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SPECIAL MEMBERS IN THE BICANONICAL PENCIL OF GODEAUX SURFACES

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Abstract

The object of this paper is to find the number of hyperelliptic curves in the bicanonical pencil of a Godeaux surface whose torsion group is $\mathbb{Z}_3$, or $\mathbb{Z}_4$, or $\mathbb{Z}_5$.

Let X be a minimal smooth projective surface of general type with $p_g = q = 0$, $K^2 = 1$. Surfaces of this type are called *Godeaux surfaces* (also they are called *numerical Godeaux surfaces*); they were discovered in the 1930s by Campedelli [2] and Godeaux [6], and are interesting in view of Castelnuovo's criterion: an irrational surface with $q = 0$ must have $P_2 \geq 1$.

The following well known results are important tools in the paper:

- Bicanonical system $|2K_X|$ gives a pencil and the fixed part of $|2K_X|$ consists of -2-curves [12];
- Tricanonical system $|3K_X|$ gives a birational map $X \to \mathbb{P}^3$ without fixed components [3], [12];
- Denote by $\mathrm{T}(X)$ the torsion subgroup of $\mathrm{H}^2(X, \mathbb{Z})$. Recall that 0, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ are the only possible values for the torsion group of a Godeaux surface [12], [13];
- A half of the number of elements in the set $\{\tau \in \mathrm{T}(X) \mid \tau \neq -\tau\}$ equals the number of base points of $|3K_X|$ [12]; and
- For each non-zero element $\tau \in \mathrm{T}(X)$, $h^0(K_X + \tau) = 1$. And $K_X + \tau_1$ intersects $K_X + \tau_2$ transversally if $\tau_1 \neq \tau_2$ [13].

In this paper, we assume that $|2K_X|$ has no fixed components and it has four simple base points. Then a general member $C \in |2K_X|$ is a non-hyperelliptic curve of genus four because $|3K_X|$ gives a birational map. Let $p \colon S \to X$ be the blowing-up of X at the base points of $|2K_X|$ (and at the base points of $|3K_X|$ if $\mathrm{T}(X) = \mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_5$) and let a fibration $f : S \to \mathbb{P}^1$ be given by the bicanonical pencil. According to the semi-positiveness and the Hirzebruch-Riemann-Roch theorem, we obtain the following [4], [11]:

- $f_*(p^*3K_X) = \mathcal{O}^{\oplus 4}$,
- $f_*(p^*6K_X) = \mathcal{O}^{\oplus 4} + \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(3)$,
- $f_*(p^*9K_X) = \mathcal{O}^{\oplus 4} + \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(2)^{\oplus 3} + \mathcal{O}(3)^{\oplus 4}$.

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Consider the following quadric sequence and the cubic sequence

$$0 \to \mathcal{K}_2 \to S^2(f_* p^* 3K_X) \to f_*(p^* 6K_X) \to \mathcal{T}_2 \to 0$$
$$0 \to \mathcal{K}_3 \to S^3(f_* p^* 3K_X) \to f_*(p^* 9K_X) \to \mathcal{T}_3 \to 0$$

where $\mathcal{K}_i$ and $\mathcal{T}_i$ are defined by the kernel sheaves and cokernel sheaves. If $|3K_X|$ has base points then consider the maps $S^2(f_*(p^*3K_X - E)) \to f_*(p^*6K_X)$, $S^3(f_*(p^*3K_X - E)) \to f_*(p^*9K_X)$ instead of the maps $S^2(f_*p^*3K_X) \to f_*(p^*6K_X)$, $S^3(f_*p^*3K_X) \to f_*(p^*9K_X)$ respectively, where E is the exceptional divisor of the base points of $|3K_X|$.

