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SUM OF DIGITS TO DIFFERENT BASES AND MUTUAL SINGULARITY OF THEIR SPECTRAL MEASURES

To Professor H. Kudo on his 60th birthday

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1. Statement of the main result

Let $s_r(n)$ denote the sum of digits in the r-adic representation of a non-negative integer n. Let $\xi(n)=e(cs_r(n))$, where $e(x)=e^{2\pi ix}$ and c is a real number such that $(r-1)c\notin \boldsymbol{Z}$. Then it is known [3] that the *covariance*

$$\gamma_\xi(m)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\xi(n+m)\overline{\xi(n)}$$

exists for any $m\in\boldsymbol{Z}$ and the *spectral measure* Λ_ξ is continuous but singular with respect to the Lebesgue measure, where Λ_ξ is the measure on $\boldsymbol{T}=\boldsymbol{R}/\boldsymbol{Z}$ such that

$$\gamma_\xi(m)=\int_{\boldsymbol{T}}e(mx)d\Lambda_\xi(x)$$

for any $m\in\boldsymbol{Z}$.

Theorem *Let p and q be two relatively prime integers not less than* 2. *Let $\alpha(n)=e(as_p(n))$ and $\beta(n)=e(bs_q(n))$, where a and b are real numbers such that $(p-1)a\notin\boldsymbol{Z}$ and $(q-1)b\notin\boldsymbol{Z}$. Then the spectral measures Λ_α and Λ_β are singular to each other.*

2. Lemmas

To prove the theorem, we may and do assume that q is an odd number. Let $e_k^r(n)$ be the k-th digit of the r-adic representation of n; that is, $e_k^r(n)\in\{0, 1, \cdots, r-1\}$ and

$$n=\sum_{k=0}^{\infty}e_k^r(n)r^k\,.$$

Lemma 1. *As m and t tend to the infinity satisfying that $m>t$, $\tau_q(p^{2m}-p^{2t})$ tends to the infinity, where $\tau_q(n)$ is the largest integer j such that there exist $2j$*

integers $0 \leqq k_1 < k_2 < \cdots < k_{2j}$ *satisfying that* $e^q_{k_{2i-1}}(n) > 0$ *and* $e^q_{k_{2i}}(n) < q-1$ *for* $i=1, 2, \cdots, j$.

Proof. Let

$$\Gamma_r(n) = \prod_{k=0}^{\infty} \cos 2\pi n r^{-k}.$$

Then by H. G. Senge and E. G. Straus [5], it holds that

$$\lim_{n\to\infty} \Gamma_\theta(n)\Gamma_\varphi(n) = 0$$

for any integers θ and φ not less than 2 such that $\log\theta/\log\varphi$ is irrational. Since

$$\inf_{m>t} |\Gamma_{p^2}(p^{2m}-p^{2t})| > 0,$$

it follows, using the above fact, that

$$\lim_{\substack{m\to\infty\\ t\to\infty\\ m>t}} \Gamma_q(p^{2m}-p^{2t}) = 0.$$

For any fixed s, there exists a constant $\delta(q, s) > 0$ such that

$$|\Gamma_q(\lambda_1 q^{k_1} + \cdots + \lambda_s q^{k_s})| \geqq \delta(q, s)$$

holds for any $\lambda_1, \cdots, \lambda_s \in \{-q+1, -q+2, \cdots, q-1\}$ and $k_1, \cdots, k_s \in \boldsymbol{N}$. If $\tau_q(n) = s$, then n can be written as

$$\lambda_1 q^{k_1} + \cdots + \lambda_{2s} q^{k_{2s}}$$

for some $\lambda_1, \cdots, \lambda_{2s} \in \{-q+1, -q+2, \cdots, q-1\}$ and $k_1, \cdots, k_{2s} \in \boldsymbol{N}$. Hence, this implies that

$$|\Gamma_q(n)| \geqq \delta(q, 2s).$$

Suppose that

$$\varliminf_{\substack{m\to\infty\\ t\to\infty\\ m>t}} \tau_q(p^{2m}-p^{2t}) = s < \infty.$$

Then we have a contradiction that

$$0 = \varlimsup_{\substack{m\to\infty\\ t\to\infty\\ m>t}} |\Gamma_q(p^{2m}-p^{2t})| \geqq \delta(q, 2s) > 0.$$

Let $\xi(n) = e(cs_r(n))$, where c is a real number such that $(r-1)c \notin \boldsymbol{Z}$. Fix n for a moment and denote $e_j = e^r_j(n)$ $(j=0, 1, \cdots)$. Let $\tau = \tau_r(n)$ and

$$\begin{aligned}
&b_0 = -1\\
&a_j = \min\{k>b_{j-1};\ e_k>0\}\\
&b_j = \min\{k>a_j;\ e_k<r-1\}\\
&(j = 1, 2, \cdots, \tau)\,.
\end{aligned}$$

Let $X_0, X_1, \cdots$ be a sequence of independent random variables on $\{0, 1, \cdots, r-1\}$ such that $P(X_k=j)=1/r$ for any $j\in\{0, 1, \cdots, r-1\}$ and $k=0, 1, \cdots$. Let

$$Y_n = \lim_{N\to\infty}\left(s_r\left(\sum_{j=0}^{N} X_j r^j + n\right) - s_r\left(\sum_{j=0}^{N} X_j r^j\right)\right),$$

where the limit exists with probability 1.

