<table>
<thead>
<tr>
<th>Title</th>
<th>On the potential taken with respect to complex-valued kernels</th>
</tr>
</thead>
<tbody>
<tr>
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ON THE POTENTIAL TAKEN WITH RESPECT TO
COMPLEX-VALUED KERNELS

MINORU MATSUDA

(Received December 24, 1971)

In the potential theory, we have two theorems called the existence theorem concerning the potential taken with respect to real-valued and symmetric kernels. They are stated as follows. Let $K(X, Y)$ be a real-valued function defined in a locally compact Hausdorff space $\Omega$, lower semi-continuous for any points $X$ and $Y$, may be $+\infty$ for $X = Y$, always finite for $X \neq Y$ and bounded from above for $X$ and $Y$ belonging to disjoint compact sets of $\Omega$ respectively. For a given positive measure $\mu$, the potential is defined by

$$K\mu(X) = \int K(X, Y) d\mu(Y),$$

and the $K$-energy of $\mu$ is defined by $\int K\mu(X) d\mu(X)$. A subset of $\Omega$ is said to be of positive $K$-transfinite diameter, when it charges a positive measure $\mu$ of finite $K$-energy with compact support, otherwise said to be of $K$-transfinite diameter zero. Let $K(X, Y)$ be symmetric: $K(X, Y) = K(Y, X)$ for any points $X$ and $Y$. Then we have two following theorems.

**Theorem A.** Let $F$ be a compact subset of positive $K$-transfinite diameter, and $f(X)$ be a real-valued upper semi-continuous function with lower bound on $F$. Then, given any positive number $a$, there exist a positive measure $\mu$ supported by $F$ and a real constant $\gamma$ such that

1. $\mu(F) = a$,
2. $K\mu(X) \geq f(X) + \gamma$ on $F$ with a possible exception of a set of $K$-transfinite diameter zero, and
3. $K\mu(X) = f(X) + \gamma$ on the support of $\mu$.

**Theorem B.** In the above theorem, suppose the further conditions: $K(X, Y) > 0$ and $\inf f(X) > 0$ for any points $X$ and $Y$ of $F$. Then, given any compact subset $F$ of positive $K$-transfinite diameter, there exists a positive measure $\mu$ supported by $F$ such that

1. $K\mu(X) \geq f(X)$ on $F$ with a possible exception of a set of $K$-transfinite diameter zero, and
(2) \( K_{\mu}(X) \leq f(X) \) on the support of \( \mu \).

Recently, N. Ninomiya ([5]) proved the existence theorems for the potential taken with respect to complex-valued and symmetric kernels and to complex-valued measures, which are the extension of the above theorems in the case of the real-valued kernels. We state them as follows. Let \( K(X, Y) \) be a complex-valued function defined in a locally compact Hausdorff space \( \Omega \). Let \( k(X, Y) = \Re K(X, Y) \) be a function lower semi-continuous, symmetric, may be \(+\infty\) for \( X = Y \), always finite for \( X \neq Y \) and bounded from above for \( X \) and \( Y \) belonging to disjoint compact sets of \( \Omega \) respectively, and \( n(X, Y) = \Im K(X, Y) \) be a finite continuous function satisfying that \( n(X, Y) = -n(Y, X) \) for any points \( X \) and \( Y \) of \( \Omega \). For any compact subset \( F \) and any positive numbers \( a \) and \( b \), denote by \( \mathcal{M}(a, F, b) \) the family of all the complex-valued measures supported by \( F \) whose real parts and imaginary parts are positive measures with total mass \( a \) and \( b \) respectively, by \( \mathcal{M}(a, F) \) the family of all the complex-valued measures supported by \( F \) whose real parts are positive measures with total mass \( a \) and imaginary parts are any positive measures, by \( \mathcal{M}(F, b) \) the family of all the complex-valued measures supported by \( F \) whose real parts are any positive measures and imaginary parts are positive measures with total mass \( b \), and by \( \mathcal{M}(F) \) the family of all the complex-valued measures supported by \( F \) whose real parts and imaginary parts are any positive measures. For any such measure \( \alpha \), the potential is defined by

\[
K\alpha(X) = \int K(X, Y) d\alpha(Y).
\]

Then we have two following theorems.

