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SOME SDES WITH DISTRIBUTIONAL DRIFT PART I: GENERAL CALCULUS

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Introduction

The aim of this paper is to study stochastic differential equations of the type

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b'(X_s)\, ds, \tag{0.1}$$

where b, σ are continuous functions such that $\sigma > 0$. In such a case the formal operator associated with X is given by $Lf = (\sigma^2/2)f'' + b'f'$.

Equation (0.1) will be considered as a martingale problem and sometimes in the weak (in law) sense.

Diffusions in a generalized sense were studied by several authors. First, we mention a classical book by N.I. Portenko ([21]) which, however, remains in the framework of semimartingales. The point of view adopted in this book is different from ours; its aim is to start from a Markov semigroup in order to construct solutions to some stochastic differential equations in a generalized sense. We choose to adopt a direct stochastic analysis perspective without using Markov properties. At this stage, comparing the two approaches appears to be a delicate challenge.

Relevant work in this area was done by H.J. Engelbert and W.M. Schmidt ([9]) who investigated solutions to stochastic differential equations with generalized drift remaining however in the class of semimartingales. More recently, H.J. Engelbert and J. Wolf ([10]) considered special cases of processes solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes. Those solutions are no longer semimartingales but Dirichlet processes. A special case of equation (0.1) with $\sigma = 1$ and continuous b was treated by P. Seignourel ([29]) without defining the stochastic analysis framework in relation with long time behaviour. This is the case of irregular medium; the case of b being a Brownian path appears also in the literature with the denomination “random medium”; for recent results we refer to [18, 19].

After finishing the paper, we found an interesting recent paper of R.F. Bass and Z-Q. Chen ([4]) which examines, from quite different techniques than ours, one-dimensional stochastic differential equations with Hölder continuous diffusion and with a drift being the derivative of a Hölder function. For that equation they establish strong existence and pathwise uniqueness.

The literature on Dirichlet processes in the framework of Dirichlet forms is huge and it is impossible to list it completely. We only want to mention some very useful monographies such as [14, 15]. The subject has shown a large development in infinite dimension starting from [2]. A later monography is [17]. Recently, the case of time-dependent Dirichlet forms has attracted a lot of interest, see [20, 30].

Our point of view of Dirichlet processes is pathwise, following [13, 5]. A (continuous) Dirichlet process is the sum of a local martingale M and a zero quadratic variation process A.

The paper is organized as follows. First we introduce the concept of a C^1-generalized solution to $Lf = \dot{l}$, where $\dot{l} \in C^0$, $f \in C^1$. Under the assumption that there exists $h \in C^1$ with $Lh = 0$, $h'(x) \neq 0$ for every x, we can show that $Lf = \dot{l}$ admits a solution for any $\dot{l} \in C^0$. $\mathcal{D}_L$ will be the subset of C^1-functions f such that $Lf = \dot{l}$ for some $\dot{l} \in C^0$. Significant examples arise when $b = \alpha\sigma^2/2 + \beta$, where $\alpha \in [0, 1]$ and β is a function of bounded variation. A particular situation arises when L is close to divergence type which means that

$$b = \frac{\sigma^2}{2} + \beta. \tag{0.2}$$

In Section 3, we present a martingale problem related to L. For it, we state an existence and uniqueness theorem, which involves a non-explosion condition. Moreover, we show that the occupation time measure always admits a density. If L is close to divergence type then it is possible to show that the martingale problem is equivalent to a stochastic differential equation in the weak sense (0.1); more precisely, the solution X to the martingale problem associated with L will solve

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A(b), \tag{0.3}$$

where $A\colon C^0(\mathbb{R}) \to \mathcal{C}$ is the unique extension of the map

$$l \mapsto \int_0^{\cdot} l'(X_s)\, ds$$

defined on $C^1(\mathbb{R})$; $\mathcal{C}$ denotes the metric space of continuous processes endowed with the ucp topology. The existence of such an extension is explained by the fact that the map $\mathcal{L}\colon \mathcal{D}_L \to C^0$, defined by

$$\mathcal{L}f(x) = \int_0^x Lf(y)\, dy$$

can be extended uniquely to $C^1(\mathbb{R})$.

In Section 3, we also prove that L is truely the infinitesimal generator associated with the solution of a martingale problem. Moreover, we treat a suitable Kolmogorov equation which allows to deduce that the law of X_t admits a density p_t, $t \geq 0$, and to examine some properties.

In part II, see [12], we examine the Lyons-Zheng structure of the process, Itô's formula under weak assumptions and a semimartingale characterization.

1. Notations and recalls

If I is a real open interval then $C(I)$ will be the F-type space (according to the notations of [7, Chapter 2]) of continuous functions on I endowed with the topology of uniform convergence on compacts. For $k \geq 0$, $C^k(I)$ will be a similar space equipped with the topology of uniform convergence of the first k derivatives. If $I = \mathbb{R}$ we will simply write C, C^k instead of $C(\mathbb{R})$, $C^k(\mathbb{R})$.

We also need to introduce the following subspaces of C^1:

$$\begin{aligned} C_0^1 &:= \{f \in C^1 : f(0) = 0\}, \\ C_{0,0}^1 &:= \{f \in C^1 : f(0) = f'(0) = 0\}. \end{aligned}$$

Furthermore, we will work with the following F-type spaces. L^2_{loc} denotes the space of all Borel functions which are square integrable when restricted to compact subsets. $W^{1,2}_{\text{loc}}$ is the space of all absolutely continuous functions f admitting a density $f' \in L^2_{\text{loc}}$. It is equipped with the distance which sums $|f(0)|$ and the distance of f' in L^2_{loc}. A subspace of $W^{1,2}_{\text{loc}}$ will be

$$W^{1,2}_{0,\text{loc}} := \{f \in W^{1,2}_{\text{loc}} : f(0) = 0\}.$$

Similarly, we can consider L^p_{loc} for $p \geq 1$. We denote the set of C^k real functions with compact support by C^k_c, $k \geq 0$. *const* will denote a generic positive constant.

T will be a fixed real number. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. All processes will be considered with index in $\mathbb{R}$. The F-type space of continuous processes equipped with the ucp topology is denoted by $\mathcal{C}$. We recall that a sequence of processes (H_n) in $\mathcal{C}$ converges ucp to H if, for every $T > 0$, $\sup_{t\in[0,T]} |(H_n - H)(t)|$ converges to zero in probability. Note that H belongs automatically to $\mathcal{C}$.

For convenience, we follow the framework of stochastic calculus introduced in [24] and continued in [25, 26, 27], [32, 33, 34] and [28]. Let $X = (X_t, t \in [0, T])$ be a continuous process and $Y = (Y_t, t \in [0, T])$ be a process with paths in L^1_{loc}. We recall in the sequel the most useful rules of calculus.

The forward integral and the covariation process are defined by the following limits in the ucp (uniform convergence in probability) sense whenever they exist

$$\int_0^t Y_s \, d^- X_s := \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon} \, ds \tag{1.1}$$

$$[X, Y]_t := \lim_{\varepsilon\to 0+} C_\varepsilon(X, Y)_t, \tag{1.2}$$

where

$$C_\varepsilon(X,Y)_t := \frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}-X_s)(Y_{s+\varepsilon}-Y_s)\,ds.$$

For $[X,X]$ we shortly write $[X]$. All stochastic integrals and covariation processes will be of course elements of $\mathcal{C}$. If $[X,Y]$, $[X,X]$, $[Y,Y]$ exist we say that (X,Y) *has all its mutual covariations.*

REMARK 1.1. $X_tY_t = X_0Y_0 + \int_0^t Y_s\,d^-X_s + \int_0^t X_s\,d^-Y_s + [X,Y]_t$ provided that two of the three integrals or covariations exist.

REMARK 1.2. a) If $[X,X]$ exists then it is always an increasing process and X is called a *finite quadratic variation process*. If $[X,X]\equiv 0$ then X is said to be a *zero quadratic variation process* (or a zero energy process).
b) Let X, Y be continuous processes such that (X,Y) has all its mutual covariations. Then $[X,Y]$ has bounded (total) variation. If $f,\ g\in C^1$ then

$$[f(X),g(Y)]_t = \int_0^t f'(X)g'(Y)\,d[X,Y].$$

c) If A is a zero quadratic variation process and X is a finite quadratic variation process then $[X,A]\equiv 0$.
d) A bounded variation process is a zero quadratic variation process.
e) (*Classical Itô formula*) If $f\in C^2$ then $\int_0^\cdot f'(X)\,d^-X$ exists and is equal to

$$f(X)-f(X_0)-\frac{1}{2}\int_0^\cdot f''(X)\,d[X].$$

f) If $f\in C^1$ and $g\in C^2$ then the forward integral $\int_0^\cdot f(X)\,d^-g(X)$ is well defined.

In this paper all filtrations are supposed to fulfill the usual conditions. If $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration, X an $\mathbb{F}$-semimartingale, Y is $\mathbb{F}$-adapted with the suitable square integrability conditions, then $\int_0^\cdot Y\,d^-X$ is the usual Itô integral. If Y is an $\mathbb{F}$-semimartingale then $\int_0^\cdot Y\,d^0X$ is the classical Fisk-Stratonovich integral and $[X,Y]$ the usual covariation process $\langle X,Y\rangle$.

An $\mathbb{F}$-*Dirichlet process* is the sum of an $\mathbb{F}$-local continuous martingale M and an $\mathbb{F}$-adapted zero quadratic variation process A, see [13, 5].

REMARK 1.3 ([28]). Let $X = M+A$ be a Dirichlet process. Remark 1.2 c) implies that $[X]=\langle M\rangle$. If $f\in C^1$ then $f(X)=M^f+A^f$ is a Dirichlet process, where

$$M^f = \int_0^\cdot f'(X_s)\,dM_s$$

and $A^f := f(X) - M^f$ has zero quadratic variation.

A sequence (τ^N) of (possibly infinite) $\mathbb{F}$-stopping times will be said to be "suitable" if

$$\bigcup_N \{\tau^N > T\}$$

has probability one. We will use the notation of stopped process as usually X^τ.

REMARK 1.4. Let X be a $\mathbb{F}$-adapted continuous process.

X is a semimartingale (resp. Dirichlet processes) if and only if the stopped processes X^{τ^N} are also semimartingales (resp. Dirichlet processes).

2. Definition of the operator L

Let σ, $b \in C^0(\mathbb{R})$ such that $\sigma > 0$. We consider formally a PDE operator of the following type:

$$Lg = \frac{\sigma^2}{2} g'' + b' g'. \tag{2.1}$$

By a mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)\,dx = 1$. We denote

$$\Phi_n(x) := n\Phi(nx), \quad \sigma_n^2 := \sigma^2 * \Phi_n, \quad b_n := b * \Phi_n.$$

We then consider

$$L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'. \tag{2.2}$$

A priori, σ_n^2, b_n and the operator L_n depend on the mollifier Φ.

DEFINITION. A function $f \in C^1(\mathbb{R})$ is said to be a *solution* to

$$Lf = \dot{l}, \tag{2.3}$$

where $\dot{l} \in C^0$, (*in the C^1-generalized sense*) if, for any mollifier Φ, there are sequences (f_n) in C^2, $(\dot{l}_n)$ in C^0 such that

$$L_n f_n = \dot{l}_n, \quad f_n \to f \text{ in } C^1, \quad \dot{l}_n \to \dot{l} \text{ in } C^0. \tag{2.4}$$

REMARK 2.1. The previous definition and notations can be adapted when $\mathbb{R}$ is replaced by a real interval $I =]a, b[$, $-\infty \le a < b \le +\infty$, σ, $b \in C^0(I)$ and (2.1) is defined on I. We extend σ, b by zero on I^c and, for $g \in C^2(I)$, we define

$$L_n g = \frac{(\sigma_n^2)_{|I}}{2} g'' + (b_n')_{|I} g'.$$

Then $f \in C^1(I)$ is a C^1-generalized solution to $Lf = \dot{l}$ if (2.3) and (2.4) hold when C^1 and C^0 are replaced by $C^1(I)$ and $C^0(I)$, respectively.

REMARK 2.2. Let I be as above. If $b' \in C^0(I)$ and $f \in C^2(I)$ is a classical solution to $Lf = \dot{l}$ then f is immediately seen to be a C^1-generalized solution.

We go on stating results for $I = \mathbb{R}$.

Proposition 2.3. *There is a solution $h \in C^1$ to $Lh = 0$ such that $h'(x) \neq 0$ for every $x \in \mathbb{R}$ if and only if*

$$\Sigma(x) := \lim_{n \to \infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy$$

exists in C^0, independently from the mollifier. Moreover, in this case, any solution f to $Lf = 0$ fulfills

$$f'(x) = e^{-\Sigma(x)} f'(0). \tag{2.5}$$

Proof. Let $h \in C^1$ be a solution to $Lh = 0$ with $h' \neq 0$ for every $x \in \mathbb{R}$. Then there are sequences $(\dot{a}_n)$ in C^0 and (h_n) in C^1 such that $\dot{a}_n \to 0$ in C^0, $h_n \to h$ in C^1 and $L_n h_n = \dot{a}_n$. Setting $g_n := h_n'$ we have $g_n'(\sigma_n^2/2) + g_n b_n' = \dot{a}_n$ and $g_n \to g = h'$ in C^0. Dividing by $(g_n \sigma_n^2)/2$, we get

$$(\log g_n)' + 2\frac{b_n'}{\sigma_n^2} = 2\frac{\dot{a}_n}{\sigma_n^2 g_n}. \tag{2.6}$$

Since $g = h' > 0$ and $g_n^{-1} \to g^{-1}$ in C^0, by integrating (2.6), Σ is well-defined and we have

$$\log g(x) = -\Sigma(x) + const. \tag{2.7}$$

This proves the direct sense of the implication; it also proves that h is of the type $h'(x) = h'(0)\exp(-\Sigma(x))$. The converse is clear choosing $\dot{a}_n \equiv 0$ and h_n fulfilling (2.6).

