

Title	Special members in the bicanonical pencil of Godeaux surfaces
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Citation	Osaka Journal of Mathematics. 42(1) P.163-P.171
Issue Date	2005-03
Text Version	publisher
URL	https://doi.org/10.18910/4464
DOI	10.18910/4464
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SPECIAL MEMBERS IN THE BICANONICAL PENCIL OF GODEAUX SURFACES

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(Received October 18, 2003)

Abstract

The object of this paper is to find the number of hyperelliptic curves in the bicanonical pencil of a Godeaux surface whose torsion group is \mathbb{Z}_3 , or \mathbb{Z}_4 , or \mathbb{Z}_5 .

Let X be a minimal smooth projective surface of general type with $p_g = q = 0$, $K^2 = 1$. Surfaces of this type are called *Godeaux surfaces* (also they are called *numerical Godeaux surfaces*); they were discovered in the 1930s by Campedelli [2] and Godeaux [6], and are interesting in view of Castelnuovo's criterion: an irrational surface with $q = 0$ must have $P_2 \geq 1$.

The following well known results are important tools in the paper:

- Bicanonical system $|2K_X|$ gives a pencil and the fixed part of $|2K_X|$ consists of -2 -curves [12];
- Tricanonical system $|3K_X|$ gives a birational map $X \rightarrow \mathbb{P}^3$ without fixed components [3], [12];
- Denote by $T(X)$ the torsion subgroup of $H^2(X, \mathbb{Z})$. Recall that $0, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ are the only possible values for the torsion group of a Godeaux surface [12], [13];
- A half of the number of elements in the set $\{\tau \in T(X) \mid \tau \neq -\tau\}$ equals the number of base points of $|3K_X|$ [12]; and
- For each non-zero element $\tau \in T(X)$, $h^0(K_X + \tau) = 1$. And $K_X + \tau_1$ intersects $K_X + \tau_2$ transversally if $\tau_1 \neq \tau_2$ [13].

In this paper, we assume that $|2K_X|$ has no fixed components and it has four simple base points. Then a general member $C \in |2K_X|$ is a non-hyperelliptic curve of genus four because $|3K_X|$ gives a birational map. Let $p: S \rightarrow X$ be the blowing-up of X at the base points of $|2K_X|$ (and at the base points of $|3K_X|$ if $T(X) = \mathbb{Z}_3, \mathbb{Z}_4$ or \mathbb{Z}_5) and let a fibration $f: S \rightarrow \mathbb{P}^1$ be given by the bicanonical pencil. According to the semi-positiveness and the Hirzebruch-Riemann-Roch theorem, we obtain the following [4], [11]:

- $f_*(p^*3K_X) = \mathcal{O}^{\oplus 4}$,
- $f_*(p^*6K_X) = \mathcal{O}^{\oplus 4} + \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(3)$,
- $f_*(p^*9K_X) = \mathcal{O}^{\oplus 4} + \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(2)^{\oplus 3} + \mathcal{O}(3)^{\oplus 4}$.

Consider the following quadric sequence and the cubic sequence

$$\begin{aligned} 0 \rightarrow \mathcal{K}_2 \rightarrow S^2(f_*p^*3K_X) \rightarrow f_*(p^*6K_X) \rightarrow \mathcal{T}_2 \rightarrow 0 \\ 0 \rightarrow \mathcal{K}_3 \rightarrow S^3(f_*p^*3K_X) \rightarrow f_*(p^*9K_X) \rightarrow \mathcal{T}_3 \rightarrow 0 \end{aligned}$$

where \mathcal{K}_i and \mathcal{T}_i are defined by the kernel sheaves and cokernel sheaves. If $|3K_X|$ has base points then consider the maps $S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X)$, $S^3(f_*(p^*3K_X - E)) \rightarrow f_*(p^*9K_X)$ instead of the maps $S^2(f_*p^*3K_X) \rightarrow f_*(p^*6K_X)$, $S^3(f_*p^*3K_X) \rightarrow f_*(p^*9K_X)$ respectively, where E is the exceptional divisor of the base points of $|3K_X|$.

