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ON RIEMANNIAN MANIFOLDS WITH A POLE

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0. Introduction

Let M be a Riemannian manifold. A point $o \in M$ is called *a pole,* if the exponential mapping at o induces a global diffeomorphism. We write (M, o) for a Riemannian manifold with the pole o and denote by $\rho_M(x)$ the distance between o and $x \in M$. By the *radial curvature* at $x \in M - \{o\}$, we mean the restriction of sectional curvature to the planes which contain the tangent vector grad $\rho_M(x)$ (At $x = o$, the radial curvature means simply the sectional curvature at o.) Let $K_M(t)$ ($t \geqq 0$) be the maximum of the values of radial curvature at $x \in M$ x varying over the points such that $\rho_M(x) = t$. It is easily seen that K_M is a continuous function on $[0, \infty)$.

The purpose of the present paper is to prove the following

Theorem. *Let (M, o) be a Riemannian manifold with a pole. Suppose that that there exists a C^1-function $y = y(t)$ which satisfies the inequality*:

$$y' + y^2 + K_M \leqq 0 \qquad \text{on } (0, \infty) \quad (\text{resp. } [0, \infty)),$$

and is positive (*resp. nonnegative*) *on $[\alpha, \infty)$ for some $\alpha \geqq 0$. Then ρ_M^2 is a strictly convex function on $\{x \in M \colon \rho_M(x) \geqq \alpha\}$.*

We recall that a C^2-function f is said to be *strictly convex* if the Hessian of f, denoted by $D^2 f$, is positive definite.

Corollary. *Let (M, o) be a Kaehler manifold with a pole. Suppose that there is a C^1-function $y = y(t)$ which satisfies the same conditions as in Theorem. Then M is a Stein manifold.*

Our results are generalizations of a result due to H. Wu, who asserts that ρ_M^2 is strictly convex everywhere on M if $K_M \leqq 0$ (cf. Proposition 1.17 in [2]). According to our Theorem, if $K_M(t) \leqslant \frac{1}{4t}$, then ρ_M^2 is strictly convex everywhere on M, since $y(t) = \frac{1}{2t}$ satisfies the assumption of the Theorem.

1. Riemannian manifolds with a pole and models

In this section, we recall several results in Greene and Wu [2].

Theorem 1 (Hessian Comparison Theorem). *Let (M, o) and (N, p) be Riemannian manifolds with a pole. Let γ_i: $[0, b]\to M$ and γ_2: $[0, b]\to N$ be normal geodesics (i.e. $|\dot\gamma_1|=|\dot\gamma_2|=1$) with $\gamma_1(0)=o$ and $\gamma_2(0)=p$. Suppose we have each radial curvature at $\gamma_2(t)\geq$ every radial curvature at $\gamma_1(t)$ for all $t\in(0, b]$. If f is a nondecreasing C^2-function on $(0, b]$, then*

$$D^2 f(\rho_N)_{\gamma_2(t)}(X_2, X_2)\leqq D^2 f(\rho_M)_{\gamma_1(t)}(X_1, X_1)$$

for all $X_1\in M_{\gamma_1(t)}$ and $X_2\in N_{\gamma_2(t)}$ with $|X_1|=|X_2|$ and $\langle X_1, \dot\gamma_1(t)\rangle=\langle X_2, \dot\gamma_2(t)\rangle$.

REMARK. This theorem was obtaind at first by Siu and Yau (cf. p. 227 in [5]) with additional assumptions that M and N are negatively curved and of the same dimension, and then by Greene and Wu in the case: $\dim N\leqq\dim M$ (cf. Theorem A in [2]). M. Itoh gives a simple proof without any restriction on the dimensions of M and N (cf. [1]).

We say (M, o) *dominates* (N, p) if each radial curvature at $x\leqq$every radial curvature at y for arbitrary $x\in M$ and $y\in N$ with $\rho_M(x)=\rho_N(y)$.