It remains to prove that any other solution to $Lf = 0$ fulfills (2.5). Let $f \in C^1$ be a solution and $x_0 \in \mathbb{R}$ with $f'(x_0) > 0$. By continuity, there is a neighbourhood I_0 of x_0 such that $f'(x) > 0$ holds for every $x \in I_0$. By the same reasoning as before, we easily verify

$$\log g(x) - \log g(x_0) = -\Sigma(x) + \Sigma(x_0)$$

for every $x \in I_0$. This establishes (2.5) on I_0.

Since f' is continuous, (2.7) holds for every x belonging to the closure of $J = \{x : f'(x) \neq 0\}$. This implies that

$$f'(x) = f'(0) \exp(-\Sigma(x)) \tag{2.8}$$

for every $x \in J$. At this point, we have two possibilities.

a) Either $f'(0) = 0$ so that $J = \emptyset$ holds according to (2.8). Thus, $f' \equiv 0$.

b) Or we have $f'(0) \neq 0$. Then J is non empty. Since J^c is open, ∂J is not empty except when $J = \mathbb{R}$. Let $a \in \partial J$. By continuity of Σ, we have

$$f'(a) = \lim_{x \to a} f'(x) = f'(0) \exp(-\Sigma(a)) \neq 0.$$

On the other hand, we observe

$$f'(a) = \lim_{x \to a, x \notin J} f'(x) = 0.$$

This contradiction implies $J = \mathbb{R}$. □

From now on, throughout the whole paper, we will suppose the existence of this function Σ. We will set

$$h'(x) := \exp(-\Sigma(x)), \quad h(0) = 0.$$

Thus, $h'(0) = 1$ holds.

REMARK 2.4. In particular, this proves the uniqueness of the problem

$$Lf = \dot{l}, \quad f \in C^1, \quad f(0) = x_0, \quad f'(0) = x_1 \tag{2.9}$$

for every $\dot{l} \in C^0$, x_0, $x_1 \in \mathbb{R}$.

REMARK 2.5. We present three examples.

a) If $b = \alpha(\sigma^2/2)$ for some $\alpha \in]0, 1]$ then

$$\Sigma(x) = \log(\sigma^{+2\alpha}(x))$$

and

$$h'(x) = \sigma^{-2\alpha}(x).$$

b) Suppose that b is of bounded variation. Then we get

$$\int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy = \int_0^x \frac{db_n(y)}{\sigma_n^2(y)} \to \int_0^x \frac{db}{\sigma^2},$$

since $db_n \to db$ weakly-$*$ and $1/\sigma^2$ is continuous.
c) If σ has bounded variation then we have

$$\Sigma(x) = -2\int_0^x b\,d\left(\frac{1}{\sigma^2}\right) + \frac{2b}{\sigma^2}(x) - \frac{2b}{\sigma^2}(0).$$

In particular, this example contains the case where $\sigma = 1$ for any b.

Lemma 2.6. *A solution to problem* (2.9) *is given by*

$$f(0) = x_0,$$
$$f'(x) = h'(x)\left(2\int_0^x \frac{\dot{l}(y)}{(\sigma^2 h')(y)}\,dy + x_1\right).$$

Proof. We define $f_n \in C^1$ such that

$$f_n(0) = x_0,$$
$$f_n'(x) = \left(2\int_0^x \frac{\dot{l}}{\sigma_n^2 h_n'}(y)\,dy + x_1\right) h_n'(x),$$

where

$$h_n(0) = 0 \quad \text{and} \quad h_n'(x) = \exp\left(-\int_0^x \frac{2b_n'}{\sigma_n^2}(y)\,dy\right).$$

Clearly, we have $L_n h_n \equiv 0$ and $h_n \to h$ in C^1. So, we observe

$$L_n f_n = \frac{\sigma_n^2}{2} f_n'' + b_n' f_n' = \dot{l}$$

and

$$f_n' \to \left(2\int_0^x \frac{\dot{l}}{\sigma^2 h'}(y)\,dy + x_1\right) h'(x)$$

in C^0. □

REMARK 2.7. Let $\dot{l} \in C^0$ and $x_0, x_1, c \in \mathbb{R}$. Then there is a unique solution in the C^1-generalized sense to

(2.10) $$Lu = \dot{l}$$

$$u(c) = x_0, \quad u'(c) = x_1.$$

The solution satisfies

$$u'(x) = h'_c(x)\left(2\int_c^x \frac{\dot{l}}{\sigma^2 h'_c}(y)\,dy + \frac{x_1}{h'_c(c)}\right),$$

where $h'_c(x) = \exp(\Sigma(c) - \Sigma(x))$.

In the case $c = 0$ this is a consequence of Lemma 2.6 and Remark 2.4. In the general case the justification is analogous.

REMARK 2.8. Let $f \in C^1(I)$. There is at most one $\dot{l} \in C^0(I)$ such that $Lf = \dot{l}$. In fact, to see this, it is enough to suppose that $f = 0$. Lemma 2.6 implies that

$$2\int_0^x \frac{\dot{l}}{\sigma^2 h'}(y)\,dy \equiv 0$$

consequently $\dot{l}$ is forced to be zero.

We will denote by $\mathcal{D}_L$ (resp. $\mathcal{D}_L(I)$) the set of all $f \in C^1(\mathbb{R})$ (resp. $C^1(I)$) such that there exists some $\dot{l} \in C^0$ with $Lf = \dot{l}$ in the C^1-generalized sense. This defines without ambiguity $L\colon \mathcal{D}_L$ (resp. $\mathcal{D}_L(I)) \to C^0$.

A direct consequence of Lemma 2.6 is the following useful result.

Lemma 2.9. *$\mathcal{D}_L(I)$ is the set of $f \in C^1(I)$ such there is $\psi \in C^1(I)$ with $f' = e^{-\Sigma}\psi$.*

In particular it gives us the following density proposition.

Proposition 2.10. *$\mathcal{D}_L$ is dense in C^1.*

Proof. It is enough to show that every C^2-function is the C^1-limit of a sequence of functions in $\mathcal{D}_L$. Let (ψ_n) be a sequence in C^1 converging to $f'e^{\Sigma}$ in C^0. It follows that

$$f_n(x) = f_n(0) + \int_0^x e^{-\Sigma}(y)\psi_n(y)\,dy, \quad x \in \mathbb{R}$$

converges to $f \in C^1$ and $f_n \in \mathcal{D}_L$. $\square$

Corollary 2.11. *$\mathcal{D}_L$ is dense in $W^{1,2}_{\mathrm{loc}}$.*

REMARK 2.12. Let us consider again the case of example a) with $b = \alpha(\sigma^2/2)$, $\alpha \in]0,1]$. Setting

$$f_n(x) = \int_0^x \sigma_n^{-2\alpha}(y)\,dy, \quad f(x) = \int_0^x \sigma^{-2\alpha}(y)\,dy,$$

we obtain $L_n f_n = 0$ and thus $Lf = 0$ in the generalized sense. Now Lf_n is a well defined distribution for each n. However, Lf_n does not converge to zero when $n \to \infty$, excepted for the case $\alpha = 1$ (divergence case).

This shows in particular that Lf cannot be defined using simply distributions theory.

We need now to discuss technical aspects of the way L and its domain $\mathcal{D}_L$ are transformed by h. We recall that $Lh = 0$ and h' is strictly positive so that we may denote the image of h by $I = \operatorname{Im} h =]a,b[$ and the inverse function by $h^{-1}\colon I \to \mathbb{R}$.

Let L^0 be the classical PDE operator

$$L^0\phi = \frac{\sigma_0^2}{2}\phi'', \tag{2.11}$$

where

$$\sigma_0(y) = \begin{cases} (\sigma h')(h^{-1}(y)) & : y \in I \\ 0 & : y \notin I. \end{cases}$$

L^0 is a classical PDE map; however we can also consider it at the formal level and introduce $\mathcal{D}_{L^0}$.

Proposition 2.13. a) $h^2 \in \mathcal{D}_L$, $Lh^2 = h'^2\sigma^2$,

b) $\mathcal{D}_{L^0}(I) = C^2(I)$,

c) *$\phi \in \mathcal{D}_{L^0}(I)$ holds if and only if $\phi \circ h \in \mathcal{D}_L$. Moreover, we have*

$$L(\phi \circ h) = (L^0\phi) \circ h \tag{2.12}$$

for every $\phi \in C^2(I)$.

Proof. a) can be easily justified by approximations. Here we only give the formal calculations:

$$\begin{aligned} Lh^2 &= \frac{\sigma^2}{2}(h^2)'' + b'(h^2)' \\ &= \sigma^2 h'' h + \sigma^2 h'^2 + 2h'b'h \\ &= 2hLh + \sigma^2 h'^2 \\ &= \sigma^2 h'^2. \end{aligned}$$

b) Since the operator L^0 has no drift, the corresponding function Σ vanishes; so the result follows immediately from Lemma 2.9.
c) Since $Lh = 0$ in the C^1 sense, we can choose a $C^2(\mathbb{R})$-sequence (h_n) such that, (h_n) converges in C^1 to h and $L_n h_n = \dot{a}_n \to 0$ in C^0 and $L_n g = (\sigma_n^2/2)g'' + b_n' g$.

If $f \in \mathcal{D}_L$ then there is a sequence (f_n) in C^2 converging in the C^1-sense to f such that $L_n f_n$ converges in C^0 to some $\dot{l}$ on C^0. We are going to prove $f \circ h^{-1} \in C^2(I)$. We evaluate

$$\begin{aligned}(f_n \circ h_n^{-1})'' &= f_n'' \circ h_n^{-1}(h_n^{-1})'^2 + f_n' \circ h_n^{-1}(h_n^{-1})'' \\ &= \frac{f_n''}{h_n'^2} \circ h_n^{-1} - \frac{f_n' h_n''}{(h_n')^3} \circ h_n^{-1} \\ &= \frac{2}{\sigma_n^2} \frac{L_n f_n}{h_n'^2} \circ h_n^{-1} - \frac{2\dot{a}_n}{\sigma_n^2} \frac{f_n'}{h_n'^3} \circ h_n^{-1}.\end{aligned}$$

Since $\dot{a}_n \to 0$ holds in C^0, the previous term converges in C^0 to

$$\frac{2\dot{l}}{\sigma^2 h'^2}(h^{-1}) = \frac{2Lf}{\sigma^2 h'^2} \circ h^{-1}.$$

Consequently, $(f_n \circ h_n^{-1})''$ is a Cauchy sequence in $C^0(I)$ and thus $f \circ h^{-1} \in C^2(I)$. Using b) we have proven the converse part of c). Moreover, recalling that $\sigma_0^2(y) = (\sigma^2 h'^2)(h^{-1}(y))$ for every $y \in I$, we have shown that

$$(f \circ h^{-1})'' = \frac{2}{\sigma_0^2}(Lf) \circ h^{-1}.$$

This entails

$$L^0(f \circ h^{-1}) = (Lf) \circ h^{-1}. \tag{2.13}$$

In order to prove the direct implication of c) we have to show that $f = \phi \circ h \in \mathcal{D}_L$ holds for $\phi \in C^2(I)$. But this is obvious because of Lemma 2.9 and the fact that $f' = e^{-\Sigma}\phi'(h)$.

This finishes the proof of b). □

We introduce now another operation which is obtained through integration of Lf. We define

$$\mathcal{L}\colon \mathcal{D}_{\mathcal{L}} \subset C^1 \to C^0$$

by

$$\mathcal{L}f := \lim_{n\to\infty} \int_0^{\cdot} L_n f_n(y)\, dy,$$

whenever this limit exists in C^0 for every mollifier Φ.

Lemma 2.14. *We have*

(i) $\mathcal{D}_L \cup C^2 \subseteq \mathcal{D}_{\mathcal{L}}$,

(ii) $\mathcal{L}f(x) = \int_0^x (Lf)(y)\,dy$ *for every* $f \in \mathcal{D}_L$,

(iii) $\mathcal{L}f(x) = \int_0^x \left(\sigma^2/2 - b\right) f''(y)\,dy + (bf')(x)$ *for every* $f \in C^2$.

Proof. The statement is clear for $f \in \mathcal{D}_L$.

Let $f \in C^2$ and (f_n) be a sequence in C^2 converging to f in C^2. Integrating by parts, we observe

$$\begin{aligned} &\int_0^{\cdot} L_n f_n(y)\,dy \\ &= \int_0^{\cdot} \left(\frac{\sigma_n^2}{2} f_n''\right)(y)\,dy + \int_0^{\cdot} (b_n' f_n')(y)\,dy \\ &= \int_0^{\cdot} \left(\frac{\sigma_n^2}{2} - b_n\right) f_n''(y)\,dy + b_n f_n' \end{aligned}$$

This converges to $\int_0^{\cdot} \left(\sigma^2/2 - b\right) f''(y)\,dy + bf'$, when n goes to ∞. $\square$

The next question concerns the closability of $\mathcal{L}$ into C^1 with values in C^0. This does not seem to be true in general. However, we are able to prove some closability of the operator with values in the space of locally bounded variation functions, as we will show in Part II, (see [12]).