Except the special members related with the elements in $T(X)$, non-hyperelliptic fibers are 3-connected. Therefore the support of the torsion sheaf consists of the points of special fibers related with $T(X)$ or the points of hyperelliptic fibers (cf. Lemma 2.1 in [4]). According to Lemma 2.1 and Proposition 2.3 in [4], each hyperelliptic curve adds the length of \mathcal{T}_2 by $2l$ and the length of \mathcal{T}_3 by $5l$ where l is the contact number of the bicanonical pencil with hyperelliptic locus. The natural surjective homomorphism $S^2(f_*p^*3K_X) \otimes f_*p^*3K_X \rightarrow S^3(f_*p^*3K_X)$ induces a homomorphism $\mathcal{K}_2^{\oplus 4} \rightarrow \mathcal{K}_3$. This homomorphism is injective and the cokernel is invertible because it is torsion free (it is done in the proof of Theorem 2.5 in [4] by embedding it in a locally free sheaf). Therefore we have the following relation between \mathcal{K}_2 and \mathcal{K}_3 ,

$$0 \rightarrow \mathcal{K}_2^{\oplus 4} \rightarrow \mathcal{K}_3 \rightarrow \mathcal{O}(a) \rightarrow 0.$$

Godeaux surfaces conjecturally depend on 8-dimensional moduli, and they possess a genus four pencil in the case where the bicanonical system has no fixed part, and no singular base points. Thus we get an 8-dimensional family of rational curves in a space of dimension 9 (Deligne-Mumford compactification of curves of genus 4, \overline{M}_4) containing a subvariety H of codimension 2 (the hyperelliptic locus). The natural question is whether the general such curve, coming from Godeaux surfaces, does not intersect the hyperelliptic locus. In the reference [4] it was shown how this question is related to the still open problem of classifying the Godeaux surfaces with 0-torsion. Another question is whether this 8-dimensional family sweeps out whole moduli space. The object of this paper is to find the degree of \mathcal{K}_2^* and the degree of a , i.e. to find the number of hyperelliptic curves in the bicanonical pencil of Godeaux surfaces for the special case of Godeaux surfaces with torsion of order 3, 4, 5. If $T(X) = \mathbb{Z}_5$, or \mathbb{Z}_4 , or \mathbb{Z}_3 , then the irreducibility of moduli space of Godeaux surfaces is known [13].

The paper concerns the following theorem and its application to deformations:

Theorem. *Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_5$, or \mathbb{Z}_4 , or \mathbb{Z}_3 . Then we have the followings:*

1. *no hyperelliptic curve in the bicanonical pencil if $T(X) = \mathbb{Z}_5$,*

2. *no hyperelliptic curve in the bicanonical pencil if $T(X) = \mathbb{Z}_4$,*
3. *one hyperelliptic curve in the bicanonical pencil if $T(X) = \mathbb{Z}_3$,*
4. *5-dimensional family of curves in \overline{M}_4 associated with $|2K_X|$ if $T(X) = \mathbb{Z}_5$,*
5. *6-dimensional family of curves in \overline{M}_4 associated with $|2K_X|$ if $T(X) = \mathbb{Z}_4$,*
6. *7-dimensional family of curves in \overline{M}_4 associated with $|2K_X|$ if $T(X) = \mathbb{Z}_3$.*

Two examples of Godeaux surfaces with $T(X) = 0$ (Barlow surface [1] and Craighero-Gattazzo-Dolgachev-Werner surface [5]) are known. For these surfaces, we have $\deg \mathcal{K}_2^* = 3$, $a = 0$ [4], [11]. It is an interesting question if we have $\deg \mathcal{K}_2^* = 3$, $a = 0$ for Godeaux surfaces with $T(X) = 0$. And we ask whether a general curve in \overline{M}_4 is associated with the bicanonical pencil of Godeaux surfaces with $T(X) = 0$ or not.

We work throughout over the complex number field \mathbb{C} .

