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SPECTRA OF RANDOM MEDIA WITH MANY RANDOMLY DISTRIBUTED OBSTACLES

Dedicated to Professor Shigetoshi Kuroda on his 60th birthday

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1. Let Ω be a bounded domain in R^d with smooth boundary $\partial\Omega$. Let $B(\varepsilon, w_i)$ $(i=1, \cdots, n)$ be balls of radius ε with centers $w_1, \cdots, w_n$. We consider the eigenvalue problem of the Laplacian in

$$\Omega_{w(m)} = \Omega \backslash \overline{\cup_{i=1}^{n} B(\varepsilon, w_i)}$$

under the Dirichlet condition on its boundary. Under some scaling limit $\varepsilon \to 0$, $n \to \infty$, $n^{\sigma}\varepsilon \to \alpha$ we know that the spectra of $-\Delta$ in $\Omega_{w(m)}$ under the Dirichlet condition on $\partial\Omega_{w(m)}$ tends to the spectra of Schrödinger operator $-\Delta + cV$ in Ω under the Dirichlet condition on $\partial\Omega$.

There are two main directions in previous research works concerning related problems. One is homogenization as was studied in [3], [7], and another direction is to calculate the eigenvalue of $-\Delta$ in $\Omega_{w(m)}$ in statistical setting, the later of which this paper concerns.

Let $V(x)$ be a positive continuous function on $\bar{\Omega}$ satisfying

$$\int_{\Omega} V(x)\, dx = 1 .$$

Then, Ω can be thought as probability space by the probability law

$$P(x \in A) = \int_A V(x)\, dx .$$

Let Ω^n be the product probability space; the corresponding probability law is denoted also by P for any n. Fix $\beta \in [d-2, d)$. Setting $\varepsilon = m^{-1}$, we take m in place of ε as a parameter. Fix and define $n = [m^{\beta}]$, $\mu_j(w(m))$ = the j-th eigenvalue of $-\Delta$ in $\Omega_{w(m)}$ under the Dirichlet condition on $\partial\Omega_{w(m)}$. Each $\mu_j(w(m))$ is viewed as a random variable on Ω^n.

Problem A. Can one say anything about the statistics of $\mu_i(w(m))$ on Ω^n when $m \to \infty$?

We know the following partial answer.

Theorem. (LLN, law of large numbers, Kac [6], Rauch-Taylor [10], Huruslov-Marchenko [5], Chavel-Feldman [2], etc.)

Assume that $d \geq 3$ and $\beta = d-2$. Then

$$\lim_{m\to\infty} P(w(m)\in\Omega^n;\ |\mu_j(w(m))-\mu_j^V|<\varepsilon) = 1\,.$$

Here μ_j^V is the j-th eigenvalue of $-\Delta+c_dV(x)$ in Ω under the Dirichlet condition on its boundary $\partial\Omega$; the constant $c_d=(d-2)|S^{d-1}|$ being the $(d-1)$ dimensional area of the unit sphere in R^d.

Recall that $c_4=2\pi^2$. More precisely we have the following LLN with remainder estimate. Let $\mu_j(V;m)$ be the j-th eigenvalue of the Schrödinger operator $-\Delta+2\pi^2 m^{\beta-2} V(x)$ in Ω under the Dirichlet condition on $\partial\Omega$. When $\beta>2$, we assume that $V(x)\equiv|\Omega|^{-1}$.

Theorem 1. *Assume that $d=4$. Fix $\beta\in[2, 12/5)$. Fix $\varepsilon>0$. Then,*

$$P(w(m)\in\Omega^n;\ m^{(6-(5/2)\beta)-\varepsilon}\ |\mu_i(w(m))-\mu_i(V;m)|<m^{\beta-2})$$

tends to 1 *as m tends to infinity.*

Remark. For $\beta\in[2, 12/5)$, $6-(5/2)\beta>0$. The above result can be thought as LLN with remainder estimate. Even in the special case $\beta=2$ (and $d=4$) Theorem 1 supply a better estimate than the theorem cited previously. It is of great interest to the author to give CLT for $d=4$. Thus, this paper can be thought as a bridgehead to the answer. The author does not know whether the random variable $m^{\xi}(\mu_j(w(m))-\mu_j(V;m))$ tends in law to some Gaussian random variable for some ξ as m tends to infinity or not, even if we know that the answer is YES when $d=3$, $\beta\in[1, 5/4)$, $\xi=1-(\beta/2)$. See [9]. To get CLT in the later case the authour used perturbative expansion, abbreviated PIA, of the Green function. In this paper too we employ PIA to prove Theorem 1.

The author considers that determing CLT (or fluctuation) result for $d\geq 4$, $\beta\in[d-2, d)$ may be a very challenging problem for the people working on analysis, probability theory and mathematical physics.

Here the author offers the following unsolved research theme. For $d=3$, $\beta\in[1, 5/4)$ we have CLT as mensioned above. Can one get CLT with the aid of Brownian motion? Analysis of Brownian motion is a strong and standard tool to attack probabilistic problem. For the problem prsented in this paper, see [1], [2], [11], [12]. We obtained LLN by using Brownian motion (also, by analytic method). However, we do not know whether Brownian motion can be a key to CLT of the above problem or not. The following question may be a good pilot for further progress. Can one get CLT for

$$\sum_{j=1}^{\infty} \exp(-t\mu_j(w(m)))\ ?$$

Main aim of this paper: lies in a systematic development of the calculus of PIA (point interaction approximation). To develop our research we encounter the situation of getting statistical properties of

$$(*) \qquad \sum_{i \neq j} G(w_i, w_j)^2 ,$$

etc. It is a standard technique to get expectation

$$E(\sum_{i \neq j} G(w_i, w_j)^2) .$$

However, the Green function $G(x, y)=(-\Delta+\lambda)^{-1}(x, y)$ is not of Hilbert-Schmidt class when $d \geq 4$. The fact that the Green function is not of Hilbert-Schmidt class is not so important in the previous papers. We must modify the argument that is previously employed. We construct large subset $\Omega^*(n)$ of Ω^n so that

$$(**) \qquad |\Omega^n \backslash \Omega^*(n)| \to 0$$

and $(*)$ does not behave very badly on $\Omega^*(n)$. A simple method of constructing $\Omega^*(n)$ suffices to this end. We set

$$\Omega^*(n) = \{(w_1, \cdots, w_n) \in \Omega^n ;\ |w_i - w_j| > m^{-\rho} \ \text{for any}\ i, j\} .$$

Here ρ is a constant satisfying $\rho > \beta/2$. Then, we see that it possesses the required properties.

Owing to the above modification our calculus involves delicate points for $d=4$.

2. PIA.

Let T be a fixed number. We put $\lambda = Tm^{\beta-(d-2)}$ and we consider the Green operator of $-\Delta+\lambda$ in $\Omega_{w(m)}$ under the Dirichlet condition on its boundary. Hereafter $d=4$, but we write d for 4 indicating the role the dimension number d plays. We consider the following condition $\mathcal{O}_1(m)$ for $w(m)=(w_1, \cdots, w_m) \in \Omega^n$.
$\mathcal{O}_1(m)$: $\sup_{K \in \mathfrak{F}_m}$ (the number of balls of radius $1/m$ with the center w_i such that ball intersect $K) \leq (\log m)^2$, where $\mathfrak{F}_m$ denotes the family of open balls of radius $m^{-\beta/d}$.
We see that

$$P(w(m) \in \Omega^n ;\ \mathcal{O}_1(m) \quad \text{holds}) \geq 1 - m^{-N}$$

for any N and any sufficiently large m depending on N. See [9]. If we suppose $\mathcal{O}_1(m)$, then we know that only one connected componnet ω of $\Omega_{w(m)}$ plays an important role and all the components other than ω are negligible in our analysis of $\mu_j(w(m))$ as $m \to \infty$. See [9].

Let $G(x,y)$ be the Green function of $-\Delta+\lambda$ in Ω under the Dirichlet condition on $\partial\Omega$. Recall that $\lambda=Tm^{\beta-(d-2)}$. Thus, $\lambda\to\infty$ when $\beta>d-2$, $m\to\infty$. Let $G(x,y;w(m))$ be the Green function of $-\Delta+\lambda$ in ω under the Dirichlet condition on $\partial\omega$. Hereafter we always assume that $w(m)$ satisfies $\mathcal{O}_1(m)$.

Let $\boldsymbol{G}$ ($\boldsymbol{G}_{w(m)}$, respectively) be the bounded linear operator on $L^2(\Omega)$ ($L^2(\omega)$, respectively) defined by

$$(\boldsymbol{G}f)(x)=\int_\Omega G(x,y)f(y)\,dy\,,$$

$$(\boldsymbol{G}_{w(m)}g)(x)=\int_\omega G(x,y;w(m))g(y)\,dy\,,$$

respectively. The eigenvalue problem of the Laplacian with respect to ω is transformed into the eigenvalue problem of $\boldsymbol{G}_{w(m)}$. As making $m\to\infty$, we see that $\mu_j(w(m))+\lambda$ is approximated by the j-th eigenvalue of the Schrödinger operator $-\Delta+2\pi^2 m^{\beta-2}V(x)+\lambda$ under the Dirichlet condition.

Let A denote the Green operator of the above Schrödinger operator. To approximate $\boldsymbol{G}_{w(m)}$ by A we introduce the following kernel. We denote by τ the constant $G^*(0,1/m,\lambda)^{-1}$ where $G^*(x,y,\lambda)$ is the Green function of $-\Delta+\lambda$ in R^d, that is, it satisfies $(-\Delta+\lambda)\,G^*(x,y,\lambda)=\delta(x-y)$. It has the asymptotic form $\tau=2\pi^2 m^{-2}\exp(-\lambda^{1/2}m^{-1})+0(\lambda^{1/2}m^{-3})$ for $d=4$.

$$\begin{aligned}(2.1)\qquad & h(x,y;w(m))\\ &=G(x,y)-\tau\sum_{i=1}^{n}G(x,w_i)\,G(w_i,y)\\ &\quad+\sum_{s=2}^{(\log m)^2}(-\tau)^s\sum\nolimits_{(s)}G(x,w_{i_1})\,G_{I(s)}\,G(w_{i_s},y)\,.\end{aligned}$$

Here $G_{I(s)}=G(w_{i_1},w_{i_2})\,G(w_{i_2},w_{i_3})\cdots G(w_{i_{s-1}},w_{i_s})$ and the indices in $\sum_{(s)}$ run over all $i_1,\cdots,i_s$ satisfying $1\le i_1,\cdots,i_s\le n$ such that $i_\nu\neq i_\mu$ when $\nu\neq\mu$. We use the notational convention $G_{I(s)}\equiv1$ when $s=1$ and $G_{ij}=G(w_i,w_j)$. It should be remarked that the product like $G(x,w_{i_1})\,G_{i_1i_2}\,G(w_{i_2},w_{i_1})\,G(w_{i_1},y)$ are excluded by the above rule. The above form (2.1) was discussed extensively when $d=3$ in [9].

