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ON RIEMANNIAN MANIFOLDS WITH A POLE

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0. Introduction

Let M be a Riemannian manifold. A point $o \in M$ is called a *pole*, if the exponential mapping at o induces a global diffeomorphism. We write (M, o) for a Riemannian manifold with the pole o and denote by $\rho_M(x)$ the distance between o and $x \in M$. By the *radial curvature* at $x \in M - \{o\}$, we mean the restriction of sectional curvature to the planes which contain the tangent vector $\text{grad } \rho_M(x)$ (At $x=o$, the radial curvature means simply the sectional curvature at o .) Let $K_M(t)$ ($t \geq 0$) be the maximum of the values of radial curvature at $x \in M$ varying over the points such that $\rho_M(x)=t$. It is easily seen that K_M is a continuous function on $[0, \infty)$.

The purpose of the present paper is to prove the following

Theorem. *Let (M, o) be a Riemannian manifold with a pole. Suppose that there exists a C^1 -function $y=y(t)$ which satisfies the inequality:*

$$y' + y^2 + K_M \leq 0 \quad \text{on } (0, \infty) \quad (\text{resp. } [0, \infty)),$$

and is positive (resp. nonnegative) on $[\alpha, \infty)$ for some $\alpha \geq 0$. Then ρ_M^2 is a strictly convex function on $\{x \in M: \rho_M(x) \geq \alpha\}$.

We recall that a C^2 -function f is said to be *strictly convex* if the Hessian of f , denoted by D^2f , is positive definite.

Corollary. *Let (M, o) be a Kaehler manifold with a pole. Suppose that there is a C^1 -function $y=y(t)$ which satisfies the same conditions as in Theorem. Then M is a Stein manifold.*

Our results are generalizations of a result due to H. Wu, who asserts that ρ_M^2 is strictly convex everywhere on M if $K_M \leq 0$ (cf. Proposition 1.17 in [2]). According to our Theorem, if $K_M(t) \leq \frac{1}{4t}$, then ρ_M^2 is strictly convex everywhere on M , since $y(t) = \frac{1}{2t}$ satisfies the assumption of the Theorem.

1. Riemannian manifolds with a pole and models

In this section, we recall several results in Greene and Wu [2].

Theorem 1 (Hessian Comparison Theorem). *Let (M, o) and (N, p) be Riemannian manifolds with a pole. Let $\gamma_i: [0, b] \rightarrow M$ and $\gamma_2: [0, b] \rightarrow N$ be normal geodesics (i.e. $|\dot{\gamma}_1| = |\dot{\gamma}_2| = 1$) with $\gamma_1(0) = o$ and $\gamma_2(0) = p$. Suppose we have each radial curvature at $\gamma_2(t) \geq$ every radial curvature at $\gamma_1(t)$ for all $t \in (0, b]$. If f is a nondecreasing C^2 -function on $(0, b]$, then*

$$D^2 f(\rho_N)_{\gamma_2(t)}(X_2, X_2) \leq D^2 f(\rho_M)_{\gamma_1(t)}(X_1, X_1)$$

for all $X_1 \in M_{\gamma_1(t)}$ and $X_2 \in N_{\gamma_2(t)}$ with $|X_1| = |X_2|$ and $\langle X_1, \dot{\gamma}_1(t) \rangle = \langle X_2, \dot{\gamma}_2(t) \rangle$.

REMARK. This theorem was obtained at first by Siu and Yau (cf. p. 227 in [5]) with additional assumptions that M and N are negatively curved and of the same dimension, and then by Greene and Wu in the case: $\dim N \leq \dim M$ (cf. Theorem A in [2]). M. Itoh gives a simple proof without any restriction on the dimensions of M and N (cf. [1]).

We say (M, o) *dominates* (N, p) if each radial curvature at $x \leq$ every radial curvature at y for arbitrary $x \in M$ and $y \in N$ with $\rho_M(x) = \rho_N(y)$.

