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ON THE POTENTIAL TAKEN WITH RESPECT TO COMPLEX-VALUED KERNELS

MINORU MATSUDA

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In the potential theory, we have two theorems called the existence theorem concerning the potential taken with respect to real-valued and symmetric kernels. They are stated as follows. Let $K(X, Y)$ be a real-valued function defined in a locally compact Hausdorff space Ω , lower semi-continuous for any points X and Y , may be $+\infty$ for $X=Y$, always finite for $X \neq Y$ and bounded from above for X and Y belonging to disjoint compact sets of Ω respectively. For a given positive measure μ , the potential is defined by

$$K\mu(X) = \int K(X, Y)d\mu(Y),$$

and the K -energy of μ is defined by $\int K\mu(X)d\mu(X)$. A subset of Ω is said to be of positive K -transfinite diameter, when it charges a positive measure μ of finite K -energy with compact support, otherwise said to be of K -transfinite diameter zero. Let $K(X, Y)$ be symmetric : $K(X, Y)=K(Y, X)$ for any points X and Y . Then we have two following theorems.

Theorem A. *Let F be a compact subset of positive K -transfinite diameter, and $f(X)$ be a real-valued upper semi-continuous function with lower bound on F . Then, given any positive number a , there exist a positive measure μ supported by F and a real constant γ such that*

- (1) $\mu(F)=a$,
- (2) $K\mu(X) \geq f(X) + \gamma$ on F with a possible exception of a set of K -transfinite diameter zero, and
- (3) $K\mu(X) \leq f(X) + \gamma$ on the support of μ .

Theorem B. *In the above theorem, suppose the further conditions : $K(X, Y) > 0$ and $\inf f(X) > 0$ for any points X and Y of F . Then, given any compact subset F of positive K -transfinite diameter, there exists a positive measure μ supported by F such that*

- (1) $K\mu(X) \geq f(X)$ on F with a possible exception of a set of K -transfinite diameter zero, and

(2) $K\mu(X) \leq f(X)$ on the support of μ .

Recently, N. Ninomiya ([5]) proved the existence theorems for the potential taken with respect to complex-valued and *symmetric* kernels and to complex-valued measures, which are the extension of the above theorems in the case of the real-valued kernels. We state them as follows. Let $K(X, Y)$ be a complex-valued function defined in a locally compact Hausdorff space Ω . Let $k(X, Y) = \Re K(X, Y)$ be a function lower semi-continuous, symmetric, may be $+\infty$ for $X = Y$, always finite for $X \neq Y$ and bounded from above for X and Y belonging to disjoint compact sets of Ω respectively, and $n(X, Y) = \Im K(X, Y)$ be a finite continuous function satisfying that $n(X, Y) = -n(Y, X)$ for any points X and Y of Ω . For any compact subset F and any positive numbers a and b , denote by $\mathfrak{M}(a, F, b)$ the family of all the complex-valued measures supported by F whose real parts and imaginary parts are positive measures with total mass a and b respectively, by $\mathfrak{M}(a, F)$ the family of all the complex-valued measures supported by F whose real parts are positive measures with total mass a and imaginary parts are any positive measures, by $\mathfrak{M}(F, b)$ the family of all the complex-valued measures supported by F whose real parts are any positive measures and imaginary parts are positive measures with total mass b , and by $\mathfrak{M}(F)$ the family of all the complex-valued measures supported by F whose real parts and imaginary parts are any positive measures. For any such measure α , the potential is defined by

$$K\alpha(X) = \int K(X, Y) d\alpha(Y).$$

Then we have two following theorems.

Theorem A'. *Let F be a compact subset of positive k -transfinite diameter, and $f(X)$ be a complex-valued function whose real part $\Re f(X)$ and imaginary part $\Im f(X)$ are upper semi-continuous functions with lower bound on F . Then, given any positive numbers a and b , there exist a measure α of $\mathfrak{M}(a, F, b)$ and a complex constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $\Re K\alpha(X) \leq \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
- (3) $\Im K\alpha(X) \geq \Im \{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $\Im K\alpha(X) \leq \Im \{f(X) + \gamma\}$ on the support of $\Im \alpha$.