Except the special members related with the elements in $\mathrm{T}(X)$, non-hyperelliptic fibers are 3-connected. Therefore the support of the torsion sheaf consists of the points of special fibers related with $\mathrm{T}(X)$ or the points of hyperelliptic fibers (cf. Lemma 2.1 in [4]). According to Lemma 2.1 and Proposition 2.3 in [4], each hyperelliptic curve adds the length of $\mathcal{T}_2$ by $2l$ and the length of $\mathcal{T}_3$ by $5l$ where l is the contact number of the bicanonical pencil with hyperelliptic locus. The natural surjective homomorphism $S^2(f_*p^*3K_X) \otimes f_*p^*3K_X \to S^3(f_*p^*3K_X)$ induces a homomorphism $\mathcal{K}_2^{\oplus 4} \to \mathcal{K}_3$. This homomorphism is injective and the cokernel is invertible because it is torsion free (it is done in the proof of Theorem 2.5 in [4] by embedding it in a locally free sheaf). Therefore we have the following relation between $\mathcal{K}_2$ and $\mathcal{K}_3$,

$$0 \to \mathcal{K}_2^{\oplus 4} \to \mathcal{K}_3 \to \mathcal{O}(a) \to 0.$$

Godeaux surfaces conjecturally depend on 8-dimensional moduli, and they possess a genus four pencil in the case where the bicanonical system has no fixed part, and no singular base points. Thus we get an 8-dimensional family of rational curves in a space of dimension 9 (Deligne-Mumford compactification of curves of genus 4, $\overline{M_4}$) containing a subvariety H of codimension 2 (the hyperelliptic locus). The natural question is whether the general such curve, coming from Godeaux surfaces, does not intersect the hyperelliptic locus. In the reference [4] it was shown how this question is related to the still open problem of classifying the Godeaux surfaces with 0-torsion. Another question is whether this 8-dimensional family sweeps out whole moduli space. The object of this paper is to find the degree of $\mathcal{K}_2^*$ and the degree of a, i.e. to find the number of hyperelliptic curves in the bicanonical pencil of Godeaux surfaces for the special case of Godeaux surfaces with torsion of order 3, 4, 5. If $\mathrm{T}(X) = \mathbb{Z}_5$, or $\mathbb{Z}_4$, or $\mathbb{Z}_3$, then the irreducibility of moduli space of Godeaux surfaces is known [13].

The paper concerns the following theorem and its application to deformations:

Theorem. *Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_5$, or $\mathbb{Z}_4$, or $\mathbb{Z}_3$. Then we have the followings*:

1. *no hyperelliptic curve in the bicanonical pencil if $\mathrm{T}(X) = \mathbb{Z}_5$,*

2. *no hyperelliptic curve in the bicanonical pencil if* $\mathrm{T}(X) = \mathbb{Z}_4$,
3. *one hyperelliptic curve in the bicanonical pencil if* $\mathrm{T}(X) = \mathbb{Z}_3$,
4. 5-*dimensional family of curves in* $\overline{M_4}$ *associated with* $|2K_X|$ *if* $\mathrm{T}(X) = \mathbb{Z}_5$,
5. 6-*dimensional family of curves in* $\overline{M_4}$ *associated with* $|2K_X|$ *if* $\mathrm{T}(X) = \mathbb{Z}_4$,
6. 7-*dimensional family of curves in* $\overline{M_4}$ *associated with* $|2K_X|$ *if* $\mathrm{T}(X) = \mathbb{Z}_3$.

Two examples of Godeaux surfaces with $\mathrm{T}(X) = 0$ (Barlow surface [1] and Craighero-Gattazzo-Dolgachev-Werner surface [5]) are known. For these surfaces, we have $\deg \mathcal{K}_2^* = 3$, $a = 0$ [4], [11]. It is an interesting question if we have $\deg \mathcal{K}_2^* = 3$, $a = 0$ for Godeaux surfaces with $\mathrm{T}(X) = 0$. And we ask whether a general curve in $\overline{M_4}$ is associated with the bicanonical pencil of Godeaux surfaces with $\mathrm{T}(X) = 0$ or not.

We work throughout over the complex number field $\mathbb{C}$.