Corollary 1. *Let (M, o) and (N, p) be Riemannian manifold with a pole. Suppose (M, o) dominates (N, o). If ρ_N^2 is strictly convex on $\{x\in N: \rho_N(x)\geqq\alpha\}$ for some $\alpha\geqq 0$, then so is ρ_M^2 on $\{x\in M: \rho_M(x)\geqslant\alpha\}$.*

Proof. We know $D^2\rho_M^2(o)=2g(o)$, where g is the Riemannian metric on M. Hence this is an immediate consequence of Theorem 1 by taking t^2 as $f(t)$.

A Riemannian manifold with a pole (N, p) is called a (Riemannian) *model* if every linear isometry Φ: $N_p\to N_p$ is realized as the differential of an isometry ϕ: $N\to N$. Let g be the Riemannian metric of a model (N, p). Since $\exp_p$: $N_p\to N$ is a diffeomorphism, $\exp_p^* g$ can be written as $\exp_p^* g=dr^2+f(r)^2 d\Theta^2$ in a geodesic polar coordinate system, where $r=\rho_N$. We remark that, by the definition of a model, $f(r)$ depends only on r but not on the angular coordinates Θ, and the radial curvature of N at $x\in N$ is a function of $r(x)$. We put $K(t)=$ radial curvature of N at any $x\in N$ such that $r(x)=t$. We call K: $[0, \infty)\to R$ the *radial curvature function* of the model (N, p). Then, it is a classical fact that f satisfies the classical Jacobi equation:

$$f''+K\cdot f=0 \qquad \text{on} \quad [0, \infty) \text{ with } f(0)=0,\ f'(0)=1\,.$$

Conversely, by Proposition 4.2 in [2] and the proof of it, we have the following

Lemma 1. *Given a continuous function K on $[0, \infty)$ such that the solution*

f: $f''+K\cdot f=0$, with $f(0)=0, f'(0)=1$ is positive on $(0, \infty)$, then, there exists a model whose metric is C^1 at the pole and C^2 elsewhere, and whose radial curvature function outside the pole is K; this model is unique up to isometry.

Lemma 2. *Let (N, p) be a model and, r and f be as above. Then, f' is positive if and only if r^2 is strictly convex.*

Proof. By the Proposition 2.20 in [2], we have $D^2r=[f'/f]\,H$ on $N-\{p\}$, where $H=g-dr\otimes dr$ and g is the Riemannian metric on N. Hence $D^2r_2=2\,dr\otimes dr+2r[f'/f]H$. Therefore f' is positive if and only if r^2 is strictly convex.

2. Review of a classical Jacobi equation

Let K be a continuous function on $[0, \infty)$ and f be the solution: $f''+K\cdot f=0$ with $f(0)=0, f'(0)=1$. On the positivity of f, we have the following

Lemma 3 (Theorem 7.2 in [4] or [6]). *Let K and f be as above. Then f is positive on $(0, \infty)$ if and only if there is a C^1-function $y=y(t)$ on $(0, \infty)$ such that $y'+y^2+K\leqq 0$ on $(0, \infty)$.*

Using this lemma, we prove the following

Lemma 4. *Let K and f be as above. If there is a C^1-function $y=y(t)$ on $(0, \infty)$ (resp. $[0, \infty)$) such that $y'+y^2+K\leqq 0$ on $(0, \infty)$ (resp. $[0, \infty)$) and $y>0$ on $[\alpha, \infty)$ (resp. $y\geqq 0$ on $[\alpha, \infty)$) for some $\alpha(0\leqq\alpha<\infty)$. Then f is positive on $(0, \infty)$ and f' is positive on $[\alpha, \infty)$.*