Theorem B'. *In the above theorem, suppose the further conditions : $k(X, Y) > 0$, $\inf \Re f(X) > 0$ and $\inf \Im f(X) > 0$ for any points X and Y of F . Then, given any positive number a such that $a|n(X, Y)| < \Im f(X)$ for points X and Y of F , there exist a measure α of $\mathfrak{M}(a, F)$ and a real constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $\Re K\alpha(X) \leq \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$,
- (3) $\Im K\alpha(X) \geq \Im f(X)$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $\Im K\alpha(X) \leq \Im f(X)$ on the support of $\Im\alpha$.

Similarly, given any positive number b such that $b|n(X, Y)| < \Re f(X)$ for points X and Y of F , there exist a measure α of $\mathfrak{M}(F, b)$ and a complex constant γ such that

- (1') $\Re K\alpha(X) \geq \Re f(X)$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2') $\Re K\alpha(X) \leq \Re f(X)$ on the support of $\Re\alpha$,
- (3') $\Im K\alpha(X) \geq \Im\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4') $\Im K\alpha(X) \leq \Im\{f(X) + \gamma\}$ on the support of $\Im\alpha$.

In this paper we are going to extend these existence theorems to the potential taken with respect to complex-valued kernels and to complex-valued measures, under an additional condition of the continuity principle for the adjoint kernel.

Let $K(X, Y)$ be a complex-valued function, not always symmetric, defined in a locally compact Hausdorff space Ω . Let $k(X, Y) = K\Re(X, Y)$ be a function lower semi-continuous, may be $+\infty$ for $X = Y$, always finite for $X \neq Y$ and $n(X, Y) = \Im K(X, Y)$ be a finite continuous function. For any positive measure μ , consider the adjoint potential defined by

$$\check{k}\mu(X) = \int \check{k}(X, Y) d\mu(Y) = \int k(Y, X) d\mu(Y).$$

Then, we have two following theorems.

Theorem 1. *Let F be a compact subset of positive k -transfinite diameter, and $f(X)$ be a complex-valued function whose real part $\Re f(X)$ and imaginary part $\Im f(X)$ are upper semi-continuous functions with lower bound on F , and a and b be two positive numbers. If the adjoint kernel $\check{k}(X, Y) = k(Y, X)$ satisfies the continuity principle¹⁾, there exist a measure α of $\mathfrak{M}(a, F, b)$ and a complex constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero,

1) We say that $k(X, Y)$ satisfies the continuity principle when for any positive measure μ with compact support, the following implication holds: (the restriction of $k\mu(X)$ to the support of μ is finite and continuous) \Rightarrow ($k\mu(X)$ is finite and continuous in the whole space Ω).

- (2) $\Re K\alpha(X) \leq \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$,
- (3) $\Im K\alpha(X) \geq \Im\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $\Im K\alpha(X) \leq \Im\{f(X) + \gamma\}$ on the support of $\Im\alpha$.

Theorem 2. *In the above theorem, suppose the further conditions : $k(X, Y) > 0$, $\inf \Re f(X) > 0$, and $\inf \Im f(X) > 0$ for any points X and Y of F . Then, there exist a measure α of $\mathfrak{M}(F)$ and a real constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $\Re K\alpha(X) \leq \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$.
- (3) $\Im K\alpha(X) \geq \Im f(X)$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $\Im K\alpha(X) \leq \Im f(X)$ on the support of $\Im\alpha$.

Similarly, there exist a measure α of $\mathfrak{M}(F)$ and a pure imaginary constant γ such that

- (1') $\Re K\alpha(X) \geq \Re f(X)$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2') $\Re K\alpha(X) \leq \Re f(X)$ on the support of $\Re\alpha$,
- (3') $\Im K\alpha(X) \geq \Im\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4') $\Im K\alpha(X) \leq \Im\{f(X) + \gamma\}$ on the support of $\Im\alpha$.

Before we prove the theorems, we prepare some lemmas.

