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SUM OF DIGITS TO DIFFERENT BASES AND MUTUAL SINGULARITY OF THEIR SPECTRAL MEASURES

To Professor H. Kudo on his 60th birthday

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1. Statement of the main result

Let $s_r(n)$ denote the sum of digits in the r -adic representation of a non-negative integer n . Let $\xi(n) = e(cs_r(n))$, where $e(x) = e^{2\pi i x}$ and c is a real number such that $(r-1)c \notin \mathbf{Z}$. Then it is known [3] that the covariance

$$\gamma_\xi(m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \xi(n+m) \overline{\xi(n)}$$

exists for any $m \in \mathbf{Z}$ and the spectral measure Λ_ξ is continuous but singular with respect to the Lebesgue measure, where Λ_ξ is the measure on $T = \mathbf{R}/\mathbf{Z}$ such that

$$\gamma_\xi(m) = \int_T e(mx) d\Lambda_\xi(x)$$

for any $m \in \mathbf{Z}$.

Theorem *Let p and q be two relatively prime integers not less than 2. Let $\alpha(n) = e(as_p(n))$ and $\beta(n) = e(bs_q(n))$, where a and b are real numbers such that $(p-1)a \notin \mathbf{Z}$ and $(q-1)b \notin \mathbf{Z}$. Then the spectral measures Λ_α and Λ_β are singular to each other.*

2. Lemmas

To prove the theorem, we may and do assume that q is an odd number. Let $e_k^r(n)$ be the k -th digit of the r -adic representation of n ; that is, $e_k^r(n) \in \{0, 1, \dots, r-1\}$ and

$$n = \sum_{k=0}^{\infty} e_k^r(n) r^k.$$

Lemma 1. *As m and t tend to the infinity satisfying that $m > t$, $\tau_q(p^{2m} - p^{2t})$ tends to the infinity, where $\tau_q(n)$ is the largest integer j such that there exist $2j$*

integers $0 \leq k_1 < k_2 < \dots < k_j$, satisfying that $e_{k_{2i-1}}^i(n) > 0$ and $e_{k_{2i}}^i(n) < q-1$ for $i=1, 2, \dots, j$.

Proof. Let

$$\Gamma_r(n) = \prod_{k=0}^{\infty} \cos 2\pi nr^{-k}.$$

Then by H. G. Senge and E. G. Straus [5], it holds that

$$\lim_{n \rightarrow \infty} \Gamma_{\theta}(n)\Gamma_{\varphi}(n) = 0$$

for any integers θ and φ not less than 2 such that $\log \theta / \log \varphi$ is irrational. Since

$$\inf_{m>t} |\Gamma_p(p^{2m}-p^{2t})| > 0,$$

it follows, using the above fact, that

$$\lim_{\substack{m \rightarrow \infty \\ t \rightarrow \infty \\ m>t}} \Gamma_q(p^{2m}-p^{2t}) = 0.$$

For any fixed s , there exists a constant $\delta(q, s) > 0$ such that

$$|\Gamma_q(\lambda_1 q^{k_1} + \dots + \lambda_s q^{k_s})| \geq \delta(q, s)$$

holds for any $\lambda_1, \dots, \lambda_s \in \{-q+1, -q+2, \dots, q-1\}$ and $k_1, \dots, k_s \in \mathbf{N}$. If $\tau_q(n)=s$, then n can be written as

$$\lambda_1 q^{k_1} + \dots + \lambda_{2s} q^{k_{2s}}$$

for some $\lambda_1, \dots, \lambda_{2s} \in \{-q+1, -q+2, \dots, q-1\}$ and $k_1, \dots, k_{2s} \in \mathbf{N}$. Hence, this implies that

$$|\Gamma_q(n)| \geq \delta(q, 2s).$$

Suppose that

$$\lim_{\substack{m \rightarrow \infty \\ t \rightarrow \infty \\ m>t}} \tau_q(p^{2m}-p^{2t}) = s < \infty.$$

Then we have a contradiction that

$$0 = \lim_{\substack{m \rightarrow \infty \\ t \rightarrow \infty \\ m>t}} |\Gamma_q(p^{2m}-p^{2t})| \geq \delta(q, 2s) > 0.$$

Let $\xi(n)=e(cs, (n))$, where c is a real number such that $(r-1)c \in \mathbf{Z}$. Fix n for a moment and denote $e_j=e_j^r(n)$ ($j=0, 1, \dots$). Let $\tau=\tau_r(n)$ and

$$\begin{aligned}
 b_0 &= -1 \\
 a_j &= \min \{k > b_{j-1}; e_k > 0\} \\
 b_j &= \min \{k > a_j; e_k < r-1\} \\
 (j &= 1, 2, \dots, \tau).
 \end{aligned}$$

Let X_0, X_1, \dots be a sequence of independent random variables on $\{0, 1, \dots, r-1\}$ such that $P(X_k=j)=1/r$ for any $j \in \{0, 1, \dots, r-1\}$ and $k=0, 1, \dots$. Let

$$Y_n = \lim_{N \rightarrow \infty} (s_r(\sum_{j=0}^N X_j r^j + n) - s_r(\sum_{j=0}^N X_j r^j)),$$

where the limit exists with probability 1.