1. Proof of Theorem

Theorem. *Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_5$, or \mathbb{Z}_4 , or \mathbb{Z}_3 .*

Then we have the followings:

1. *no hyperelliptic curve in the bicanonical pencil if $T(X) = \mathbb{Z}_5$,*
2. *no hyperelliptic curve in the bicanonical pencil if $T(X) = \mathbb{Z}_4$,*
3. *one hyperelliptic curve in the bicanonical pencil if $T(X) = \mathbb{Z}_3$,*
4. *5-dimensional family of curves in \overline{M}_4 associated with $|2K_X|$ if $T(X) = \mathbb{Z}_5$,*
5. *6-dimensional family of curves in \overline{M}_4 associated with $|2K_X|$ if $T(X) = \mathbb{Z}_4$,*
6. *7-dimensional family of curves in \overline{M}_4 associated with $|2K_X|$ if $T(X) = \mathbb{Z}_3$.*

Proof.

CASE 1. $T(X) = \mathbb{Z}_5$. Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_5$. Then each curve, in the bicanonical pencil of X , is a stable curve of genus four [11]. Let $\lambda, \delta_0, \delta_1, \delta_2$ be the standard generators of $\text{Pic}(\overline{M}_4)$ and let $\mathbb{P}^1 = \mathbb{P}(H^0(X, \mathcal{O}(2K_X)))$. If one expresses Θ_{null} with λ, δ_i in \overline{M}_4 , then

$$\Theta_{\text{null}} = 34\lambda - 4\delta_0 - 14\delta_1 - 18\delta_2 \quad [14].$$

Therefore $\Theta_{\text{null}, \mathbb{P}^1} = 0$ by the numerical data $\lambda, \mathbb{P}^1 = 4$, $\delta_0, \mathbb{P}^1 = 25$, $\delta_1, \mathbb{P}^1 = 0$ and $\delta_2, \mathbb{P}^1 = 2$ [11]. It implies that there is no hyperelliptic curve in the bicanonical pencil. So the support of the torsion sheaf consists of two points associated with $\delta_2, \mathbb{P}^1 = 2$. Let $p: S \rightarrow X$ be the blowing-up of X at the base points of $|2K_X|$ and at two base points of $|3K_X|$. And let E_1, E_2 be the exceptional divisors of the base points of $|3K_X|$. The quadric sequence and the cubic sequence are the following:

$$\begin{aligned} 0 \rightarrow \mathcal{K}_2 \rightarrow S^2(f_*(p^*3K_X - E_1 - E_2)) \rightarrow f_*(p^*6K_X) \rightarrow \mathcal{T}_2 \rightarrow 0 \\ 0 \rightarrow \mathcal{K}_3 \rightarrow S^3(f_*(p^*3K_X - E_1 - E_2)) \rightarrow f_*(p^*9K_X) \rightarrow \mathcal{T}_3 \rightarrow 0. \end{aligned}$$

The map $S^2(f_*(p^*3K_X - E_1 - E_2)) \rightarrow f_*(p^*6K_X)$ is factorized through $S^2(f_*(p^*3K_X - E_1 - E_2)) \rightarrow f_*(p^*6K_X - 2E_1 - 2E_2) \rightarrow f_*(p^*6K_X)$. It implies that the length $\mathcal{T}_2 = 3 + 3 = 6$;

Since $|3K_X|$ has two simple base points p_1, p_2 , the natural map $H^0(S, \mathcal{O}_S(p^*3K_X - E_i)) \rightarrow H^0(E_i, \mathcal{O}_{E_i}(-E_i))$ is surjective. On the other hand, since $|6K_X|$ is very ample, the natural map $H^0(S, \mathcal{O}_S(p^*6K_X)) \rightarrow H^0(S, \mathcal{O}_S(p^*6K_X)/\mathcal{O}_S(p^*6K_X - 2E_i))$ is surjective with three-dimensional image.

By the similar computation, length $\mathcal{T}_3 = 6 + 6 = 12$. It implies that $\deg \mathcal{K}_2^* = 1$ and $a = -6$. In [10], [11], there is an explicit description of five dimensional family of curves in \overline{M}_4 associated with the bicanonical pencil of Godeaux surfaces with $T(X) = \mathbb{Z}_5$. Let (x, y, z, w) be a coordinate of \mathbb{P}^3 . Then quadrics associated with a bicanonical pencil lie in \mathbb{P}^5 with coordinate xy, xz, xw, yz, yw, zw . Since $\deg \mathcal{K}_2^* = 1$, we have 4 singular quadrics associated with the bicanonical pencil. By $\Theta_{\text{null}} \cdot \mathbb{P}^1 = 0$, each special member $(K_X + \tau_1 \cup_{q_1} K_X + \tau_4, K_X + \tau_2 \cup_{q_2} K_X + \tau_3)$ is counted as two.