Lemma 1. *Let μ be a positive measure with compact support. If the adjoint kernel $\check{k}(X, Y)$ satisfies the continuity principle, the set $E = \{X \mid k_\mu(X) = +\infty\}$ of Ω is of k -transfinite diameter zero.*

Lemma 2. *Let F be a compact subset, and $f(X)$ be a complex-valued function whose real part $\Re f(X)$ and imaginary part $\Im f(X)$ are upper semi-continuous functions with lower bound on F respectively, and a and b be two positive numbers. If the real part $k(X, Y)$ of $K(X, Y)$ is a finite continuous function defined in Ω , there exist a measure α of $\mathfrak{M}(a, F, b)$ and a complex constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F ,
- (2) $\Re K\alpha(X) = \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$,
- (3) $\Im K\alpha(X) \geq \Im\{f(X) + \gamma\}$ on F , and
- (4) $\Im K\alpha(X) = \Im\{f(X) + \gamma\}$ on the support of $\Im\alpha$.

Lemma 3. *In above Lemma 2, suppose the further conditions : $k(X, Y) > 0$, $\inf \Re f(X) > 0$, and $\inf \Im f(X) > 0$ for any points X and Y of F and both $\Re f(X)$ and $\Im f(x)$ are finite and continuous. Then, there exist a measure α of $\mathfrak{M}(F)$ and a real constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F ,
- (2) $\Re K\alpha(X) = \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$,
- (3) $\Im K\alpha(X) \geq \Im f(X)$ on F , and
- (4) $\Im K\alpha(X) = \Im f(X)$ on the support of $\Im\alpha$.

Lemma 4. *In above Theorem 2, suppose the further conditions : both $\Re f(X)$ and $\Im f(X)$ are finite and continuous. Then, there exist a measure α of $\mathfrak{M}(F)$ and a real constant γ such that*

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $\Re K\alpha(X) \leq \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$,
- (3) $\Im K\alpha(X) \geq \Im f(X)$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $\Im K\alpha(X) \leq \Im f(X)$ on the support of $\Im\alpha$.

Proof of Lemma 1. Let the set E be of positive k -transfinite diameter. $\check{k}(X, Y)$ satisfying the continuity principle, there exists a positive measure σ such that

- (a) the compact support of σ is contained in the set E , and
- (b) $\check{k}\sigma(X)$ is finite and continuous in the whole space Ω .

Hence we have

$\int k\mu(X)d\sigma(X) = +\infty$, that is, $\int \check{k}\sigma(X)d\mu(X) = +\infty$, which is a contradiction.

Proof of Lemma 2. For any positive number c , denote by $m(c, F)$ the set of all positive measures supported by F with total mass c . We define the point-to-set mapping φ on the product space $m(a, F) \times m(b, F)$ into $\mathfrak{F}(m(a, F) \times m(b, F))$ which is the family of all closed convex subsets in $m(a, F) \times m(b, F)$. For any $\alpha = \mu + i\nu$, that is, $\alpha = (\mu, \nu)$ of $m(a, F) \times m(b, F)$, φ is defined as follows.

$$\begin{aligned} \varphi((\mu, \nu)) &= \{(\lambda, \tau) \in m(a, F) \times m(b, F) \mid \\ &\quad \int (k\mu(X) - n\nu(X) - \Re f(X))d\lambda(X) + \int (k\nu(X) + n\mu(X) - \Im f(X))d\tau(X) \\ &\quad = \inf \{ \int (k\mu(X) - n\nu(X) - \Re f(X))d\xi(X) + \int (k\nu(X) + n\mu(X) - \Im f(X))d\eta(X) \mid \\ &\quad (\xi, \eta) \in m(a, F) \times m(b, F) \} \}. \end{aligned}$$

Obviously $\varphi((\mu, \nu)) \neq \emptyset$. For, putting

$$d = \inf \{ \int (k\mu(X) - n\nu(X) - \Re f(X))d\xi(X) + \int (k\nu(X) + n\mu(X) - \Im f(X))d\eta(X) \mid (\xi, \eta) \in m(a, F) \times m(b, F) \},$$