Lemma 2.

$$\gamma_\xi(n) = E(e(cY_n))\,.$$

Proof. Clear.

Lemma 3. *$\gamma_\xi(n)$ tends to 0 as n tends to the infinity satisfying that $\tau_r(n)\to\infty$.*

Proof. Define random variables $\tilde{c}_1, \tilde{c}_2, \cdots$ by

$$\{b_{2j};\ X_{b_{2j}} = 0\} = \{\tilde{c}_1<\tilde{c}_2<\cdots\}\,.$$

For

$$\{c_1<c_2<\cdots<c_k\}\subset\{b_{2j};\ j = 1, 2, \cdots\}\,,$$

define a stochastic event

$$I(c_1, \cdots, c_k) = \{\tilde{c}_1 = c_1, \cdots, \tilde{c}_k = c_k\}\,.$$

Let

$$\begin{aligned}
\varepsilon &\equiv 1-P\Big(\bigcup_{c_1\cdots c_k} I(c_1, \cdots, c_k)\Big)\\
&= 1-P(|\{j;\ X_{b_{2j}} = 0\}|\geqq k)\\
&= 1-\sum_{j=k}^{[\tau/2]}\binom{[\tau/2]}{j}\left(\frac{1}{r}\right)^j\left(\frac{r-1}{r}\right)^{[\tau/2]-j}.
\end{aligned}$$

Then $\varepsilon\to 0$ as $\tau\to\infty$ satisfying that $k\sim\dfrac{\tau}{2r+1}$. On each event $I(c_1, \cdots, c_k)$, define

$$Z_h = s_r\Big(\sum_{i\in]c_{h-1},c_h]} X_i r^i + d_h\Big) - s_r\Big(\sum_{i\in]c_{h-1},c_h]} X_i r^i\Big)$$

for $h=1, 2, \cdots, k$, where $c_0=-1$ and

$$d_h = \sum_{i\in]c_{h-1},c_h]} e_i r^i > 0\,.$$

Define also

$$Z_{k+1}=\lim_{N\to\infty}(s_r(\sum_{i\in]c_k,N]}X_i r^i+d_{k+1})-s_r(\sum_{i\in]c_k,N]}X_i r^i))\,,$$

where

$$d_{k+1}=\sum_{i>c_k}e_i r^i\,.$$

Then on each event $I(c_1,\cdots,c_k)$, the random variables $Z_1, Z_2, \cdots, Z_{k+1}$ are independent and it holds that $Y_n=\sum_{h=1}^{k+1}Z_h$. Let $h\in\{1, 2, \cdots, k\}$. Let

$$j=\min\{i\in]c_{h-1}, c_h];\, e_i>0 \text{ and } e_{i+1}<r-1\}$$

and $g=e_j$. Then we have, putting $I=I(c_1,\cdots,c_k)$,

$$\begin{aligned}
&E(e(cZ_h)\,|\,I)\\
&\leqq\frac{r^2-2}{r^2}+E(e(cZ_h)\chi_{X_j\in\{0,r-g\}}\chi_{X_{j+1}=0}\,|\,I)\\
&\leqq\frac{r^2-2}{r^2}+\frac{1}{r^2}\{E(e(cZ_h)\,|\,I,\,X_j=0,\,X_{j+1}=0)+E(e(cZ_h)\,|\,I,\,X_j=r-g,\,X_{j+1}=0)\}\\
&=\frac{r^2-2}{r^2}+\frac{1}{r^2}(e((r-1)c)+1)E(e(cZ_h)\,|\,I,\,X_j=r-g,\,X_{j+1}=0)\\
&\leqq\frac{r^2-2+e((r-1)c)+1}{r^2}\equiv\delta<1\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E(e(cY_n))\\
&\leqq\sum_{c_1\cdots c_k}|E(e(cY_n)\,|\,I(c_1,\cdots,c_k))|P(I(c_1,\cdots,c_k))+\varepsilon\\
&=\sum_{c_1\cdots c_k}\prod_{h=1}^{k+1}|E(e(cZ_h)\,|\,I(c_1,\cdots,c_k))|P(I(c_1,\cdots,c_k))+\varepsilon\\
&\leqq\delta^k+\varepsilon.
\end{aligned}$$

Thus $E(e(cY_n))\to0$ as $\tau\to\infty$ satisfying that $k\sim\dfrac{\tau}{2r+1}$.

Lemma 4. *It holds that*

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}T^{p^{2n}}\beta\right\|=0\,,$$

where T is the shift of arithmetic functions and for an arithmetic function η,

$$\|\eta\|=\left(\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=0}^{N-1}|\eta(n)|^2\right)^{1/2}.$$

Proof. It holds that

$$\left\|\frac{1}{N}\sum_{n=1}^{N}T^{p^{2n}}\beta\right\|^2=\frac{1}{N^2}\sum_{m,t=1}^{N}\gamma_\beta(p^{2m}-p^{2t}).$$

Thus, lemma 4 follows from lemma 1 and 3.