We put

$$(\boldsymbol{H}_{w(m)}f)(x)=\int_\omega h(x,y;w(m))f(y)\,dy\,,\quad x\in\omega$$

and

$$(\tilde{\boldsymbol{H}}_{w(m)}g)(x)=\int_\Omega h(x,y;w(m))g(y)\,dy\,,\quad x\in\Omega\,.$$

We have the following Theorem.

Theorem 2. *Fix $\beta\in[2,4)$. Fix an arbitrary $\varepsilon>0$. Then, there exists a constant T in λ such that*

(2.2) $$P(\|G_{w(m)}-H_{w(w)}\|_{\mathcal{L}(L^2(\omega),L^2(\omega))}\leq m^{3\varepsilon-3+\beta}\lambda^{1/2})\geq 1-m^{-\varepsilon}$$

holds.

This formula gives an asymptotic behaviour of $G_{w(m)}$ as $m\to\infty$ in probabilistic context. Let χ_ω be the characteristic function of ω. We can deduce from Theorem 2 that $\|(G_{w(m)}-H_{w(m)})(\chi_\omega f)\|_{L^2(\omega)}$ for $f\in L^2(\Omega)$ is a remainder in some sense for $\beta<12/5$. An application of Theorem 2 on spectral result is given in the section eight. The reader who want to know the reason why we can deduce Theorem 2 from Proposition 3.8 may be reffered to §8.

3. **L^p-method.** Hereafter $d=4$. In this section we want to study

(3.1) $$\|G_{w(m)}-H_{w(m)}\|_{L^p(\omega)}\,.$$

It should be noticed that we are assuming $\mathcal{O}_1(m)$.

Lemma 3.1. *Fix $p>2$. If $u\in C^\infty(\omega)\cap C^0(\overline{\omega})$ satisfies*

$$(-\Delta+\lambda)\,u(x)=0\,,\quad x\in\omega$$
$$u(x)=0\,,\quad x\in\partial\Omega\cap\partial\omega$$

and

$$\max\{|u(x)|\,;\,x\in\partial B_r\cap\partial\omega\}=M_r\,,\quad r=1,\cdots,n\,.$$

Here M_r is zero when $\partial B_r\cap\partial\omega=\phi$. Then,

$$\|u\|_{L^p(\omega)}\leq C\,m^{-4/p}\sum_{r=1}^{n}M_r\,.$$

Proof. By the Hopf maximum principle

$$|u(x)|\leq C\,m^{-2}\sum_{r=1}^{n}\exp(-\lambda^{1/2}\,|x-w_r|/C')\,|x-w_r|^{-2}M_r$$

for some constant C, C'. We have

$$\left(\int_{1/m}^{k}r^{-2p}\,r^3\,dr\right)^{1/p}\leq C\,m^{2-(4/p)}$$

for $p>2$. Thus, we get the desired result by Minkowski's inequality.

Lemma 3.2. *There is a constant C such that*

$$\|Gf\|_{L^\infty(\omega)}\leq C\,\lambda^{(2/p)-1}\,\|f\|_{L^p(\Omega)}\quad(p>2)$$

and

$$\|Gf\|_{L^\infty(\omega)}\leq C\,\lambda^{-1}\,\|f\|_{L^\infty(\Omega)}$$

holds.

As a corollary of Lemma 3.1 and the fact that $u=(G_{w(m)}-H_{w(m)})\,(\chi_\omega f)$

satisfies Lemma 3.1 we get the following.

Proposition 3.3. *Fix $f\in C^\infty(\bar{\Omega})$. Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then,*

$$\|(\boldsymbol{G}_{w(m)}-\boldsymbol{H}_{w(m)})(\chi_\omega f)\|_{L^p(\omega)}\leq C\, m^{-4/p}\sum_{r=1}^{n} M_r$$

for $p>2$.

As was discussed in [9] it is very useful to introduce the rearrangement of the Green function. For $s=0$, we put

$$(I_r^0 f)(x)=(\boldsymbol{G}f)(x)-\tau\, G(x, w_r)(\boldsymbol{G}f)(w_r)\,.$$

For $s\geq 1$, we set $I_r^s f$ as the following term

$$\begin{aligned}&\textstyle\sum'_{(s)} G(x, w_{i_1})\, G_{i_1 i_2}\cdots G_{i_{s-1}i_s}(\boldsymbol{G}f)(w_{i_s})\\ &\quad -\tau \textstyle\sum'_{(s)} G(x, w_r)\, G_{r i_1}\, G_{i_1 i_2}\cdots G_{i_{s-1}i_s}(\boldsymbol{G}f)(w_{i_s})\,.\end{aligned}$$

Here the indices $i_1, \cdots, i_s$ in $\sum'_{(s)}$ run over all $i_1, \cdots, i_s$ satisfying $i_\nu\neq i_\mu$ if $\nu\neq\mu$ and $i_\nu\neq r$ for $\nu=1, \cdots, s$. For $s\geq 2$ we set $\tilde{I}_r^s f$ as the following term

$$\textstyle\sum''_{(s)} G(x, w_{i_1})\, G_{i_1 i_2}\cdots G_{i_{s-1}i_s}(\boldsymbol{G}f)(w_{i_s})\,.$$

Here the indices $i_1, \cdots, i_s$ in $\sum''_{(s)}$ run over all $i_1, \cdots, i_s$ satisfying $i_\nu\neq i_\mu$ if $\nu\neq\mu$ and exactly one of $i_\nu(\nu\geq 2)$ is equal to r. We have the rearrangement.

$$\begin{aligned}(\boldsymbol{H}_{w(m)}g)(x)&=\sum_{s\geq 0}(-\tau)^s(I_r^s g)(x)\\ &\quad+\sum_{s\geq 2}(-\tau)^s(\tilde{I}_r^s g)(x)+(-\tau)^{m^*}(Z_r^{m^*}g)(x)\,,\end{aligned}$$

where

$$(Z_r^{m^*}g)(x)=(\textstyle\sum'_{(m^*)}+\sum''_{(m^*)})\, G(x, w_{i_1})\cdots(\boldsymbol{G}g)(w_{i_{m^*}})\,.$$

We put

$$G(x, y)-G^*(x, y, \lambda)=S(x, y)\,.$$

We need the following Lemma 3.4 whose proof is as in the proof of Lemma 3.4 in [9]. Note that $d=4$ in the following.

Lemma 3.4. *Fix $\beta\in[2, 4)$. Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then, there exists a constant C independent of m such that* (3.2) *and* (3.3) *hold.*

$$\max_{x\in\partial B_r\cap\partial\omega}|G(x, w_i)-G(w_r, w_i)|\leq C\, m^{-1}\,\Phi_3(w_i, w_r, \lambda/C) \tag{3.2}$$

$$\max_{x\in\partial B_r\cap\partial\omega}|S(x, w_r)\, G(w_r, w_i)|\leq C(\log m)^2\, m\Phi_3(w_i, w_r, \lambda/C) \tag{3.3}$$

where $\Phi_\theta(x, y, \lambda)$ denote $\exp(-\lambda^{1/2}|x-y|)\,|x-y|^{-\theta}$.

To get a bound for $\sum M_r$ we need some lemmas on $I_r^s f$. First we have the following

Proposition 3.5. *Fix $f\in C^{\infty}(\Omega)$. Fix $\varepsilon>0$ and $p>2$. We assume $\mathcal{O}_1(m)$. Then, there exists a constant C independent of f, m such that*

$$\begin{aligned}(3.4)\qquad &\sum_r \max_{\partial B_r\cap\partial\omega} |I_r^0(\chi_\omega f)(x)| \\ &\quad\le C(m^{\varepsilon-1+\beta}\lambda^{-1/2}+m^{\pi})\,||f||_{L^p(\Omega)}\end{aligned}$$

holds where $\pi=-((2-(4/p))\,\beta/4)+(\beta/p')+\varepsilon$.

Proof. Denote $\chi_\omega f$ by f_ω. We put

$$\begin{aligned}J_r^1 &= (Gf_\omega)(x)-(Gf_\omega)(w_r)\\ J_r^2 &= -\tau S(x, w_r)(Gf_\omega)(w_r)\,.\end{aligned}$$

Let $B(r;*)$ be the ball $\{x;\,|x-w_r|<3m^{-\beta/4}\}$. Then, we have

$$\begin{aligned}(3.5)\qquad &\sum_r \max_{\partial B_r} |J_r^1| \\ &\le 4\sum_r \max_{\bar{B}_r} |G(\chi_{B(r\,;\,*)}f)(x)| \\ &+C\,m^{-1}\sum_r \int_{B(r,*)^C\cap\omega} \Phi_3(w_r, y, \lambda/C)\,|f_\omega(y)|\,dy\end{aligned}$$

observing $|G(x, y)-G(w_r, y)|\le C\,\Phi_3(w_r, y, \lambda/C)$ for $y\in B(r, *)^C\cap\omega$. By the Hölder inequality we see that the first term in the right hand side of (3.5) does not exceed

$$m^{-\xi}\sum_r \Big(\int_\Omega |f(y)|^p\,\chi_{B(r\,;\,*)}(y)\,dy\Big)^{1/p}$$

for $\xi=((4/p')-2)\,(\beta/4)$ observing

$$\Big(\int_{B(r\,;\,*)} G(x, y)^{p'}\,dy\Big)^{1/p'}\le C\,m^{\xi}\,.$$

Therefore, it is estimated by

$$\begin{aligned}&C\,m^{-\xi+(\beta/p')}\Big(\sum_r \int_\Omega |(f(y)|^p\,\chi_{B(r\,;\,*)}(y)\,dy\Big)^{1/p}\\ &\quad\le C\,m^{-\xi+(\beta/p')}\Big(\int_\Omega |f(y)|^p\,\Big(\sum_r \chi_{B(r\,;\,*)}(y)\Big)\,dy\Big)^{1/p}\,.\end{aligned}$$