there exist sequences of

$\xi_n \in m(a, F)$ and $\eta_n \in m(b, F)$ such that

$\int(k\mu(X) - n\nu(X) - \Re f(X))d\xi_n(X) + \int(k\nu(X) + n\mu(X) - \Im f(X))d\eta_n(X) \rightarrow d$. As we have vaguely convergent subnets $\xi_{n_k} \in m(a, F)$ and $\eta_{n_k} \in m(b, F)$ such that $\xi_{n_k} \rightarrow \xi_0$ and $\eta_{n_k} \rightarrow \eta_0$, there holds $\varphi((\mu, \nu)) \ni (\xi_0, \eta_0)$. Moreover $\varphi((\mu, \nu))$ is upper semi-continuous in the following sense : if nets $\{\delta_\alpha \mid \alpha \in D, \text{ a directed set}\}$ and $\{\zeta_\alpha \mid \alpha \in D\}$ converge to δ and ζ with respect to the product topology respectively, and if $\delta_\alpha \in \varphi(\zeta_\alpha)$ for any $\alpha \in D$, then $\delta \in \varphi(\zeta)$. In fact, if we put $\delta_\alpha = (\lambda_\alpha, \tau_\alpha)$, $\zeta_\alpha = (\sigma_\alpha, \gamma_\alpha)$, $\delta = (\lambda_0, \tau_0)$, and $\zeta = (\sigma_0, \gamma_0)$, we have

$$\begin{aligned} & \int(k\sigma_\alpha(X) - n\gamma_\alpha(X) - \Re f(X))d\lambda_\alpha(X) + \int(k\gamma_\alpha(X) + n\sigma_\alpha(X) - \Im f(X))d\tau_\alpha(X) \\ & \leq \int(k\sigma_\alpha(X) - n\gamma_\alpha(X) - \Re f(X))d\xi(X) + \int(k\gamma_\alpha(X) + n\sigma_\alpha(X) - \Im f(X))d\eta(X) \end{aligned}$$

for any $(\xi, \eta) \in m(a, F) \times m(b, F)$. By the limit process, we have

$$\begin{aligned} & \int(k\sigma_0(X) - n\gamma_0(X) - \Re f(X))d\lambda_0(X) + \int(k\gamma_0(X) + n\sigma_0(X) - \Im f(X))d\tau_0(X) \\ & \leq \int(k\sigma_0(X) - n\gamma_0(X) - \Re f(X))d\xi(X) + \int(k\gamma_0(X) + n\sigma_0(X) - \Im f(X))d\eta(X) \end{aligned}$$

for any $(\xi, \eta) \in m(a, F) \times m(b, F)$. Then we have $\delta \in \varphi(\zeta)$. Consequently, by the fixed point theorem of Fan and Glicksberg ([1]), there exists an element $\alpha = (\mu, \nu) \in m(a, F) \times m(b, F)$ such that $\varphi((\mu, \nu)) \ni (\mu, \nu)$. Hence we have

$$\begin{aligned} & \int(k\mu(X) - n\nu(X) - \Re f(X))d\mu(X) + \int(k\nu(X) + n\mu(X) - \Im f(X))d\nu(X) \\ & \leq \int(k\mu(X) - n\nu(X) - \Re f(X))d\xi(X) + \int(k\nu(X) + n\mu(X) - \Im f(X))d\eta(X) \end{aligned}$$

for any $(\xi, \eta) \in m(a, F) \times m(b, F)$. If we put

$$\gamma_1 = \frac{1}{a} \int (k\mu(X) - n\nu(X) - \Re f(X))d\mu(X),$$

and

$$\gamma_2 = \frac{1}{b} \int (k\nu(X) + n\mu(X) - \Im f(X))d\nu(X), \text{ we have}$$

$$\int(k\mu(X) - n\nu(X) - \Re f(X) - \gamma_1)d\xi(X) + \int(k\nu(X) + n\mu(X) - \Im f(X) - \gamma_2)d\eta(X) \geq 0$$

for any $(\xi, \eta) \in m(a, F) \times m(b, F)$. The existence of a positive measure $\xi_0 \in m(a, F)$ with $\int(k\mu(X) - n\nu(X) - \Re f(X) - \gamma_1)d\xi_0(X) < 0$ leads us to a contradiction as follows. For any signed measure τ_0 supported by F with total mass zero such that $\eta = \nu + \varepsilon\tau_0$ is a positive measure for any positive number $\varepsilon (< 1)$, we have

$$\begin{aligned} & \int(k\mu(X) - n\nu(X) - \Re f(X) - \gamma_1)d\xi_0(X) \\ & + \varepsilon \int(k\nu(X) + n\mu(X) - \Im f(X) - \gamma_2)d\tau_0(X) \geq 0. \end{aligned}$$