**Theorem A'**. Let \( F \) be a compact subset of positive \( k \)-transfinite diameter, and \( f(X) \) be a complex-valued function whose real part \( \Re f(X) \) and imaginary part \( \Im f(X) \) are upper semi-continuous functions with lower bound on \( F \). Then, given any positive numbers \( a \) and \( b \), there exist a measure \( \alpha \) of \( \mathcal{M}(a, F, b) \) and a complex constant \( \gamma \) such that

1. \( \Re K\alpha(X) \geq \Re \{f(X) + \gamma\} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,
2. \( \Re K\alpha(X) \leq \Re \{f(X) + \gamma\} \) on the support of \( \Re \alpha \),
3. \( \Im K\alpha(X) \geq \Im \{f(X) + \gamma\} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
4. \( \Im K\alpha(X) \leq \Im \{f(X) + \gamma\} \) on the support of \( \Im \alpha \).

**Theorem B'**. In the above theorem, suppose the further conditions: \( k(X, Y) > 0, \inf \Re f(X) > 0 \) and \( \inf \Im f(X) > 0 \) for any points \( X \) and \( Y \) of \( F \). Then, given any positive number \( a \) such that \( a|n(X, Y)| < \Im f(X) \) for points \( X \) and \( Y \) of \( F \), there exist a measure \( \alpha \) of \( \mathcal{M}(a, F) \) and a real constant \( \gamma \) such that
(1) \( \Re K\alpha(X) \geq \Re \{ f(X) + \gamma \} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,

(2) \( \Re K\alpha(X) \leq \Re \{ f(X) + \gamma \} \) on the support of \( \Re \alpha \),

(3) \( \Im K\alpha(X) \geq \Im \{ f(X) + \gamma \} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and

(4) \( \Im K\alpha(X) \leq \Im \{ f(X) + \gamma \} \) on the support of \( \Im \alpha \).

Similarly, given any positive number \( b \) such that \( b|\n(X, Y)| < \Re f(X) \) for points \( X \) and \( Y \) of \( F \), there exist a measure \( \alpha \) of \( \mathcal{M}(F, b) \) and a complex constant \( \gamma \) such that

(1') \( \Re K\alpha(X) \geq \Re f(X) \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,

(2') \( \Re K\alpha(X) \leq \Re f(X) \) on the support of \( \Re \alpha \),

(3') \( \Im K\alpha(X) \geq \Im \{ f(X) + \gamma \} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and

(4') \( \Im K\alpha(X) \leq \Im \{ f(X) + \gamma \} \) on the support of \( \Im \alpha \).

In this paper we are going to extend these existence theorems to the potential taken with respect to complex-valued kernels and to complex-valued measures, under an additional condition of the continuity principle for the adjoint kernel.

Let \( K(X, Y) \) be a complex-valued function, not always symmetric, defined in a locally compact Hausdorff space \( \Omega \). Let \( k(X, Y) = K(X, Y) \) be a function lower semi-continuous, may be \(+\infty\) for \( X = Y \), always finite for \( X \neq Y \) and \( \n(X, Y) = \Im K(X, Y) \) be a finite continuous function. For any positive measure \( \mu \), consider the adjoint potential defined by

\[
\kappa_{\mu}(X) = \int k(X, Y)d\mu(Y) = \int k(Y, X)d\mu(Y).
\]

Then, we have two following theorems.

**Theorem 1.** Let \( F \) be a compact subset of positive \( k \)-transfinite diameter, and \( f(X) \) be a complex-valued function whose real part \( \Re f(X) \) and imaginary part \( \Im f(X) \) are upper semi-continuous functions with lower bound on \( F \), and \( a \) and \( b \) be two positive numbers. If the adjoint kernel \( \kappa(X, Y) = k(Y, X) \) satisfies the continuity principle\(^1\), there exist a measure \( \alpha \) of \( \mathcal{M}(a, F, b) \) and a complex constant \( \gamma \) such that

(1) \( \Re K\alpha(X) \geq \Re \{ f(X) + \gamma \} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,

---

\(^1\) We say that \( k(X, Y) \) satisfies the continuity principle when for any positive measure \( \mu \) with compact support, the following implication holds: (the restriction of \( k\mu(X) \) to the support of \( \mu \) is finite and continuous) \( = (k\mu(X) \) is finite and continuous in the whole space \( \Omega \)).
(2) $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$.
(3) $\Im K\alpha(X) \leq \Im \{f(X) + \gamma\}$ on $F$ with a possible exception of a set of $k$-transfinite diameter zero, and
(4) $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$.

