4$ by (4.2) and (4.2′) we see that

$$\{M_2\wedge P, S^4\} \cong Z_2 \quad \text{generated by} \quad (S^2\bar{\eta})(1_M\wedge\pi_P)\,.$$

On the other hand, discussing an exact sequence (1.5) for $S^1\wedge P\to M_2\wedge P\to S^2\wedge P$ by (8.2) we see that

$$\{M_2\wedge P, S^4\} \cong Z_2 \quad \text{generated by} \quad \pi_2\wedge\xi\,.$$

Thus the lemma follows.

By Lemma 11.6 the element $b_q\in\{N_2, SM_q\}$ can be written as

$$b_q = \varepsilon\cdot(Si_q)\bar{\eta}\pi_{0,2}+\varepsilon'\cdot(Si_q)\xi\pi_1$$

with $\varepsilon\in Z_2$ and $\varepsilon'\in Z_q$. Then, by (11.5)

$$\begin{aligned}&\rho_{q,2}\mu_q-\mu_2(\rho_{q,2}\otimes\rho_{q,2})\\ &\quad= \varepsilon\cdot\sigma^{-2}\gamma_W((i_q\wedge i_q)\bar{\eta}\pi_{0,2})^{**}(1\wedge T_q\wedge 1)^*\mu\\ &\qquad+\varepsilon'\cdot\sigma^{-2}\gamma_W((i_q\wedge i_q)\xi\pi_1)^{**}(1\wedge T_q\wedge 1)^*\mu\,.\end{aligned}$$

Here

$$\begin{aligned}&\sigma^{-2}\gamma_W((i_q\wedge i_q)\bar{\eta}\pi_{0,2})^{**}(1\wedge T_q\wedge 1)^*\mu\\ &\quad= \sigma^{-2}\bar{\eta}^{**}(1_X\wedge T_1\wedge 1_S)^*\mu(\sigma\delta_{q,0}\otimes\sigma\delta_{q,0}) &&\text{by Lemma 5.2, (i)}\\ &\quad= \sigma^{-2}\bar{\eta}^{**}\sigma^2\mu(\delta_{q,0}\otimes\delta_{q,0})\,,\end{aligned}$$

where $T_1=T(Y, S^1)$ and $S=S^1$, and

$$\begin{aligned}&\sigma^{-2}\gamma_W((i_q\wedge i_q)\xi\pi_1)^{**}(1\wedge T_q\wedge 1)^*\mu\\ &\quad= \sigma^{-4}\tilde{\gamma}_{W\wedge M}(1_W\wedge p_0)^*\sigma^2(\xi\pi_1)^{**}\sigma^2\mu(\delta_{q,0}\otimes\delta_{q,0}) &&\text{by Lemma 9.2}\\ &\quad= \sigma^{-4}\tilde{\gamma}_{W\wedge M}((S^2\xi)(S^2\pi_1)p_0)^{**}\sigma^4\mu(\delta_{q,0}\otimes\delta_{q,0})\\ &\quad= \sigma^{-4}\gamma_{W\wedge M}((S^2\xi)(1_P\wedge\pi_2)T)^{**}\sigma^4\mu(\delta_{q,0}\otimes\delta_{q,0}) &&\text{by (8.10) for } T=T(M_2, P)\\ &\quad= \sigma^{-4}\gamma_{W\wedge M}(\pi_2\wedge\xi)^{**}\sigma^4\mu(\delta_{q,0}\otimes\delta_{q,0}) &&\text{since } T(S^2, S^2)\simeq 1\\ &\quad= \sigma^{-4}\gamma_{W\wedge M}(1_M\wedge\pi_P)^{**}(S^2\bar{\eta})^{**}\sigma^4\mu(\delta_{q,0}\otimes\delta_{q,0}) &&\text{by Lemma 11.7}\\ &\quad= \sigma^{-2}\bar{\eta}^{**}\sigma^2\mu(\delta_{q,0}\otimes\delta_{q,0}) &&\text{by Lemma 9.1, (i).}\end{aligned}$$