1. Proof of Theorem

Theorem. *Let X be a general Godeaux surface with* $\mathrm{T}(X) = \mathbb{Z}_5$, *or* $\mathbb{Z}_4$, *or* $\mathbb{Z}_3$. *Then we have the followings*:
1. *no hyperelliptic curve in the bicanonical pencil if* $\mathrm{T}(X) = \mathbb{Z}_5$,
2. *no hyperelliptic curve in the bicanonical pencil if* $\mathrm{T}(X) = \mathbb{Z}_4$,
3. *one hyperelliptic curve in the bicanonical pencil if* $\mathrm{T}(X) = \mathbb{Z}_3$,
4. 5-*dimensional family of curves in* $\overline{M_4}$ *associated with* $|2K_X|$ *if* $\mathrm{T}(X) = \mathbb{Z}_5$,
5. 6-*dimensional family of curves in* $\overline{M_4}$ *associated with* $|2K_X|$ *if* $\mathrm{T}(X) = \mathbb{Z}_4$,
6. 7-*dimensional family of curves in* $\overline{M_4}$ *associated with* $|2K_X|$ *if* $\mathrm{T}(X) = \mathbb{Z}_3$.

Proof.

CASE 1. $\mathrm{T}(X) = \mathbb{Z}_5$. Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_5$. Then each curve, in the bicanonical pencil of X, is a stable curve of genus four [11]. Let λ, δ_0, δ_1, δ_2 be the standard generators of $\mathrm{Pic}(\overline{M_4})$ and let $\mathbb{P}^1 = \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}(2K_X)))$. If one expresses Θ_{null} with λ, δ_i in $\overline{M_4}$, then

$$\Theta_{\mathrm{null}} = 34\lambda - 4\delta_0 - 14\delta_1 - 18\delta_2 \quad [14].$$

Therefore $\Theta_{\mathrm{null}}.\mathbb{P}^1 = 0$ by the numerical data $\lambda.\mathbb{P}^1 = 4$, $\delta_0.\mathbb{P}^1 = 25$, $\delta_1.\mathbb{P}^1 = 0$ and $\delta_2.\mathbb{P}^1 = 2$ [11]. It implies that there is no hyperelliptic curve in the bicanonical pencil. So the support of the torsion sheaf consists of two points associated with $\delta_2.\mathbb{P}^1 = 2$. Let $p\colon S \to X$ be the blowing-up of X at the base points of $|2K_X|$ and at two base points of $|3K_X|$. And let E_1, E_2 be the exceptional divisors of the base points of $|3K_X|$. The quadric sequence and the cubic sequence are the following:

$$0 \to \mathcal{K}_2 \to S^2\big(f_*(p^*3K_X - E_1 - E_2)\big) \to f_*(p^*6K_X) \to \mathcal{T}_2 \to 0$$
$$0 \to \mathcal{K}_3 \to S^3\big(f_*(p^*3K_X - E_1 - E_2)\big) \to f_*(p^*9K_X) \to \mathcal{T}_3 \to 0.$$

The map $S^2(f_*(p^*3K_X - E_1 - E_2)) \to f_*(p^*6K_X)$ is factorized through $S^2(f_*(p^*3K_X - E_1 - E_2)) \to f_*(p^*6K_X - 2E_1 - 2E_2) \to f_*(p^*6K_X)$. It implies that the length $\mathcal{T}_2 = 3+3 = 6$;
Since $|3K_X|$ has two simple base points $p_1,\ p_2$, the natural map $H^0(S, \mathcal{O}_S(p^*3K_X - E_i)) \to H^0(E_i, \mathcal{O}_{E_i}(-E_i))$ is surjective. On the other hand, since $|6K_X|$ is very ample, the natural map $H^0(S, \mathcal{O}_S(p^*6K_X)) \to H^0(S, \mathcal{O}_S(p^*6K_X)/\mathcal{O}_S(p^*6K_X - 2E_i))$ is surjective with three-dimensional image.

By the similar computation, length $\mathcal{T}_3 = 6+6 = 12$. It implies that $\deg \mathcal{K}_2^* = 1$ and $a = -6$. In [10], [11], there is an explicit description of five dimensional family of curves in $\overline{M_4}$ associated with the bicanonical pencil of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_5$. Let (x, y, z, w) be a coordinate of $\mathbb{P}^3$. Then quadrics associated with a bicanonical pencil lie in $\mathbb{P}^5$ with coordinate $xy,\ xz,\ xw,\ yz,\ yw,\ zw$. Since $\deg \mathcal{K}_2^* = 1$, we have 4 singular quadrics associated with the bicanonical pencil. By $\Theta_{\mathrm{null}}.\mathbb{P}^1 = 0$, each special member $(K_X + \tau_1 \cup_{q_1} K_X + \tau_4,\ K_X + \tau_2 \cup_{q_2} K_X + \tau_3)$ is counted as two.