Corollary 1. *Let (M, o) and (N, p) be Riemannian manifold with a pole. Suppose (M, o) dominates (N, o) . If ρ_N^2 is strictly convex on $\{x \in N: \rho_N(x) \geq \alpha\}$ for some $\alpha \geq 0$, then so is ρ_M^2 on $\{x \in M: \rho_M(x) \geq \alpha\}$.*

Proof. We know $D^2 \rho_M^2(o) = 2g(o)$, where g is the Riemannian metric on M . Hence this is an immediate consequence of Theorem 1 by taking t^2 as $f(t)$.

A Riemannian manifold with a pole (N, p) is called a (Riemannian) *model* if every linear isometry $\Phi: N_p \rightarrow N_p$ is realized as the differential of an isometry $\phi: N \rightarrow N$. Let g be the Riemannian metric of a model (N, p) . Since $\exp_p: N_p \rightarrow N$ is a diffeomorphism, $\exp_p^* g$ can be written as $\exp_p^* g = dr^2 + f(r)^2 d\Theta^2$ in a geodesic polar coordinate system, where $r = \rho_N$. We remark that, by the definition of a model, $f(r)$ depends only on r but not on the angular coordinates Θ , and the radial curvature of N at $x \in N$ is a function of $r(x)$. We put $K(t) =$ radial curvature of N at any $x \in N$ such that $r(x) = t$. We call $K: [0, \infty) \rightarrow \mathbb{R}$ the *radial curvature function* of the model (N, p) . Then, it is a classical fact that f satisfies the classical Jacobi equation:

$$f'' + K \cdot f = 0 \quad \text{on } [0, \infty) \text{ with } f(0) = 0, f'(0) = 1.$$

Conversely, by Proposition 4.2 in [2] and the proof of it, we have the following

Lemma 1. *Given a continuous function K on $[0, \infty)$ such that the solution*

$f: f'' + K \cdot f = 0$, with $f(0) = 0, f'(0) = 1$ is positive on $(0, \infty)$, then, there exists a model whose metric is C^1 at the pole and C^2 elsewhere, and whose radial curvature function outside the pole is K ; this model is unique up to isometry.

Lemma 2. *Let (N, p) be a model and, r and f be as above. Then, f' is positive if and only if r^2 is strictly convex.*

Proof. By the Proposition 2.20 in [2], we have $D^2r = [f'/f] H$ on $N - \{p\}$, where $H = g - dr \otimes dr$ and g is the Riemannian metric on N . Hence $D^2r = 2 dr \otimes dr + 2r[f'/f]H$. Therefore f' is positive if and only if r^2 is strictly convex.

2. Review of a classical Jacobi equation

Let K be a continuous function on $[0, \infty)$ and f be the solution: $f'' + K \cdot f = 0$ with $f(0) = 0, f'(0) = 1$. On the positivity of f , we have the following

Lemma 3 (Theorem 7.2 in [4] or [6]). *Let K and f be as above. Then f is positive on $(0, \infty)$ if and only if there is a C^1 -function $y = y(t)$ on $(0, \infty)$ such that $y' + y^2 + K \leq 0$ on $(0, \infty)$.*

Using this lemma, we prove the following

Lemma 4. *Let K and f be as above. If there is a C^1 -function $y = y(t)$ on $(0, \infty)$ (resp. $[0, \infty)$) such that $y' + y^2 + K \leq 0$ on $(0, \infty)$ (resp. $[0, \infty)$) and $y > 0$ on $[\alpha, \infty)$ (resp. $y \geq 0$ on $[\alpha, \infty)$) for some $\alpha (0 \leq \alpha < \infty)$. Then f is positive on $(0, \infty)$ and f' is positive on $[\alpha, \infty)$.*