By $\mathcal{O}_1(m)$ we see that

$$\sum_r \chi_{B(r\,;\,*)}(x)\le C\,(\log m)^2\,.$$

The second term in the right hand side of (3.5) does not exceed

$$(3.6)\qquad m^{-1}\int_\Omega |f(y)|\,\Big(\sum_r \Phi_3(w_r, y)\Big)\,dy\,.$$

Here the sum for r in (3.5) run over r such that

$$|y-w_r|>3m^{-\beta/4}.$$

We are now going to estimate $\sum_r \Phi_3(w_r, y)$. We put

$$F_k=\{r;\, k\, m^{-\beta/4}\le|y-w_r|\le(k+1)\, m^{-\beta/4}\}$$

for $k=0, 1, 2, \cdots$. Then,

$$\begin{aligned}(3.7)\qquad & \sum_{k=3}\sum_{r\in F_k}\Phi_3(w_r, y)\\ & \le C\sum_{k=3}\exp(-\lambda^{1/2}\, km^{-\beta/4})\,(km^{-\beta/4})^{-3}\, k^3\,(\log m)^2\end{aligned}$$

observing that $\#(r;\, w_r\in F_k)\le Ck^3(\log m)^2$. Thus, we see that (3.7) does not exceed $Cm^\beta(\log m)^2\,\lambda^{-1/2}$ observing the fact that $m^{(3/4)\beta}$ is less than the bound for $\beta\in[2, 4)$. We have similar result for J_r^2.

Summing up these facts we get the desired result.

Propositions 3.6, 3.7 gave estimates for $I_r^s f$, $\tilde{I}_r^s f$. These are probabilistic results. Expectation over Ω^n is modified so that we can avoid divergence when we consider Propostion 3.7 that follows.

Proposition 3.6. *We assume the same assumption as in Proposition* 3.5. *Fix* $\varepsilon>0$. *Then, there exists C such that*

$$(3.8)\qquad P\Big(\sum_r \max_{\partial B_r\cap\partial\omega}|I_r^s(\chi_\omega f)(x)|\le m^\varepsilon(m^\beta/C\lambda)^s\, m^{\beta-1}\,\lambda^{1/2}\, D\Big)\ge 1-m^{-\varepsilon}$$

where $D=\|G(\chi_\omega f)\|_{L^\infty}$.

Proof. By Lemma 3.4 we see that the left hand side of (3.8) does not exceed D times

$$(3.9)\qquad m^{-1}(\log m)^2\sum_r\sum_I\Phi_3(w_r, w_{i_1})\, G_{I(s)}\,.$$

Here $r\cup I$ is self-avoiding observing the definition of $I_r^s f$. We have

$$E\Big(\sum_r\sum_I \text{ in (3.9)}\Big)\le C'\,(C\lambda^{-1})^{s-(1/2)}\,(m^\beta)^{s+1}\,.$$

Proposition 3.7. *Fix* $\varepsilon>0$. *Under the same assumption as in Proposition* 3.5 *it holds that*

$$\begin{aligned}(3.10)\qquad & P\Big(\sum_r \max_{\partial B_r\cap\partial\omega}|\tilde{I}_r^s(\chi_\omega f)(x)|\\ & \le m^{2\varepsilon}(m^\beta/C\lambda)^s\,\lambda^2\cdot D\Big)\ge 1-m^{-\varepsilon}\end{aligned}$$

for some constant C.

Proof. We see that the left hand side of (3.10) does not exceed $\|Gf_\omega\|_{L^\infty}$ times

$$\sum_r \sum_I G_{ri_1} G_{i_1 i_2} \cdots G_{i_{s-1} i_s}, \tag{3.11}$$

where I is self-avoiding and exactly one of I is equal to r. We use calculation $\mathcal{E}_{\mathcal{R}}$ which is presented in section four. Let us assume that $i_2=r$. Then, (3.11) is equal to

$$\sum_{r, i_1, i_3} G_{ri_1}^2 G_{ri_3} \cdots .$$

We have

$$\begin{aligned} &\mathcal{E}_{\mathcal{R}}(G_{ri_1}^2 G_{ri_3}\cdots) \\ &\le \Big(\int_{\Pi(i,r)} G_{ri_1}^2 \tilde{d}w_{i_1}\Big) \max_r \int G_{ri_3} \cdots \tilde{d}w_{i_3} \cdots \tilde{d}w_{i_s} . \end{aligned}$$

Thus, we get the desired result by Lemma 4.3 and the fact that the number of distinct indices are at most $(m^\beta)^s$. When $i_\nu=r$ ($\nu\ge 3$), then we get

$$\begin{aligned} &\mathcal{E}_{\mathcal{R}}(G_{ri_1}\cdots G_{i_{\nu-1}r} G_{ri_{\nu+1}} \cdots G_{i_{s-1}i_s}) \\ &\le \mathcal{E}_{\mathcal{R}}(G_{ri_1}\cdots G_{i_{\nu-1}r}) \max_r \int G_{ri_{\nu+1}} \cdots \tilde{d}w_{i_{\nu+1}} \cdots . \end{aligned} \tag{3.12}$$

By Lemma 4.4, we get the desired result.

Summing up these facts we get the following.

Proposition 3.8. *Fix $\varepsilon>0$. Fix $p>2$. Then,*

$$P(w(m)\in\Omega^n;\ w(m) \text{ satisfies } \mathcal{O}_1(m) \text{ and } (3.13)) \ge 1-m^{-\varepsilon}$$

holds, where

$$\begin{aligned} &\|(G_{w(m)}-H_{w(m)})(\chi_\omega f)\|_{L_p(\omega)} \\ &\le C\, m^{2\varepsilon-(4/p)} \{(m^{\varepsilon-1+\beta} \lambda^{-1/2} + m^{\varepsilon}) \|f\|_{L^p(\Omega)} \\ &\quad + m^{\beta-1} \lambda^{1/2} \|G(\chi_\omega f)\|_{L^\infty} \\ &\quad + \lambda^2 \|G(\chi_\omega f)\|_{L^\infty}\} . \end{aligned} \tag{3.13}$$

It should be remarked by our argument that the constant C in the inequality (3.13) is f-independent. Take $f=\chi_\omega f$. Thus, by duality argument $(L^p)'=L^{p'}$ and interpolation inequality gives Theorem 2.

4. **Calculation involving G_I.** We put $\mathcal{R}(m)=\{w(m)\in\Omega^n;\ |w_i-w_j|> m^{-\rho}$ for any $i,j\,(i\neq j)\}$. Then, we have the following

Lemma 4.1. *Fix $\rho>\beta/2$. There exists a constant $\varepsilon>0$ such that* (4.1) *holds for sufficiently large m*:

$$P(w(m)\in\Omega^n;\ w(m)\in\mathcal{R}(m))>1-m^{-\varepsilon}\,. \tag{4.1}$$

Proof. Fix x. Then, the measure of the set $\{w_i;\ |x-w_i|<m^{-\rho}\}$ is of order $m^{-4\rho}$. If we get $n^2 m^{-4\rho}\to 0$, then (4.1) holds.

We put $\Pi(i_1, \cdots, i_h)=\{(w_{i_1}, \cdots, w_{i_h});\ |w_{i_\nu}-w_{i_\mu}|>m^{-\rho}\}$ for $1\le\nu<\mu\le h$. Then, we define

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(f(w_{i_1}, \cdots, w_{i_h}))\\ =\int_{\Pi(i_1,\cdots,i_h)} f(w_{i_1}, \cdots, w_{i_h})\, \tilde{d}w_{i_1}\cdots\tilde{d}w_{i_h}\,,\end{aligned} \tag{4.2}$$

if the variables $w_{i_1}, \cdots, w_{i_h}$ are all distinct. Here $\tilde{d}w_i=V(w_i)\,dw_i$. The following inequality is essential to consider probabilistic result.

$$\begin{aligned}&\int_{(\Pi(i_1,\cdots,i_s,\cdots,i_k)} |g(w_{i_1}, \cdots, w_{i_s}, \cdots, w_{i_k})|\ \tilde{d}w_{i_1}\cdots\tilde{d}w_{i_k}\\ &\le\int_{\Pi(i_{s+1},\cdots,i_k)}\Big(\int_{\Pi(i_1,\cdots,i_s)} |g(w_{i_1}, \cdots, w_{i_s}, \cdots)|\ (\tilde{d}w_{i_1}\cdots\tilde{d}w_{i_s})\,\tilde{d}w_{i_{s+1}}\cdots\end{aligned} \tag{4.3}$$

We introduce the integral over $\mathcal{R}(m)$.
We put

$$\begin{aligned}\mathcal{E}_{\mathcal{R}(m)}\,(f(w_{i_1}, \cdots, w_{i_h}))\\ =\int_{\mathcal{R}(m)} f(w_{i_1}, \cdots, w_{i_h})\, \tilde{d}w_1\,\tilde{d}w_2\cdots\tilde{d}w_n\,.\end{aligned}$$

Note that $\mathcal{E}_{\mathcal{R}(m)}\,(f(w_{i_1}, \cdots, w_{i_h}))\le\mathcal{E}_{\mathcal{R}}(f(w_{i_1}, \cdots, w_{i_h}))$, if $f\ge 0$. Thus, we used the above inequality to get some probabilistic bound. Delicateness of analysis using $\mathcal{E}_{\mathcal{R}(m)}$ is presented in the section six.

In this section we want to study $\mathcal{E}_{\mathcal{R}}(\sum_{I,J} G_I\, G_J)$. The result we presented is important to study $\mathcal{E}_{\mathcal{R}}(\|\boldsymbol{H}_{w(m)}-A'\|^2_{L^2(\omega)})$. We write the term $G_{i_1i_2}\cdots G_{i_{s-1}i_s}$ for G_I. We also write the term $G_{j_1j_2}\cdots G_{j_{t-1}j_t}$ for G_J. If we want to get a bound for $E(G_I\, G_J)$ we must classify the indices $I\cup J$.