Therefore, for $x\in\tilde{h}^i(X; Z_q)$ and $y\in\tilde{h}^j(Y; Z_q)$,

$$\begin{aligned}&\rho_{q,2}\mu_q(x\otimes y)-\mu_2(\rho_{q,2}x\otimes\rho_{q,2}y)\\ &\quad= \varepsilon_r\cdot\sigma^{-2}\bar{\eta}^{**}\sigma^2\mu(\delta_{q,0}x\otimes\delta_{q,0}y)\\ &\quad= \varepsilon_r\cdot\mu(\mu(\delta_{q,0}x\otimes\delta_{q,0}y)\otimes\bar{\eta}^*1)\\ &\quad= \varepsilon_r\cdot\mu_L(\mu(\delta_{q,0}x\otimes\delta_{q,0}y)\otimes a)\\ &\quad= \varepsilon_r\cdot\mu_2(\mu_2(\delta_{q,2}x\otimes\delta_{q,2}y)\otimes a)\,,\end{aligned}$$

where $\varepsilon_r=\varepsilon+(\varepsilon' \bmod 2)\in Z_2$ and a is an element of $\tilde{h}^{-2}(S^0; Z_2)$ such that $a=\bar{\eta}^*1$ *via* the identification $\tilde{h}^{-2}(S^0; Z_2)=\tilde{h}^0(M_2)$. Since $\delta_{2,0}a=\sigma\eta^*1=0$, there exists an element $a_0\in\tilde{h}^{-2}(S^2)$ such that $a=\rho_2a_0$. Thus we obtain

Proposition 11.8. *Let $q=2^r$, $r\geq 2$, $x\in\tilde{h}^i(X;Z_q)$ and $y\in\tilde{h}^j(Y;Z_q)$. There holds the relation*

$$\begin{aligned}&\rho_{q,2}\mu_q(x\otimes y)-\mu_2(\rho_{q,2}x\otimes\rho_{q,2}y)\\&\quad=\varepsilon_r\cdot\mu_2(\mu_2(\delta_{q,2}x\otimes\delta_{q,2}y)\otimes\rho_2 a_0)\end{aligned}$$

for an element $a_0\in\tilde{h}^{-2}(S^0)$ (independent of r) and $\varepsilon_r\in Z_2$. If $\bar{\eta}^{**}=0$ in $\tilde{h}$, then the right hand side becomes zero.

The case p odd prime. By (11.10) the element $b_q\in\{N_p, SM_q\}$ can be written as

$$b_q=\varepsilon_q\cdot(Si_q)\pi'_p$$

with $\varepsilon_q\in Z_q$. Then, by (11.5)

$$\begin{aligned}&\rho_{q,p}\mu_q-\mu_p(\rho_{q,p}\otimes\rho_{q,p})\\&\quad=\varepsilon_q\cdot\sigma^{-2}\gamma_W\pi'^{**}_p(i_q\wedge i_q)^{**}(1\wedge T_q\wedge 1)^*\mu\\&\quad=\varepsilon_q\cdot\sigma^{-2}i'^{**}_p\pi'^{**}_p(i_q\wedge i_q)^{**}(1\wedge T_q\wedge 1)^*\mu&&\text{by }(5.4')\\&\quad=0&&\text{by }(5.2).\end{aligned}$$

Thus we obtain

Proposition 11.9. *Let p be an odd prime and $q=p^r$, $r\geq 2$. There holds the relation*

$$\rho_{q,p}=\mu_p(\rho_{q,p}\otimes\rho_{q,p}).$$

11.7. μ_p gives a multiplication m_1 on $\tilde{E}^*(\cdot Z_p)$ by putting $m_1=\mu_p$. Since δ_p is a derivation for μ_p, d_1 is also so for m_1; hence m_1 induces a multiplication m_2 in $\tilde{E}^*_2(\,;Z_p)$ passing to quotients. Assume that m_i, $2\leq i\leq r$, is defined so that m_i induces m_{i+1} passing to quotient $1\leq i\leq r-1$. Propositions 11.1, 11.8 and 11.9 show that

$$\begin{aligned}&l_r\mu_q(x\otimes y)-m_r(l_rx\otimes l_ry)\\&\quad=\begin{cases}0&\text{if }p\text{ odd}\\\varepsilon_r\cdot m_r(m_r(d_rl_rx\otimes d_rl_ry)\otimes q_ra_0)&\text{if }q=2\end{cases}\end{aligned}\tag{11.11}$$

for any $x\in\tilde{h}^i(X;Z_q)$ and $y\in\tilde{h}^j(Y;Z_q)$, where $q=p^r$, $\varepsilon_r\in Z_2$ and a_0 is an element of $\tilde{h}^{-2}(S^0)$. The right hand side of (11.11) vanishes if $d_rl_rx=0$ or $d_rl_ry=0$; in particular

$$\begin{aligned}&m_r(m_r(d_rl_rx\otimes d_rl_ry)\otimes q_ra_0)\\&\quad=m_r(m_r(l_r\delta_qx\otimes l_r\delta_qy)\otimes l_r\rho_qa_0)&&\text{by Proposition 11.1}\\&\quad=l_r\mu_q(\mu_q(\delta_qx\otimes\delta_qy)\otimes\rho_qa_0)\\&\quad=l_r\delta_q\mu_q(\mu_q(x\otimes\delta_qy)\otimes\rho_qa_0)\\&\quad=d_rm_r(m_r(l_rx\otimes d_rl_ry)\otimes q_ra_0).\end{aligned}$$