CASE 2. $T(X) = \mathbb{Z}_4$. Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_4$. Then there are two special members related with the elements in $T(X)$ i.e. $K_X + \tau_1 \cup_q K_X + \tau_3$ (q is the base point of $|3K_X|$) and $2(K_X + \tau_2)$ where τ_i is a nonzero torsion element in $T(X)$. The image of $K_X + \tau_2$ by $|3K_X|$ is a double line in $\mathbb{P}^3 = \mathbb{P}(H^0(X, \mathcal{O}_X(3K_X)))$. Let (x, y, z, w) be a coordinate of \mathbb{P}^3 . Consider the quadric linear system in \mathbb{P}^3 :

$$\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(2))) = \mathbb{P}^9.$$

Then each quadric associated with a bicanonical pencil contains the line $(x = w = 0)$ because it intersects this line with at least four points. Therefore we have a morphism

$$\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^6 \subset \mathbb{P}^9,$$

where \mathbb{P}^6 is given by $x^2, xy, xz, xw, yw, zw, w^2$. Let $p: S \rightarrow X$ be the blowing-up of X at the base points of $|2K_X|$ and at the base point of $|3K_X|$. And let E be the exceptional divisor of the base point of $|3K_X|$. The quadric sequence and the cubic sequence are the following:

$$\begin{aligned} 0 &\rightarrow \mathcal{K}_2 \rightarrow S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X) \rightarrow \mathcal{T}_2 \rightarrow 0 \\ 0 &\rightarrow \mathcal{K}_3 \rightarrow S^3(f_*(p^*3K_X - E)) \rightarrow f_*(p^*9K_X) \rightarrow \mathcal{T}_3 \rightarrow 0. \end{aligned}$$

There is a two dimensional family of quadrics containing the double line (two times the line $x = w = 0$) and the map $S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X)$ is factorized through $S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X - 2E) \rightarrow f_*(p^*6K_X)$. It implies that the length $\mathcal{T}_2 \geq 2 + 3 = 5$. By the similar computation, length $\mathcal{T}_3 \geq 5 + 6 = 11$. It implies that $\deg \mathcal{K}_2^* \leq 2$ and that the possibility of number of hyperelliptic curve is zero or one. If there is one hyperelliptic curve then $\deg \mathcal{K}_2^* \leq 0$. But then there is no singular quadric

associated with the bicanonical pencil. It gives a contradiction because the number of singular quadrics for each special member $(K_X + \tau_1 \cup_q K_X + \tau_3, 2(K_X + \tau_2))$, hyperelliptic curve) is counted at least as two. Therefore we have $\text{deg } \mathcal{K}_2^* = 2$ and $a = -3$. It also implies that we have a six dimensional family of curves in \overline{M}_4 associated with the bicanonical pencil of Godeaux surfaces with $T(X) = \mathbb{Z}_4$. There is another proof in [8].

CASE 3. $T(X) = \mathbb{Z}_3$. Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_3$. Then there is one special member related with the elements in $T(X)$ i.e. $K_X + \tau_1 \cup_q K_X + \tau_2$ (q is the base point of $|3K_X|$) where τ_i is a nonzero torsion element in $T(X)$.

First we will show that there is a hyperelliptic curve in the bicanonical pencil. Reid [13] gave an explicit description of the construction of Godeaux surfaces with $T(X) = \mathbb{Z}_3$ by using the canonical ring of unbranched cover. Let Y be the unbranched \mathbb{Z}_3 -cover of X , and let

$$\bar{F} = \text{Proj } R = \mathbb{P}(1, 1, 2, 2, 2)$$

where $R = k[x_1, x_2, y_0, y_1, y_2]$, $x_i \in H^0(X, \mathcal{O}_X(K_X + \tau_i))$ for $i = 1, 2$ and $y_i \in H^0(X, \mathcal{O}_X(2K_X + \tau_i))$ for $i = 0, 1, 2$. Then \bar{F} is embedded into \mathbb{P}^5 with coordinate $\{u_0, u_1, u_2, u_3, u_4, u_5\}$ by $x_1^2, x_2^2, x_1x_2, y_0, y_1, y_2$, and it is the cone on the Veronese surface. The natural desingularization of \bar{F} is a rational scroll