So far, we have learnt how to eliminate the first order term in a PDE operator through a transformation which is called of Zvonkin type (see [35]). Now we would like to introduce a transformation which puts the PDE operator in a divergence form.

Let L be a PDE operator which is formally of type (2.1)

$$Lg = \frac{\sigma^2}{2} g'' + b'g'.$$

Of course always at a formal level, it can be written such that the second order part appears in a divergence form. This reads

$$L^d g = \left(\frac{\sigma^2}{2} g'\right)' + (b^d)'g', \tag{2.14}$$

where

$$b^d = b - \frac{\sigma^2}{2}. \tag{2.15}$$

Clearly, we can introduce the concept of a C^1-generalized solution for $L^d f = \dot{l}$ in a rigorous way. It is also clear that f is a C^1-generalized solution to $Lf = \dot{l}$ if and only if $L^d f = \dot{l}$.

Obviously, $\Sigma(x) = \lim_{n\to\infty} 2\int_0^x (b_n'/\sigma_n^2)(y)\,dy$ exists in C^0 if and only if $\Sigma^d(x) = \lim_{n\to\infty} 2\int_0^x ((b_n^d)'/\sigma_n^2)(y)\,dy$ exists. In that case we have

$$\Sigma^d = \Sigma(x) + \log(\sigma^{-2}(x)). \tag{2.16}$$

Thus, we actually may identify L and L^d and use the same notation L.

We consider a C^1-function $k\colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{L}k = 0$, $k'(x) \neq 0$ for every $x \in \mathbb{R}$ and $\tilde{L}g = (\sigma^2/2)g'' - (b^d)'g'$ in the C^1-generalized sense. Such a function exists since Σ^d exists. Clearly, we have $\tilde{\Sigma}(x) = -\Sigma(x) + \log \sigma^2(x)$. We can choose k such that

$$k(0) = 0 \quad \text{and} \quad k'(x) = \exp(-\tilde{\Sigma}(x)) = \sigma^{-2}(x)\exp(\Sigma(x)).$$

REMARK 2.15. If there is no drift term then we have $k'(x) = \sigma^{-2}(x)$.

Lemma 2.16. *Under the usual assumptions we choose $k \in C^1$ such that $k'(x) = \sigma^{-2}(x)\exp(\Sigma(x))$. We consider the formal PDE operator given by*

$$L^1 g = \left(\frac{\sigma_1^2}{2} g'\right)' \tag{2.17}$$

where

$$\sigma_1(z) = \begin{cases} (\sigma k')\circ k^{-1}(z) : & z \in J \\ 0 : & z \notin J \end{cases}$$

J being the image of k. Then

(i) *$g \in \mathcal{D}_{L^1(J)}$ if and only if $g \circ k \in \mathcal{D}_L$,*

(ii) *for every $g \in \mathcal{D}_{L^1(J)}$ we have $L^1 g = L(g\circ k)\circ k^{-1}$.*

Proof. Let σ_n^2, b_n be the usual regularizations of σ^2, b^2. We set

$$\begin{aligned} L_n f &:= \frac{\sigma_n^2}{2} f'' + b_n' f', \\ \tilde{L}_n f &:= \frac{\sigma_n^2}{2} f'' - (b_n^d)' f' \end{aligned}$$

for each $n \in \mathbb{N}$.

Let (k_n) be a sequence in C^1 such that $\tilde{L}_n k_n \to 0$ in C^0 and $k_n \to k$ in C^1. Let $g \in C^1(J)$ such that $g \circ k \in \mathcal{D}_L$. Let (g_n) be a sequence of functions in $C^1(J)$ converging to g and ensuring that the sequence $(\dot{\lambda}_n) \subset C^0(J)$, defined by

$$\dot{\lambda}_n \circ k_n^{-1} = \tilde{L}_n(g_n \circ k_n),$$

converges in $C^0(J)$ to some $\dot\lambda$.

We now calculate

$$\begin{aligned}\dot\lambda_n \circ k_n^{-1} &= \left(\frac{\sigma_n^2}{2}(g_n \circ k_n)'\right)' + (b_n^d)'(g_n \circ k)' \\ &= \left(\frac{\sigma_n^2}{2} g_n'(k_n)\frac{k_n'^2}{k_n'}\right)' + (b_n^d)'(g_n \circ k)' \\ &= \left(\frac{\sigma_{1,n}^2}{2}(k_n)\frac{g_n'(k_n)}{k_n'}\right)' + (b_n^d)' g_n'(k_n)k_n',\end{aligned}$$

where $\sigma_{1,n} = (\sigma_n k_n')(k_n^{-1})$. We continue to compute

$$\begin{aligned}\dot\lambda_n \circ k_n^{-1} &= \left(\frac{\sigma_{1,n}^2}{2} g_n'\right)'(k_n) - \frac{\sigma_{1,n}^2}{2}(k_n)\frac{k_n''}{k_n'^2} g_n'(k_n) + (b_n^d)' g_n'(k_n)k_n' \\ &= \left(\frac{\sigma_{1,n}^2}{2} g_n'\right)'(k_n) - g_n'(k_n)\left(\frac{\sigma_n^2}{2}k_n'' - (b_n^d)'k_n'\right) \\ &= \left(\frac{\sigma_{1,n}^2}{2} g_n'\right)'(k_n) - g_n'(k_n)\tilde L_n k_n.\end{aligned}$$

We have shown that

$$\dot\lambda \circ k^{-1} = \lim_{n\to\infty} \dot\lambda_n \circ k_n^{-1} = \lim_{n\to\infty}\left(\frac{\sigma_{1,n}^2}{2} g_n'\right)'(k_n)$$

in C^0, because $\tilde L k = 0$ holds in the generalized C^1-sense. Consequently, in $C^0(J)$ we have

$$\dot\lambda = \lim_{n\to\infty} \dot\lambda_n = \lim_{n\to\infty}\left(\frac{\sigma_{1,n}^2}{2} g_n'\right)'.$$

Setting $\dot\mu_n := ((\sigma_{1,n}^2/2)g_n')'$ and integrating, we get

$$g_n'(y) = \frac{2}{\sigma_{1,n}^2(y)}\left(\int_0^y \dot\mu_n(z)\,dz + g_n'(0)\sigma_{1,n}^2(0)\right).$$

Since $g_n \to g$ in $C^1(J)$, $\sigma_{1,n}^2 \to {\sigma_1}^2$ and $\dot\mu_n \to \dot\lambda$, we obtain

$$g'(y) = \frac{2}{{\sigma_1}^2(y)}\left(\int_0^y \lambda(z)\,dz + g'(0){\sigma_1}^2(0)\right).$$

Using Lemma 2.6 and the uniqueness of the C^1-generalized solution (Remark 2.4), we conclude

$$L^1 g = \lambda$$

and so $g \in \mathcal{D}_{L^1}(J)$. On the other hand, we have also proven

$$L(g \circ k) = \dot{\lambda} \circ k. \tag{2.18}$$

This establishes the converse implication of i). The direct one follows by symmetric analogous arguments.

Statement (ii) follows from (2.18). □

We make still some comments on the operator L in situations related to divergence form.

In general, we do not even know if $L\colon \mathcal{D}_L \subset C^1 \to C^0$ is closable. We consider

$$\mathcal{L}\colon \mathcal{D}_L \subset C^1 \to C_0^0 := \{f \in C^0(\mathbb{R}) : f(0) = 0\},$$

defined by

$$\mathcal{L}\colon f \mapsto \int_0^{\cdot} Lf(y)\,dy.$$

A priori, $\mathcal{L}$ is not closable in this context. Under some particular assumptions we know more.

Proposition 2.17. *Suppose that we are given*

$$Lf = \left(\frac{\sigma^2}{2} f'\right)' + \beta' f', \tag{2.19}$$

where β is a continuous function of bounded variation.

(i) *$\mathcal{L}$ admits a continuous extension from $\mathcal{D}_L$ to C^1, denoted by $\hat{\mathcal{L}}$.*

(ii) *Let $T\colon C_0^1 \to C_{0,0}^1 := \{f \in C^1 : f(0) = f'(0) = 0\}$ be defined by $Tl = f$, where $f \in C_{0,0}^1$ is the unique solution to $Lf = l'$. Then T admits a continuous extension to C_0^0 which we denote by $\hat{T}$.*

(iii) *The restriction of $\hat{\mathcal{L}}$ to $C_{0,0}^1$ is invertible on C_0^0 and $\hat{\mathcal{L}}^{-1} = \hat{T}$.*

(iv) *The operator $\mathcal{L}\colon \mathcal{D}_L \subset W_{\mathrm{loc}}^{1,2} \to L_{\mathrm{loc}}^2$ also admits a continuous extension $\tilde{\mathcal{L}}$ to the whole space $W_{\mathrm{loc}}^{1,2}$.*

(v) *The restriction of $\tilde{\mathcal{L}}$ to*

$$W_{0,\mathrm{loc}}^{1,2} := \{f \in W_{\mathrm{loc}}^{1,2} : f(0) = 0\}$$

is also invertible; $\tilde{T} = \tilde{\mathcal{L}}^{-1}$ extends $\hat{T}$.

REMARK 2.18. a) If L satisfies assumption (2.19) then we say that it *is close to the divergence type*.
b) $\hat{\mathcal{L}}$ coincides with the expression of $\mathcal{L}$ in C^2, see Lemma 2.14 (iii).
c) To avoid overcharge of notations, in the sequel we will denote the extension of $\mathcal{L}$ to $W^{1,2}_{\mathrm{loc}}$ also by $\hat{\mathcal{L}}$.

Proof of Proposition 2.17. i) We first evaluate $\mathcal{L}f$ for $f \in \mathcal{D}_L$. In that case, we consider a sequence (f_n) of C^2-functions converging to f in C^1 such that, with the usual notations, $L_n f_n = ((\sigma_n^2/2) f_n')' + \beta_n' f_n'$ converge to Lf in C^0. Then we have

$$\begin{aligned}\mathcal{L}f(x) &= \lim_{n\to\infty} \int_0^x L_n f_n(y)\,dy \\ &= \lim_{n\to\infty} \left(\frac{\sigma_n^2}{2} f_n'(x) + \int_0^x f_n'\,d\beta_n \right) \\ &= \frac{\sigma^2}{2} f'(x) + \int_0^x f'\,d\beta.\end{aligned}$$

This shows that the linear map $\mathcal{L}$ is continuous on $\mathcal{D}_L$ with respect to the topology of C^1. Therefore, $\mathcal{L}$ can be extended to C^1. Thus, we get

$$\hat{\mathcal{L}}f(x) = \frac{\sigma^2}{2} f'(x) + \int_0^x f'\,d\beta. \tag{2.20}$$

ii) If $l \in C_0^1$ and $f = Tl$, using Lemma 2.6, we can write

$$f'(x) = \exp(-\Sigma(x)) 2 \int_0^x \frac{l'(y)\exp(\Sigma(y))}{\sigma^2(y)}\,dy. \tag{2.21}$$

In particular, we have

$$\begin{aligned}\Sigma(x) &= \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy \\ &= \log\sigma^2(x) - \log\sigma^2(0) + 2\int_0^x \frac{d\beta}{\sigma^2}.\end{aligned}$$

Therefore, we get

$$h'(x) = \exp(-\Sigma(x)) = \frac{\sigma^2(0)}{\sigma^2(x)} \exp\left(-2\int_0^x \frac{d\beta}{\sigma^2}\right),$$

which solves in particular $Lh = 0$. Now (2.21) takes the form

$$f'(x) = \frac{1}{\sigma^2(x)} \exp\left(-2\int_0^x \frac{d\beta}{\sigma^2}\right) \int_0^x l'(y) \exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right) dy \tag{2.22}$$

$$f'(x) = \frac{1}{\sigma^2(x)}\Big(l(x) - \exp\Big(-2\int_0^x \frac{d\beta}{\sigma^2}\Big)$$

$$\cdot \int_0^x l(y)\exp\Big(2\int_0^y \frac{d\beta}{\sigma^2}\Big)\frac{1}{\sigma^2(y)}\,d\beta(y)\Big). \tag{2.23}$$

The right term of (2.23) is continuous with respect to $l \in C_0^0$. This allows to define immediately the extension $\hat{T}l$.

iii) By construction, we know

$$\mathcal{L}Tl = l$$

for every $l \in C_0^1$ and

$$T\mathcal{L}f = f$$

for every $f \in \mathcal{D}_L \cap C_{0,0}^0$. Furthermore, $\hat{\mathcal{L}}\hat{T}$ can be extended from C_0^1 to C_0^0 with values in C_0^0 and $\hat{T}\hat{\mathcal{L}}$ admits a continuous extension from $\mathcal{D}_L \cap C_{0,0}^1$ to $C_{0,0}^1$ with values in $C_{0,0}^1$. Therefore, we have $\hat{\mathcal{L}}\hat{T} = id$ on C_0^0 and $\hat{T}\hat{\mathcal{L}} = id$ on $C_{0,0}^1$. This establishes (iii).

iv) The expression (2.20) can be extended to $W_{\mathrm{loc}}^{1,2}$ because the right member of (2.20) admits a continuous extension to $W_{\mathrm{loc}}^{1,2}$.

v) The expression (2.23) can be extended to L_{loc}^2. We emphasize that C_0^0 is dense in L_{loc}^2. So, $C_{0,0}^1$ is dense in $W_{\mathrm{loc}}^{1,2}$. Thus, (2.23) defines $\tilde{T}\colon W_{0,\mathrm{loc}}^{1,2} \to L_{\mathrm{loc}}^2$. A similar reasoning as in iii) now completes the proof. ☐

Corollary 2.19. *In particular, if $f(x) = x$ then*

$$\hat{\mathcal{L}}f(x) = \int_0^x f'\,d\beta + \frac{\sigma^2}{2}f'(x) = \frac{\sigma^2(x)}{2} + \beta(x) = b(x).$$

We need now to solve the equation $Lu = u$ in the C^1-generalized sense.