Proof. Let $y = y(t)$ be as above, defined on $(0, \infty)$. We put $u(t) = \exp \int_c^t \times y(s) ds$, where c is any positive constant. Then u is positive on $(0, \infty)$ and satisfies an inequality: $u'' + K \cdot u \leq 0$ on $(0, \infty)$. Let $f_s (0 \leq s < \infty)$ be the family of solutions: $f_s'' + K \cdot f_s = 0$ with $f_s(s) = 0, f_s'(s) = 1$. We fix any $s > 0$. Then for $t \in (s, \infty)$, we get

$$\begin{aligned} 0 &\leq \int_s^t \{u(r)(f_s''(r) + K(r)f_s(r)) - f_s(r)(u''(r) + K(r)u(r))\} dr \\ &= \int_s^t \{(u(r)f_s'(r))' - (f_s(r)u'(r))'\} dr \\ &= u(t)f_s'(t) - u(s)f_s'(s) - f_s(t)u'(t) + f_s(s)u'(s). \end{aligned}$$

Hence we have

$$(1) \quad 0 \leq u(t)f_s'(t) - u(s) - f_s(t)u'(t)$$

for any $t \in (s, \infty)$. Since $u > 0$ and $u' = y \cdot u$, we see

$$y(t)f_s(t) < f_s'(t).$$

By the continuity of solutions on initial conditions, we have

$$y(t)f(t) = \lim_{s \rightarrow 0} y(t)f_s(t) \leq \lim_{s \rightarrow 0} f'_s(t) = f'(t).$$

By Lemma 3., we know $f(t)$ is positive on $(0, \infty)$. Thus $y > 0$ on $[\alpha, \infty)$ implies $f'(t) > 0$ on $[\alpha, \infty)$. Similarly, in the case where y is defined on $[0, \infty)$, we have (1). Taking $s=0$, we obtain

$$u(0) + f(t)y(t)u(t) \leq u(t)f'(t).$$

Since f is positive on $(0, \infty)$, $y \geq 0$ on $[\alpha, \infty)$ implies $f'(t) > 0$ on $[\alpha, \infty)$.

Corollary 2 ([6]). *Let K and f be as above. If K satisfies an inequality: $\int_t^\infty K^+(s)ds \leq \frac{1}{4t}$ on $(0, \infty)$, where $K^+ = \max\{K, 0\}$, or an inequality: $\left(\int_t^\infty K\right)^2 \leq \frac{K(t)}{4}$, then f and f' are positive on $(0, \infty)$.*

Proof. For the former case, set $y(t) = 2 \int_t^\infty K^+(s)ds + \frac{1}{4t}$. For the latter, set $y(t) = 2 \int_t^\infty K(s)ds$.

REMARK. In Lemma 3, if $K \geq 0$ and $K \neq 0$ near ∞ , it is easily verified that $f > 0$ on $(0, \infty)$ implies $f' > 0$ on $(0, \infty)$.

3. Proof of Theorem and Corollary

Let (M, o) be a Riemannian manifold with a pole. Let y be a C^1 -function in Theorem. Then by Lemma 4 and Lemma 1, there exists a model (N, p) whose metric is C^1 at p and C^2 elsewhere, and whose radial curvature function outside the pole p is K_M , where K_M is a continuous function on $[0, \infty)$ defined in Introduction. Moreover (N, p) is dominated by (M, o) and, by Lemma 2, $r^2(r = \rho_N)$ is strictly convex on $\{x \in N: r(x) \geq \alpha\}$. Therefore (M, o) and (N, p) satisfy all the conditions of Corollary 1. That is, ρ_M^2 is strictly convex on $\{x \in M: \rho_M(x) \geq \alpha\}$.

As for the proof of Corollary, we note that, in general, a (strictly) convex C^2 -function on a Kaehler manifold is a (strictly) plurisubharmonic C^2 -function. Let (M, o) be a Kaehler manifold with a pole. Let y be a C^1 -function in Corollary. Then by Theorem we can see that ρ_M^2 is strictly plurisubharmonic outside a compact set. Since M is diffeomorphic to C^m ($m = \dim_c M$), the arguments in [3] (p. 87) shows that M is a Stein manifold.

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