Hence

$$\begin{aligned} d_r m_r(l_r x \otimes l_r y) &= l_r \delta_q \mu_q(x \otimes y) \\ &= l_r \mu_q(\delta_q x \otimes y) + (-1)^i l_r \mu_q(x \otimes \delta_q y) \\ &= m_r(d_r l_r x \otimes l_r y) + (-1)^i m_r(l_r x \otimes d_r l_r y), \end{aligned}$$

where $\deg x = i$. Since l_r is epimorphic by Proposition 11. 2, (iii), the above formula shows that d_r is a derivation for m_r. Therefore m_r induces a multiplication m_{r+1} in $\tilde{E}^*_{r+1}(\ ; Z_p)$ by passing to quotients. Thus, by an induction on r, m_r is defined for all $r \geq 1$.

We saw also that d_r is a derivation for m_r and for all $r \geq 1$.

Next we shall discuss the commutativity of m_r for $p=2$ and $r \geq 2$. Since μ_q is commutative for $r \geq 2$, by (11. 11) we have

$$\begin{aligned} & m_r(l_r x \otimes l_r y) + \varepsilon_r \cdot m_r(m_r(d_r l_r x \otimes d_r l_r y) \otimes q_r a_0) \\ &= l_r \mu_q(x \otimes y) \\ &= l_r T^* \mu_q(y \otimes x) = T^* l_r \mu_q(y \otimes x) \\ &= T^* m_r(l_r y \otimes l_r x) + \varepsilon_r \cdot m_r(T^* m_r(d_r l_r y \otimes d_r l_r x) \otimes q_r a_0), \end{aligned}$$

where $T = T(X, Y)$ and the naturalities of l_r and m_r are used. This formula shows first that the commutativity relation holds if $d_r l_r x = 0$ or $d_r l_r y = 0$; in particular

$$T^* m_r(d_r l_r y \otimes d_r l_r x) = m_r(d_r l_r x \otimes d_r l_r y).$$

Thus

$$m_r(l_r x \otimes l_r y) = T^* m_r(l_r y \otimes l_r x)$$

for any $x \in \tilde{h}^i(X; Z_q)$ and $y \in \tilde{h}^j(Y; Z_q)$, i.e., m_r is commutative for $p=2$ and $r \geq 2$ by Proposition 11. 2, (iii).

Summarizing the above discussions we have

Theorem 11. 10. *Let $\tilde{h}$ be given with a commutative and associative multiplication μ, and assume that $\eta^{**}=0$ in $\tilde{h}$ in case $p=2$. For every prime p a suitable admissible multiplication μ_p induces a multiplication*

$$m_r : \tilde{E}^i_r(X; Z_p) \otimes \tilde{E}^j_r(Y; Z_p) \to \tilde{E}^{i+j}_r(X \wedge Y; Z_p)$$

(*in the sense that it is defined for any i, j, X and Y such that* i) *linear,* ii) *natural and* iii) *has a bilateral unit $1 \in \tilde{E}^0_r(S^0; Z_p)$*) *for each $r \geq 1$. m_r is compatible with μ in the sense that*

$$m_r(q_r \otimes q_r) = q_r \mu.$$

d_r is a derivation for m_r and m_r induces m_{r+1} by passing to quotients ($m_1 = \mu_p$). m_r is commutative for $r \geq 2$. (If p is odd or if $\bar{\eta}^{**}=0$ in $\tilde{h}$

then m_1 *is also commutative.) If* $p \neq 3$ *or if* $(\nu\pi_3)^{**}=0$ *in* $\tilde{h}$ *then* m_r *is associative.*

11.8. Finally consider the case $h=K$. Since $\tilde{K}^{\#}$ (a sphere) is torsion-free, every Bockstein spectral sequence of a sphere for $\tilde{K}$ collapses by Theorem 11.5, i.e., denoting the Z-graded r-th term by $E_r^{\#}$ in this case, we have

(11.12) $$\tilde{E}_r^{\#}(S^n; Z_p) \cong \tilde{K}_r^{\#}(S^n; Z_p) \qquad \text{for} \quad 1 \leq r \leq \infty .$$