Lemma 2.

$$\gamma_\xi(n) = E(e(cY_n)).$$

Proof. Clear.

Lemma 3. $\gamma_\xi(n)$ tends to 0 as n tends to the infinity satisfying that $\tau_r(n) \rightarrow \infty$.

Proof. Define random variables $\tilde{c}_1, \tilde{c}_2, \dots$ by

$$\{b_{2j}; X_{b_{2j}} = 0\} = \{\tilde{c}_1 < \tilde{c}_2 < \dots\}.$$

For

$$\{c_1 < c_2 < \dots < c_k\} \subset \{b_{2j}; j = 1, 2, \dots\},$$

define a stochastic event

$$I(c_1, \dots, c_k) = \{\tilde{c}_1 = c_1, \dots, \tilde{c}_k = c_k\}.$$

Let

$$\begin{aligned}
 \varepsilon &\equiv 1 - P(\cup_{c_1 \dots c_k} I(c_1, \dots, c_k)) \\
 &= 1 - P(|\{j; X_{b_{2j}} = 0\}| \geq k) \\
 &= 1 - \sum_{j=k}^{\lceil \tau/2 \rceil} \binom{\lceil \tau/2 \rceil}{j} \left(\frac{1}{r}\right)^j \left(\frac{r-1}{r}\right)^{\lceil \tau/2 \rceil - j}.
 \end{aligned}$$

Then $\varepsilon \rightarrow 0$ as $\tau \rightarrow \infty$ satisfying that $k \sim \frac{\tau}{2r+1}$. On each event $I(c_1, \dots, c_k)$, define

$$Z_h = s_r(\sum_{i \in [c_{h-1}, c_h]} X_i r^i + d_h) - s_r(\sum_{i \in [c_{h-1}, c_h]} X_i r^i)$$

for $h=1, 2, \dots, k$, where $c_0 = -1$ and

$$d_h = \sum_{i \in [c_{h-1}, c_h]} e_i r^i > 0.$$

Define also

$$Z_{k+1} = \lim_{N \rightarrow \infty} (s_r(\sum_{i \in]c_k, N[} X_i r^i + d_{k+1}) - s_r(\sum_{i \in]c_k, N[} X_i r^i)),$$

where

$$d_{k+1} = \sum_{i > c_k} e_i r^i.$$

Then on each event $I(c_1, \dots, c_k)$, the random variables Z_1, Z_2, \dots, Z_{k+1} are independent and it holds that $Y_n = \sum_{h=1}^{k+1} Z_h$. Let $h \in \{1, 2, \dots, k\}$. Let

$$j = \min \{i \in]c_{h-1}, c_h[; e_i > 0 \text{ and } e_{i+1} < r-1\}$$

and $g = e_j$. Then we have, putting $I = I(c_1, \dots, c_k)$,

$$\begin{aligned} & E(e(cZ_h) | I) \\ & \leq \frac{r^2-2}{r^2} + E(e(cZ_h) \mathcal{X}_{X_j \in (0, r-g)} \mathcal{X}_{X_{j+1}=0} | I) \\ & \leq \frac{r^2-2}{r^2} + \frac{1}{r^2} \{E(e(cZ_h) | I, X_j = 0, X_{j+1} = 0) + E(e(cZ_h) | I, X_j = r-g, X_{j+1} = 0)\} \\ & = \frac{r^2-2}{r^2} + \frac{1}{r^2} (e((r-1)c) + 1) E(e(cZ_h) | I, X_j = r-g, X_{j+1} = 0) \\ & \leq \frac{r^2-2 + e((r-1)c) + 1}{r^2} \equiv \delta < 1. \end{aligned}$$

Therefore,

$$\begin{aligned} & E(e(cY_n)) \\ & \leq \sum_{c_1 \dots c_k} |E(e(cY_n) | I(c_1, \dots, c_k))| P(I(c_1, \dots, c_k)) + \varepsilon \\ & = \sum_{c_1 \dots c_k} \prod_{h=1}^{k+1} |E(e(cZ_h) | I(c_1, \dots, c_k))| P(I(c_1, \dots, c_k)) + \varepsilon \\ & \leq \delta^k + \varepsilon. \end{aligned}$$

Thus $E(e(cY_n)) \rightarrow 0$ as $\tau \rightarrow \infty$ satisfying that $k \sim \frac{\tau}{2r+1}$.