Making $\varepsilon \rightarrow 0$, we have a contradiction. So we have

$$\int(k\mu(X) - n\nu(X) - \Re f(X) - \gamma_1)d\xi(X) \geq 0 \text{ for any } \xi \in m(a, F).$$

By the same way as above, we have

$$f(kv(X) + n\mu(X) - \Im f(X) - \gamma_2)d\eta(X) \geq 0 \quad \text{for any } \eta \in m(b, F).$$

By these inequalities, we have

- (1) $k\mu(X) - n\nu(X) \geq \Re f(X) + \gamma_1$ on F ,
- (2) $k\mu(X) - n\nu(X) = \Re f(X) + \gamma_1$ on the support of μ ,
- (3) $kv(X) + n\mu(X) \geq \Im f(X) + \gamma_2$ on F , and
- (4) $kv(X) + n\mu(X) = \Im f(X) + \gamma_2$ on the support of ν .

Consequently for a complex-valued measure $\alpha = \mu + i\nu$ of $\mathfrak{M}(a, F, b)$ and a complex constant $\gamma = \gamma_1 + i\gamma_2$, we have

- (1) $\Re K\alpha(X) \geq \Re \{f(X) + \gamma\}$ on F ,
- (2) $\Re K\alpha(X) = \Re \{f(X) + \gamma\}$ on the support of $\Re \alpha$,
- (3) $\Im K\alpha(X) \geq \Im \{f(X) + \gamma\}$ on F , and
- (4) $\Im K\alpha(X) = \Im \{f(X) + \gamma\}$ on the support of $\Im \alpha$.

Thus the proof is completed.

Proof of Lemma 3. Putting $k'(X, Y) = k(X, Y)/\Im f(X)$ and $n'(X, Y) = n(X, Y)/\Im f(X)$, $k'(X, Y)$ and $n'(X, Y)$ are finite continuous functions, and $k'(X, Y) > 0$ for any points X and Y of F . Taking a positive number a which is less than

$$\frac{\min\{k(X, Y) \mid X \in F, Y \in F\} \cdot \min\{\Im f(X) \mid X \in F\}}{\max\{|n(X, Y)| \mid X \in F, Y \in F\} \cdot \max\{\Im f(X) \mid X \in F\}},$$

we have $\int(k'\nu(X) + n'\mu(X))d\nu(X) > 0$ for any $(\mu, \nu) \in m(a, F) \times m(1, F)$. For this positive number a we consider the point-to-set mapping φ defined on $m(a, F) \times m(1, F)$ into $\Im(m(a, F) \times m(1, F))$ which is the family of all closed convex subsets in $m(a, F) \times m(1, F)$. For any $(\mu, \nu) \in m(a, F) \times m(1, F)$, φ is defined as follows.

$$\begin{aligned} \varphi((\mu, \nu)) = \{(\lambda, \tau) \in m(a, F) \times m(1, F) \mid \\ \int(k\mu(X) - n\nu(X) - \int(k'\nu(X) + n'\mu(X))d\nu(X) \cdot \Re f(X))d\lambda(X) + \\ \int(k'\nu(X) + n'\mu(X))d\tau(X) = \inf(\int(k\mu(X) - n\nu(X) - \\ \int(k'\nu(X) + n'\mu(X))d\nu(X) \cdot \Re f(X))d\xi(X) + \\ \int(k'\nu(X) + n'\mu(X))d\eta(X) \mid (\xi, \eta) \in m(a, F) \times m(1, F))\} \end{aligned}$$

Obviously $\varphi((\mu, \nu))$ is a non-empty closed convex subset and φ is upper semi-continuous as in Lemma 2. Hence, by the fixed point theorem of Fan and Glicksberg, there exists an element $(\mu_0, \nu_0) \in m(a, F) \times m(1, F)$ such that $\varphi((\mu_0, \nu_0)) \ni (\mu_0, \nu_0)$. Then we have