In the present case, since $\eta^{**}=\nu^{**}=0$ in $\tilde{K}$, $\tilde{E}_r^{\#}(\ ; Z_p)$ has an associative multiplication m_r for every p and $r \geq 1$ with properties described in Theorem 11.10. Through the isomorphism (11.12) we have a natural map

(11.13) $$\beta_{p,r}: \tilde{E}_r^i(X; Z_p) \to \tilde{E}_r^{i-2}(X; Z_p), \qquad 1 \leq r \leq \infty,$$

defined by

(11.13′) $$\beta_{p,r} = m_r(\ \otimes q_r\sigma^{-2}g)$$

for any i and X, where g is the reduced Hopf bundle over S^2. Clearly

(11.14) $\beta_{p,1} = \beta_p$, *the* mod p *Bott isomorphism,*

(11.15) $d_r\beta_{p,r} = \beta_{p,r}d_r$ *and* $\beta_{p,r}$ *induces* $\beta_{p,r+1}$ *by passing to quotients.*

Since β_p is an isomorphism ((6.2)), (11.14)–(11.15) imply

(11.16) $\beta_{p,r}$ *is an isomorphism for every* p *and* $r \geq 1$.

From (6.4) follows

(11.17) $$m_r(\beta_{p,r} \otimes 1) = m_r(1 \otimes \beta_{p,r}) = \beta_{p,r}m_r .$$

Thus, identifying $\tilde{E}_r^i(X; Z_p)$ with $\tilde{E}_r^{i-2}(X; Z_p)$ by $\beta_{p,r}$, we obtain a functor of Z_2-graded spectral sequences

$$\tilde{E}_r^*(\ ; Z_p) = \tilde{E}_r^0(\ ; Z_p) \oplus \tilde{E}_r^1(\ ; Z_p), \qquad r \geq 1,$$

for each prime p with $\tilde{E}_1^*(\ ; Z_p) = \tilde{K}^*(\ ; Z_p)$ and with a multiplication $\{m_r^*\}$ induced by $\{m_r\}$, which coincides of course with the mod p Bockstein spectral sequence for $\tilde{K}^*$. Since d_r is a derivation for m_r^*, m_{r+1}^* is induced by m_r^* and $m_1^* = \mu_p^*$, the Künneth isomorphism (Theorem 6.2) implies inductively

Theorem 11.11. *In case* $h=K$, m_r^* *induces an isomorphism*

$$\tilde{E}_r^*(X; Z_p) \otimes \tilde{E}_r^*(Y; Z_p) \cong \tilde{E}_r^*(X \wedge Y; Z_p)$$

for any X and Y, $1 \leq r \leq \infty$, *and for each prime p.*

12. Appendix.

12.1. We consider here some properties of the maps $\bar{a}$ of (2.5).

Lemma 12.1. *Let a be any integer. The element* $\bar{a} \in \{M_2, M_2\}$ *satisfies the relation*

$$\bar{a} = a^2 \cdot 1_M = \begin{cases} 1_M & \text{if } a \text{ is odd} \\ 0 & \text{if } a \text{ is even}. \end{cases}$$

Proof. Let P^n be real projective n-spaces and $i: M_2 = P^2 \subset P^3$ the inclusion. Then, for $\bar{a}: M_2 \to M_2$

$$\{i\bar{a}\} \in [M_2, P^3] \cong [M_2, P^\infty] \cong H^1(M_2; Z_2) \cong Z_2.$$

Thus

$$i\bar{a} \simeq i\bar{b} \quad \text{if and only if} \quad a \equiv b \pmod 2.$$

The homomorphism

$$\{M_2, M_2\} \to \mathrm{Hom}(\widetilde{KO}(M_2), \widetilde{KO}(M_2))$$

defined by the assignment $\alpha \to \alpha^*$ is an isomorphism, and the map $i^*: \widetilde{KO}(P^3) \to \widetilde{KO}(M_2)$ is also an isomorphism [1]. Hence $a \equiv b \pmod 2$ implies that $(i\bar{a})^* = (i\bar{b})^*$, whence $\bar{a}^* = \bar{b}^*$, whence $\bar{a} = \bar{b}$. Here, taking $b = 0$ or 1 we obtain the lemma.