Lemma 5. *It holds that*

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}T^{p^{2n}}\alpha-K\alpha\right\|=0,$$

where

$$K=\frac{(p-1)e(pa)}{pe((p-1)a)-1}\neq 0.$$

Proof. Let $r=p$. Note that

$$\left\|\frac{1}{N}\sum_{n=1}^{N}T^{p^{2n}}\alpha-K\alpha\right\|^2=E\left(\left|\frac{1}{N}\sum_{n=1}^{N}e(aY_{p^{2n}})-K\right|^2\right).$$

It holds that

$$\begin{aligned}&\lim_{N\to\infty}E\left(\left|\frac{1}{N}\sum_{n=1}^{N}e(aY_{p^{2n}})-K\right|^2\right)\\&=\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,t=1}^{N}E((e(aY_{p^{2n}})-K)\overline{(e(aY_{p^{2t}})-K)})\\&=0,\end{aligned}$$

since

$$\begin{aligned}&\lim_{\substack{m\to\infty\\t\to\infty\\m-t\to\infty}}|E((e(aY_{p^{2m}})-K)\overline{(e(aY_{p^{2t}})-K)})|\\&=\lim_{n\to\infty}|E((e(aY_{p^{2n}})-K)\overline{(e(aY_1)-K)})|\\&\leqq\lim_{n\to\infty}\left|\sum_{k=2}^{2n-1}E((e(aY_{p^{2n}})-K)\overline{(e(aY_1)-K)}\,|\,J_k)P(J_k)\right|\\&=\lim_{n\to\infty}\left|\sum_{k=2}^{2n-1}E(e(aY_{p^{2n}})-K\,|\,J_k)E(\overline{e(aY_1)-K}\,|\,J_k)P(J_k)\right|\\&=\lim_{n\to\infty}|(2n-2)E(e(aY_1)-K)E(\overline{e(aY_1)-K}\,|\,J_k)P(J_k)|\\&=0,\end{aligned}$$

where for $k=2,3,\cdots,2n-1$,

$$J_k=\{X_2\neq 0,\cdots,X_{k-1}\neq 0,X_k=0\}.$$

3. Proof of the theorem

For an arithmetic function η, let $\|\eta\|$ be the norm in lemma 4. Let

$\mathcal{S}=\{\eta;\ \|\eta\|<\infty\}$, $\mathcal{N}=\{\eta;\ \|\eta\|=0\}$ and $\mathcal{B}=\mathcal{S}/\mathcal{N}$. Then it is known [2] that $\mathcal{B}$ is a Banach space. Since $T\mathcal{N}\subset\mathcal{N}$ and $T^{-1}\mathcal{N}\subset\mathcal{N}$, T can be considered as an invertible transformation on $\mathcal{B}$. In this sense, it is clear that T is an isometry. For $\eta\in\mathcal{B}$, let $H(\eta)$ be the closed subspace of $\mathcal{B}$ generated by $\{T^n\eta;\ n\in\boldsymbol{Z}\}$. For η and ζ in B, define an *inner product*

$$(\eta,\zeta)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\eta(n)\overline{\zeta(n)}$$

if this limit exists. It is clear that if $\gamma_\eta(m)$ exists for any $m\in\boldsymbol{Z}$, then the inner product always exists in $H(\eta)$ and $H(\eta)$ becomes a Hilbert space. By A. N. Kolmogorov [4], to prove the theorem, it is sufficient to prove that $H(\alpha)\perp H(\beta)$ and $\alpha\in H(\alpha+\beta)$. It was proved by J. Besineau [1] that $(\alpha,\beta)=0$. His proof works as well to prove that $(T^n\alpha, T^m\alpha)=0$ for any $n, m\in\boldsymbol{Z}$. Thus we have $H(\alpha)\perp H(\beta)$. On the other hand, since

$$\lim_{N\to\infty}\left\|\frac{1}{NK}\sum_{n=1}^{N}T^{p^{2n}}(\alpha+\beta)-\alpha\right\|=0$$

by lemma 4 and 5, $\alpha\in H(\alpha+\beta)$ holds. Thus we complete the proof.

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References

[1] J. Bésineau: *Indépendance statistique d'ensembles liés à la fonction "somme des chiffres,"* Acta Arith. **XX** (1972).

[2] J.P. Bertrandias: *Espace de fonctions bornées et continues en moyenne asymptotique d'ordre p*, Bull. Soc. Math. France, Memoire **5** (1966).

[3] J. Coquet, T. Kamae and M. Mendès France: *Sur la mesure spectrale de certaines suites arithmétiques*, Bull. Soc. Math. France **105** (1977).

[4] A.N. Kolmogorov: *Stationary sequences in Hilbert spaces* (Russian), Bull. Math. Univ. Moscow **2-6** (1941).

[5] H.G. Senge and E.G. Straus: *PV-number and of multiplicity*, Proc. Washington State Univ. Conf. on Number Theory (1971).