**Theorem 2.** In the above theorem, suppose the further conditions: $k(X, Y) > 0$, $\inf \Re f(X) > 0$, and $\inf \Im f(X) > 0$ for any points $X$ and $Y$ of $F$. Then, there exist a measure $\alpha$ of $\mathcal{M}(F)$ and a real constant $\gamma$ such that

(1) $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on $F$ with a possible exception of a set of $k$-transfinite diameter zero,
(2) $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
(3) $\Im K\alpha(X) \geq \Im \{f(X) + \gamma\}$ on $F$ with a possible exception of a set of $k$-transfinite diameter zero, and
(4) $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$.

Similarly, there exist a measure $\alpha$ of $\mathcal{M}(F)$ and a pure imaginary constant $\gamma$ such that

(1') $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on $F$ with a possible exception of a set of $k$-transfinite diameter zero,
(2') $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
(3') $\Im K\alpha(X) \geq \Im \{f(X) + \gamma\}$ on $F$ with a possible exception of a set of $k$-transfinite diameter zero, and
(4') $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$.

Before we prove the theorems, we prepare some lemmas.

**Lemma 1.** Let $\mu$ be a positive measure with compact support. If the adjoint kernel $k(X, Y)$ satisfies the continuity principle, the set $E = \{X | k_\mu(X) = +\infty\}$ of $\Omega$ is of $k$-transfinite diameter zero.

**Lemma 2.** Let $F$ be a compact subset, and $f(X)$ be a complex-valued function whose real part $\Re f(X)$ and imaginary part $\Im f(X)$ are upper semi-continuous functions with lower bound on $F$ respectively, and $a$ and $b$ be two positive numbers. If the real part $k(X, Y)$ of $K(X, Y)$ is a finite continuous function defined in $\Omega$, there exist a measure $\alpha$ of $\mathcal{M}(a, F, b)$ and a complex constant $\gamma$ such that

(1) $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on $F$,
(2) $\Re K\alpha(X) = \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
(3) $\Im K\alpha(X) \geq \Im \{f(X) + \gamma\}$ on $F$, and
(4) $\Re K\alpha(X) = \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$.

**Lemma 3.** In above Lemma 2, suppose the further conditions: $k(X, Y) > 0$, $\inf \Re f(X) > 0$, and $\inf \Im f(X) > 0$ for any points $X$ and $Y$ of $F$ and both $\Re f(X)$ and $\Im f(X)$ are finite and continuous. Then, there exist a measure $\alpha$ of $\mathcal{M}(F)$ and a real constant $\gamma$ such that
POTENTIAL TAKEN WITH RESPECT TO COMPLEX-VALUED KERNELS

539

(1) $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on $F$,
(2) $\Re K\alpha(X) = \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
(3) $\Im K\alpha(X) \geq \Im f(X)$ on $F$, and
(4) $\Im K\alpha(X) = \Im f(X)$ on the support of $\Im \alpha$.

Lemma 4. In above Theorem 2, suppose the further conditions: both
\( Rf(X) \) and $\Im f(X)$ are finite and continuous. Then, there exist a measure $\alpha$ of
\( R(\Omega) \) and a real constant $\gamma$ such that
(1) $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on $F$ with a possible exception of a set of $k$-transfinite
diameter zero,
(2) $\Re K\alpha(X) = \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
(3) $\Im K\alpha(X) \geq \Im f(X)$ on $F$ with a possible exception of a set of $k$-transfinite
diameter zero, and
(4) $\Im K\alpha(X) = \Im f(X)$ on the support of $\Im \alpha$.

Proof of Lemma 1. Let the set $E$ be of positive $k$-transfinite diameter.
$\hat{k}(X, Y)$ satisfying the continuity principle, there exists a positive measure $\sigma$
such that
(a) the compact support of $\sigma$ is contained in the set $E$, and
(b) $\hat{k}\sigma(X)$ is finite and continuous in the whole space $\Omega$.
Hence we have
\[ f(k\mu(X)d\sigma(X) = +\infty, \text{ that is, } \int \hat{k}\sigma(X)d\mu(X) = +\infty, \text{ which is a contradic-
tion.} \]