$$F = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)).$$

Let $\pi: F \rightarrow \mathbb{P}^1$ be a natural projection. Let $\mathcal{O}_F(A), \mathcal{O}_F(B)$ be $\pi^*\mathcal{O}(1)$, the tautological line bundle of F respectively. Choose two irreducible divisors Q and C in F with $Q \in |6A + 2B|, C \in |6A + 3B|$. Then $Q \cap C = \tilde{Y} + \sum_{i=1}^3 Q_i$ where \tilde{Y} is the blowing-up of Y and Q_i is a fiber of the map $Q \rightarrow \mathbb{P}^1$. The detailed construction is in [13]. Lift the bicanonical pencil of X to Y then the pencil is given by the equation

$$x_1x_2 + y_0 = 0.$$

The proper transform of the intersection \bar{F} and $y_0 = 0$ (i.e. $u_3 = 0$) in \mathbb{P}^5 under the map $F \rightarrow \bar{F}$ is the rational scroll \mathbb{P}^2 over \mathbb{P}^1 . So the special member $(y_0 = 0)$ associated with the pencil of $Y, x_1x_2 + y_0 = 0$, is the curve that has a fibration over \mathbb{P}^1 with six points in each fiber. Then there is a natural induced \mathbb{Z}_3 -action on each fiber of unbranched cover $Y \rightarrow X$. So the image of this curve in X is a hyperelliptic curve and it is in the bicanonical pencil of X .

Let p be the blowing-up at the base points of $|2K_X|$ and the base point of $|3K_X|$. And let E be the exceptional divisor of the base point of $|3K_X|$. The quadric sequence and the cubic sequence are the following:

$$\begin{aligned} 0 &\rightarrow \mathcal{K}_2 \rightarrow S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X) \rightarrow \mathcal{T}_2 \rightarrow 0 \\ 0 &\rightarrow \mathcal{K}_3 \rightarrow S^3(f_*(p^*3K_X - E)) \rightarrow f_*(p^*9K_X) \rightarrow \mathcal{T}_3 \rightarrow 0. \end{aligned}$$

Since there is at least one hyperelliptic curve in the bicanonical pencil and the map $S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X)$ is factorized through $S^2(f_*(p^*3K_X - E)) \rightarrow f_*(p^*6K_X - 2E) \rightarrow f_*(p^*6K_X)$, $\text{length } \mathcal{T}_2 \geq 2 + 3 = 5$. By the similar computation, $\text{length } \mathcal{T}_3 \geq 5 + 6 = 11$. It implies that $\text{deg } \mathcal{K}_2^* \leq 2$ and that the possibility of number of hyperelliptic curve is one or two. If there are two hyperelliptic curves then $\text{deg } \mathcal{K}_2^* \leq 0$. But then there is no singular quadric associated with the bicanonical pencil. It gives a contradiction because the number of singular quadrics for each special member ($K_X + \tau_1 \cup_q K_X + \tau_2$, two hyperelliptic curves) is counted at least as two. Therefore we have $\text{deg } \mathcal{K}_2^* = 2$ and $a = -3$.

Let (x, y, z, w) be a coordinate of \mathbb{P}^3 and let the quadric equation of the special member be $xw = 0$. Consider the quadric linear system in \mathbb{P}^3 :

$$\mathbb{P}(\mathbf{H}^0(\mathbb{P}^3, \mathcal{O}(2))) = \mathbb{P}^9.$$

Then each quadric associated with a bicanonical pencil contains two points (the image of the base points of $|2K_X|$ under $|3K_X|$). Therefore we have a morphism

$$\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^7 \subset \mathbb{P}^9,$$

where \mathbb{P}^7 is given by $x^2, xy, xz, xw, yz, yw, zw, w^2$. It implies that we have at most seven dimensional family of curves in \overline{M}_4 associated with the bicanonical pencil of Godeaux surfaces with $\mathbf{T}(X) = \mathbb{Z}_3$. Consider the explicit construction of Godeaux surfaces with $\mathbf{T}(X) = \mathbb{Z}_3$ as above. Then it is easy to check that there is a six dimensional family of curves associated to the special member ($xw = 0$) because its preimage in Y is a union of two curves of genus four intersecting at three points with a free \mathbb{Z}_3 -action. Therefore we have a seven dimensional family of curves associated with the bicanonical pencil of Godeaux surfaces with $\mathbf{T}(X) = \mathbb{Z}_3$. \square