Theorem 12.2. *Let a be any integer. The element* $\bar{a} \in \{M_q, M_q\}$ *satisfies the relation*

$$\bar{a} = \begin{cases} a \cdot 1_M & \text{if } q \not\equiv 2 \pmod 4 \\ a \cdot 1_M + (a(a-1)/2) \cdot i_q \eta \pi_q & \text{if } q \equiv 2 \pmod 4. \end{cases}$$

Proof. By an exact sequence (1.5) for $S^1 \to M_q \to S^2$ we see easily that

$$\bar{a} = a \cdot 1_M + x \cdot i_q \eta \pi_q$$

for $x \in Z_2$, where $x = 0$ in case q odd. In case q even, for $\overline{q/2}: M_q \to M_2$,

$$\begin{aligned}(\overline{q/2})\bar{a} &= a \cdot (\overline{q/2}) + x \cdot (\overline{q/2}) i_q \eta \pi_q \\ &= a \cdot (\overline{q/2}) + x \cdot i_2 \eta \pi_q.\end{aligned}$$

On the other hand

$$(\overline{q/2})\bar{a} = \bar{a}(\overline{q/2}) = a^2 \cdot (\overline{q/2})$$

by Lemma 12.1. Thus

$$x\cdot i_2\pi_q = a(a-1)\cdot(\overline{q/2})\,.$$

Here

$$2\cdot(\overline{q/2}) = \begin{cases} 0 & \text{if} \quad q\equiv 0 \pmod 4 \\ i_2\eta\pi_q & \text{if} \quad q\equiv 2 \pmod 4 \end{cases}$$

by (4.13′). Therefore

$$x = \begin{cases} 0 & \text{if} \quad q\equiv 0 \pmod 4 \\ a(a-1)/2 \pmod 2 & \text{if} \quad q\equiv 2 \pmod 4)\,. \end{cases} \qquad \text{q.e.d.}$$

12. 2. From the above theorem we obtain the following corollaries.

Corollary 12. 3. *If* $q\not\equiv 2$ (mod 4) *or if* $\eta^{**}=0$ *in* $\tilde{h}$, *then*

$$r_*(x) = r\cdot x$$

for any integer r *and* $r_*: \tilde{h}^*(X; Z_q)\rightarrow\tilde{h}^*(X; Z_q)$.

Corollary 12. 4. *Let* $d=(q, r)$ *and* $\{M_q, M_r\}\ni\overline{q/d}$ (*a generator*). *For* $\overline{a(q/d)}\in\{M_q, M_r\}$, *we have*

$$\overline{a(q/d)} = \begin{cases} a\cdot\overline{q/d} & \textit{if} \quad q\not\equiv 2 \textit{ or if } r\not\equiv 2 \pmod 4 \\ a\cdot\overline{q/d}+(a(a-1)/2)\cdot i_r\eta\pi_q & \textit{if} \quad q\equiv r\equiv 2 \pmod 4)\,. \end{cases}$$

Corollary 12. 5. *For* $\bar{a}$, $\bar{b}$ *and* $\overline{a+b}$ *of* $\{M_q, M_r\}$,

$$\overline{a+b} = \begin{cases} \bar{a}+\bar{b} & \textit{if} \quad q\not\equiv 2 \textit{ or if } r\not\equiv 2 \pmod 4 \\ \bar{a}+\bar{b}+ab\cdot i_r\eta\pi_q & \textit{if} \quad q\equiv r\equiv 2 \pmod 4)\,. \end{cases}$$

Corollary 12. 6. *Propositions* 2.4 *and* 2.5 *hold under the assumption that* $q\not\equiv 2$ *or* $r\not\equiv 2$ (mod 4) *or* $\eta^{**}=0$ *in* $\tilde{h}$.

Corollary 12. 7. *The terms* $\tilde{E}_r^*$, $r\geq 2$, *of* mod 2 *Bockstein spectral sequences are* Z_2*-modules.* (*If* p *is odd or if* $\eta^{**}=0$ *in* $\tilde{h}$, *then* $\tilde{E}_r^*(\ ; Z_p)$ *for* $\tilde{h}$ *are* Z_p*-modules for* $r\geq 1$.)

Because: for $r\geq 2$,

$$\begin{aligned} l_r^{-1}(0) &\supset 2_*\tilde{h}^i(X; Z_{2^{r-1}}) \\ &\supset 2_*\rho_{2^r,2^{r-1}}\tilde{h}^i(X; Z_{2^r}) = 2\cdot\tilde{h}^i(X; Z_{2^r})\,, \end{aligned}$$

whence Proposition 11.2, (iii), proves Corollary 12.7.

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References

[1]-[10] are listed at the end of Part I.
[11] W. S. Massey, *Exact couples in algebraic topology* (Parts I and II), Ann. of Math. **56** (1952), 363-396.