Proposition 2.20. *Let $c \in \mathbb{R}$ and consider the solution to*

$$Lv = 1, \quad v(c) = 0, \quad v'(c) = 0. \tag{2.24}$$

- *There is a unique solution to the equation*

$$Lu = u, \quad u(c) = 1, \quad u'(c) = 0. \tag{2.25}$$

- *u is non-negative and strictly decreasing* (resp. *increasing*) *on* $]-\infty, c]$ (resp. $[c, +\infty[$).
-

$$1 + v(x) \le u(x) \le \exp(v(x)), \quad \forall x \in \mathbb{R} \tag{2.26}$$

Proof. Without loss of generality, we may suppose $c = 0$. According to Lemma 2.6, we can write

$$v'(x) = 2\exp(-\Sigma(x)) \int_0^x \frac{\exp(\Sigma(y))}{\sigma^2(y)}\, dy.$$

We set $u_0 \equiv 1$ and, for $n \in \mathbb{N}$, we define recursively

$$\begin{aligned} u_n'(x) &= 2\exp(-\Sigma(x)) \int_0^x \frac{\exp(\Sigma(y))}{\sigma^2(y)} u_{n-1}(y)\, dy \\ u_n(0) &= 0 \end{aligned} \tag{2.27}$$

which means $Lu_n = u_{n-1}$, $u_n(0) = u_n'(0) = 0$. The u_n are easily seen to be non-negative, strictly increasing on $\mathbb{R}_+$ and strictly decreasing on $\mathbb{R}_-$. We can show by induction that

$$u_n(x) \leq \frac{v^n(x)}{n!} \tag{2.28}$$

for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Indeed, (2.28) is valid for $n = 0$. Assuming that it is true for $n - 1 \geq 0$ and using (2.27) we get for $x \geq 0$

$$\begin{aligned} u_n(x) &\leq \int_0^x dy\ \exp(-\Sigma(y)) 2 \int_0^y dz\, \frac{\exp(\Sigma(z))}{\sigma^2(z)} u_{n-1}(z) \\ &\leq \frac{1}{(n-1)!} \int_0^x dy\ \exp(-\Sigma(y)) 2 v^{n-1}(y) \int_0^y dz\, \frac{\exp(\Sigma(z))}{\sigma^2(z)} \\ &= \frac{1}{(n-1)!} \int_0^x dv(y)\, v^{n-1}(y) \\ &= \frac{v^n(x)}{n!} \end{aligned}$$

for each $n \in \mathbb{N}$. This implies that

$$\sum_{n=0}^{\infty} u_n'(x)$$

converges absolutely and uniformly on compact real intervals. Another consequence is that so does

$$\sum_{n=0}^{\infty} u_n(x).$$

The function $u(x) := \sum_{n=0}^{\infty} u_n(x)$ clearly belongs to C^1 and we have

$$u'(x) = \sum_{n=0}^{\infty} u_n'(x).$$

Summing up (2.27), we get

$$u'(x) = \exp(-\Sigma(x))2\int_0^x \frac{\exp(\Sigma(y))}{\sigma^2(y)}u(y)\,dy.$$

Since u is the sum of u_n, it is non-negative and strictly increasing (resp. decreasing) on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$).

Lemma 2.6 now implies that $Lu = u$ holds in the C^1-generalized sense.

Given two solutions u^1 and u^2 of (2.24), it is possible to show $u^1 = u^2$ using similar arguments and Gronwall with Lemma 2.6.

The relation (2.26) obviously follows from

$$1 + v(x) = 1 + u_1(x) \le \sum_{n=0}^{\infty} u_n(x) = u(x) \le \sum_{n=0}^{\infty} \frac{v^n(x)}{n!} = \exp(v(x)). \qquad \square$$

Similarly to problem 5.27 and 5.28 of [16], we need the following result.

Lemma 2.21. *Let v_c be the solution to $Lv = 1$, $v_c(c) = v_c'(c) = 0$, $v \equiv v_0$.*

(i) *If $h(+\infty) = +\infty$ then $v_c(+\infty) = +\infty$ holds for every $c \in \mathbb{R}$.*

(ii) *If $h(-\infty) = -\infty$ then $v_c(+\infty) = +\infty$ holds for every $c \in \mathbb{R}$.*

(iii) $v_c(x) = v_c(a) + v_c'(a)\int_a^x \exp(-\Sigma(y))\,dy + v_a(x)$ *holds for every $a, c \in \mathbb{R}$.*

(iv) *We have $v_c(\pm\infty) < \infty$ if and only if $v_0(\pm\infty) < \infty$.*

Proof. i) For $x \ge c + 1$, we have

$$\begin{aligned} v_c(x) &= \int_c^x dy\, h'(y) \int_c^y \frac{2}{h'(z)\sigma^2(z)}\,dz \\ &\ge \int_{c+1}^x dy\, h'(y) \int_c^{c+1} \frac{2}{h'(z)\sigma^2(z)}\,dz \\ &= \int_c^{c+1} \frac{2}{h'(z)\sigma^2(z)}\,dz\,(h(x) - h(c+1)). \end{aligned}$$

If $h(+\infty) = +\infty$ then $v_c(+\infty) = +\infty$.

Statement ii) follows similarly to (i), whereas (iii) is a consequence of the explicit expression

$$v_c(x) = \int_c^x dy\, h'(y) \int_c^y \frac{2}{h'(z)\sigma^2(z)}\,dz.$$

For the proof of (iv), we rewrite (iii) as

$$v_c(x) - v_a(x) = v_c'(a)(h(x) - h(a)).$$

If $v_c(+\infty) < +\infty$ then $h(+\infty) < +\infty$ holds by i), thus showing $v_a(+\infty) < +\infty$. $\square$

3. A suitable martingale problem

In this section, we consider a PDE operator satisfying the same properties as in the previous section, i.e.

$$Lg = \frac{\sigma^2}{2} g'' + b' g', \tag{3.1}$$

where $\sigma > 0$ and b are continuous. In particular, we assume that

$$\Sigma(x) = \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy \tag{3.2}$$

exists in C^0, independently from the chosen mollifier. Then h defined by $h'(x) := \exp(-\Sigma(x))$ and $h(0) = 0$, is a solution to $Lh = 0$ with $h' \neq 0$.

DEFINITION. A process X is said to solve *the martingale problem* related to L with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)\, ds$$

is a local martingale for $f \in \mathcal{D}_L$ and $X_0 = x_0$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq 0)$ solves the martingale problem related to L with initial value x at time s if

(i) $X_s^{s,x} = x$,

(ii) for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\, dr, \quad t \geq s$$

is a local martingale.

We remark that $X^{s,x}$ solves the martingale problem at time s if and only if $X_t := X_{t-s}^{s,x}$ solves the martingale problem at time 0.

REMARK 3.1. (i) In general, $f(x) = x$ does not belong to $\mathcal{D}_L$. In part II, see [12] we will give necessary and sufficient conditions on b so that X is a semimartingale.

(ii) We are interested in the operators

$$\mathcal{A}\colon \mathcal{D}_L \to \mathcal{C}, \text{ given by } \quad \mathcal{A}(f) = \int_0^{\cdot} Lf(X_s)\, ds$$

and

$$A\colon C^1 \to \mathcal{C}, \text{ given by } \quad A(l) = \int_0^{\cdot} l'(X_s)\, ds.$$

We may ask whether $\mathcal{A}$ and A are closable in C^1 and in C^0, respectively. We will see that $\mathcal{A}$ even admits a continuous extension to C^1. However, A can be extended to C^0 continuously when L is close to divergence type.
(iii) For the moment, we continue to work with the domains C^1 or C^0 because we do not need to examine in detail the fundamental solutions related to L which will be in fact the densities of the laws of the considered processes. Once, we will take into account the information of those densities. Then $\mathcal{A}$ will be extended to $W^{1,2}_{\mathrm{loc}}$; if L is close to divergence type, A will be extended to L^2_{loc}.

The first result on solutions to the martingale problem related to L is the following

Proposition 3.2. *Let $I =]a, b[$ be the image of h, $-\infty \leq a < b \leq +\infty$. A process X solves the martingale problem related to L if and only if $Y = h(X)$ is a local martingale with values in I which solves weakly the stochastic differential equation*

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s)\, dW_s, \tag{3.3}$$

where $Y_0 = h(X_0)$ and $\sigma_0(y) = (\sigma h')(h^{-1}(y))$.

REMARK 3.3. (i) Y always stays in the interval I.
(ii) Let $T > 0$ and $(Z_t)_{t\geq 0}$ be a process. We denote by $\mathbb{F} = \mathbb{F}_Z$ the natural forward filtration of Z, given by $\mathcal{F}_t = \sigma(Z_s : s \leq t)$, clearly, we have $\mathcal{F}_Y = \mathcal{F}_X$.
(iii) Since Y is a local martingale, we know from Remark 1.3 that $X = h^{-1}(Y)$ is a Dirichlet process with martingale part

$$M^X_t = \int_0^t (h^{-1})'(Y_s)\, dY_s = \int_0^t \sigma(X_s)\, dW_s.$$

In particular, X is a finite quadratic variation process with

$$[X, X] = [M^X, M^X]_t = \int_0^t \sigma^2(X_s)\, ds.$$

Proof of Propositon 3.2. First, let X be a solution to the martingale problem related to L. Since $h \in \mathcal{D}_L$ and $Lh = 0$, we know that $Y = h(X)$ is a local martingale. In order to calculate its bracket we recall that $h^2 \in \mathcal{D}_L$ and $Lh^2 = \sigma^2(h')^2$ hold by Proposition 2.13 a). Thus,

$$h^2(X_t) - \int_0^t (\sigma h')^2(X_s)\, ds$$

is a local martingale. This implies

$$[Y]_t = \int_0^t (\sigma h')^2(h^{-1}(Y_s))\,ds = \int_0^t \sigma_0^2(Y_s)\,ds.$$

Finally, Y solves weakly the SDE (3.3) with respect to the standard $\mathcal{F}_Y$-Brownian motion W given by

$$W_t = \int_0^t \frac{1}{\sigma_0(Y_s)}\,dY_s.$$

Now, let $Y = h(X)$ be a solution to (3.3) and $f \in \mathcal{D}_L$. Proposition 2.13 b) says that $\phi := f \circ h^{-1} \in \mathcal{D}_{L^0} \equiv C^2$, where

$$L^0\phi = \frac{\sigma_0^2}{2}\phi'' = (Lf) \circ h. \tag{3.4}$$

So we can apply Itô formula to evaluate $\phi(Y)$ which coincides with $f(X)$. This gives

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s)\,dY_s + \frac{1}{2}\int_0^t \phi''(Y_s)\,d[Y_s].$$

Using $d[Y]_s = \sigma_0^2(Y_s)\,ds$ and taking into account 3.4, we conclude

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\,dW_s + \int_0^t Lf(X_s)\,ds. \tag{3.5}$$

This establishes the proposition. $\square$

Corollary 3.4. *The map $\mathcal{A}$ admits a continuous extension from $\mathcal{D}_L$ to C^1 with values in $\mathcal{C}$ which we will denote again by $\mathcal{A}$. Moreover, $\mathcal{A}(f)$ is a zero quadratic variation process for every $f \in C^1$.*

Proof. $\mathcal{A}$ has a continuous extension because of (3.5). $\mathcal{A}(f)$ is a zero quadratic variation process because X is a Dirichlet process with martingale part $\int_0^\cdot \sigma(X_s)\,dW_s$ and because of Remark 1.3. $\square$

REMARK 3.5. The extension of (3.5) to C^1 gives

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\,dW_s + \mathcal{A}(f). \tag{3.6}$$

Choosing $f = id$ in (3.6), we get

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + \mathcal{A}(id).$$

We would now like to ask if $\mathcal{A}(f)$ corresponds to $A(l)$ for some $l \in C^0$. In that case, X would turn out to solve a stochastic differential equation with diffusion σ and generalized drift l'. Unfortunately, for the moment, we cannot answer the question in such a general framework. However, we will provide an answer if L is close to divergence type. Moreover, even if L is not of that type, we get results on local time.

Proposition 3.6. *If X solves the martingale problem with respect to L then it admits a local time* (*as a density of an occupation time measure*).