DEFINITION. Assume that both $I=I(s)\ni i_1, \cdots, i_s$ and $J=J(t)\ni j_1, \cdots, j_t$ are self-avoiding. If there are exactly q couples of $(h(k), p(k))$ $(k=1, \cdots, q)$ such that $i_{h(k)}=j_{p(k)}$, we say that $(i_1, \cdots, i_s)$ and $(j_1, \cdots, j_t)$ have q-intersections and it is denoted by $\sharp(I\cap J)=q$.

Lemma 4.2. *Fix $\rho>\beta/2$ and $\varepsilon>0$. Assume that $\sharp(I\cap J)=q$. Then,*

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(G_I\, G_J)\le C'm^{\varepsilon(q-1)}\,(C\lambda)^{2(q-1)-(s+t-2)} &\quad (q\ge 1)\\ C'(C\lambda)^{2-(s+t)} &\quad (q=0)\,.\end{aligned} \tag{4.4}$$

The folloiwng Lemmas 4.3, 4.4 enable us to get Lemma 4.2.

Lemma 4.3. *We have*

$$\int_{\Pi(r,i)} G_{ri}^2 \, dw_i \leq C(\log m) . \tag{4.5}$$

Proof. It is easy to see (4.5).

Let $G_{(h)}(x, y)$ be the iterated kernel of the Green function defined by

$$G_{(1)}(x, y) = G(x, y) ,$$
$$G_{(k+1)}(x, y) = \int_\Omega G_{(k)}(x, z)\, G(z, y)\, \tilde{d}z \quad (k = 1, \cdots) .$$

Lemma 4.4. *We have*

$$\max_i \int_{\Pi(i,j)} G_{(h)}(w_i, w_j)^2 \, \tilde{d}w_j \leq C(\log m)\,(C\lambda^{-1})^{2h-2} . \tag{4.6}$$

Proof. By observing the singularity of the iteration of the resolvent kernel function we get (4.6).

Proof of Lemma 4.2. First we consider the case where $\#(I \cap J)=q\geq 2$. We assume that $i_{h(k)}=j_{p(k)}$ for $k=1, \cdots, q$. We define $r(k)$ as in the proof of Lemma 3.7 in [9]. When $q\geq 2$, we define the contracted term G_I^{co} and G_J^{co} by

$$G_I^{co} = \prod_{k=1}^{q-1} G_{h(k+1)-h(k)}\,(w_{i_{h(k)}}, w_{i_{h(k+1)}})$$

$$G_J^{co} = \prod_{k=1}^{q-1} G_{r(k+1)-r(k)}\,(w_{j_{r(k)}}, w_{j_{r(k+1)}}) .$$

Then, $\mathcal{E}_{\mathfrak{R}}(G_I\, G_J)$ does not exceed

$$\mathcal{E}_{\mathfrak{R}}(G_I^{co}\, G_J^{co}) \times (C\lambda^{-1})^{h(1)-1+(s-h(q))+(r(1)-1)+(t-r(q))}$$

observing

$$\int G(w, z)\, \tilde{d}w \leq C\lambda^{-1} .$$

Since $\mathcal{E}_{\mathfrak{R}}(G_I^{co})^2$ is estimated by Lemma 4.4 we get the desired result. We have (4.2) for the case $q=0, 1$.

5. Lemmas on integration over small set ω^C. In this section we obtain certain estimate for the integrand of the following form. The positive functions Φ and $\tilde{\Phi}$ are chosen as below. We consider

$$\int_{\omega^c} \Phi(w_i, x)\, \tilde{\Phi}(x, w_j)\, dx . \tag{5.1}$$

Since $|\omega^C| \to 0$ as m tends to infinity, (5.1) is smaller than

(5.2) $$\int_{\Omega} \Phi(w_i, x)\, \tilde{\Phi}(x, w_j)\, dx\,.$$

Since ω^C has small measure, (5.1) is very small compared to (5.2). We want to make rigorous proof for this intuition.

Let us begin with the following integral (5.3), where $\Phi(x, y)$ is a positive function depending only on $|x-y|$, that is $\Phi(x, y)=\Phi^*(|x-y|)$ for some Φ^*. We assume that $\Phi^*(r)$ is a decreasing function. We assume that $|\Phi^*(r)|<+\infty$ for $r>0$, $|\Phi^*(0)|\leq M\leq+\infty$ and

$$\int_0^\infty \Phi^*(r)\, r^{d-1}\, dr<+\infty\,.$$

Consider

(5.3) $$\max_i \int_{\omega^C} \Phi(w_i, x)\, dx\,.$$

We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. It should be noted that there is connected component decomposition

$$\omega^C = \bigcup_{j=1}^{N} \omega^{(j)}$$

such that diam $(\omega^{(j)})\leq 3\, m^{-1}(\log m)^2$ and $\overline{\omega^{(j)}}\ni w_k$ for some k. Then,

$$\overline{\omega^C}\subset \bigcup_i \{x;\ |x-w_i|\leq 3m^{-1}(\log m)^2\}\,.$$

Therefore,

(5.4) $$|\omega^C|\leq C'\, m^{\beta-4}(\log m)^8\,.$$

The term (5.3) can be estimated by the following method. We put

$$A_k^{(i)} = \omega^C \cap \{x;\ km^{-\beta/4}\leq |x-w_i|\leq (k+1)\, m^{-\beta/4}\}$$

for $k=0, 1, 2, \cdots$. Fix i. Then,

$$\int_{A_0^{(i)}} \Phi(w_i, x)\, dx$$
$$\leq(\text{the number of } p \text{ such that } |w_p-w_i|\leq 2\, m^{-\beta/4})$$
$$\times\int_0^{3m^{-1}(\log m)^2} \Phi^*(r)\, r^3\, dr\,.$$

Next we have

$(5.5)_k$ $$\int_{A_k^{(i)}} \Phi(w_i, x)\, dx\leq |A_k^{(i)}|\,\Phi^*(km^{-\beta/4})$$
$$\leq Cm^{-4}(\log m)^{10}\, \Phi^*(km^{-\beta/4})\, k^3\,.$$

Therefore,

$$\sum_{k=1}^{\infty}(5.5)_k \leq C'(\log m)^{10}\, m^{\beta-4}(m^{-\beta/4})\sum_{k=1}^{\infty}(km^{-\beta/4})^3\,\Phi^*(km^{-\beta/4}) \tag{5.6}$$
$$\leq C'm^{\beta-4}(\log m)^{10}\int_0^\infty \Phi^*(r)\,r^3\,dr\,.$$

Summing up these facts we get the following.

Lemma 5.1. *Under the same assumption as above we get*

$$(5.3) \leq C\,\{(\log m)^2\int_0^{3(\log m)^2/m}\Phi^*(r)\,r^3\,dr \tag{5.7}$$
$$+m^{\beta-4}(\log m)^{10}\int_0^\infty \Phi^*(r)\,r^3\,dr\}\,.$$

As a corollary of Lemma 5.1 we have

Lemma 5.2. *Under the same assumption as above we get*

$$\max_i \int_{\omega^C}\Phi_\theta(w_i, x, \lambda/C)\,dx \tag{5.8}$$
$$\leq C''((m^{-1}(\log m)^2)^{4-\theta}(\log m)^2$$
$$+m^{\beta-4}(\log m)^{10}(\lambda^{-1/2})^{4-\theta}$$

holds for $0\leq\theta<4$.

We want to examine (5.1). We assume that $\Phi(x, w_i)=\Phi_\theta(x, w_i, \lambda/C)$, $\tilde{\Phi}(x, w_j)=\Phi_{\theta'}(w_j, x, \lambda/C)$. We do not take care the constant C in $\Phi(x, y, \lambda/C)$, since the constant C does not make any important role when we estimate formulas of this paper. We put

$$D_1=\omega^C\cap\{x;\ |x-w_i|<(2/3)\ |w_i-w_j|\}$$
$$D_2=\omega^C\cap\{x;\ |x-w_j|<(2/3)\ |w_i-w_j|\}$$
$$D_3=\omega^C\cap(D_1\cup D_2)^C\,.$$

We have

$$\int_{D_1}\Phi_\theta(w_i, x)\,\Phi_{\theta'}(x, w_j)\,dx$$
$$\leq C\,(\int_{D_1}\Phi_\theta(w_i, x)\,dx)\,\Phi_{\theta'}(w_i, w_j, \lambda/C')$$
$$\leq C(5.8)\,\Phi_{\theta'}(w_i, w_j)\,.$$

We have a bound for the integral over D_2.

We put

$$D_4=D_3\cap\{x;\ |w_j-x|<8m^{-\beta/4}\}$$
$$D_5=D_3\cap\{x;\ |w_i-x|<8m^{-\beta/4}\}$$

and

$$D_6 = D_3 \cap \{x;\ |w_j - x| > 8m^{-\beta/4}, \\ |w_i - x| > 8m^{-\beta/4}\}.$$

Then, $D_3 = D_4 \cup D_5 \cup D_6$. We have

$$(5.9) \qquad \left|\int_{D_4} \Phi(w_i, x)\, \tilde{\Phi}(x, w_j)\, dx\right| \le \Phi_\theta(w_i, w_j, \lambda/C'') \int_{D_4} \tilde{\Phi}(x, w_j)\, \tilde{d}x .$$

Observing the fact that the number of distinct connected component of ω^C in D_4 is at most $C'(\log m)^2$ we see that the right hand side of (5.9) does not exceed

$$\Phi_\theta(w_i, w_j, \lambda/C')\,(\log m)^2\,(m^{-1}(\log m)^2)^{4-\theta'} .$$

We also have a bound for the integral over D_5.