Lemma 4. *It holds that*

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{p^{2n}} \beta \right\| = 0,$$

where T is the shift of arithmetic functions and for an arithmetic function η ,

$$\|\eta\| = \left(\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\eta(n)|^2 \right)^{1/2}.$$

Proof. It holds that

$$\left\| \frac{1}{N} \sum_{n=1}^N T^{p^{2n}} \beta \right\|^2 = \frac{1}{N^2} \sum_{m,t=1}^N \gamma_{\beta}(p^{2m} - p^{2t}).$$

Thus, lemma 4 follows from lemma 1 and 3.

Lemma 5. *It holds that*

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{p^{2n}} \alpha - K\alpha \right\| = 0,$$

where

$$K = \frac{(p-1)e(pa)}{pe((p-1)a)-1} \neq 0.$$

Proof. Let $r=p$. Note that

$$\left\| \frac{1}{N} \sum_{n=1}^N T^{p^{2n}} \alpha - K\alpha \right\|^2 = E \left(\left| \frac{1}{N} \sum_{n=1}^N e(aY_{p^{2n}}) - K \right|^2 \right).$$

It holds that

$$\begin{aligned} & \lim_{N \rightarrow \infty} E \left(\left| \frac{1}{N} \sum_{n=1}^N e(aY_{p^{2n}}) - K \right|^2 \right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{m,t=1}^N E((e(aY_{p^{2m}}) - K)(\overline{e(aY_{p^{2t}}) - K})) \\ &= 0, \end{aligned}$$

since

$$\begin{aligned} & \lim_{\substack{m \rightarrow \infty \\ t \rightarrow \infty \\ m-t \rightarrow \infty}} |E((e(aY_{p^{2m}}) - K)(\overline{e(aY_{p^{2t}}) - K}))| \\ &= \lim_{n \rightarrow \infty} |E((e(aY_{p^{2n}}) - K)(\overline{e(aY_1) - K}))| \\ &\leq \lim_{n \rightarrow \infty} \left| \sum_{k=2}^{2n-1} E((e(aY_{p^{2n}}) - K)(\overline{e(aY_1) - K}) | J_k) P(J_k) \right| \\ &= \lim_{n \rightarrow \infty} \left| \sum_{k=2}^{2n-1} E(e(aY_{p^{2n}}) - K | J_k) E(\overline{e(aY_1) - K} | J_k) P(J_k) \right| \\ &= \lim_{n \rightarrow \infty} |(2n-2)E(e(aY_1) - K) E(\overline{e(aY_1) - K} | J_k) P(J_k)| \\ &= 0, \end{aligned}$$

where for $k=2, 3, \dots, 2n-1$,

$$J_k = \{X_2 \neq 0, \dots, X_{k-1} \neq 0, X_k = 0\}.$$

3. Proof of the theorem

For an arithmetic function η , let $\|\eta\|$ be the norm in lemma 4. Let

$\mathcal{S} = \{\eta; \|\eta\| < \infty\}$, $\mathcal{N} = \{\eta; \|\eta\| = 0\}$ and $\mathcal{B} = \mathcal{S}/\mathcal{N}$. Then it is known [2] that \mathcal{B} is a Banach space. Since $T\mathcal{N} \subset \mathcal{N}$ and $T^{-1}\mathcal{N} \subset \mathcal{N}$, T can be considered as an invertible transformation on \mathcal{B} . In this sense, it is clear that T is an isometry. For $\eta \in \mathcal{B}$, let $H(\eta)$ be the closed subspace of \mathcal{B} generated by $\{T^n\eta; n \in \mathbf{Z}\}$. For η and ζ in \mathcal{B} , define an *inner product*

$$(\eta, \zeta) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \eta(n) \overline{\zeta(n)}$$

if this limit exists. It is clear that if $\gamma_\eta(m)$ exists for any $m \in \mathbf{Z}$, then the inner product always exists in $H(\eta)$ and $H(\eta)$ becomes a Hilbert space. By A. N. Kolmogorov [4], to prove the theorem, it is sufficient to prove that $H(\alpha) \perp H(\beta)$ and $\alpha \in H(\alpha + \beta)$. It was proved by J. Besineau [1] that $(\alpha, \beta) = 0$. His proof works as well to prove that $(T^n\alpha, T^m\alpha) = 0$ for any $n, m \in \mathbf{Z}$. Thus we have $H(\alpha) \perp H(\beta)$. On the other hand, since

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{NK} \sum_{n=1}^N T^{p^{2^n}}(\alpha + \beta) - \alpha \right\| = 0$$

by lemma 4 and 5, $\alpha \in H(\alpha + \beta)$ holds. Thus we complete the proof.

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