CASE 2. $\mathrm{T}(X) = \mathbb{Z}_4$. Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_4$. Then there are two special members related with the elements in $\mathrm{T}(X)$ i.e. $K_X + \tau_1 \cup_q K_X + \tau_3$ (q is the base point of $|3K_X|$) and $2(K_X + \tau_2)$ where τ_i is a nonzero torsion element in $\mathrm{T}(X)$. The image of $K_X + \tau_2$ by $|3K_X|$ is a double line in $\mathbb{P}^3 = \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(3K_X)))$. Let (x, y, z, w) be a coordinate of $\mathbb{P}^3$. Consider the quadric linear system in $\mathbb{P}^3$:

$$\mathbb{P}\big(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(2))\big) = \mathbb{P}^9.$$

Then each quadric associated with a bicanonical pencil contains the line $(x = w = 0)$ because it intersects this line with at least four points. Therefore we have a morphism

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^6 \subset \mathbb{P}^9,$$

where $\mathbb{P}^6$ is given by $x^2,\ xy,\ xz,\ xw,\ yw,\ zw,\ w^2$. Let $p\colon S \to X$ be the blowing-up of X at the base points of $|2K_X|$ and at the base point of $|3K_X|$. And let E be the exceptional divisor of the base point of $|3K_X|$. The quadric sequence and the cubic sequence are the following:

$$0 \to \mathcal{K}_2 \to S^2\big(f_*(p^*3K_X - E)\big) \to f_*(p^*6K_X) \to \mathcal{T}_2 \to 0$$
$$0 \to \mathcal{K}_3 \to S^3\big(f_*(p^*3K_X - E)\big) \to f_*(p^*9K_X) \to \mathcal{T}_3 \to 0.$$

There is a two dimensional family of quadrics containing the double line (two times the line $x = w = 0$) and the map $S^2(f_*(p^*3K_X - E)) \to f_*(p^*6K_X)$ is factorized through $S^2(f_*(p^*3K_X - E)) \to f_*(p^*6K_X - 2E) \to f_*(p^*6K_X)$. It implies that the length $\mathcal{T}_2 \geq 2+3 = 5$. By the similar computation, length $\mathcal{T}_3 \geq 5+6 = 11$. It implies that $\deg \mathcal{K}_2^* \leq 2$ and that the possibility of number of hyperelliptic curve is zero or one. If there is one hyperelliptic curve then $\deg \mathcal{K}_2^* \leq 0$. But then there is no singular quadric associated with the bicanonical pencil. It gives a contradiction because the number of singular quadrics for each special member ($K_X+\tau_1 \cup_q K_X+\tau_3$, $2(K_X+\tau_2)$, hyperelliptic curve) is counted at least as two. Therefore we have $\deg \mathcal{K}_2^* = 2$ and $a = -3$. It also implies that we have a six dimensional family of curves in $\overline{M_4}$ associated with the bicanonical pencil of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_4$. There is another proof in [8].

CASE 3. $\mathrm{T}(X) = \mathbb{Z}_3$. Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_3$. Then there is one special member related with the elements in $\mathrm{T}(X)$ i.e. $K_X + \tau_1 \cup_q K_X + \tau_2$ (q is the base point of $|3K_X|$) where τ_i is a nonzero torsion element in $\mathrm{T}(X)$.

First we will show that there is a hyperelliptic curve in the bicanonical pencil. Reid [13] gave an explicit description of the construction of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_3$ by using the canonical ring of unbranched cover. Let Y be the unbranched $\mathbb{Z}_3$-cover of X, and let

$$\overline{F} = \operatorname{Proj} R = \mathbb{P}(1,1,2,2,2)$$

where $R = k[x_1, x_2, y_0, y_1, y_2]$, $x_i \in \mathrm{H}^0(X, \mathcal{O}_X(K_X + \tau_i))$ for $i = 1, 2$ and $y_i \in \mathrm{H}^0(X, \mathcal{O}_X(2K_X + \tau_i))$ for $i = 0, 1, 2$. Then $\overline{F}$ is embedded into $\mathbb{P}^5$ with coordinate $\{u_0, u_1, u_2, u_3, u_4, u_5\}$ by x_1^2, x_2^2, x_1x_2, y_0, y_1, y_2, and it is the cone on the Vernose surface. The natural desingulariszation of $\overline{F}$ is a rational scroll

$$F = \mathbb{P}\big(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)\big).$$