Proof of Lemma 2. For any positive number $c$, denote by $m(c, F)$ the set
of all positive measures supported by $F$ with total mass $c$. We define the
point-to-set mapping $\varphi$ on the product space $m(a, F) \times m(b, F)$ into $\mathbb{F}(m(a, F) \times m(b, F))$ which is the family of all closed convex subsets in $m(a, F) \times m(b, F)$. For any $\alpha = \mu + iv$
that is, $\alpha = (\mu, \nu)$ of $m(a, F) \times m(b, F)$, $\varphi$ is defined as
follows.
\[ \varphi((\mu, \nu)) = \{(\lambda, \tau) \in m(a, F) \times m(b, F) \mid \int (k\mu(X) - nv(X) - Rf(X))d\lambda(X) + \int (kv(X) + n\mu(X) - \Im f(X))d\tau(X) \]
\[ = \inf \{(k\mu(X) - nv(X) - \Re f(X))d\xi(X) + \int (kv(X) + n\mu(X) - \Im f(X))d\eta(X) \mid \]
\[ (\xi, \eta) \in m(a, F) \times m(b, F)) \}. \]

Obviously $\varphi((\mu, \nu)) \neq \phi$. For, putting
\[ d = \inf \{(k\mu(X) - nv(X) - \Re f(X))d\xi(X) + \int (kv(X) + n\mu(X) - \Im f(X))d\eta(X) \mid \]
\[ (\xi, \eta) \in m(a, F) \times m(b, F)) \}, \]
there exist sequences of
$\xi_n \in m(a, F)$ and $\eta_n \in m(b, F)$ such that
As we have vaguely convergent subnets \( \xi_{n_k} \in m(a, F) \) and \( \eta_{n_k} \in m(b, F) \) such that \( \xi_{n_k} \rightarrow \xi_0 \) and \( \eta_{n_k} \rightarrow \eta_0 \), there holds \( \varphi((\mu, \nu)) \equiv (\xi_0, \eta_0) \). Moreover \( \varphi((\mu, \nu)) \) is upper semi-continuous in the following sense: if nets \( \{\delta_J^\alpha, a \text{ directed set} \} \) and \( \{\zeta_a^I^a \} \) converge to \( \delta \) and \( \zeta \) with respect to the product topology respectively, and if \( \delta_a \in \varphi(\zeta_a) \) for any \( \alpha \in D \), then \( \delta \in \varphi(\xi) \). In fact, if we put \( \delta_a = (\lambda_a, \tau_a), \zeta_a = (\sigma_a, \gamma_a), \delta = (\lambda_0, \tau_0), \) and \( \zeta = (\sigma_0, \gamma_0) \), we have

\[
\begin{align*}
\int (k\sigma_a(X) - n\gamma_a(X) - Rf(X))d\lambda_a(X) + f(k\gamma_a(X) + n\sigma_a(X) - \Re f(X))d\tau_a(X) \leq \int (k\sigma_a(X) - n\gamma_a(X) - Rf(X))d\xi(X) + f(k\gamma_a(X) + n\sigma_a(X) - \Re f(X))d\gamma(X)
\end{align*}
\]

for any \( (\xi, \eta) \in m(a, F) \times m(b, F) \). By the limit process, we have

\[
\begin{align*}
\int (k\sigma(X) - n\gamma(X) - Rf(X))d\lambda(X) + f(k\gamma(X) + n\sigma(X) - \Re f(X))d\tau(X) \leq \int (k\sigma(X) - n\gamma(X) - Rf(X))d\xi(X) + f(k\gamma(X) + n\sigma(X) - \Re f(X))d\gamma(X)
\end{align*}
\]

for any \( (\xi, \eta) \in m(a, F) \times m(b, F) \). Then we have \( \delta \in \varphi(\xi) \). Consequently, by the fixed point theorem of Fan and Glicksberg ([1]), there exists an element \( \alpha = (\mu, \nu) \in m(a, F) \times m(b, F) \) such that \( \varphi((\mu, \nu)) \equiv (\mu, \nu) \). Hence we have

\[
\begin{align*}
\int (k\mu(X) - n\nu(X) - Rf(X))d\mu(X) + f(k\nu(X) + n\mu(X) - \Re f(X))d\nu(X) \leq \int (k\mu(X) - n\nu(X) - Rf(X))d\xi(X) + f(k\nu(X) + n\mu(X) - \Re f(X))d\gamma(X)
\end{align*}
\]

for any \( (\xi, \eta) \in m(a, F) \times m(b, F) \). If we put

\[
\gamma_1 = \frac{1}{a} \int (k\mu(X) - n\nu(X) - Rf(X))d\mu(X),
\]

and

\[
\gamma_2 = \frac{1}{b} \int (k\nu(X) + n\mu(X) - \Re f(X))d\nu(X),
\]

we have

\[
\int (k\mu(X) - n\nu(X) - Rf(X) - \gamma_1)d\xi(X) + \int (k\nu(X) + n\mu(X) - \Re f(X) - \gamma_2)d\gamma(X) \geq 0
\]