$$\begin{aligned} \int(k\mu_0(X) - n\nu_0(X) - \int(k'\nu_0(X) + n'\mu_0(X))d\nu_0(X) \cdot \Re f(X))d\mu_0(X) + \\ \int(k'\nu_0(X) + n'\mu_0(X))d\nu_0(X) \leq \int(k\mu_0(X) - n\nu_0(X) - \\ \int(k'\nu_0(X) + n'\mu_0(X))d\nu_0(X) \cdot \Re f(X))d\xi(X) + \int(k'\nu_0(X) + n'\mu_0(X))d\eta(X) \end{aligned}$$

for any $(\xi, \eta) \in m(a, F) \times m(1, F)$. Putting

$$\gamma_1 = \frac{1}{a} \cdot \int (k\mu_0(X) - n\nu_0(X) - \int (k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X) \cdot \Re f(X)) d\mu_0(X),$$

and

$$\begin{aligned} \gamma_2 &= \int (k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X), \text{ we have} \\ \int (k\mu_0(X) - n\nu_0(X) - \int (k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X) \cdot \Re f(X) - \gamma_1) d\xi(X) + \\ \int (k'\nu_0(X) + n'\mu_0(X) - \gamma_2) d\eta(X) &\geq 0 \end{aligned}$$

for any $(\xi, \eta) \in m(a, F) \times m(1, F)$. By the same way as Lemma 2, we have two following inequalities.

- (1) $\int (k\mu_0(X) - n\nu_0(X) - \int (k'\nu_0(X) + n'\mu_0(X)) d\nu_0(X) \cdot \Re f(X) - \gamma_1) d\xi(X) \geq 0$ for any $\xi \in m(a, F)$, and
- (2) $\int (k'\nu_0(X) + n'\mu_0(X) - \gamma_2) d\eta(X) \geq 0$ for any $\eta \in m(1, F)$.

From these inequalities we have

- (1) $k\mu_0(X) - n\nu_0(X) - \gamma_2 \cdot \Re f(X) \geq \gamma_1$ on F ,
- (2) $k\mu_0(X) - n\nu_0(X) - \gamma_2 \cdot \Re f(X) = \gamma_1$ on the support of μ_0 ,
- (3) $k'\nu_0(X) + n'\mu_0(X) \geq \gamma_2$ on F , and
- (4) $k'\nu_0(X) + n'\mu_0(X) = \gamma_2$ on the support of ν_0 .

By the property of the number a , γ_2 is strictly positive. Putting $\mu = \frac{\mu_0}{\gamma_2}$, $\nu = \frac{\nu_0}{\gamma_2}$

and $\gamma = \frac{\gamma_1}{\gamma_2}$, we have

- (1) $k\mu(X) - n\nu(X) \geq \Re f(X) + \gamma$ on F ,
- (2) $k\mu(X) - n\nu(X) = \Re f(X) + \gamma$ on the support of μ ,
- (3) $k\nu(X) + n\mu(X) \geq \Im f(X)$ on F , and
- (4) $k\nu(X) + n\mu(X) = \Im f(X)$ on the support of ν .

Thus, the measure $\alpha = \mu + i\nu$, and the real constant γ are what Lemma 3 needs.

Proof of Lemma 4. As $k(X, Y)$ is a lower semi-continuous function such that $\inf \{k(X, Y) \mid (X, Y) \in F \times F\} = 2p > 0$, there exists an increasing net $\{k_m(X, Y) \mid m \in D, \text{ a directed set}\}$ of finite continuous functions such that $\lim_m k_m(X, Y) = k(X, Y)$ and $k_m(X, Y) > p$ for any points X and Y of F . Taking a positive number a which is less than

$$\frac{p \cdot \min \{\Im f(X) \mid X \in F\}}{\max \{\Re f(X) \mid X \in F\} \cdot \max \{|n(X, Y)| \mid (X, Y) \in F \times F\}},$$

by Lemma 3, there exist measures $\alpha_m = \mu_m + i\nu_m \in \mathfrak{M}(a, F, 1)$ and real constants γ_m and γ'_m such that

- (1) $k_m\mu_m(X) - n\nu_m(X) - \gamma'_m \cdot \Re f(X) \geq \gamma_m$ on F ,
- (2) $k_m\mu_m(X) - n\nu_m(X) - \gamma'_m \cdot \Re f(X) = \gamma_m$ on the support of μ_m ,

- (3) $k'_m \nu_m(X) + n' \mu_m(X) \geq \gamma'_m$ on F , and
 (4) $k'_m \nu_m(X) + n' \mu_m(X) = \gamma'_m$ on the support of ν_m .