Finally we get

$$\left|\int_{D_6} \Phi(w_i, x)\, \tilde{\Phi}(x, w_j)\, \tilde{d}x\right| \\ \le C(m^{-1}(\log m)^2)^4 \sum_r{}' \Phi_\theta(w_i, w_r)\, \Phi_{\theta'}(w_r, w_j) ,$$

where the indices r in $\sum'$ run over all r with respect to D_6. It is checked by the fact that diam $\omega^{(j)} \ll m^{-\beta/4}$ and that the number of p satisfying $w_p \in \{x;\ |x-y| < m^{-\beta/4}\} \le (\log m)^2 + 1$,

$$\sum_r{}' \Phi_\theta(w_i, w_r)\, \Phi_{\theta'}(w_r, w_j) \\ \le C\, m^\beta (\log m)^2 \int_\Omega \Phi_\theta(w_i, z)\, \Phi_{\theta'}(z, w_j)\, \tilde{d}z .$$

We combine above facts and we get the following.

Lemma 5.3. *Under the same assumption as above we get*

$$(5.10) \qquad (5.1) \le CP(m, \theta)\, \Phi_{\theta'}(w_i, w_j) \\ + CP(m, \theta')\, \Phi_\theta(w_i, w_j) \\ + Cm^{\beta-4}(\log m)^{10} \int_\Omega \Phi_\theta(w_i, z)\, \Phi_{\theta'}(z, w_j)\, \tilde{d}z ,$$

where

$$P(m, \theta) = (m^{-1}(\log m)^2)^{4-\theta}(\log m)^2 \\ + m^{\beta-4}(\log m)^{10}(\lambda^{-1/2})^{4-\theta} .$$

6. Probabilistic approach to Hilbert-Schmidt norm estimation.

In this section we want to prove the following Proposition 6.1. We put

$$|||f|||_\omega^2 = \int_\omega \int_\omega f(x, y)^2\, \tilde{d}x\, \tilde{d}y .$$

which is an aubse of integral kernel and the integral operator defined by f.

Let A' denote the operator given by

$$A' = \boldsymbol{G} + \sum_{s=1} (-\tau m^{\beta})^{s} J_{s} \boldsymbol{G}(V\boldsymbol{G})^{s},$$

where $J_s=(1-(1/n))\cdots(1-(s-1)/n)$. We study A' in place of A, since $\|A-A'\|_{L^2(\Omega)}\leq Cm^{-\beta}\lambda^{-3}$ and its difference is negligible to consider spectral result.

Proposition 6.1. *Fix an arbitrary* $\varepsilon>0$. *Then, there exists a constant* $T\gg1$ *such that*

$$P(w(m)\in\Omega^n;\ w(m) \text{ satisfies } \mathcal{O}_1(m) \text{ and } (6.2)) \geq 1-m^{-\varepsilon} \tag{6.1}$$

holds as m tends to infinity, where

$$|||\boldsymbol{H}_{w(m)}-A'|||_{\omega}^{2}\leq m^{4\varepsilon}\, m^{\beta-4}(\log m)^{10}\,\lambda^{\varepsilon}\,. \tag{6.2}$$

Proof.

It is easy to see that

$$|||\boldsymbol{H}_{w(m)}-A'|||_{\omega}^{2}\leq(\log m)^{4}\sum_{s\geq1}|||\nabla_{s}|||_{\omega}^{2}\,, \tag{6.3}$$

where

$$|||\nabla_{s}|||_{\omega}^{2}=\int_{\omega}\int_{\omega}\{\tau^{s}\sum_{I}G(x,w_{i_1})\,G_{I}\,G(w_{i_s},y) -(\tau^{s}\,m^{\beta s})\,J_{s}\,G(VG)^{s}\,(x,y)\}^{2}\,\tilde{d}x\,\tilde{d}y\,.$$

For the sake of simplicity we first discuss the case $s=1$.

First step. When $s=1$, $G_I\equiv1$. We have the following.

$$\begin{aligned} &\Big|\,|||\nabla_{I}|||_{\omega}^{2}-\Big(\sum_{\substack{i,j\\ i\neq j}}\tau^{2}\,G_{(2)}(w_i,w_j)^{2} \\ &\quad-2\tau^{2}\,m^{\beta}\sum_{I}G(VG)^{3}\,(w_i,w_i) \\ &\quad+\tau^{2}\,m^{2\beta}\int_{\Omega}G(VG)^{3}\,(x,x)\,\tilde{d}x\Big)\Big| \\ &\leq C\Big(m^{\beta-4}\,(\log m)^{8}\,\lambda^{\varepsilon/2} \\ &\quad+m^{-6}\,(\log m)^{10}\sum_{\substack{i,j\\ i\neq j}}G(w_i,w_j)\,G_{(2)}(w_i,w_j) \\ &\quad+m^{\beta-8}\,(\log m)^{10}\sum_{\substack{i,j\\ i\neq j}}G_{(2)}(w_i,w_j)^{2} \\ &\quad+m^{2\beta-6}\,(\log m)^{10}\,\tilde{G}_{(3)}\Big)\,, \end{aligned} \tag{6.4}$$

where $\tilde{G}_{(h)}=\max_{x,y}G_{(h)}(x,y)$.

We put

$$L_1 = \sum_{\substack{i,j \\ i\neq j}} \tau^2 G(x, w_i) G(w_i, y) G(x, w_j) G(w_j, y)$$
$$L_2 = -2\tau^2 m^\beta \sum_i G(x, w_i) G(w_i, y) (G(VG))(x, y)$$
$$L_3 = \tau^2 m^{2\beta} G(VG)(x, y)^2$$
$$L_4 = \tau^2 \sum_i G(x, w_i)^2 G(w_i, y)^2 .$$

By definition we have

$$|||\nabla_1|||_\omega^2 = \sum_{k=1}^4 \iint_{\omega\times\omega} L_k \tilde{d}x \tilde{d}y . \tag{6.5}$$

We have the following general formula.

$$\begin{aligned} \iint_{(\omega\times\omega)^C} |f(x, y)| \tilde{d}x \tilde{d}y & \\ \leq \int_{\omega^C} \Big(\int_\Omega |f(x, y)| \tilde{d}y \Big) \tilde{d}x & \\ + \int_{\omega^C} \Big(\int_\Omega |f(x, y)| \tilde{d}x \Big) \tilde{d}y . & \end{aligned} \tag{6.6}$$

We see that the left hand side of (6.4) does not exceed

$$\sum_{k=1}^3 \int_{(\omega\times\omega)^C} |L_k| \tilde{d}x \tilde{d}y + \int_{\omega\times\omega} |L_4| \tilde{d}x \tilde{d}y .$$

By the formula (6.6), we get

$$\int_{(\omega\times\omega)^C} |L_1| \tilde{d}x \tilde{d}y \leq \tau^2 \sum_{\substack{i,j \\ i\neq j}} G_{(2)}(w_i, w_j) (G\chi_{\omega^C} VG)(w_i, w_j) . \tag{6.7}$$

By Lemma 5.3 we obtained

$$\begin{aligned} |(G\chi_{\omega^C} VG)(w_i, w_j)| \leq C(m^{-2}(\log m)^{10} G(w_i, w_j) & \\ + m^{\beta-4}(\log m)^{10} G_{(2)}(w_i, w_j)) . & \end{aligned} \tag{6.8}$$

By (6.7), (6.8) we have a bound for (6.7) (the left hand side).

By the formula (6.6) we get

$$\begin{aligned} \iint_{(\omega\times\omega)^C} |L_2| \tilde{d}x \tilde{d}y & \leq C\tau^2 m^\beta \sum_i \int_{\omega^C} (G(VG)^2)(y, w_i) G(w_i, y) dy \\ & \leq C\tau^2 m^\beta \tilde{G}_{(3)} \sum_i \int_{\omega^C} G(w_i, y) dy . \end{aligned} \tag{6.9}$$

Therefore, by Lemma 5.2 we have a bound for (6.9) which is

$$C\tau^2 m^\beta \tilde{G}_{(3)} m^{\beta-2} (\log m)^{10} .$$

We have

$$\iint_{(\omega\times\omega)^C} |L_3|\, dx\, dy$$
$$\leq C\, m^{2\beta-4} \int_{\omega^C} \Big(\int_\Omega (\log |x-y|)^2\, e^{-\lambda^{1/2}|x-y|/C}\, dx\Big)\, dy$$
$$\leq C\, m^{2\beta-4}\, |\omega^C|\, (\lambda^{-1/2})^{4-\varepsilon}\,.$$

We know that $|\omega^C|$ does not exceed $m^{\beta-4}(\log m)^8$.

We also have

$$\iint_{\omega\times\omega} |L_4|\, dx\, dy \leq C\tau^2\, m^\beta (\log m)^2$$
$$= O(m^{\beta-4}(\log m)^2)\,.$$

Summing up these facts we obtain (6.4).

Second step. When $s\geq 2$, we have the following. Here we write w_{i_k} as i_k and w_{j_p} as j_p for the sake of simplicity. But, we usually use w_{i_k} in this paper.

$$(6.10)\quad \Big|\, |||\nabla_s|||^2_\omega - \sum_{\sharp(I\cap J)=0} \tau^{2s}\, G_{(2)}(i_1, j_1)\, G_{(2)}(i_s, j_s)\, G_{IJ}$$
$$-2\,\tau^{2s}\, m^{\beta s} J_s \sum_I G(VG)^{s+2}(i_1, i_s)\, G_I$$
$$+\tau^{2s}\, m^{2\beta s} J_s^2 \iint_{\Omega^2} G(VG)^s\,(x,y)^2\, dx\, dy\,\Big|$$
$$\leq \tau^{2s}\Big\{ m^{-2}(\log m)^{10} \sum_{\sharp(I\cap J)=0} G_{(2)}(i_1, j_1)\, G_{IJ}\, G(i_s, j_s)$$
$$+ m^{\beta-4}(\log m)^{10} \sum_{\sharp(I\cap J)=0} G_{(2)}(i_1, j_1)\, G_{(2)}(i_s, j_s)\, G_{IJ}$$
$$+ m^{-2}(\log m)^{10} \sum_{\sharp(I\cap J)=0} G(i_1, j_1)\, G_{(2)}(i_s, j_s)\, G_{IJ}\Big\}$$
$$+\tau^{2s}\, m^{\beta s}(m^{-2}(\log m)^{10}\, (C\lambda^{-1})^{s-\varepsilon} \sum_I G_I \exp(-\lambda^{1/2}|i_1 - i_s|/C')$$
$$+\tau^{2s}\, m^{\beta s-\beta}(\log m)^{10}\, (C\lambda^{-1})^{s-\varepsilon} \sum_I G_I\, G(i_1, i_s)$$
$$+\tau^{2s}\, m^{2\beta s}\, m^{\beta-4}(\log m)^8\, (C\lambda^{-1})^{2s-\varepsilon}$$
$$+ C(\log m)^2\, \tau^{2s} \sum_{\sharp(I\cap J)\geq 2} G_{IJ}$$
$$+ C(\log m)\, \tau^{2s} \sum_{\substack{\sharp(I\cap J)\geq 1\\ i_1\neq j_1}} G_{(2)}(i_1, j_1)\, G_{IJ}$$
$$+ C\tau^{2s} \sum_{\substack{\sharp(I\cap J)\geq 1\\ i_1\neq j_1, i_s\neq j_s}} G_{(2)}(i_1, j_1)\, G_{(2)}(i_s, j_s)\, G_{IJ}\,.$$