for any \( (\xi, \eta) \in m(a, F) \times m(b, F) \). The existence of a positive measure \( \xi_0 \in m(a, F) \) with \( \int (k\mu(X) - n\nu(X) - Rf(X) - \gamma_1)d\xi_0(X) < 0 \) leads us to a contradiction as follows. For any signed measure \( \tau_0 \) supported by \( F \) with total mass zero such that \( \eta = \nu + \varepsilon\tau_0 \) is a positive measure for any positive number \( \varepsilon < 1 \), we have

\[
\begin{align*}
\int (k\mu(X) - n\nu(X) - Rf(X) - \gamma_1)d\xi_0(X) + \varepsilon f(k\nu(X) + n\mu(X) - \Re f(X) - \gamma_2)d\tau_0(X) \geq 0.
\end{align*}
\]

Making \( \varepsilon \to 0 \), we have a contradiction. So we have

\[
\int (k\mu(X) - n\nu(X) - Rf(X) - \gamma_1)d\xi(X) \geq 0 \text{ for any } \xi \in m(a, F).
\]

By the same way as above, we have
\[ f(k\mu(X) + n\mu(X) - \Re f(X) - \gamma_2) d\eta(X) \geq 0 \quad \text{for any } \eta \in m(b, F). \]

By these inequalities, we have

1. \( k\mu(X) - nv(X) \geq \Re f(X) + \gamma_1 \text{ on } F, \)
2. \( k\mu(X) - nv(X) = \Re f(X) + \gamma_1 \text{ on the support of } \mu, \)
3. \( k\nu(X) + n\mu(X) \geq \Im f(X) + \gamma_2 \text{ on } F, \)
4. \( k\nu(X) + n\mu(X) = \Im f(X) + \gamma_2 \text{ on the support of } \nu. \)

Consequently for a complex-valued measure \( \alpha = \mu + iv \) of \( \mathcal{M} (a, F, b) \) and a complex constant \( \gamma = \gamma_1 + i\gamma_2, \) we have

1. \( \Re K\alpha(X) \geq \Re \{ f(X) + \gamma \} \text{ on } F, \)
2. \( \Re K\alpha(X) = \Re \{ f(X) + \gamma \} \text{ on the support of } \Re \alpha, \)
3. \( \Im K\alpha(X) \geq \Im \{ f(X) + \gamma \} \text{ on } F, \)
4. \( \Im K\alpha(X) = \Im \{ f(X) + \gamma \} \text{ on the support of } \Im \alpha. \)

Thus the proof is completed.

Proof of Lemma 3. Putting \( k'(X, Y) = k(X, Y) / \Re f(X) \) and \( n'(X, Y) = n(X, Y) / \Im f(X), \) \( k'(X, Y) \) and \( n'(X, Y) \) are finite continuous functions, and \( k'(X, Y) > 0 \) for any points \( X \) and \( Y \) of \( F. \) Taking a positive number \( a \) which is less than

\[
\frac{\min \{ k(X, Y) | X \in F, Y \in F \} \cdot \min \{ \Im f(X) | X \in F \}}{\max \{ n(X, Y) | X \in F, Y \in F \} \cdot \max \{ \Im f(X) | X \in F \}},
\]

we have \( f(k'\nu(X) + n'\mu(X)) d\nu(X) > 0 \) for any \( (\mu, \nu) \in m(a, F) \times m(1, F). \) For this positive number \( a \) we consider the point-to-set mapping \( \varphi \) defined on \( m(a, F) \times m(1, F) \) into \( \mathfrak{F}(m(a, F) \times m(1, F)) \) which is the family of all closed convex subsets in \( m(a, F) \times m(1, F). \) For any \( (\mu, \nu) \in m(a, F) \times m(1, F), \) \( \varphi \) is defined as follows.

\[
\varphi((\mu, \nu)) = \{ (\lambda, \tau) \in m(a, F) \times m(1, F) | \}
\]

\[
f(k\mu(X) - nv(X) - f(k'\nu(X) + n'\mu(X)) d\nu(X) \cdot \Re f(X)) d\lambda(X) +
\]

\[
f(k'\nu(X) + n'\mu(X)) d\tau(X) = \inf (f(k\mu(X) - nv(X) -
\]

\[
f(k'\nu(X) + n'\mu(X)) d\nu(X) \cdot \Re f(X)) d\xi(X) +
\]

\[
f(k'\nu(X) + n'\mu(X)) d\eta(X) | (\xi, \eta) \in m(a, F) \times m(1, F)) \}
\]