Proof. Let $l \in C^0$. For h defined as before, we have

$$\begin{aligned}\int_0^t l(X_s)\,ds &= \int_0^t l \circ h^{-1}(Y_s)\,ds \\ &= \int_0^t \Phi(Y(s))\,d[Y,Y]_s,\end{aligned}$$

where $\Phi(y) := (l \circ h^{-1})/((\sigma h')^2 \circ h^{-1})(y)$. Using the occupation time density formula, we get

$$\int_0^t l(X_s)\,ds = \int L_t^Y(y)\Phi(y)\,dy, \tag{3.7}$$

where L^Y is the local time of Y (in the sense of Tanaka formula). Then, (3.7) becomes

$$\int L_t^Y(h(x))\frac{l(x)}{(\sigma h')^2(x)}h'(x)\,dx = \int \mathbb{L}_t^X(x)l(x)\,dx,$$

where $\mathbb{L}_t^X(x) = L_t^Y(h(x))/(\sigma^2(x)h'(x))$. $\square$

Now the following question arises. Under which conditions on b is $\mathbb{L}^X$ a good Bouleau-Yor integrator? In other words, under which conditions does $d\mathbb{L}_t^X$ integrate continuous functions? For this, we would need to extend the operator A to the whole space $C^0(\mathbb{R})$.

REMARK 3.7. If L is close to divergence type then $A\colon l \mapsto \int_0^{\cdot} l'(X_s)\,ds$ admits a continuous extension to C_0^0 and therefore to C^0 because of $A(l) \equiv A(l + const)$.

In fact, if $l \in C_0^1$ then $A(l) = \mathcal{A}(Tl)$, where T is defined in Proposition 2.17 (i). Since T admits a continuous extension $\hat{T}$ to C_0^0 with values in $C_{0,0}^1$, the operator A can be extended to C_0^0 by $\mathcal{A} \circ \hat{T}$. We still denote this extension by A.

An example of a process X solving a martingale problem with respect to L, where L is close to divergence type, is given by a solution of a stochastic differential equation of the following type,

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \gamma(X_s)\, ds,$$

where σ is Lipschitz, positive and $\gamma \in L^\infty_{\text{loc}}$.

Let X be a stochastic process for which there is a mapping $A\colon C^0 \to \mathcal{C}$ which extends continuously $l \to \int_0^\cdot l'(X_s)\, ds$ from C^1. X is said to fulfill the *Bouleau-Yor property* if $\int_0^\cdot g(X) d^- A(l)$ exists for every $g \in C^2$ and every $l \in C^0$.

REMARK 3.8. Let S be a continuous semimartingale. We recall that Bouleau and Yor [6] have proved the existence of an integral

$$\int f(a) L_t^S(da)$$

for f bounded Borel real function. This procedure allowed them to extend the map $f \mapsto \int_0^t f'(S) d\langle S\rangle$ from C^1 to bounded Borel functions.

Lemma 3.9. *If X is a solution to a martingale problem related to a PDE operator which is close to divergence type, then it fulfills the Bouleau-Yor property.*

Proof. Let $l \in C^0$. There is $f \in C^1$ such that $\hat{\mathcal{L}} f = l$. Since $f(X)$ equals a local martingale plus $A(l)$, it remains to show that

$$\int_0^\cdot g(X) d^- f(X) \tag{3.8}$$

exists for any $g \in C^2$. Integrating by parts previous integral, (3.8) equals

$$(gf)(X_\cdot) - (gf)(X_0) - \int_0^\cdot f(X) d^- g(X) - [f(X), g(X)].$$

Remark 1.2 b), f) tells that the right member is well-defined. □

Lemma 3.10. *Let X be a process having the Bouleau-Yor property. Then, for every $g \in C^2$ and every $l \in C^0$, we have*

$$\int_0^\cdot g(X) d^- A(l) = A(\Phi(g, l)) \tag{3.9}$$

where

$$\Phi(g, l)(x) = (gl)(x) - (gl)(0) - \int_0^x (lg')(y)\, dy \tag{3.10}$$

Proof. The Banach-Steinhaus theorem for F-spaces (see [7, ch. 2]) implies that, for every $g \in C^2$

$$l \mapsto \int_0^{\cdot} g(X)\, d^- A(l) \tag{3.11}$$

is continuous from C^0 to $\mathcal{C}$. Note that Φ is a continuous bilinear map from $C^1 \times C^0$ to C^0. Since $A\colon C^0 \to \mathcal{C}$ is continuous, the mapping $l \to A(\Phi(g,l))$ is also continuous from C^0 to $\mathcal{C}$. In order to conclude the proof, we need to check the identity (3.9) for $l \in C^1$. In that case, by differentiation of l and Φ both members of (3.9) equal

$$\int_0^{\cdot} (gl')(X_s)\, ds. \qquad \square$$

We are now going to investigate the relation between the martingale problem associated with L and stochastic differential equations with distributional drift.

Proposition 3.11. *Suppose that L is close to divergence form. If X solves the martingale problem with respect to L then it is a solution to the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + A(b), \tag{3.12}$$

where $b = \sigma^2/2 + \beta$.

Proof. If X solves the martingale problem related to L then, by (3.5),

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + A(\mathcal{L}f) \tag{3.13}$$

holds for every $f \in \mathcal{D}_L$. Remark 3.7 and Proposition 2.17 allow us to extend (3.13) to any $f \in C^1$. Then $A(\mathcal{L}f)$ is replaced with $\mathcal{A}(\hat{\mathcal{L}}f)$.

If $f = id$ then $\hat{\mathcal{L}}f = b$ holds in view of Corollary 2.19. $\square$

At this stage, it seems natural to ask whether the converse of Proposition 3.11 is true. In other words, if X solves (3.12), is it a solution to the martingale problem related to L? The answer is not immediate. We still suppose that L is close to divergence type. We know the answer only if X fulfills the Bouleau-Yor property. Let $f \in C^2$. By Corollary 3.4 and Proposition 2.17, we know that $A(b)$ has zero quadratic variation. Since X solves (3.12) and $\int_0^{\cdot} f'(X_s)\, d^- X_s$ always exists by the classical Itô formula (see Remark 1.2 e) of Chapter 1) we know that $\int_0^{\cdot} f'(X)\, d^- A(b)$ also exists and is equal to $\int_0^\cdot f'(X)\,d^-X - \int_0^\cdot (f'\sigma)(X)\,dW$. Therefore, this Itô formula says that

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\sigma(X_s)\,dW_s + \int_0^t f'(X)\,d^-A(b) \\ &\quad + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\,ds \end{aligned}$$

holds.

Let L_n be the PDE operator defined in Section 2 by $L_n g = (\sigma_n^2/2)g'' + b_n' g'$. By Lemma 3.10, the linearity of mapping A and Lemma 2.14 we get

$$\begin{aligned} &\int_0^t f'(X)d^-A(b) + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)\,ds \\ = &\quad A\left(\Phi(f',b)\right)_t + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)\,ds \\ = &\int_0^t (\sigma^2 - b)(X_s) f''(X_s)\,ds + A(bf') = A(\mathcal{L}f) \end{aligned}$$

This shows

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\,dW_s + A(\mathcal{L}f). \tag{3.14}$$

Because of Remark 2.18 c), the previous expression can of course be prolongated to any $f \in C^1$. Taking $f \in \mathcal{D}_L$, it follows that X fulfills a martingale problem with respect to L.

Corollary 3.12. *Let the PDE operator L be close to divergence type. Then X solves the martingale problem related to L if and only if it solves the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + A(b) \tag{3.15}$$

and X has the Bouleau-Yor property.

Proof. The statement follows from Lemma 3.9, Proposition 3.11, and from the considerations above. □

Now we are going to examine existence and uniqueness (and non-explosion).

Proposition 3.13. *Let v be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$. Then for any horizon $T > 0$, there exists a unique solution to the martingale problem* *related to L with prescribed initial condition $x_0 \in \mathbb{R}$ if and only if*

$$v(-\infty) = v(+\infty) = +\infty. \tag{3.16}$$

REMARK 3.14. The previous result is a generalization of the Feller test for explosion stated for instance in [16, Theorem 5.29].

Proof of Proposition 3.13. Let X be a solution to the martingale problem related to L. Then Proposition 3.2 says that $Y := h(X)$ solves the stochastic differential equation

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s)\, dW_s \tag{3.17}$$

for $y_0 = h(x_0) \in I = \operatorname{Im} h$.

At this level, we can apply the results of [9] stated also in [16, Theorem 5.7] (Engelbert-Schmidt theorem). According to their notations, we have $Z(\sigma_0) = I^c$, which means that the set of zeros of σ_0 is I^c. On the other hand, the set

$$I(\sigma_0) = \left\{ x \in \mathbb{R} : \int_{(-\varepsilon,\varepsilon)} \frac{dy}{\sigma_0^2(x+y)} = +\infty \right\}$$

is equal to I^c. In fact, since σ is strictly positive and continuous and I is open, we have $I \subset I(\sigma_0)^c$. If $x \in I^c$ then σ_0 is zero in some neighbourhood of x and so x belongs to $I(\sigma_0)$. Thus, we have $I(\sigma_0) = Z(\sigma_0) = I^c$ so that the Engelbert-Schmidt theorem ensures that (3.17) has a unique solution.

Let Y be the solution to (3.17). We remark that this solution cannot explode, see [16, Problem 5.2]. So, if $I = \operatorname{Im} h = \mathbb{R}$, Proposition 3.2 will yield existence for the martingale problem related to L. However, Y could reach I^c or equivalently ∂I. The following lemma now completes the proof. □

Lemma 3.15. *For $y_0 \in I$, the solution Y to* (3.17) *remains in I a.s. if and only if* (3.16) *holds.*

Proof. We recall that Y remains in I if and only if $X = h^{-1}(Y)$ is always finite, where h is extended to $\overline{\mathbb{R}}$ with values in $\overline{I}$.

For $m, n \in \mathbb{N}$, we define

$$X_t^{m,n} = X_{t\wedge\tau_m\wedge\phi_n} \quad \text{and} \quad Y_t^{m,n} = Y_{t\wedge\tau_m\wedge\phi_n},$$

where

$$\tau_m := \inf\{t \geq 0 : X_t \leq -m\},$$

$$\phi_n := \inf\{t \geq 0 : X_t \geq n\}$$

Let $u \in \mathcal{D}_L$; we know $\tilde{u} = u \circ h^{-1} \in \mathcal{D}_{L^0} \equiv C^2$ in view of Proposition 2.13. Then, by the classical Itô formula, we calculate

$$Z_t := \tilde{u}(Y_t) = \tilde{u}(y_0) + \int_0^t \sigma_0 \tilde{u}'(Y_s)\, dW_s + \int_0^t L^0 \tilde{u}(Y_s)\, ds.$$

Setting $Z_t^{m,n} := Z_{t \wedge \tau_m \wedge \phi_n}$, we get

$$Z_t^{m,n} = \tilde{u}(y_0) + \int_0^{t\wedge\tau_m\wedge\phi_n} \sigma_0 \tilde{u}'(Y_s)\, dW_s + \int_0^{t\wedge\tau_m\wedge\phi_n} L^0 \tilde{u}(Y_s)\, ds.$$

Using Proposition 2.13, for $Z_t^{m,n} = u(X_{t\wedge\tau_m\wedge\phi_n})$, we obtain

$$Z_t^{m,n} = u(x_0) + \int_0^{t\wedge\tau_m\wedge\phi_n} \sigma u'(X_s)\, dW_s + \int_0^{t\wedge\tau_m\wedge\phi_n} Lu(X_s)\, ds.$$

Let us now suppose $Lu = u$ according to Proposition 2.20. Integrating $M_t^{m,n} := \exp(-t \wedge \tau_m \wedge \phi_n) Z_t^{m,n}$ by parts yields

$$\begin{aligned} M_t^{m,n} &= M_0^{m,n} + \int_0^{t\wedge\tau_m\wedge\phi_n} \exp(-s) u'(X_s) \sigma(X_s)\, dW_s \\ &= M_0^{m,n} + \int_0^t \exp(-s) u'(X_s^{m,n}) \sigma(X_s^{m,n})\, dW_s. \end{aligned}$$

Therefore $M^{m,n}$ is a local martingale which, by definition, is non-negative. Hence, $M^{m,n}$ is a supermartingale.

We consider the stopping times $\phi := \lim_{n\to\infty} \phi_n$ and $\tau := \lim_{m\to\infty} \tau_m$. We observe that the processes

$$\begin{aligned} M_t^m &:= \lim_{n\to\infty} M_t^{m,n} = \exp(-t \wedge \phi \wedge \tau_m) u(X_{t\wedge\phi\wedge\tau_m}), \\ M_t &:= \lim_{n\to\infty} M_t^{n,n} = \exp(-t \wedge \phi \wedge \tau) u(X_{t\wedge\phi\wedge\tau}) \end{aligned}$$

are also supermartingales. Therefore, for every $m \geq 0$,

$$M_\infty^m = \lim_{t\to\infty} M_t^m \text{ a.s.,} \tag{3.18}$$

$$M_\infty = \lim_{t\to\infty} M_t \text{ a.s.} \tag{3.19}$$

exist and are finite.

After these preliminaries, we suppose first that (3.16) holds. Then (2.26) in Proposition 2.20 implies $u(\pm\infty) = +\infty$. By (3.19), $M_\infty = +\infty$ holds on $\{\phi \wedge \tau = +\infty\}$. This entails $\mathbb{P}(\{\tau \wedge \phi < +\infty\}) = 0$. Hence, Y remains in I a.s.