We want to prove (6.10). We put

$$K_1 = \sum_{\sharp(I\cap J)=0} \tau^{2s}\, G(x, w_{i_1})\, G(x, w_{j_1})\, G(w_{i_s}, y)\, G(w_{j_s}, y)\, G_{IJ}$$
$$K_2 = -2\,\tau^{2s}\, m^{\beta s} J_s \sum_I G(x, w_{i_1})\, G_I\, G(y, w_{i_s})\, G(VG)^s\,(x,y)$$
$$K_3 = \tau^{2s}\, m^{2\beta s} J_s^2 (G(VG)^s\,(x,y))^2\,.$$

Then, (6.10) does not exceed

$$\sum_{j=1}^{3}\iint_{(\omega\times\omega)^C}|K_j|\,\tilde{d}x\,dy+\iint_{\omega\times\omega}|K_4|\,\tilde{d}x\,\tilde{d}y\,,$$

where

$$K_4=\tau^{2s}\sum_{\sharp(I\cap J)\geq 1}G(x,w_{i_1})\,G(x,w_{j_1})\,G(w_{i_s},y)\,G(w_{j_s},y)\,G_{IJ}\,.$$

We first examine

$$\iint_{(\omega\times\omega)^C}|K_1|\,\tilde{d}x\,\tilde{d}y\,. \tag{6.11}$$

By (6.6) we see that (6.11) does not exceed

$$\sum_{\sharp(I\cap J)=0}\tau^{2s}(G_{(2)}(w_{i_1},w_{j_1})\,G_{IJ}\,(\int_{\omega^C}G(w_{i_s},y)\,G(y,w_{j_s})\,\tilde{d}y) \tag{6.12}$$
$$+G_{(2)}(w_{i_s},w_{j_s})\,G_{IJ}\,(\int_{\omega^C}G(x,w_{i_1})\,G(w_{j_1},x)\,\tilde{d}x))\,.$$

By Lemma 5.3. we have a bound for (6.11). The terms which comes from (6.11) are represented as the first, the second and the third terms in the right hand side of the inequality in (6.10).

We have

$$\iint_{(\omega\times\omega)^C}|K_2|\,\tilde{d}x\,\tilde{d}y \tag{6.13}$$
$$\leq C\tau^{2s}\,m^{\beta s}\sum_I(\int_{\omega^C}G(VG)^{s+1}(y,w_{i_1})\,G(w_{i_s},y)\,dy)\,G_I\,.$$

Since $s\geq 2$, we see that

$$(G(VG)^{s+1})\,(x,y)\leq C'(C\lambda^{-1})^{s-\varepsilon}\exp(-\lambda^{1/2}|x-y|/C)\,. \tag{6.14}$$

And we have

$$G(VG)^{s+1}(y,w_{i_1})\,G(w_{i_s},y)$$
$$\leq C\exp(-\lambda^{1/2}\,|w_{i_1}-w_{i_s}|/C')\,(G(VG)^{s+1})\,(y,w_{i_1},\lambda/C'')\,G(w_{i_s},y,\lambda/C)\,.$$

Here $G(x,y,\lambda/C)$ is the Green function associated with $-\Delta+(\lambda/C)$. By Lemma 5.3 we get a bound for (6.13) which are represented as the fourth, the fifth terms in the right hand side of the inequality in (6.10).

We have

$$\iint_{(\omega\times\omega)^C}|K_3|\,\tilde{d}x\,\tilde{d}y \tag{6.15}$$
$$\leq\tau^{2s}\,m^{2\beta s}\,|\omega^C|\,\max_y\int_\Omega G(VG)^s\,(x,y)^2\,dx$$
$$\leq\tau^{2s}\,m^{2\beta s}\,m^{\beta-4}\,(\log m)^8\,(C\lambda^{-1})^{2s-\varepsilon}$$

for $\varepsilon>0$.

Finally we examine

$$Q=\iint_{\omega\times\omega}|K_4|\,\tilde{d}x\,\tilde{d}y\,.$$

There are four cases.
(i) $i_1=j_1,\ i_s=j_s$
(ii) $i_1\neq j_1,\ i_s=j_s$
(iii) $i_1=j_1,\ i_s\neq j_s$
(iv) $i_1\neq j_1,\ i_s\neq j_s$.
When we consider the case (i), we have

$$\left|\int_\omega G(x,w_i)^2\,dx\right|\le C(\log m)\,.$$

Therefore,

$$|Q|\le C(\log m)^2\,\tau^{2s}\sum_{\sharp(I\cap J)\ge 2}G_{IJ}\,.$$

when we consider the case (ii), we have

$$|Q|\le C(\log m)\,\tau^{2s}\sum_{\substack{\sharp(I\cap J)\ge 1\\ i_1\neq j_1}}G_{(2)}(w_{i_1},w_{j_1})\,G_{IJ}\,.$$

The case (iii) is similar to the case (ii). For the case (iv) we have

$$|Q|\le C\tau^{2s}\sum_{\substack{\sharp(I\cap J)\ge 1\\ i_1\neq j_1,i_s\neq j_s}}G_{(2)}(w_{i_1},w_{j_1})\,G_{(2)}(w_{i_s},w_{j_s})\,G_{IJ}\,.$$

Summing up these facts we get the desired result.

Integration over $\mathcal{R}(m)$. (as the thrid step).

To get probabilistic result we reconsider the integral over $\mathcal{R}(m)$. If f is a positive function, then we see that

$$\mathcal{E}_{\mathcal{R}(m)}(f(w_{i_1},\cdots,w_{i_h}))\le\mathcal{E}_{\mathcal{R}}(f(w_{i_1},\cdots,w_{i_h}))\,,$$

where

$$\begin{aligned}&\mathcal{E}_{\mathcal{R}(m)}(f(w_{i_1},\cdots,w_{i_h}))\\ &\qquad=\int_{\Omega^n}f(w_{i_1},\cdots,w_{i_h})\,\chi_{\mathcal{R}(m)}\,\tilde{d}w_1\cdots\tilde{d}w_n\,.\end{aligned}$$

If we want to know the exact value (with remainder) of

$$\mathcal{E}_{\mathcal{R}(m)}(f(w_{i_1},\cdots,w_{i_h}))\,,\tag{6.16}$$

its calculation involves some delicateness. Consider the integral

$$\int_{\Omega^h}f(w_{i_1},\cdots,w_{i_h})\,\tilde{d}w_{i_1}\cdots\tilde{d}w_{i_h}\,.\tag{6.17}$$

The difference (6.16)–(6.17) is calculated as the remainder and (6.17) is a value to be desired. It should be noticed that $f(w_{i_1}, \cdots, w_{i_k})$ is a function of $w_{i_1}, \cdots, w_{i_k}$. However, if we want to calculate $\mathcal{E}_{\mathcal{R}(m)}(f)$, we should be careful to consider hidden variables which are other than $w_{i_1}, \cdots, w_{i_k}$.

Let us begin with estimating

$$\mathcal{E}_{\mathcal{R}(m)}(\sum_i f(w_i))-n\int_\Omega f(x)\,dx\,.$$

Assume that $|f|\leq C<+\infty$ for the sake of completeness. Then,

$$\int_{\Omega^n} f(w_i)\,(1-\chi_{\mathcal{R}(m)})\,(w)\,dw_1\cdots dw_n$$
$$\leq(1-|\mathcal{R}(m)|)\,\|f\|_{L^\infty}\leq n^2\,m^{-4\rho}\,\|f\|_{L^\infty}\,.$$

Thus,

$$|\mathcal{E}_{\mathcal{R}(m)}(\sum_i f(w_i))-n\int_\Omega f(x)\,dx|\leq n^3\,m^{-4\rho}\,\|f\|_{L^\infty}\,. \tag{6.18}$$

This inequality can be generalized when f is locally uniformly integrable in the sense that

$$\sup_w \int_{|x-w|\leq s} |f(x)|\,dx\leq C_s<+\infty$$

for any $s>0$. In this case

$$|\mathcal{E}_{\mathcal{R}(m)}(\sum_i f(w_i))-n\int_\Omega f(x)\,dx|\leq n^3\,C_{m^{-\rho}}\,.$$

Assume the following local integrability for f:

$$\max_{w,x}\int_{|y-w|\leq s} |f(x,y)|\,dy\leq D_s<+\infty$$

and the similar inequality for $g(y,x)=f(x,y)$ for any $s\geq 0$. We have the following estimate

$$\begin{aligned}|\mathcal{E}_{\mathcal{R}(m)}(\sum_{\substack{i,j\\ i\neq j}} f(w_i,w_j)-n(n-1)\iint_{\Omega^2} f(x,y)\,dx\,dy|\\ \leq n^4\,D_{m^{-\rho}}\,,\end{aligned} \tag{6.19}$$

observing that

$$\begin{aligned}&|\int_{\mathcal{R}(m)} f\,dw-\int_{\Omega^n} f\,dw| \quad (dw=\prod_{i=1}^n dw_i)\\ &\leq \sum_{j=2}\int_{\Omega^{n-1}}(\int_{|w_1-w_j|\leq m^{-\rho}} |f(w_1,w_2)|\,d\tilde{w}_1)\,dw_2\cdots dw_n\\ &\quad+\sum_{\substack{j=1\\ j\neq 2}}\int_{\Omega^{n-1}}(\int_{|w_2-w_j|\leq m^{-\rho}} |f(w_1,w_2)|\,dw_2)\,dw_1\cdots dw_n\end{aligned}$$

$$+\sum_{\substack{j=1\\ j\neq 3}}\int_{\Omega^{n-1}}\Big(\int_{|w_3-w_j|\leq m^{-\rho}}|f(w_1,w_2)|\,\tilde{d}w_3\Big)\,\tilde{d}w_1\,\tilde{d}w_2\,\tilde{d}w_4\cdots\tilde{d}w_n$$
$$+\cdots.$$
$$\leq n^2 D_{m^{-\rho}}.$$