Obviously \( \varphi((\mu, \nu)) \) is a non-empty closed convex subset and \( \varphi \) is upper semi-continuous as in Lemma 2. Hence, by the fixed point theorem of Fan and Glicksberg, there exists an element \((\mu_0, \nu_0) \in m(a, F) \times m(1, F)\) such that \( \varphi((\mu_0, \nu_0)) \ni (\mu_0, \nu_0). \) Then we have

\[
f(k\mu_0(X) - nv_0(X) - f(k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X) \cdot \Re f(X)) d\mu_0(X) +
\]

\[
f(k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X) \leq f(k\mu_0(X) - nv_0(X) -
\]

\[
f(k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X) \cdot \Re f(X)) d\xi(X) + f(k'\nu_0(X) + n'\mu_0(X)) d\eta(X)
\]
for any \((\xi, \eta) \in m(a, F) \times m(1, F)\). Putting

\[
\gamma_1 = \frac{1}{a} \cdot \int (k \mu_\circ (X) - n \nu_\circ (X) - f(k' \nu_\circ (X) + n' \mu_\circ (X)) \nu_\circ (X) \cdot \Re f(X) \mu_\circ (X),
\]

and

\[
\gamma_2 = \int (k' \nu_\circ (X) + n' \mu_\circ (X)) \nu_\circ (X),
\]

we have

\[
\int (k \mu_\circ (X) - n \nu_\circ (X) - f(k' \nu_\circ (X) + n' \mu_\circ (X)) \nu_\circ (X) \cdot \Re f(X) \mu_\circ (X)
\]

for any \((\xi, \eta) \in m(a, F) \times m(1, F)\). By the same way as Lemma 2, we have two following inequalities.

(1) \(\int (k \mu_\circ (X) - n \nu_\circ (X) - f(k' \nu_\circ (X) + n' \mu_\circ (X)) \nu_\circ (X) \cdot \Re f(X) \mu_\circ (X) \geq 0\)

for any \(\xi \in m(a, F)\), and

(2) \(\int (k' \nu_\circ (X) + n' \mu_\circ (X) - \gamma_2) \nu_\circ (X) \geq 0\)

for any \(\eta \in m(1, F)\).

From these inequalities we have

(1) \(k \mu_\circ (X) - n \nu_\circ (X) \geq \gamma_1 \cdot \Re f(X) \mu_\circ (X)\) on \(F\),

(2) \(k \mu_\circ (X) - n \nu_\circ (X) = \gamma_1 \cdot \Re f(X) \mu_\circ (X)\) on the support of \(\mu_\circ\),

(3) \(k' \nu_\circ (X) + n' \mu_\circ (X) \geq \gamma_2\) on \(F\),

(4) \(k' \nu_\circ (X) + n' \mu_\circ (X) = \gamma_2\) on the support of \(\nu_\circ\).

By the property of the number \(a\), \(\gamma_2\) is strictly positive. Putting \(\mu = \frac{\mu_\circ}{\gamma_2}, \nu = \frac{\nu_\circ}{\gamma_2}\) and \(\gamma = \gamma_1\), we have

(1) \(k \mu (X) - n \nu(X) \geq \Re f(X) + \gamma\) on \(F\),

(2) \(k \mu (X) - n \nu(X) = \Re f(X) + \gamma\) on the support of \(\mu\),

(3) \(k' \nu (X) + n' \mu(X) \geq \Re f(X)\) on \(F\), and

(4) \(k' \nu (X) + n' \mu(X) = \Re f(X)\) on the support of \(\nu\).

Thus, the measure \(\alpha = \mu + i \nu\), and the real constant \(\gamma\) are what Lemma 3 needs.

Proof of Lemma 4. As \(k(X, Y)\) is a lower semi-continuous function such that \(\inf \{k(X, Y) | (X, Y) \in F \times F\} = 2p > 0\), there exists an increasing net \(\{k_m(X, Y) | m \in D\}\) of finite continuous functions such that \(\lim k_m (X, Y) = k(X, Y)\) and \(k_m(X, Y) > p\) for any points \(X\) and \(Y\) of \(F\). Taking a positive number \(a\) which is less than

\[
p \cdot \min \{\Re f(X) | X \in F\}
\]

by Lemma 3, there exist measures \(\alpha_m = \mu_m + i \nu_m \in \mathfrak{M}(a, F, 1)\) and real constants \(\gamma_m\) and \(\gamma'_m\) such that

(1) \(k_m \mu_m(X) - n \nu_m(X) - \gamma'_m \cdot \Re f(X) \geq \gamma_m\) on \(F\),

(2) \(k_m \mu_m(X) - n \nu_m(X) - \gamma'_m \cdot \Re f(X) = \gamma_m\) on the support of \(\mu_m\),
(3) \( k_m'\nu_m(X) + n'\mu_m(X) \geq \gamma'_m \) on \( F \), and
(4) \( k_m'\nu_m(X) + n'\mu_m(X) = \gamma'_m \) on the support of \( \nu_m \).