Let $\pi : F \to \mathbb{P}^1$ be a natural projection. Let $\mathcal{O}_F(A)$, $\mathcal{O}_F(B)$ be $\pi^*\mathcal{O}(1)$, the tautological line bundle of F respectively. Choose two irreducible divisors Q and C in F with $Q \in |6A + 2B|$, $C \in |6A + 3B|$. Then $Q \cap C = \tilde{Y} + \sum_{i=1}^{3} Q_i$ where $\tilde{Y}$ is the blowing-up of Y and Q_i is a fiber of the map $Q \to \mathbb{P}^1$. The detailed construction is in [13]. Lift the bicanonical pencil of X to Y then the pencil is given by the equation

$$x_1x_2 + y_0 = 0.$$

The proper transform of the intersection $\overline{F}$ and $y_0 = 0$ (i.e. $u_3 = 0$) in $\mathbb{P}^5$ under the map $F \to \overline{F}$ is the rational scroll $\mathbb{P}^2$ over $\mathbb{P}^1$. So the special member ($y_0 = 0$) associated with the pencil of Y, $x_1x_2 + y_0 = 0$, is the curve that has a fibration over $\mathbb{P}^1$ with six points in each fiber. Then there is a natural induced $\mathbb{Z}_3$-action on each fiber of unbranched cover $Y \to X$. So the image of this curve in X is a hyperelliptic curve and it is in the bicanonical pencil of X.

Let p be the blowing-up at the base points of $|2K_X|$ and the base point of $|3K_X|$. And let E be the exceptional divisor of the base point of $|3K_X|$. The quadric sequence and the cubic sequence are the following:

$$0 \to \mathcal{K}_2 \to S^2\big(f_*(p^*3K_X - E)\big) \to f_*(p^*6K_X) \to \mathcal{T}_2 \to 0$$
$$0 \to \mathcal{K}_3 \to S^3\big(f_*(p^*3K_X - E)\big) \to f_*(p^*9K_X) \to \mathcal{T}_3 \to 0.$$

Since there is at least one hyperelliptic curve in the bicanonical pencil and the map $S^2(f_*(p^*3K_X - E)) \to f_*(p^*6K_X)$ is factorized through $S^2(f_*(p^*3K_X - E)) \to f_*(p^*6K_X - 2E) \to f_*(p^*6K_X)$, length $\mathcal{T}_2 \geq 2 + 3 = 5$. By the similar computation, length $\mathcal{T}_3 \geq 5 + 6 = 11$. It implies that $\deg \mathcal{K}_2^* \leq 2$ and that the possibility of number of hyperelliptic curve is one or two. If there are two hyperelliptic curves then $\deg \mathcal{K}_2^* \leq 0$. But then there is no singular quadric associated with the bicanonical pencil. It gives a contradiction because the number of singular quadrics for each special member ($K_X+\tau_1 \cup_q K_X+\tau_2$, two hyperelliptic curves) is counted at least as two. Therefore we have $\deg \mathcal{K}_2^* = 2$ and $a = -3$.

Let (x, y, z, w) be a coordinate of $\mathbb{P}^3$ and let the quadric equation of the special member be $xw = 0$. Consider the quadric linear system in $\mathbb{P}^3$:

$$\mathbb{P}\big(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(2))\big) = \mathbb{P}^9.$$

Then each quadric associated with a bicanonical pencil contains two points (the image of the base points of $|2K_X|$ under $|3K_X|$). Therefore we have a morphism