We apply $\mathcal{E}_{\mathfrak{R}(m)}$ to the terms in the left hand side of the inequalities (6.4) and we get

$$\mathcal{E}_{\mathfrak{R}(m)}\Big(|||\nabla_1|||_\omega^2-\Big(\sum_{i,j}\tau^2 G_{(2)}^2+\cdots+\tau^2 m^{2\beta}\int_\Omega G(VG)^3\,\tilde{d}x\Big)\Big)$$
$$=-\tau^2 n\int_\Omega G(VG)^3(x,x)\,\tilde{d}x+O(m^{4\beta-4-4\rho}(\log m)^2\tilde{G}_{(4)}) \tag{6.20}$$

observing that $n(n-1)-n^2=-n$ and (6.18), (6.19). Since

$$\mathcal{E}_{\mathfrak{R}(m)}\text{(the terms in the right hand side of the inequality (6.4))}$$
$$\leq \mathcal{E}_{\mathfrak{R}}(\;''\;)$$
$$\leq m^{2\beta-6}(\log m)^{10}\iint_{\Omega^2}G(x,y)\,G_{(2)}(x,y)\,\tilde{d}x\,\tilde{d}y$$
$$+m^{3\beta-8}(\log m)^{10}\iint_{\Omega^2}G_{(2)}(x,y)^2\,\tilde{d}x\,\tilde{d}y$$
$$+m^{2\beta-6}(\log m)^{10}\tilde{G}_{(3)}+m^{\beta-4}(\log m)^8\lambda^{\varepsilon/2},$$

we get the following

$$\mathcal{E}_{\mathfrak{R}(m)}(|||\nabla_1|||_\omega^2)$$
$$=O\big(m^{4\beta-4-4\rho}(\log m)^2\tilde{G}_{(4)}+m^{\beta-4}(\log m)^8\lambda^{\varepsilon/2}$$
$$+m^{2\beta-6}(\log m)^{10}\tilde{G}_{(3)}+m^{3\beta-8}(\log m)^{10}\tilde{G}_{(4)}$$
$$+m^{\beta-4}\bar{G}_{(4)}\big) \tag{6.21}$$

for any fixed $\varepsilon>0$.

We are going to apply $\mathcal{E}_{\mathfrak{R}(m)}$ to (6.10). Before doing it we study some inequality. We denote $\Phi_\theta(x,y,\lambda/100C)$ as $\Psi_\theta(x,y)$ and we consider the following integral

$$\int_S \Psi_\theta(w_l,w_j)\,\Psi_{\theta'}(w_j,w_k)\,\tilde{d}w, \tag{6.22}$$

where $S=\{w_j;\ |w_k-w_j|\leq m^{-\rho}$ or $|w_l-w_j|\leq m^{-\rho}\}$. When $|w_k-w_l|>3m^{-\rho}$ we see that (6.22) dones not exceed

$$C\,\Psi_\theta(w_l,w_k)\int_0^{7m^{-\rho}}r^{3-\theta'}\,dr+\cdots$$
$$\leq C''(m^{-(4-\theta')\rho}\,\Psi_\theta(w_l,w_k)+m^{-(4-\theta)\rho}\,\Psi_{\theta'}(w_l,w_k)). \tag{6.23}$$

Here we used the fact that $|w_j-w_k|\geq C'|w_l-w_k|$, $(|w_l-w_k|/C')^{-\theta}=C'^{\theta}$

$|w_l-w_k|^{-\theta}$. The same inequality holds for the case $|w_k-w_l|\le 3m^{-\rho}$. By geometric observation we can say more about it. We have

$$(6.24)\qquad \sup_z \Big|\int_{|w_j-z|\le m^{-\rho}} \Psi_\theta(w, w_j)\, \Psi_\theta(w_j, w_k)\, \tilde{d}w_j\Big| \\ \le C''(m^{-(4-\theta')\rho}\, \Psi_\theta + m^{-(4-\theta)\rho}\, \Psi_{\theta'})\,.$$

Our claim in calculating $\mathcal{E}_{\mathfrak{R}(m)}$ ((6.10)) is the following.

$$(6.25)\quad \mathcal{E}_{\mathfrak{R}(m)}\big(\sum_{\sharp(I\cap J)=0} \cdots, \text{ in the left hand side of the inequality of (6.10)}\big) \\ = J_{2s}\, \tau^{2s}\, m^{2\beta s} \iint_{\Omega^2} G(VG)^s\,(x, y)^2\, \tilde{d}x\, \tilde{d}y \\ +O((\tau m^\beta)^{2s}\, m^{2\beta}(m^{-(4-\varepsilon)\rho}\, |\Psi_2 * G_{(2s-1)}| \\ +m^{-2\rho}\, |\Psi_\varepsilon * G_{(2s-1)}| + m^{-2\rho}\, |G_{(2s)} * \Psi_2|)$$

Here

$$|f*g| = \max_{x,y} \int_\Omega |f(x, z)\, g(z, y)|\, \tilde{d}z\,.$$

Assume that $i_1=1, \cdots, i_k=k, \cdots, i_s=s$ and $j_1=1', \cdots, j_k=k', \cdots, j_s=s'$. We put

$$P_{j,k} = \int_{\Omega^{n-1}} \Big(\int_{|w_j-w_k|\le m^{-\rho}} G_{(2)}(w_1, w_{1'})\, G_{(2)}(w_s, w_{s'})\, G_{IJ}\Big)\, \tilde{d}w_j \\ d(v.o.t.\ w_j)\,.$$

Here *v.o.t.* means *variables other than.* There are four cases. (i) $j=1$ or $j=1'$, (ii) $j=s$ or $j=s'$, (iii) ($j\in(1, \cdots, s)$ but $j\neq 1, j\neq s$) or ($j\notin(i', \cdots, s')$ but $j\neq 1'$, $j\neq s'$), (iv) $j\notin(1, \cdots, s)\cup(1', \cdots, s')$.

First we study the case (i). We assume that $j=1$. We know that $|G_{(2)}(x, y)|\le C\, \Psi_\varepsilon(x, y)$, $|G(x, y)|\le C\, \Psi_2(x, y)$ for any $\varepsilon>0$. Therefore,

$$(6.26)\quad \int_{|w_1-w_k|\le m^{-\rho}} G_{(2)}(w_i, w_{1'})\, \Phi_2(w_1, w_2)\, (G_{i_2 i_3}\cdots)\, G_{(2)}(w_s, w_{s'})\, G_J\, \tilde{d}w_1 \\ \le C'(m^{-(4-\varepsilon)\rho}\, \Psi_2(w_{1'}, w_2) + m^{-2\rho}\, \Psi_\varepsilon(w_{1'}, w_2))\, G_{i_2 i_3}\cdots) \\ \times G_{(2)}(w_s, w_{s'})\, G_J\,.$$

Therefore,

$$|P_{j,k}| \le \int_{\Omega^{n-1}} (6.26)\, \tilde{d}w_2\, \tilde{d}w_3 \cdots \tilde{d}w_n \\ \le C \iint_{\Omega^2} ((m^{-(4-\varepsilon)\rho}\, \Psi_2(w_{1'}, w_2) + m^{-2\rho}\, \Psi_\varepsilon(w_{1'}, w_2)) \\ \times G_{(2s-1)}(w_{1'}, w_2)\, \tilde{d}w_{1'}\, \tilde{d}w_2\,.$$

We also have the same inequality for $P_{j,k}$ for the case (ii).

We now consider the case (iii). Assume that $j\in(1, \cdots, s)$. Then,

$$\int_{|w_j-w_k|\le m^{-\rho}} G_{j-1,j}\, G_{j,j+1}\, \tilde{d}w_j \le C\, m^{-2\rho}\, \Psi_2(w_{j-1}, w_{j+1})\,.$$

Therefore, we have

$$|P_{jk}| \le C\, m^{-2\rho}\, |(G_{(2s)} * \Psi_2)|\,.$$

Summing up these facts we get (6.25).

For the sake of simplicity we put

$$L(s) = \iint_{\Omega^2} (G(VG)^s)\,(x, y)^2\, \tilde{d}x\, \tilde{d}y\,.$$

We want to show

$$\begin{aligned}&\mathcal{E}_{\mathfrak{R}(m)}\,(-2\tau^{2s}\, m^{\beta s}\, J_s \sum_I G_{(s+3)}\,(w_{i_1}, w_{i_s})\, G_I) \\ &\quad = -2\tau^{2s}\, m^{2\beta s}\, J_s^2\, L(s) + 0(\tau^{2s}\, m^{2\beta s}\, m^{3\beta-2\rho}(C\lambda^{-1})^{s-2}\, \tilde{G}_{(s+3)})\,.\end{aligned}$$

We write

$$Q_{j,k} = \int_{\Omega^{n-1}} (\int_{|w_j-w_k|\le m^{-\rho}} G_{(s+3)}\,(w_1, w_{i_s})\, G_I\, \tilde{d}w_j)\, \tilde{d}(v.o.t.\ w_j)$$

We assume that $i_k = k$. There are three cases (i), (ii), (iii).

(i) $j=1$ or $j=s$,

(ii) $j\in(1, \cdots, s)$ but $j\neq 1$, $j\neq s$,

(iii) $j\notin(1, \cdots, s)$.

We have

$$\begin{aligned}|Q_{j,k}| &\le \tilde{G}_{(s+3)} \int_{\Omega^{n-1}} (\int_{|w_1-w_k|\le m^{-\rho}} G_{12}\, G_{23} \cdots G_{s-1s}\, \tilde{d}w_1)\, \tilde{d}(v.o.t.) \\ &\le \tilde{G}_{(s+3)}\, m^{-2\rho}(C\lambda^{-1})^{s-2}\,.\end{aligned}$$

We have the same estimate for the case (ii).