In the first place, we are going to see the boundedness of the net \( \{ \gamma'_m \mid m \in D \} \).

Obviously \( \gamma'_m > 0 \) for any \( m \). Supposing that \( \lim_m \gamma'_m = +\infty \), we can take a subnet \( \{ \gamma'_m \mid m \in D' \} \), a directed set such that \( \nu_m \to \nu \), \( \mu_m \to \mu \), \( \gamma'_m \to +\infty \), and \( k_m(X, Y) \to k(X, Y) \) along \( D' \) for any points \( X \) and \( Y \) of \( F \). \( k(X, Y) \) satisfying the continuity principle, we have, by the above inequality (3),

\[
\lim_{m_i} k_{m_i}'\nu_{m_i}(X) + \lim_{m_i} n_{m_i}'\mu_{m_i}(X) \geq \lim_{m_i} \gamma '_{m_i} = +\infty
\]
on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero. Then we have that \( k\nu(X) = +\infty \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, which is a contradiction by Lemma 1. Using the boundedness of the net \( \{ \gamma'_m \mid m \in D \} \), we can see the boundedness of the net \( \{ \gamma_m \mid m \in D \} \) by the same way as above. Consequently, considering an adequate directed set \( E \), we have that \( \gamma'_1 \to \gamma'_2 \), \( \gamma'_1 \to \gamma'_3 \), \( \mu_1 \to \mu_0 \), \( \nu_1 \to \nu_0 \), and \( k_1(X, Y) \to k(X, Y) \) along \( E \). Hence we have, by the same way as M. Kishi ([2] and [3])

(1) \( k\mu_0(X) - n\nu_0(X) - \gamma_2 \Re f(X) \geq \gamma_1 \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,
(2) \( k\mu_0(X) - n\nu_0(X) - \gamma_2 \Re f(X) \leq \gamma_1 \) on the support of \( \mu_0 \),
(3) \( k'\nu_0(X) + n'\mu_0(X) \geq \gamma_2 \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
(4) \( k'\nu_0(X) + n'\mu_0(X) \leq \gamma_2 \) on the support of \( \nu_0 \).

By the property of the number \( a \), \( \gamma_2 \) is strictly positive. Putting \( \mu = \mu_0 + i\nu_0 \), \( \nu = \nu_0 \), \( \gamma = \gamma_2 \), and \( \gamma = \gamma_2 \), we have

(1) \( k\mu(X) - n\nu(X) \geq \Re f(X) + \gamma \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,
(2) \( k\mu(X) - n\nu(X) \leq \Re f(X) + \gamma \) on the support of \( \mu \),
(3) \( k\nu(X) + n\mu(X) \geq \Im f(X) \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
(4) \( k\nu(X) + n\mu(X) \leq \Im f(X) \) on the support of \( \nu \).

Thus, the measure \( \alpha = \mu + i\nu \), and the real constant \( \gamma \) are what Lemma 4 needs. Finally, we prove the theorems.

Proof of Theorem 1. As \( k(X, Y) \) is a lower semi-continuous function such that \(-\infty < k(X, Y) \leq +\infty \), there exists an increasing net \( \{ k_m(X, Y) \mid m \in D \} \), a directed set\{ of finite continuous functions such that \( \lim_m k_m(X, Y) = k(X, Y) \) for any points \( X \) and \( Y \) of \( F \). Then, by Lemma 2, there exist measures \( \alpha_m = \mu_m + i\nu_m \) of \( \Re(a, F, b) \) and complex constants \( \gamma_m = \gamma_m' + i\gamma_m'' \) such that
\( k_{m\mu_m}(X) - n\nu_m(X) \geq \Re f(X) + \gamma_m \) on \( F \),
\( k_{m\mu_m}(X) - n\nu_m(X) = \Re f(X) + \gamma_m' \) on the support of \( \mu_m \),
\( k_{m\nu_m}(X) + n\mu_m(X) \geq \Im f(X) + \gamma_m'' \) on \( F \), and
\( k_{m\nu_m}(X) + n\mu_m(X) = \Im f(X) + \gamma_m'' \) on the support of \( \nu_m \).