Conversely, let us suppose that X does not explode and (3.16) fails, for instance suppose $v(+\infty) < +\infty$. Let $c \in \mathbb{Z}$ such that $x_0 > c$. By Lemma 2.21 (iv), $v_c(+\infty)$ is finite. This implies that the unique solution u to $Lu = u$, $u(c) = 1$, $u'(c) = 0$ fulfills $u(+\infty) < \infty$, see (2.26) in Proposition 2.20. The continuous process

$$M_t^c = \exp(-t \wedge \tau_c \wedge \phi)u(X_{t\wedge\tau_c\wedge\phi}), \quad t \geq 0,$$

is a bounded supermartingale. But it is also a local martingale and hence, a martingale in L^1. The convergence (3.18) holds also in L^1. Consequently, we have

$$u(x_0) = \mathbb{E}(\exp(-\tau_c \wedge \phi)u(X_{\tau_c\wedge\phi})).$$

Since $\phi = +\infty$ a.s., X being always finite, we have

$$\begin{aligned} u(x_0) &= \mathbb{E}(\exp(-\tau_c)u(X_{\tau_c})) \\ &= \mathbb{E}(\mathbf{1}_{\{\tau_c < +\infty\}} \exp(-\tau_c)u(X_{\tau_c})) \leq u(c). \end{aligned}$$

This contradicts the fact that u is strictly increasing on $[c, x]$. Therefore, $v(+\infty) = +\infty$ holds. A similar reasoning works for $v(-\infty) = -\infty$. $\square$

We would like to finish this section with two considerations. The first one concerns in which sense L can be looked upon as the *extended infinitesimal generator* of a process X solving the martingale problem related to L. The second one concerns the *Kolmogorov equation* associated with the law of X.

a) **The extended infinitesimal generator.** We recall the notation C_c^k standing for the set of C^k-functions with compact support.

Lemma 3.16. *For every $f \in \mathcal{D}_L$ satisfying $Lf \in C_c^0$ there is a sequence (f_n) in $\mathcal{D}_L \cap C_c^1$ with $\lim_{n\to\infty} f_n = f$ in the sense of the graph norm.*

Proof. Let $f \in \mathcal{D}_L$. Then $f \circ h^{-1} \in C^2$ holds by Proposition 2.13. Since $C_0^2(I)$ is dense in $C^2(I)$, where $I = \operatorname{Im} h$, we find a sequence $(\tilde{f}_n)$ in $C_0^2(I)$ such that $\tilde{f}_n \to f \circ h^{-1}$ in C^2. Clearly, $f_n = \tilde{f}_n \circ h$ is a sequence in $\mathcal{D}_L \cap C_c^1$ which tends to f in C^1. Moreover, $Lf_n = (\sigma_0^2 \tilde{f}_n''/2) \circ h$ are continuous functions with compact support and converge to $(L^0 f)(f) = Lf$. $\square$

Let $(X_t^x, t \geq 0, x \in \mathbb{R})$ be a random field which is measurable in (t, x, ω) such that $X_0^x = x$. We say that L is its *infinitesimal generator* if

$$Lf(x) = \lim_{t\to 0} \frac{1}{t}\mathbb{E}(f(X_t^x) - f(x)) \tag{3.20}$$

holds for every $f \in \mathcal{D}_L \cap C_c^0$.

REMARK 3.17. If $(X_t^x, t \geq 0)$ solves the martingale problem related to L with initial condition x then we know that $Y_t^x := h(X_t^x)$ solves the stochastic differential equation

$$Y_t^x = h(x) + \int_0^t (\sigma h') \circ h^{-1}(Y_s^x)\, dW_s. \tag{3.21}$$

By the classical theory of stochastic differential equations there is a version which is measurable with respect to (t, x, ω); so the same holds for X.

REMARK 3.18. X^x solves the martingale problem related to L with initial condition x if and only if

$$f(X_t) - f(x) - \int_0^t (Lf)(X_s)\, ds \tag{3.22}$$

is a martingale for every $f \in C_c^1 \cap \mathcal{D}_L$.

This follows from the fact that, by Lemma 3.16, $\mathcal{D}_L \cap C_c^1$ is dense in $\mathcal{D}_L$ and from (3.5) in the proof of Proposition 3.2 which says that (3.22) equals $\int_0^t (f'\sigma)(X_s)\, dW_s$ and so it is truely a martingale if $f \in C_c^1$.

Proposition 3.19. *Let $(X_t^x, t \geq 0)$ be a random field as above such that X^x solves the martingale problem related to L for every initial condition x. Then L is its infinitesimal generator.*

Proof. Let $f \in C_c^1 \cap \mathcal{D}_L$. Taking the expectation in the martingale (3.22), we get

$$\frac{1}{t}\mathbb{E}(f(X_t) - f(x)) = \frac{1}{t}\int_0^t ds\, \mathbb{E}((Lf)(X_s)), \quad t > 0, \tag{3.23}$$

where $X = X^x$. We denote the law of X_t^x by ν_t. Now (3.23) can be rewritten as

$$\frac{1}{t}\int_0^t ds\, \nu_s(Lf). \tag{3.24}$$

But $t \mapsto \nu_t(g)$ is a continuous function for every $g \in C^0$ so that (3.24) equals $\nu_{\tilde{s}}(Lf)$, $\tilde{s} \in [0, t]$. Therefore, the previous term converges to $Lf(x)$. ☐

b) **The Kolmogorov equation.** Now we want to discuss the Kolmogorov equation corresponding to a random field $(X_t^{s,x}, t \geq s \geq 0, x \in \mathbb{R})$ such that $X^{s,x}$ solves the martingale problem related to L with initial condition x at time s. Again, L_n will be the same regularizing PDE operators as in Section 2.

We define the set $\mathcal{U}_L$ of bounded functions $u = (u(t,x), t \geq 0, x \in \mathbb{R})$ in $C^0([0,T] \times \mathbb{R})$ such that there are bounded functions v^a, $v^b \in C^0([0,T] \times \mathbb{R})$ and a sequence $u_n = (u_n(t,x), t \geq 0, x \in \mathbb{R})$ in $C^{1,2}([0,T] \times \mathbb{R})$ satisfying

(i) $u_n \to u$,

(ii) $\partial u_n / \partial t \to v^a$,

(iii) $L_n u_n \to v^b$

pointwise. In this case we say that $(\partial_t + L)(u) = v^a + v^b$ holds in the C^0-generalized sense.

REMARK 3.20. If $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$ with $u(t,\cdot) \in \mathcal{D}_L$ for every $t \geq 0$ then $u \in \mathcal{U}_L$ holds.

It is also possible to consider the case of Dirichlet boundary conditions.

Given a bounded interval D, we define similarly to the above definition, the set $\mathcal{U}_L(D)$ of functions $u = (u(t,x), t \geq 0, x \in D)$ such that there are bounded functions v^a, $v^b \in C^0([0,T] \times \overline{D})$ and a sequence $u_n = (u_n(t,x), t \in [0,T], x \in \overline{D})$ in $C^{1,2}([0,T] \times \overline{D})$ with zero Dirichlet boundary conditions fulfilling points (i), (ii), (iii) above. In this case we say that $(\partial_t + L)(u) = v^a + v^b$ holds (in the C^0-generalized sense) with zero Dirichlet boundary conditions.

Theorem 3.21. *Suppose that $X^{s,z}$ solves the martingale problem related to L with initial condition z at time s.*

(i) *Let $u = (u(t,z), t \in [0,T], z \in \mathbb{R})$ in $\mathcal{U}_L$. Then we have*

$$u(s,z) = \mathbb{E}(u(T, X_T^{s,z})) + \int_s^T \mathbb{E}\Big((\partial_t + L)(u(s, X_r^{s,z}))\Big)\, dr, \tag{3.25}$$

(ii) *Let $u = (u(t,z), t \in [0,T], z \in \mathbb{R})$ be in $\mathcal{U}_L(D)$ such that $(\partial_t + L)(u) = 0$ holds with zero Dirichlet boundary conditions. Then we have*

$$u(s,z) = \mathbb{E}\left(u(T, X_T^{s,z}) 1_{\{X_t^{s,z} \in D, \forall t \in [0,T]\}}\right).$$

Proof. In the case of (ii), we can prolongate u with zero outside D to get a function in $\mathcal{U}_L$.

We set $\tilde{u}_n(t,y) := u_n(t, h_n^{-1}(y))$, where $(h_n) \subset C^2$ satisfies $L_n h_n \to 0$ in C^0, $h_n \to h$ in C^1, $h_n(0) = 0$, $h_n'(0) = 1$. We can apply the classical Itô formula to $\tilde{u}_n(t, Y_t)$, where $Y = h(X)$ and $X = X^{s,z}$. We recall that

$$Y_t = h(z) + \int_s^t \sigma_0(Y_s)\, dW_s \tag{3.26}$$

holds, where $\sigma_0 = (\sigma h') \circ h^{-1}$. Therefore, we have

$$\tilde{u}_n(t, Y_t) = \tilde{u}_n(s, h(z)) + \int_s^t \frac{\partial \tilde{u}_n}{\partial r}(r, Y_r)\, dr + \int_s^t \frac{\partial \tilde{u}_n}{\partial x}(r, Y_r)\sigma_0(Y_r)\, dW_r + \frac{1}{2}\int_s^t \frac{\partial^2 \tilde{u}_n}{\partial x^2}(r, h_n(X_r))\sigma_0^2(Y_r)\, dr,$$

where L^0 only acts on y. Coming back to X and setting $i_n := h_n \circ h^{-1}$, we calculate

$$\begin{aligned} u_n(t, i_n(X_t)) = u_n(s, i_n(X_s)) &+ \int_s^t \frac{\partial u_n}{\partial r}(r, i_n(X_r))\, dr \\ &+ \int_s^t \frac{\partial \tilde{u}_n}{\partial x}(r, h_n(X_r))\sigma_0(h_n(X_r))\, dW_r \\ &+ \frac{1}{2}\int_s^t L_n u_n(r, i_n(X_r))\frac{\sigma_0^2}{\sigma_{0,n}^2}(i_n(X_r))\, dr, \end{aligned} \tag{3.27}$$

where $\sigma_{0,n} = (\sigma_n h_n') \circ h_n^{-1}$. The last integral could be transformed using Proposition 2.13.

Given a bounded interval Δ containing z, we define the stopping time

$$\tau := \inf\{t \in [s, T] : X_t \notin \Delta\} \wedge (T + 1).$$

Stopping the process X at time τ, we obtain

$$\begin{aligned} u_n(t \wedge \tau, i_n(X_{t\wedge\tau})) = u_n(s, i_n(X_s^\tau)) &+ \int_s^{t\wedge\tau} \frac{\partial u_n}{\partial r}(r, i_n(X_r))\mathbf{1}_{\{X_r \in \Delta\}}\, dr \\ &+ \int_s^{t\wedge\tau} \frac{\partial \tilde{u}_n}{\partial x}(r, h_n(X_r))\sigma_0(h_n(X_r))\mathbf{1}_{\{X_r \in \Delta\}}\, dW_r \\ &+ \frac{1}{2}\int_{s\wedge\tau}^{t\wedge\tau} L_n u_n(r, i_n(X_r))\frac{\sigma_0^2}{\sigma_{0,n}^2}\mathbf{1}_{\{X_r \in \Delta\}}\, dr. \end{aligned}$$

Since the stochastic integrand with respect to the Brownian motion is bounded, its expectation is zero. Therefore, we get

$$\begin{aligned} &\mathbb{E}\Big(u_n(\tau \wedge T, h_n(X_{\tau\wedge T})) - u_n(0, i_n(X_s))\Big) \\ &\quad = \mathbb{E}\bigg(\int_{s\wedge\tau}^{T\wedge\tau} \mathbf{1}_{\{X_r \in D\}}\bigg(\frac{\partial u_n}{\partial r}(r, i_n(X_r)) + L_n u_n(r, i_n(X_r))\frac{\sigma_0^2}{\sigma_{0,n}^2}(X_r)\bigg)\, dr\bigg). \end{aligned}$$

We remark that the expectation exists since all integrands are bounded. Passing to the limit $n \to \infty$ and using $\sigma_0^2/\sigma_{0,n}^2 \to 1$ in C^0, we obtain

$$\mathbb{E}(u(\tau \wedge T, X_{\tau\wedge T})) - u(s, z) = \mathbb{E}\left(\int_s^{T\wedge\tau} (\partial_t u + Lu)(r, X_r^{s,z})\, dr\right). \tag{3.28}$$

(i) For $N > 0$, we set $\Delta := [-N, N]$, $\tau = \tau^N$, the sequence (τ^N) defines a "suitable" sequence of stopping times in the sense defined before Remark 1.3. We let $N \to \infty$ in (3.28) and the result follows.
(ii) We set $\Delta := D$. According to the assumption we get

$$\begin{aligned} u(s, z) &= \mathbb{E}\left(u(\tau \wedge T, X_{\tau\wedge T})\right) \\ &= \int_{\{X_t^{s,z} \in D,\, \forall t \in [0,T]\}} u(T, X_T^{s,z})\, dP \\ &\quad + \int_{\{\tau^n \le T\}} \underbrace{u(\tau, X_\tau^{s,z})}_{0}\, dP \end{aligned}$$

This allows to conclude. $\square$

Corollary 3.22. *Let $u = (u(t, x), t \geq 0, x \in \mathbb{R})$ be of class C^1 and g continuous and bounded such that*
(i) $u(t, \cdot) \in \mathcal{D}_L$ *for every* $t \geq 0$,
(ii) $u(T, z) = g(z)$,
(iii) u *solves the parabolic PDE*

$$\frac{\partial u}{\partial t}(t, \cdot) + Lu(t, \cdot) = 0 \tag{3.29}$$

in the C^1-generalized sense.
Then we have

$$u(s, z) = \mathbb{E}(g(X_T^{s,z})),$$

where $X^{s,z}$ solves the martingale problem related to L with initial condition z at s.