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^7 \subset \mathbb{P}^9,$$

where $\mathbb{P}^7$ is given by x^2, xy, xz, xw, yz, yw, zw, w^2. It implies that we have at most seven dimensional family of curves in $\overline{M_4}$ associated with the bicanonical pencil of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_3$. Consider the explicit construction of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_3$ as above. Then it is easy to check that there is a six dimensional family of curves associated to the special member ($xw = 0$) because its preimage in Y is a union of two curves of genus four intersecting at three points with a free $\mathbb{Z}_3$-action. Therefore we have a seven dimensional family of curves associated with the bicanonical pencil of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_3$. $\square$

Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_3$. Assume that each curve, in the bicanonical pencil of X, is a stable curve of genus four. Let λ, δ_0, δ_1, δ_2 be the standard generators of $\mathrm{Pic}(\overline{M_4})$ and let $\mathbb{P}^1 = \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(2K_X)))$. The number $\Theta_{\mathrm{null}}.\mathbb{P}^1 = 14$ is obtained by the numerical data $\lambda.\mathbb{P}^1 = 4$, $\delta_0.\mathbb{P}^1 = 26$, $\delta_1.\mathbb{P}^1 = 0$ and $\delta_2.\mathbb{P}^1 = 1$ [11]. Since $\deg \mathcal{K}_2^* = 2$, we have 8 singular quadrics associated with the bicanonical pencil. The number 8 can be decomposed as 2 (from the special member $K_X + \tau_1 \cup_q K_X + \tau_2$) + 2 (from the hyperelliptic curve) + 4 (from other singular quadrics).

2. Its application to deformations

Let X be a Godeaux surface and let C be a general member in $|2K_X|$. Consider the following commutative diagram of exact sequences of deformations and obstructions;

$$\begin{array}{ccccc}
\mathrm{H}^0(C, \mathcal{O}_C(C)) & \longrightarrow & \mathrm{T}^1_{X,C} & \longrightarrow & \mathrm{T}^1_X \\
\big\downarrow{\scriptstyle \mathrm{ob}} & & \big\downarrow{\scriptstyle \mathrm{ob}} & & \big\downarrow{\scriptstyle \mathrm{ob}} \\
\mathrm{H}^1(C, \mathcal{O}_C(C)) & \longrightarrow & \mathrm{T}^2_{X,C} & \longrightarrow & \mathrm{T}^2_X.
\end{array}$$

Since $\mathrm{H}^1(C, \mathcal{O}_C(C)) = 0$, the forgetful functor $\mathrm{T}^1_{X,C} \to \mathrm{T}^1_X$ is smooth with one dimensional fiber. Let q be a natural map from $\mathbb{P}^1$ to $\overline{M_4}$ induced by the bicanonical pencil of X. The first order deformation of pairs $\mathrm{T}^1_{X,C}$ can be obtained by a computation of $q^*T_{\overline{M_4}}$. According to the deformation of morphisms of curves [9],

$$\dim \mathrm{Hom}_q(\mathbb{P}^1, \overline{M_4}) \geq -K_{\overline{M_4}}.\mathbb{P}^1 + 9\chi(\mathcal{O}_{\mathbb{P}^1}).$$

Since $\delta_1.\mathbb{P}^1 = 0$, we have the following intersection number:

$$K_{\overline{M_4}}.\mathbb{P}^1 = (13\lambda - 2\delta_0 - 2\delta_2).\mathbb{P}^1 = -2 \quad [7].$$

The above computation gives $\dim \mathrm{Hom}_q(\mathbb{P}^1, \overline{M_4}) \geq 11$.

$$q^*T_{\overline{M_4}} = \mathcal{O}(2) + q^*N_{\mathbb{P}^1|\overline{M_4}},$$

and by the classical lemma of Grothendieck $q^*N_{\mathbb{P}^1|\overline{M_4}} = \sum_{i=1}^8 \mathcal{O}(a_i)$. Then $K_{\overline{M_4}}.\mathbb{P}^1 = -2$ implies that $a_1 + \cdots + a_8 = 0$.

If $\mathrm{T}(X) = \mathbb{Z}_5$, or $\mathbb{Z}_4$, or $\mathbb{Z}_3$, then the unobstructedness and the irreducibility of moduli space of Godeaux surfaces are known [13]. It implies that $a_i \geq -1$ for all $i = 1, \ldots, 8$ and $\dim \mathrm{Hom}_q(\mathbb{P}^1, \overline{M_4}) = 11$.