By the same way as Lemma 4, there exist a measure \( \alpha = \mu + iv \) of \( \mathfrak{M}(a, F, b) \) and a complex constant \( \gamma = \gamma_1 + i\gamma_2 \) such that

1. \( \Re K\alpha(X) \geq \Re \{ f(X) + \gamma \} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,
2. \( \Re K\alpha(X) \leq \Re \{ f(X) + \gamma \} \) on the support of \( \Re \alpha \),
3. \( \Im K\alpha(X) \geq \Im \{ f(X) + \gamma \} \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
4. \( \Im K\alpha(X) \leq \Im \{ f(X) + \gamma \} \) on the support of \( \Im \alpha \).

**Proof of Theorem 2.** Let \( \{ f_m(X) \mid m \in D \} \) and \( \{ g_m(X) \mid m \in D \} \) be decreasing nets of positive finite continuous functions on \( F \) such that \( f_m(X) \downarrow \Re f(X) \) and \( g_m(X) \downarrow \Im f(X) \). Taking an adequate positive number \( a \), by Lemma 4, there exist measures \( \alpha_m = \mu_m + iv_m \) of \( \mathfrak{M}(a, F, 1) \) and real constants \( \gamma_m \) and \( \gamma_m'' \) such that

1. \( k\mu_m(X) - n\nu_m(X) - \gamma_m'M \cdot f_m(X) \geq \gamma_m' \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,
2. \( k\mu_m(X) - n\nu_m(X) - \gamma_m'M \cdot f_m(X) \leq \gamma_m' \) on the support of \( \mu_m \),
3. \( k\nu_m(X) + n\mu_m(X) \geq \gamma_m'' \cdot g_m(X) \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
4. \( k\nu_m(X) + n\mu_m(X) \leq \gamma_m'' \cdot g_m(X) \) on the support of \( \nu_m \).

By the same way as Lemma 4, there exist a measure \( \alpha = \mu + iv \) of \( \mathfrak{M}(F) \) and a real constant \( \gamma \) such that

1. \( k\mu(X) - n\nu(X) \geq \Re f(X) + \gamma \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero,
2. \( k\mu(X) - n\nu(X) \leq \Re f(X) + \gamma \) on the support of \( \mu \).
3. \( k\nu(X) + n\mu(X) \geq \Im f(X) \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
4. \( k\nu(X) + n\mu(X) \leq \Im f(X) \) on the support of \( \nu \).

Thus, the measure \( \alpha = \mu + iv \), and the real constant \( \gamma \) are what Theorem 2 needs. The analogous arguments will give us the latter part of Theorem 2.

**Corollary.** Let \( F \) be a compact subset of positive \( k \)-transfinite diameter, and \( f(X) \) be a real-valued upper semi-continuous function with lower bound on \( F \), and \( a \) be a positive number. If the adjoint kernel \( \hat{k}(X, Y) \) satisfies the continuity principle, then there exist a measure \( \mu \) of \( m(a, F) \) and a real constant \( \gamma \) such that

1. \( k\mu(X) \geq f(X) + \gamma \) on \( F \) with a possible exception of a set of \( k \)-transfinite diameter zero, and
2. \( k\mu(X) \leq f(X) + \gamma \) on the support of \( \mu \).
REMARK. In above Theorem 2, we can not always reduce the constant $\gamma$ to zero. We may consider the following example: let $\Omega$ be a finite space consisting of two points $X_1$ and $X_2$, and $\Re K(X, Y)$ and $\Im K(X, Y)$ be given by the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively, and $\Re f(X)$ and $\Im f(X)$ be equal to 1 everywhere. Then, for the compact set $F = \Omega$, we have no measure $\alpha$ such that

1. $\Re K \alpha(X) \geq \Re f(X)$ on $F$,
2. $\Re K \alpha(X) = \Re f(X)$ on the support of $\Re \alpha$,
3. $\Im K \alpha(X) \geq \Im f(X)$ on $F$, and
4. $\Im K \alpha(X) = \Im f(X)$ on the support of $\Im \alpha$.

REMARK. Putting $n(X, Y) = \Re K(X, Y) \equiv 0$, we can assert that our Theorem 2 contains the existence theorem obtained by M. Kishi and M. Nakai ([2], [3] and [4]).

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References