Let X be a general Godeaux surface with $\mathbf{T}(X) = \mathbb{Z}_3$. Assume that each curve, in the bicanonical pencil of X , is a stable curve of genus four. Let $\lambda, \delta_0, \delta_1, \delta_2$ be the standard generators of $\text{Pic}(\overline{M}_4)$ and let $\mathbb{P}^1 = \mathbb{P}(\mathbf{H}^0(X, \mathcal{O}_X(2K_X)))$. The number $\Theta_{\text{null}, \mathbb{P}^1} = 14$ is obtained by the numerical data $\lambda \cdot \mathbb{P}^1 = 4, \delta_0 \cdot \mathbb{P}^1 = 26, \delta_1 \cdot \mathbb{P}^1 = 0$ and $\delta_2 \cdot \mathbb{P}^1 = 1$ [11]. Since $\text{deg } \mathcal{K}_2^* = 2$, we have 8 singular quadrics associated with the bicanonical pencil. The number 8 can be decomposed as 2 (from the special member $K_X + \tau_1 \cup_q K_X + \tau_2$) + 2 (from the hyperelliptic curve) + 4 (from other singular quadrics).

2. Its application to deformations

Let X be a Godeaux surface and let C be a general member in $|2K_X|$. Consider the following commutative diagram of exact sequences of deformations and obstruc-

tions;

$$\begin{array}{ccccc} H^0(C, \mathcal{O}_C(C)) & \longrightarrow & T_{X,C}^1 & \longrightarrow & T_X^1 \\ \downarrow \text{ob} & & \downarrow \text{ob} & & \downarrow \text{ob} \\ H^1(C, \mathcal{O}_C(C)) & \longrightarrow & T_{X,C}^2 & \longrightarrow & T_X^2. \end{array}$$

Since $H^1(C, \mathcal{O}_C(C)) = 0$, the forgetful functor $T_{X,C}^1 \rightarrow T_X^1$ is smooth with one dimensional fiber. Let q be a natural map from \mathbb{P}^1 to \overline{M}_4 induced by the bicanonical pencil of X . The first order deformation of pairs $T_{X,C}^1$ can be obtained by a computation of $q^*T_{\overline{M}_4}$. According to the deformation of morphisms of curves [9],

$$\dim \text{Hom}_q(\mathbb{P}^1, \overline{M}_4) \geq -K_{\overline{M}_4} \cdot \mathbb{P}^1 + 9\chi(\mathcal{O}_{\mathbb{P}^1}).$$

Since $\delta_1 \cdot \mathbb{P}^1 = 0$, we have the following intersection number:

$$K_{\overline{M}_4} \cdot \mathbb{P}^1 = (13\lambda - 2\delta_0 - 2\delta_2) \cdot \mathbb{P}^1 = -2 \quad [7].$$

The above computation gives $\dim \text{Hom}_q(\mathbb{P}^1, \overline{M}_4) \geq 11$.

$$q^*T_{\overline{M}_4} = \mathcal{O}(2) + q^*N_{\mathbb{P}^1|\overline{M}_4},$$

and by the classical lemma of Grothendieck $q^*N_{\mathbb{P}^1|\overline{M}_4} = \sum_{i=1}^8 \mathcal{O}(a_i)$. Then $K_{\overline{M}_4} \cdot \mathbb{P}^1 = -2$ implies that $a_1 + \dots + a_8 = 0$.

If $T(X) = \mathbb{Z}_5$, or \mathbb{Z}_4 , or \mathbb{Z}_3 , then the unobstructedness and the irreducibility of moduli space of Godeaux surfaces are known [13]. It implies that $a_i \geq -1$ for all $i = 1, \dots, 8$ and $\dim \text{Hom}_q(\mathbb{P}^1, \overline{M}_4) = 11$.