Proof. Let $y=y(t)$ be as above, defined on $(0, \infty)$. We put $u(t)=\exp\int_c^t \times y(s)\,ds$, where c is any positive constant. Then u is positive on $(0, \infty)$ and satisfies an inequality: $u''+K\cdot u\leqq 0$ on $(0, \infty)$. Let f_s $(0\leq s<\infty)$ be the family of solutions: $f_s''+K\cdot f_s=0$ with $f_s(s)=0, f_s'(s)=1$. We fix any $s>0$. Then for $t\in(s, \infty)$, we get

$$\begin{aligned}0 &\leqq \int_s^t \{u(r)(f_k''(r)+K(r)f_s(r))-f_s(r)(u''(r)+K(r)u(r))\}\,dr\\ &= \int_s^t \{(u(r)f_s'(r))'-(f_s(r)u'(r))'\}\,dr\\ &= u(t)f_s'(t)-u(s)f_s'(s)-f_s(t)u'(t)+f_s(s)u'(s)\,.\end{aligned}$$

Hence we have

$$0\leqq u(t)f_s'(t)-u(s)-f_s(t)u'(t) \tag{1}$$

for any $t\in(s, \infty)$. Since $u>0$ and $u'=y\cdot u$, we see

$$y(t)f_s(t)<f_s'(t)\,.$$

By the continuity of solutions on initial conditions,we have

$$y(t)f(t) = \lim_{s\to 0} y(t)f_s(t) \leqq \lim_{s\to 0} f'_s(t) = f'(t)\,.$$

By Lemma 3., we know $f(t)$ is positive on $(0, \infty)$. Thus $y>0$ on $[\alpha, \infty)$ implies $f'(t)>0$ on $[\alpha, \infty)$. Similaly, in the case where y is defined on $[0, \infty)$, we have (1). Taking $s=0$, we obtain

$$u(0)+f(t)y(t)u(t) \leqq u(t)f'(t)\,.$$

Since f is positive on $(0, \infty)$, $y\geqq 0$ on $[\alpha, \infty)$ implies $f'(t)>0$ on $[\alpha, \infty)$.

Corollary 2 ([6]). *Let K and f be as above. If K satisfies an inequality: $\int_t^\infty K^+(s)ds \leq \frac{1}{4t}$ on $(0, \infty)$, where $K^+=\max\{K, 0\}$, or an inequality: $\left(\int_t^\infty K\right)^2 \leqq \frac{K(t)}{4}$, then f and f' are positive on $(0, \infty)$.*

Proof. For the former case, set $y(t)=2\int_t^\infty K^+(s)ds+\frac{1}{4t}$. For the latter, set $y(t)=2\int_t^\infty K(s)ds$.

REMARK. In Lemma 3, if $K\geqq 0$ and $K\not\equiv 0$ near ∞, it is easily verified that $f>0$ on $(0, \infty)$ implies $f'>0$ on $(0, \infty)$.

3. Proof of Theorem and Corollary

Let (M, o) be a Riemannian manifold with a pole. Let y be a C^1-function in Theorem. Then by Lemma 4 and Lemma 1, there exists a model (N, p) whose metric is C^1 at p and C^2 elsewhere, and whose radial curvature function outside the pole p is K_M, where K_M is a continuous function on $[0, \infty)$ defined in Introduction. Moreover (N, p) is dominated by (M, o) and, by Lemma 2, $r^2 (r=\rho_N)$ is strictly convex on $\{x\in N: r(x)\geqslant\alpha\}$. Therefore (M, o) and (N, p) satisfy all the conditions of Corollary 1. That is, ρ_M^2 is strictly convex on $\{x\in M: \rho_M(x)\geqslant\alpha\}$.

As for the proof of Corollary, we note that, in general, a (strictly) convex C^2-function on a Kaehler manifold is a (strictly) plurisubharmonic C^2-function. Let (M, o) be a Kaehler manifold with a pole. Let y be a C^1-function in Corollary. Then by Theorem we can see that ρ_M^2 is strictly plurisubharmonic outside a compact set. Since M is diffeomorphic to C^m ($m=\dim_C M$), the arguments in [3] (p. 87) shows that M is a Stein manifold.

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