4. A general result on finite quadratic variation processes

Let X be the solution to a martingale problem related to L with initial condition x_0. In the following section, we are also interested in the Dirichlet-Fukushima structure of $f(X)$, $f \in W^{1,2}_{\mathrm{loc}}$. Under some assumptions on σ and b it should be possible to obtain a Fukushima-decomposition of $f(X)$ using a Dirichlet form method. However, since we are potentially interested in applications to non-Markovian processes, we will implement here a technique which is based on "pathwise calculus" and on the existence of the density with respect to Lebesgue measure. We proceed in two steps.
I) We construct a general tool related to finite quadratic variations processes.
II) We observe that the law of X_t, $t > 0$ has a density which fulfills some basic estimates, fitting to I).

Point I) will be a refinement of Remark 1.3. It will be the object of this section.

Point II) will be treaten in the next section.

In this section we consider a continuous process $(X_t)_{t\geq 0}$ adapted to some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We make the following assumptions on the law of (X_s, X_t).
H1) For each $0 < s < t$, the law of (X_s, X_t) has a density $(q_{s,t}(x_1, x_2))$.
H2) For every $T > 0$, $\delta > 0$, we suppose that the following quantity is well-defined for $x_1 \neq x_2$

$$\phi_\delta(x_1, x_2) = \int_0^T ds \quad q_{s,s+\delta}(x_1, x_2).$$

H3) For every $D > 0$ such that $|x_0| \leq D$ there is a constant C such that

$$x_1 \neq x_2, \quad |x_1|, \ |x_2| \leq D \Rightarrow \sup_{0<\delta\leq 1} \sqrt{\delta}\, \phi_\delta(x_1, x_2) \leq C.$$

We start with a technical lemma.

Lemma 4.1. *Let $(X_t)_{t\geq 0}$ be a process fulfilling* H1), H2), H3). *Let Y be a continuous process such that (X, Y) has all its mutual covariations. Then*

$$[f(X), Y]_t = \int_0^t f'(X_s)\, d[X, Y]_s$$

holds for every $f \in W^{1,2}_{\mathrm{loc}}$.

Proof. We observe that $L^2_{\mathrm{loc}}(\mathbb{R})$ is an F-type space which includes $C^1(\mathbb{R})$ as a dense subset. We consider the maps

$$T_\varepsilon, \ T : L^2_{\mathrm{loc}} \longrightarrow \mathcal{C}$$

defined by

$$\begin{aligned} T_\varepsilon(f) &= C_\varepsilon(f(X), Y) \\ T(f) &= \int_0^\cdot f'(X_s)\, d[X, Y]_s \end{aligned}$$

provided previous expressions make sense. We aim at applying Banach-Steinhaus theorem, see [7], Chapter 2.1, to conclude that for every $f \in L^2_{\mathrm{loc}}$, $T_\varepsilon(f) \longrightarrow T(f)$ in the ucp topology. For this, it remains to show the following.
a) T_ε, T are well-defined. On C^1, they are defined because of Remark 1.2 b).
b) For every $f \in C^1$

$$[f(X), Y]_t = \int_0^t f'(X_s)\, d[X, Y]_s$$

c) For every fixed $f \in W^{1,2}_{\mathrm{loc}}$, $(T_\varepsilon(f))$ is a bounded sequence in the F-type space $\mathcal{C}$.

We start with c). Cauchy-Schwarz implies that

$$C_\varepsilon(f(X), Y) \le C_\varepsilon(Y, Y) C_\varepsilon(f(X), f(X)) \tag{4.1}$$

so that we will prove that $(C_\varepsilon(f(X), f(X))$ is bounded in $\mathcal{C}$. Since X has locally bounded paths, by localisation it will be enough to show that $C_\varepsilon(f(X), f(X))$ is bounded on the set $\Omega_M = \sup_{t \le T+1} |X_t| \le M/2$ for each $M > 0$.

To prove that $C_\varepsilon(f(X), f(X))\mathbf{1}_{\Omega_M}$ is bounded in $\mathcal{C}$, it will be enough to show that $C_\varepsilon(f(X), f(X))_T \mathbf{1}_{\Omega_M}$ is bounded in probability. We will even show

$$\sup_{\varepsilon > 0} \mathbb{E}\Big(\mathbf{1}_{\Omega_M} \int_0^T (f(X_{s+\varepsilon}) - f(X_s))^2 \frac{ds}{\varepsilon} \Big) < \infty.$$

The expectation above is bounded by

$$\int_0^T \frac{ds}{\varepsilon} \int_0^1 da \quad \mathbb{E}\Big(\mathbf{1}_{\Omega_M} f'^2(X_s + a(X_{s+\varepsilon} - X_s))(X_{s+\varepsilon} - X_s)^2 \Big). \tag{4.2}$$

We set $g = f'$. The expectation in (4.2) is bounded by

$$\int_{-M}^M dx \int_{-M}^M dy \; g^2(x + a(y-x))(y-x)^2 q_{s,s+\varepsilon}(x, y).$$

So (4.2) gives

$$\int_0^1 da \int_{-M}^M dx \int_{-M}^M dy \; g^2(x + a(y+x)) \Big(\frac{y-x}{\sqrt{\varepsilon}} \Big)^2 \phi_\varepsilon(x, y).$$

Using assumption H3) there is a constant C such that previous expression is bounded by

$$\frac{C}{\sqrt{\varepsilon}} \int_0^1 da \int_{-M}^M dx \int_{-M}^M dy \; g^2(x + a(y-x)) \Big(\frac{y-x}{\sqrt{\varepsilon}} \Big)^2.$$

We replace g with $g^M = g\mathbf{1}_{[-M,M]}$, $g^M \in L^2$. Therefore defining $\psi(z) = C\mathbf{1}_{[-2M,2M]}(z)$, previous term is bounded by

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 da \int dx \int dy \; (g^M)^2(x + a(y-x)) \frac{(y-x)^2}{\varepsilon} \psi(y-x).$$

Setting $\tilde{y} = (y-x)/\sqrt{\varepsilon}$ we get $y = \sqrt{\varepsilon}\,\tilde{y} + x$, $dy = \sqrt{\varepsilon}\, d\tilde{y}$ and thus one obtains

$$\int_0^1 da \int dx \int d\tilde{y} \; (g^M)^2(x + a\sqrt{\varepsilon}\,\tilde{y})\tilde{y}^2 \psi(\tilde{y}^2)$$

$$= \int_0^1 da \int d\tilde{y}\ \tilde{y}^2 \psi(\tilde{y}^2) \int dx\ (g^M)^2(x).$$

This establishes point c) because previous quantity is finite.

b) is contained in Remark 1.2 b).

The proof of a) has common features with the one of c). Concerning the existence of $T_\varepsilon(f)$, $f \in W^{1,2}_{\mathrm{loc}}$, (4.1) says that it is enough to show that $C_\varepsilon(f(X), f(X))$ exists. This is a byproduct of c).

Then we observe that for $a, b \in \mathbb{R}_+$,

$$|[X,Y]_b - [X,Y]_a| \le ([X]_b - [X]_a)([Y]_b - [Y]_a).$$

So, to prove that $T(f)$ exists it is enough to show that

$$\int_0^T f'^2(X_s)\, d[X]_s < \infty, \quad a.s. \tag{4.3}$$

For (4.3) similar arguments as for c) will again apply. $\square$

A consequence of Lemma 4.1 and polarization is the following.

Corollary 4.2. *For every $f, g \in W^{1,2}_{\mathrm{loc}}$, we have*

$$[f(X), g(X)]_t = \int_0^t f'(X_s) g'(X_s)\, d[X]_s \tag{4.4}$$

This leads us to the following Dirichlet characterization:

Proposition 4.3. *Let $(X_t)_{t\ge 0}$ be an $(\mathbb{F})$-Dirichlet process with M as local martingale part. We make again assumptions* H1), H2), H3) *on the law of X. Let $f \in W^{1,2}_{\mathrm{loc}}$. Then $(f(X_t))_t$ is again a Dirichlet process with martingale part $M_t^f = \int_0^t f'(X_s)\, dM_s$.*

Proof. We aim at proving that

$$A_t^f := f(X_t) - M_t^f$$

is a zero quadratic variation process. Using the bilinearity of the covariation, we have

$$[A^f] = [f(X)]_t - 2[f(X), M^f] + [M^f]_t;$$

we recall that $[X] = \langle M \rangle$.

(4.4) implies

$$[f(X)]_t = \int_0^t f'(X_s)^2\, d[X]_s = \int_0^t f'(X_s)^2 d\langle M\rangle_s.$$

Corollary 4.2 says that

$$[f(X), M^f]_t = \int_0^t f'(X_s)\, d[X, M^f]_s.$$

Since

$$\begin{aligned}[X, M^f] &= [M, M^f]_t = \langle M, M^f\rangle_t \\ &= \int_0^t f'(X_s)\, d\langle M\rangle_s.\end{aligned}$$

it follows that $[f(X), M^f]_t = \int_0^t f'(X_s)^2\, d\langle M\rangle_s$. Classical stochastical calculus also says that

$$[M^f]_t = \langle M^f\rangle_t = \int_0^t f'(X_s)^2\, d\langle M\rangle_s.$$

This concludes the proof of the Proposition. □

5. Dirichlet structure of the solution to a martingale problem

Let X be the solution to a martingale problem related to L with initial condition x_0.

We first suppose that L is of divergence type which means

$$b = \frac{\sigma^2}{2} \quad \text{such that} \quad Lg = \left(g' \frac{\sigma^2}{2}\right)'. \tag{5.1}$$

We recall the fundamental lemma in this situation.

Lemma 5.1. *We suppose* $0 < c \le \sigma^2 \le C$. *Let* σ_n, $n \in \mathbb{N}$, *be smooth functions such that* $0 < c \le \sigma_n^2 \le C$ *and* $\sigma_n^2 \to \sigma^2$ *in* C^0 *as at the beginning of* Section 2. *We set* $L_n g = (\{\sigma_n^2/2\}g')'$. *There exists a family of probability measures* $(\nu_t(dx, y), t \ge 0, y \in \mathbb{R})$, resp. $(\nu_t^n(dx, y), t \ge 0, y \in \mathbb{R})$, *enjoying the following properties*:

(i) $\nu_t(dx, y) = p_t(x, y)\, dx$, $\nu_t^n(dx, y) = p_t^n(x, y)\, dy$.

(ii) (*Aronson estimates*) *There exists* $M > 0$ *with*

$$\frac{1}{M\sqrt{t}} \exp\left(-\frac{M|x-y|^2}{t}\right) \le p_t(x, y) \le \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right).$$

(iii) *We have*

$$\frac{\partial \nu_t}{\partial t}(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y \tag{5.2}$$

and

$$\frac{\partial \nu_t^n}{\partial t}(\cdot, y) = L_n \nu_t^n(\cdot, y), \quad \nu_0^n(\cdot, y) = \delta_y.$$

ν (resp. ν^n) *is called the fundamental solution related to the previous parabolic linear equation.*

(iv) *We have*

$$\begin{aligned}\frac{\partial}{\partial t}\nu_t(x, \cdot) &= L\nu(x, \cdot)\\ \frac{\partial}{\partial t}\nu_t^n(x, \cdot) &= L_n\nu^n(x, \cdot)\end{aligned}$$

(v) *The map* $(t, x, y) \mapsto p_t(x, y)$ *is continuous from* $]0, \infty[\times \mathbb{R}^2$ *to* $\mathbb{R}$.

(vi) *The* p^n *are smooth on* $]0, \infty[\times \mathbb{R}^2$.

(vii) *We have* $\lim_{n\to\infty} p_t^n(x, y) = p_t(x, y)$ *uniformly on each compact subset of* $]0, \infty[\times \mathbb{R}^2$.

(viii) $p_t(x, y) = p_t(y, x)$ *holds for every* $t > 0$ *and every* $x, y \in \mathbb{R}$.

(ix) *The semigroup property holds, i.e. for positive* s, t *we have*

$$\int p_t(x, y) p_s(y, z)\, dy = p_{t+s}(x, z).$$

REMARK 5.2. (i) Aronson estimates, which were established in [1], imply in particular

$$\lim_{|y|\to\infty} p_t(x, y) = \lim_{|x|\to\infty} p_t(x, y) = 0.$$

(ii) The continuity of $y \mapsto p_t(x, y)$ ($t > 0, x \in \mathbb{R}$) entails that

$$t \mapsto \int \nu_t(dx, y)\, f(x)$$

is continuous for every bounded Borel function f.