Corollary. *We have the followings:*

1. $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(-1)^{\oplus 4}$ if $T(X) = \mathbb{Z}_5$,
2. $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}(1)^{\oplus 3} + \mathcal{O}(-1)^{\oplus 3} + \mathcal{O}^{\oplus 2}$ if $T(X) = \mathbb{Z}_4$,
3. $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}(1)^{\oplus 2} + \mathcal{O}(-1)^{\oplus 2} + \mathcal{O}^{\oplus 4}$ if $T(X) = \mathbb{Z}_3$.

Proof.

CASE 1. $T(X) = \mathbb{Z}_5$. Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_5$. If we fix an element $[C] \in \overline{M}_4$ associated with the bicanonical pencil, then there is a four dimensional family of Godeaux surfaces with $T(X) = \mathbb{Z}_5$ that contain $[C]$, [11]. By this argument we may assume that a_1, a_2, a_3, a_4 are at least 1 [9], and that $a_j < 0$ for $j = 5, 6, 7, 8$ by Theorem. So $a_5 = a_6 = a_7 = a_8 = -1$ and we obtain the conclusion $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}(1)^{\oplus 4} + \mathcal{O}(-1)^{\oplus 4}$ by the equality $a_1 + \dots + a_8 = 0$.

CASE 2. $T(X) = \mathbb{Z}_4$. Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_4$. If we fix an element $[C] \in \overline{M}_4$ associated with the bicanonical pencil, then there is a

three dimensional family of Godeaux surfaces with $T(X) = \mathbb{Z}_4$ that contain $[C]$, [8]. By this argument, we may assume that a_1, a_2, a_3 are at least 1 and that $a_j \leq 0$ for $j = 4, 5, 6, 7, 8$, [9]. By Theorem and by the unobstructedness of moduli space, we may assume that $a_4 = a_5 = 0$ and that $a_6 = a_7 = a_8 = -1$. Therefore we obtain the conclusion $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}(1)^{\oplus 3} + \mathcal{O}(-1)^{\oplus 3} + \mathcal{O}^{\oplus 2}$ by the equality $a_1 + \dots + a_8 = 0$.

CASE 3. $T(X) = \mathbb{Z}_3$. Let X be a general Godeaux surface with $T(X) = \mathbb{Z}_3$. If we fix an element $[C] \in \overline{M}_4$ associated with the bicanonical pencil, then there is a two dimensional family of Godeaux surfaces with $T(X) = \mathbb{Z}_3$ that contain $[C]$. By this argument, we may assume that a_1, a_2 are at least 1 and that $a_j \leq 0$ for $j = 3, 4, 5, 6, 7, 8$, [9]. By Theorem and by the unobstructedness of moduli space, we may assume that $a_3 = a_4 = a_5 = a_6 = 0$ and that $a_7 = a_8 = -1$. It implies that $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}(1)^{\oplus 2} + \mathcal{O}(-1)^{\oplus 2} + \mathcal{O}^{\oplus 4}$ by the equality $a_1 + \dots + a_8 = 0$. \square

For general two points $[C_1], [C_2] \in \mathbb{P}_X^1 \subset \overline{M}_4$, if there is another pencil \mathbb{P}_Y^1 , associated with the bicanonical pencil of Y which is in the same component of the moduli space of X , intersecting \mathbb{P}_X^1 with $[C_1], [C_2]$, then there is $a_i \geq 2$ in $q^*N_{\mathbb{P}^1|\overline{M}_4}$. But this is not possible by Corollary if $T(X) = \mathbb{Z}_5$, or \mathbb{Z}_4 , or \mathbb{Z}_3 . It is an interesting question if we have $q^*N_{\mathbb{P}^1|\overline{M}_4} = \mathcal{O}^{\oplus 8}$ for Godeaux surfaces with $T(X) = 0$. If this is true, then there is no hyperelliptic curve in the bicanonical pencil of a general Godeaux surface with $T(X) = 0$ because the hyperelliptic locus in \overline{M}_4 is codimension two.

ACKNOWLEDGEMENTS. The author would like to thank the referee for several corrections.

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