For the case (iii) we have

$$|Q_{j,k}| \le C'\, m^{-4\rho}(C\lambda^{-1})^{s-1}\, \tilde{G}_{(s+3)}\,.$$

Summing up these result we get the desired result.

It is easy to see that

$$\mathcal{E}_{\mathfrak{R}(m)}\,(L(s)) = L(s) + O(n^2\, m^{-4\rho}\, L(s))\,.$$

As a consequence of the above results we get the following

$$\begin{aligned}|||\nabla_s|||_\omega^2 &= \tau^{2s}\, m^{2\beta s}\, (J_{2s} - J_s^2)\, L(s) \\ &\quad + \tau^{2s}\, m^{2\beta s+2\beta}\, 0(H(s)) \\ &\quad + O(\mathcal{E}_{\mathfrak{R}}\ \text{(the right hand side of the inequality in (6.10))}\,,\end{aligned}$$

where

$$\begin{aligned}H(s) = m^{-2\rho}(|G_{(2s-1)}*\Psi_s|+|G_{(2s)}*\Psi_2|)\\+m^{-(4-\varepsilon)\rho}\,|G_{(2s-1)}*\Psi_2|\\+m^{\beta-2\rho}(C\lambda^{-1})^{s-2}\,\tilde{G}_{(s+3)}+m^{-4\rho}\,L(s)\,.\end{aligned}$$

From now on we discuss $\mathcal{E}_{\mathfrak{R}}$ (the right hand side of (6.10)).

There are nine terms in the right hand side of the inequality in (6.10). We use

$$\mathcal{E}_{\mathfrak{R}}\,(\text{the term})\leq E(\text{the term})$$

for the first $\sim$ the seventh terms in the right hand side of the inequality in (6.10). Here E denotes the expectation over Ω^{s}. Then, we get

$$\begin{aligned}&\mathcal{E}_{\mathfrak{R}}\,(\sum_{k=1}^{6}\text{the } k^{\text{th}} \text{ term in the right hand of the inequality in (6.10)})\\&\leq C\,\tau^{2s}\,m^{2\beta s}\{m^{-2}(\log m)^{10}\,\tilde{G}_{(2s+1)}+m^{\beta-4}(\log m)^{10}\,\tilde{G}_{(2s+2)}\\&\quad+m^{-2}(\log m)^{10}\,(C\lambda^{-1})^{s-\varepsilon}\,|G_{(s-1)}*\Phi_0|\\&\quad+m^{\beta-4}(\log m)^{8}\,(C\lambda^{-1})^{2s-\varepsilon}\\&\quad+m^{-\beta}(\log m)^{10}\,(C\lambda^{-1})^{s-\varepsilon}\,|G_{(s-1)}*\Phi_2|\}.\end{aligned}$$

We see that

$$\begin{aligned}&|\text{the } (7^{\text{th}}+8^{\text{th}}+9^{\text{th}}) \text{ terms}|\\&\leq C(\log m)^2\,\tau^{2s}\sum_{\sharp(I\cap J)\geq 1}G_{IJ}\end{aligned}$$

observing that $G_{(2)}(w_{i_1}, w_{j_1})\leq C(\log m)$ for $|w_{i_1}-w_{j_1}|\geq m^{-\rho}$. By Lemma 4.2 we see that

$$\begin{aligned}&\mathcal{E}_{\mathfrak{R}}\,(\sum_{k=7}^{9}\text{the } k^{\text{th}} \text{ term})\\&\leq C(\log m)^2\,(\tau\lambda^{-1}\,m^{\beta})^{2s}\,(\lambda^2\,m^{-\beta})\end{aligned}$$

using that

$$\sum_{q\geq 1}(m^{-\beta+\varepsilon}(C\lambda^{-1})^{-2})^{q}$$

converges.

Summing up these facts we get

$$\begin{aligned}&\mathcal{E}_{\mathfrak{R}(m)}\,(\text{the right hand side of the inequality in (6.10)})\\&\leq C'(\tau m^{\beta}/C\lambda)^{2s}\{m^{-2}(\log m)^{10}\,\lambda^{1+\varepsilon}+m^{\beta-4}(\log m)^{10}\,\lambda^{\varepsilon}\\&\quad+m^{-\beta}(\log m)^{10}\,\lambda^{2+\varepsilon}\}\end{aligned}$$

observing that

$$\tilde{G}_{(s)} \leq C'(C\lambda^{-1})^{s-2-\varepsilon}$$

for $s \geq 3$, $\varepsilon > 0$ and

$$\begin{aligned} |G_{(s)}*\Phi_0| &= \max_{x,y} \left| \int_\Omega G_{(s)}(x, z)\, \Phi_0(z, y)\, dz \right| \\ &\leq C'(C\lambda^{-1})^{s-\varepsilon}. \end{aligned}$$

Summing up these facts we get

$$\begin{aligned} (6.27) \qquad & \mathcal{E}_{\mathcal{R}(m)}(|||\nabla_s|||^2) \\ &= (\tau m^\beta / C\lambda)^{2s} (m^{-\beta} (\log m)^4 \lambda^\varepsilon + m^{2\beta-2\rho} \lambda^{1+\varepsilon} + m^{2\beta-(4-\varepsilon)\rho} \lambda^{2+\varepsilon} \\ &\quad + m^{3\beta-2\rho} \lambda^{1+\varepsilon} + m^{2\beta-4\rho} \lambda^\varepsilon + m^{-2} (\log m)^{10} \lambda^{1+\varepsilon} \\ &\quad + m^{\beta-4} (\log m)^{10} \lambda^\varepsilon + m^{-\beta} (\log m)^{10} \lambda^{2+\varepsilon}). \end{aligned}$$

As a conclusin of (6.3), (6.21) and (6.27), we get our proof of Proposition 6.1 when we take sufficiently large ρ.

We here make a comment on our argument. We do not take E over Ω^n, since $E(G_{IJ})$ can be divergent when $\sharp(I \cap J) \geq 1$. Thus, we used $\mathcal{E}_{\mathcal{R}(m)}$ to avoid this divergence. Owing to the usage of $\mathcal{E}_{\mathcal{R}(m)}$ our calculation becomes very long. The author hopes some simplification of our calculus.

7. On A'.

In this section we want to examine

$$\|\chi_\omega A' \chi_\omega - A'\|_{L^2(\Omega)}.$$

It does not exceed

$$(7.1) \qquad \|(1-\chi_\omega) A' \chi_\omega\|_{L^2(\Omega)} + \|A'(1-\chi_\omega)\|_{L^2(\Omega)}.$$

By the duality argument we have to estimate

$$\|(1-\chi_\omega) A'\|_{L^2(\Omega)}$$

to get a bound for (7.1). Fix $f \in L^2(\Omega)$. Then,

$$\begin{aligned} &\left| \int_{\omega^c} (A'f)(x)^2\, dx \right| \\ &\leq \int_{\omega^c} \left(\int_\Omega G(x, y) f(y)\, dy \right)^2 dx. \\ &\leq C\lambda^{-1} \int_{\omega^c} \left(\int_\Omega G(x, y) f(y)^2\, dy \right) dx \\ &\leq C\lambda^{-1} \max_y \left(\int_{\omega^c} G(x, y)\, dx \right) \|f\|^2_{L^2(\Omega)}. \end{aligned}$$

As we studied before, we get

$$\max_{y} \int_{\omega^C} G(x, y)\, dx \leq C\, m^{-2}(\log m)^{10}$$

when $w(m)$ satisfies $\mathcal{O}_1(m)$.
As a conclusion we get the following

Proposition 7.1. *Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then, there exists a constant C such that*

(7.2) $$||\chi_\omega A' \chi_\omega - A'||_{L^2(\Omega)} \leq C \lambda^{-1/2} m^{-1}(\log m)^5$$

holds.

8. Spectral result. Proof of Theorem 1. We want to get spectral properties of $G_{w(m)}$. Let $\lambda_j^{(i)}(w(m))$ $(i=1, 2, 3)$ and $\lambda_j^{(4)}$ be the j-th eigenvalue of the operators $\boldsymbol{G}_{w(m)}$ $(i=1)$, $\boldsymbol{H}_{w(m)}$ $(i=2)$, $\chi_\omega A' \chi_\omega$ $(i=3)$, $A'(i=4)$.

By the spectral theory of operators applied to Theorem 2, Proposition 6.1, Proposition 7.1 we see that the measure of the set $w(m)$ satisfying

$$|\lambda_j^{(1)} - \lambda_j^{(3)}| \leq m^{5\varepsilon}(m^{(3/2)\beta-4} + m^{(\beta/2)-2}(\log m)^5 \lambda^{\varepsilon/2})$$
$$|\lambda_j^{(3)} - \lambda_j^{(4)}| \leq C\, m^{5\varepsilon-(\beta/2)}(\log m)^3$$

since $G_{\omega(m)}$ and $\chi_\omega A' \chi_\omega$ are positive compact operators. Hereafter we assume that $V \equiv |\Omega|^{-1}$ when $\beta > 2$. We consider the case $\lambda = Tm^{\beta-2}$. Then, $\lambda_j^{(1)}(w(m)) = (\mu_j(w(m)) + Tm^{\beta-2})^{-1}$ and $\lambda_j^{(4)}(w(m)) = (\mu_j(V; m) + Tm^{\beta-2})^{-1}$ with $\mu_j(V; m) = \mu_j + 2\pi^2 m^{\beta-2}|\Omega|^{-1}$ for $\beta > 2$. Therefore,

$$\begin{aligned} &|\mu_j(w(m)) - \mu_j(V; m)| \\ &\quad \leq Cm^{5\varepsilon+2(\beta-2)}\,(m^{(3/2)\beta-4} + m^{(\beta/2)-2}(\log m)^5 \lambda^{\varepsilon/2} \\ &\qquad + m^{-\beta/2}(\log m)^5)\,. \end{aligned}$$

As a conclusion we get Theorem 1. It is a natural question to ask the optimal $\delta_0(\beta)$ and the fluctuation associated with optimal exponent $\delta_0(\beta)$.

ADDENDUM. We list the papers [13], [14].

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