Corollary. *We have the followings*:

1. $q^*N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(-1)^{\oplus 4}$ *if* $\mathrm{T}(X) = \mathbb{Z}_5$,
2. $q^*N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}(1)^{\oplus 3} + \mathcal{O}(-1)^{\oplus 3} + \mathcal{O}^{\oplus 2}$ *if* $\mathrm{T}(X) = \mathbb{Z}_4$,
3. $q^*N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}(1)^{\oplus 2} + \mathcal{O}(-1)^{\oplus 2} + \mathcal{O}^{\oplus 4}$ *if* $\mathrm{T}(X) = \mathbb{Z}_3$.

Proof.

CASE 1. $\mathrm{T}(X) = \mathbb{Z}_5$. Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_5$. If we fix an element $[C] \in \overline{M_4}$ associated with the bicanonical pencil, then there is a four dimensional family of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_5$ that contain $[C]$, [11]. By this argument we may assume that a_1, a_2, a_3, a_4 are at least 1 [9], and that $a_j < 0$ for $j = 5, 6, 7, 8$ by Theorem. So $a_5 = a_6 = a_7 = a_8 = -1$ and we obtain the conclusion $q^*N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(-1)^{\oplus 4}$ by the equality $a_1 + \cdots + a_8 = 0$.

CASE 2. $\mathrm{T}(X) = \mathbb{Z}_4$. Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_4$. If we fix an element $[C] \in \overline{M_4}$ associated with the bicanonical pencil, then there is a three dimensional family of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_4$ that contain $[C]$, [8]. By this argument, we may assume that a_1, a_2, a_3 are at least 1 and that $a_j \leq 0$ for $j = 4, 5, 6, 7, 8$, [9]. By Theorem and by the unobstructedness of moduli space, we may assume that $a_4 = a_5 = 0$ and that $a_6 = a_7 = a_8 = -1$. Therefore we obtain the conclusion $q^* N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}(1)^{\oplus 3} + \mathcal{O}(-1)^{\oplus 3} + \mathcal{O}^{\oplus 2}$ by the equality $a_1 + \cdots + a_8 = 0$.

CASE 3. $\mathrm{T}(X) = \mathbb{Z}_3$. Let X be a general Godeaux surface with $\mathrm{T}(X) = \mathbb{Z}_3$. If we fix an element $[C] \in \overline{M_4}$ associated with the bicanonical pencil, then there is a two dimensional family of Godeaux surfaces with $\mathrm{T}(X) = \mathbb{Z}_3$ that contain $[C]$. By this argument, we may assume that a_1, a_2 are at least 1 and that $a_j \leq 0$ for $j = 3, 4, 5, 6, 7, 8$, [9]. By Theorem and by the unobstructedness of moduli space, we may assume that $a_3 = a_4 = a_5 = a_6 = 0$ and that $a_7 = a_8 = -1$. It implies that $q^* N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}(1)^{\oplus 2} + \mathcal{O}(-1)^{\oplus 2} + \mathcal{O}^{\oplus 4}$ by the equality $a_1 + \cdots + a_8 = 0$. $\square$

For general two points $[C_1], [C_2] \in \mathbb{P}^1_X \subset \overline{M_4}$, if there is another pencil $\mathbb{P}^1_Y$, associated with the bicanonical pencil of Y which is in the same component of the moduli space of X, intersecting $\mathbb{P}^1_X$ with $[C_1]$, $[C_2]$, then there is $a_i \geq 2$ in $q^* N_{\mathbb{P}^1|\overline{M_4}}$. But this is not possible by Corollary if $\mathrm{T}(X) = \mathbb{Z}_5$, or $\mathbb{Z}_4$, or $\mathbb{Z}_3$. It is an interesting question if we have $q^* N_{\mathbb{P}^1|\overline{M_4}} = \mathcal{O}^{\oplus 8}$ for Godeaux surfaces with $\mathrm{T}(X) = 0$. If this is true, then there is no hyperelliptic curve in the bicanonical pencil of a general Godeaux surface with $\mathrm{T}(X) = 0$ because the hyperelliptic locus in $\overline{M_4}$ is codimension two.

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