In the first place, we are going to see the boundedness of the net $\{\gamma'_m | m \in D\}$. Obviously $\gamma'_m > 0$ for any m . Supposing that $\lim_m \gamma'_m = +\infty$, we can take a subnet $\{\gamma'_{m_i} | m_i \in D', \text{ a directed set}\}$ such that $\nu_{m_i} \rightarrow \nu$, $\mu_{m_i} \rightarrow \mu$, $\gamma'_{m_i} \rightarrow +\infty$, and $k_{m_i}(X, Y) \uparrow k(X, Y)$ along D' for any points X and Y of F . $k'(X, Y)$ satisfying the continuity principle, we have, by the above inequality (3),

$$k'\nu(X) + n'\mu(X) \geq \lim_{m_i} k'_{m_i} \nu_{m_i}(X) + \lim_{m_i} n' \mu_{m_i}(X) \geq \lim_{m_i} \gamma'_{m_i} = +\infty$$

on F with a possible exception of a set of k -transfinite diameter zero. Then we have that $k\nu(X) = +\infty$ on F with a possible exception of a set of k -transfinite diameter zero, which is a contradiction by Lemma 1. Using the boundedness of the net $\{\gamma'_m | m \in D\}$, we can see the boundedness of the net $\{\gamma_m | m \in D\}$ by the same way as above. Consequently, considering an adequate directed set E , we have that $\gamma'_{l_i} \rightarrow \gamma_2$, $\gamma_{l_i} \rightarrow \gamma_1$, $\mu_{l_i} \rightarrow \mu_0$, $\nu_{l_i} \rightarrow \nu_0$, and $k_{l_i}(X, Y) \uparrow k(X, Y)$ along E . Hence we have, by the same way as M. Kishi ([2] and [3])

- (1) $k\mu_0(X) - n\nu_0(X) - \gamma_2 \cdot \Re f(X) \geq \gamma_1$ on F with a possible exception of a set of k -transfinite diameter zero,
 (2) $k\mu_0(X) - n\nu_0(X) - \gamma_2 \cdot \Re f(X) \leq \gamma_1$ on the support of μ_0 ,
 (3) $k\nu_0(X) + n'\mu_0(X) \geq \gamma_2$ on F with a possible exception of a set of k -transfinite diameter zero, and
 (4) $k\nu_0(X) + n'\mu_0(X) \leq \gamma_2$ on the support of ν_0 .

By the property of the number a , γ_2 is strictly positive. Putting $\mu = \frac{\mu_0}{\gamma_2}$, $\nu = \frac{\nu_0}{\gamma_2}$,

and $\gamma = \frac{\gamma_1}{\gamma_2}$, we have

- (1) $k\mu(X) - n\nu(X) \geq \Re f(X) + \gamma$ on F with a possible exception of a set of k -transfinite diameter zero,
 (2) $k\mu(X) - n\nu(X) \leq \Re f(X) + \gamma$ on the support of μ ,
 (3) $k\nu(X) + n\mu(X) \geq \Im f(X)$ on F with a possible exception of a set of k -transfinite diameter zero, and
 (4) $k\nu(X) + n\mu(X) \leq \Im f(X)$ on the support of ν .

Thus, the measure $\alpha = \mu + i\nu$, and the real constant γ are what Lemma 4 needs. Finally, we prove the theorems.

Proof of Theorem 1. As $k(X, Y)$ is a lower semi-continuous function such that $-\infty < k(X, Y) \leq +\infty$, there exists an increasing net $\{k_m(X, Y) | m \in D, \text{ a directed set}\}$ of finite continuous functions such that $\lim_m k_m(X, Y) = k(X, Y)$ for any points X and Y of F . Then, by Lemma 2, there exist measures $\alpha_m = \mu_m + i\nu_m$ of $\mathfrak{M}(a, F, b)$ and complex constants $\gamma_m = \gamma'_m + i\gamma''_m$ such that

- (1) $k_m\mu_m(X) - n\nu_m(X) \geq \Re f(X) + \gamma'_m$ on F ,
- (2) $k_m\mu_m(X) - n\nu_m(X) = \Re f(X) + \gamma'_m$ on the support of μ_m ,
- (3) $k_m\nu_m(X) + n\mu_m(X) \geq \Im f(X) + \gamma''_m$ on F , and
- (4) $k_m\nu_m(X) + n\mu_m(X) = \Im f(X) + \gamma''_m$ on the support of ν_m .

By the same way as Lemma 4, there exist a measure $\alpha = \mu + i\nu$ of $\mathfrak{M}(a, F, b)$ and a complex constant $\gamma = \gamma_1 + i\gamma_2$ such that

- (1) $\Re K\alpha(X) \geq \Re\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $\Re K\alpha(X) \leq \Re\{f(X) + \gamma\}$ on the support of $\Re\alpha$,
- (3) $\Im K\alpha(X) \geq \Im\{f(X) + \gamma\}$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $\Im K\alpha(X) \leq \Im\{f(X) + \gamma\}$ on the support of $\Im\alpha$.

Proof of Theorem 2. Let $\{f_m(X) | m \in D\}$ and $\{g_m(X) | m \in D\}$ be decreasing nets of positive finite continuous functions on F such that $f_m(X) \downarrow \Re f(X)$ and $g_m(X) \downarrow \Im f(X)$. Taking an adequate positive number a , by Lemma 4, there exist measures $\alpha_m = \mu_m + i\nu_m$ of $\mathfrak{M}(a, F, 1)$ and real constants γ'_m and γ''_m such that

- (1) $k\mu_m(X) - n\nu_m(X) - \gamma''_m \cdot f_m(X) \geq \gamma'_m$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $k\mu_m(X) - n\nu_m(X) - \gamma''_m \cdot f_m(X) \leq \gamma'_m$ on the support of μ_m ,
- (3) $k\nu_m(X) + n\mu_m(X) \geq \gamma''_m \cdot g_m(X)$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $k\nu_m(X) + n\mu_m(X) \leq \gamma''_m \cdot g_m(X)$ on the support of ν_m .

By the same way as Lemma 4, there exist a measure $\alpha = \mu + i\nu$ of $\mathfrak{M}(F)$ and a real constant γ such that

- (1) $k\mu(X) - n\nu(X) \geq \Re f(X) + \gamma$ on F with a possible exception of a set of k -transfinite diameter zero,
- (2) $k\mu(X) - n\nu(X) \leq \Re f(X) + \gamma$ on the support of μ .
- (3) $k\nu(X) + n\mu(X) \geq \Im f(X)$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (4) $k\nu(X) + n\mu(X) \leq \Im f(X)$ on the support of ν .

Thus, the measure $\alpha = \mu + i\nu$, and the real constant γ are what Theorem 2 needs. The analogous arguments will give us the latter part of Theorem 2.

Corollary. Let F be a compact subset of positive k -transfinite diameter, and $f(X)$ be a real-valued upper semi-continuous function with lower bound on F , and a be a positive number. If the adjoint kernel $\check{k}(X, Y)$ satisfies the continuity principle, then there exist a measure μ of $m(a, F)$ and a real constant γ such that

- (1) $k\mu(X) \geq f(X) + \gamma$ on F with a possible exception of a set of k -transfinite diameter zero, and
- (2) $k\mu(X) \leq f(X) + \gamma$ on the support of μ .

REMARK. In above Theorem 2, we can not always reduce the constant γ to zero. We may consider the following example: let Ω be a finite space consisting of two points X_1 and X_2 , and $\Re K(X, Y)$ and $\Im K(X, Y)$ be given by the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively, and $\Re f(X)$ and $\Im f(X)$ be equal to 1 everywhere. Then, for the compact set $F = \Omega$, we have no measure α such that

- (1) $\Re K\alpha(X) \geq \Re f(X)$ on F ,
- (2) $\Re K\alpha(X) = \Re f(X)$ on the support of $\Re \alpha$,
- (3) $\Im K\alpha(X) \geq \Im f(X)$ on F , and
- (4) $\Im K\alpha(X) = \Im f(X)$ on the support of $\Im \alpha$.

REMARK. Putting $n(X, Y) = \Im K(X, Y) \equiv 0$, we can assert that our Theorem 2 contains the existence theorem obtained by M. Kishi and M. Nakai ([2], [3] and [4]).

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