(iii) (5.2) has to be understood in the following distributional way:

$$\int \nu_t(dx, y)\, f(x) = f(y) - \int_0^t ds \int dx\, \frac{\sigma^2}{2}(x)\frac{\partial}{\partial x}(p_s(x, y)) f'(x). \tag{5.3}$$

(iv) We can replace δ_y with any probability measure μ_0. The solution to (5.2) is then given by

$$\nu_t(dy) = \int d\mu_0(x)\, p_t(x, y).$$

(v) The maps $(t, y) \mapsto p_t(x, y)$ are in $L^2(]0, T] \times \mathbb{R}^2)$ again because of the Aronson estimates.

Proof. The proof is essentially contained in [31, ch. II.3] and the references therein. Aronson estimates are established in [1]. Statement (viii) is a consequence of the fact that L is self-adjoint. □

Let X solve the martingale problem related to L with initial condition x. For $t \geq 0$, we denote the law of X_t by ν_t. Our aim is now to show that its law has a density $(p_t(x, y), t > 0, x, y \in \mathbb{R})$ enjoying the property of Lemma 5.1.

Proposition 5.3. *Let L be of divergence type* (*see* (5.1)), *$g \in C_c^1 \cap \mathcal{D}_L$ such that $Lg \in C_c^0$. We use the same notation as in* Lemma 5.1 *and define*

$$v(t, z) = \int \nu_t(dx, z)g(x).$$

Then u: $(t, z) \mapsto v(T - t, z)$ belongs to $\mathcal{U}_L$. Moreover, $\partial_t u + Lu = 0$ holds in the C^0-generalized sense.

Proof. First of all, $u \in C^0([0, T] \times \mathbb{R})$ follows from Remark 5.2 ii) and v) because $v \in C^0([0, T] \times \mathbb{R})$. Moreover, v is bounded because of

$$\int p_t(x, y)\, dx = 1. \tag{5.4}$$

Let (g_n) such that $L_n g_n = Lg$, $g_n(0) = g(0)$, $g_n'(0) = g'(0)$. Then by Lemma 2.6, g_n converges to g in C^0. We define

$$v_n(t, z) = \int \nu_t^n(dx, z)\, g_n(x) \tag{5.5}$$

are smooth because so are p^n. Moreover, by Lemma 5.1 (vii), we have

$$v_n(t, z) = \int p_t^n(x, z)g_n(x)\, dx \to \int p_t(x, z)g(x)\, dx \tag{5.6}$$

since g is bounded.

It remains to prove that $\partial_t v_n$ and $L_n v_n$ converge pointwise to some continuous and bounded functions v^a and v^b on $[0, T] \times \mathbb{R}$. For this, we calculate

$$\begin{aligned} \partial_t \left(\int_{\mathbb{R}} p_t^n(x, z)g_n(x)\, dx \right) &= \int \partial_t p_t^n(z, x)g_n(x)\, dx \\ &= \int L_{n,x} p_t^n(z, x)g_n(x)\, dx \\ &= \int_{\mathbb{R}} p_t^n(z, x)L_n g_n(x)\, dx. \end{aligned}$$

This quantity converges pointwise to

$$v^a(t,z) = \int_{[0,T]\times\mathbb{R}} p_t(z,x) Lg(x)\,dx. \tag{5.7}$$

Again v^a is bounded because of (5.4). Moreover, it is continuous.

The proof of the convergence of $L_n v_n$ to $v^b = v^a$ follows from previous verification and Lemma 5.1 (iii). Therefore, we have $v \in \mathcal{U}_L$ and $\partial_t u + Lu = 0$ in the C^0-generalized sense.

Corollary 5.4. *Let L be of divergence type as in* (5.1) *with $0 < c \le \sigma^2 \le C$. Let X be the solution to the martingale problem related to L with initial condition z. Then the law of X_t, $t > 0$, has a density which we denote by $p_t(x,z)$. Moreover, $p_t(x,z)$ coincides with the density introduced in* Lemma 5.1.

Proof. We start with $p_t(x,z)$ introduced in Lemma 5.1 and $g \in \mathcal{D}_L \cap C_c^1$ such that Lg has compact support. The function

$$\tilde{v}_T(t,z) = \int p_{T-t}(x,z) g(x)\,dx \tag{5.8}$$

coincides with

$$\mathbb{E}(g(X_t^z)) \tag{5.9}$$

by Theorem 3.21. Since $\{g \in \mathcal{D}_L \cap C_c^1 : Lg \in C_c^0\}$ is dense in $\mathcal{D}_L$ which is dense in C^1, the law of X_t^z is completely determined by equality (5.8) and (5.9). □

In the following lines, we will show that the law of X_t, $t > 0$, has always a density if X solves the martingale problem related to any L satisfying the conditions of Section 2, with a supplementary technical assumption.

In the sequel of this section we will use the same notations as in Section 2. σ, b will be continuous functions such that $\sigma > 0$, σ_n^2, b_n will be regularizations of σ^2 and b with the same mollifier. L_n will stand for

$$L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'.$$

We suppose that

$$\Sigma(x) = \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy \tag{5.10}$$

exists in C^0. We recall that, by Proposition 2.3, there is a unique $h \in C^1$ such that

$Lh = 0$ and $h(0) = 0$, $h'(1) = 0$. It can be represented as

$$h'(x) = \exp(-\Sigma(x)). \tag{5.11}$$

A family $(p_t(x, \cdot), t > 0, x \in \mathbb{R})$ of probability densities is said to fulfill the *local Aronson estimates* if, for every continuous function χ with compact support, there is some $M > 0$ such that

$$\begin{aligned} &\frac{1}{M\sqrt{t}} \exp\left(-\frac{|x-y|^2 M}{t}\right) \chi(x-y) \\ &\quad \le p_t(x, y)\chi(x-y) \\ &\quad \le \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right) \chi(x-y). \end{aligned} \tag{5.12}$$

Let X be the solution to the martingale problem related to L with initial condition x_0.

At this level, we need to formulate a technical assumption (TA). It will suppose there are positive constants c, C, such that

$$c \le \frac{e^{\Sigma}}{\sigma} \le C. \tag{TA}$$

REMARK 5.5. If the condition of non explosion (3.16) stated in Proposition 3.13 is fulfilled then an easy calculation will show that (TA) is verified.

We will show that, under (TA), for $t > 0$, the law of X_t admits a density fulfilling the local Aronson estimates. However we first need to recall some notations and facts from Sections 2 and 3.

By Section 3, we know that, for $y_0 := h(x_0)$, we have $Y_t = y_0 + \int_0^t \sigma_0(Y_s)\, dW_s$, where W is a $\mathcal{F}_Y$-Brownian motion and $\sigma_0 = (\sigma h') \circ h^{-1}$. By the classical Itô formula, Y solves the martingale problem related to L^0, where

$$L^0 f = \frac{1}{2}\sigma_0^2 (h')^2 f''.$$

By Proposition 3.19, L^0 is also the infinitesimal generator of Y.

We denote again by I the image set of h. We consider again the C^1 application k defined at Lemma 2.16. We define $j = k \circ h^{-1}$; j maps I on J: we can easily establish that $j(0) = 0$ and $j'(y) = (1/\sigma_0^2)(y)$. We consider again the formal PDE operator

$$L^1 f = \left(\frac{\sigma_1^2}{2} f'\right)', \text{ where } \sigma_1 = (\sigma k') \circ k^{-1} = (\sigma_0 j) \circ j^{-1}.$$

The assumption (TA) on L implies that σ_1^2 is lower and upper bounded by a positive constant; so it fullfills the basic assumption of Lemma 5.1.

We set again $Z := k(X)$, so that $Z = j(Y)$.

Lemma 5.6. *Z solves the martingale problem related to L^1 with initial condition $z_0 := k(x_0) = j(y_0)$.*

Proof. Let $\tilde{f} \in \mathcal{D}_{L^1}(J)$. We know that $f = \tilde{f} \circ j \in C^2(I)$ by Proposition 2.13. Therefore we get

$$f(Y_t) = f(y_0) + \int_0^t f'(Y_s)\, dY_s + \int_0^t (L^0 f)(Y_s)\, ds.$$

Since $Y_t = j^{-1}(Z_t)$ and $\tilde{f} = f \circ j$, we conclude

$$\begin{aligned}\tilde{f}(Z_t) = \tilde{f}(z_0) &+ \int_0^t (\tilde{f} \circ j)'(Y_s)\, dY_s \\ &+ \int_0^t L^1(\tilde{f} \circ j)(j^{-1}(Z_s))\, ds,\end{aligned}$$

which completes the proof of the Lemma. □

Theorem 5.7. *Suppose that* (TA) *is verified.*
(i) *For every $t > 0$, the law of X_t has a density $p = p_t(x_0, \cdot)$.*
(ii) *p satisfies the local Aronson estimates and $(t, x, y) \mapsto p_t(x, y)$ is continuous from $]0, \infty[\times \mathbb{R}^2$ to $\mathbb{R}$.*

REMARK 5.8. Fabes and Kenig ([11]) prove the existence of a diffusion (with inhomogenous diffusion term) whose law density is singular with respect to Lebesgue measure (even if it is non-atomic). Theorem 5.7 tells us that this is not possible in the case of homogeneous coefficients.

Proof. The law of $Z_t^{z_0}$ $(t > 0)$ has a density $r_t(z_0, \cdot)$ by Corollary 5.4. Since $X = (h \circ j^{-1})(Z)$ and $Y = j^{-1}(Z)$, for $t > 0$ the law of X_t, resp. of Y_t, has a density $p_t(x_0, \cdot)$, resp. $q_t(y_0, \cdot)$, where $j(y_0) = (k \circ h)(x_0) = z_0$. Those densities can be calculated. In fact, if $f \in C^0$ is bounded and $Y^{y_0} = Y$, $Z^{z_0} = Z$, we get

$$\begin{aligned}\mathbb{E}(f(Y_t^{y_0})) &= \mathbb{E}(f \circ j^{-1}(Z_t^{z_0})) \\ &= \int (f \circ j^{-1})(z) r_t(z_0, z)\, dz \\ &= \int f(y) q_t(y_0, y)\, dy,\end{aligned}$$

where

$$q_t(y_0, y) = r_t(j(y_0), j(y))j'(y) = r_t(j(y_0), j(y))\frac{1}{\sigma_0^2(y)}. \tag{5.13}$$

In the same way, we verify

$$p_t(x_0, x) = r_t(k(x_0), k(x))k'(x). \tag{5.14}$$

This establishes (i) and (ii) of the theorem. □

Concerning the Dirichlet-Fukushima decomposition, it is possible to relax the technical assumption (TA). For $M > 0$, and a real function f we set

$$f^M(x) = \begin{cases} f(x) & \text{if } |x| \le M \\ f(M) & \text{if } x \ge M \\ f(-M) & \text{if } x \le -M \end{cases}$$

We can show that

$$\lim_{n\to\infty} \int_0^{\cdot} \frac{(b_n^M)'}{(\sigma_n^M)^2}(y)\, dy$$

is well-defined in C^0 (independently of the mollifier) and it equals Σ^M. It is obvious that for the PDE map $L(M)$, defined formally by

$$L(M)g = \frac{(\sigma^M)^2}{2} g'' + (b^M)' g',$$

the assumption (TA) is fulfilled.

We consider the event

$$\Omega_M = \{\omega : X_t(\omega) \in [-M, M], \forall t \in [0, T]\}$$

and the stopping time

$$\tau^M = \inf\{t \in [0, T] \mid X_t \notin [-M, M]\} \wedge (T + 1)$$

(τ^M) is a "suitable" sequence of stopping times.

Lemma 5.9. *Let $M > 0$ such that $x_0 \in\,]-M, M[$. On Ω_M, the process X coincides with the stopped processes X^{τ^M}. On the same event, this one coincides with a stopped process $X(M)^{\tau^M}$ where $X(M)$ is the solution to the martingale problem related to $L(M)$ with initial condition x_0.*

Proof. Proposition 3.2 allows us to consider the stochastic differential equation

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s)\,dW_s,$$

which is solved by $Y := h(X)$. The time changed process

$$B_t := Y_{T_t},$$

where $T_t = A_t^{-1}$ is the inverse of $A_t := \int_0^t \sigma_0^2(Y_s)\,ds$, is easily checked to be a Brownian motion. Furthermore, by [8, Proposition 5.2], we know

$$T_t = \int_0^t \frac{1}{\sigma_0^2}(B_s)\,ds.$$

Now we define

$$\sigma_0^{(M)}(y) = \begin{cases} \sigma_0(y) & \text{if } |y| \le h(M) \\ \sigma_0(M) & \text{if } y \ge h(M) \\ \sigma_0(-M) & \text{if } y \le h(-M) \end{cases}$$

and consider

$$T_t^{(M)} := \int_0^t \frac{1}{(\sigma_0^{(M)})^2}(B_s)\,ds$$

and $A_t^{(M)} := T_t^{(M)^{-1}}$. By [8, Proposition 5.2], the process $Y(M)_t := B_{A_t^{(M)}}$ then solves the stochastic differential equation

$$Y(M)_t = Y_0 + \int_0^t \sigma_0^{(M)}(Y(M)_s)\,d\tilde{W}_s$$

for some Brownian motion $\tilde{W}$. From $B_t = Y_{T_t}$ we deduce $T_t^{(M)} = T_t$ on $\{t < A_{\tau_m}\}$, hence

$$A_t = A_t^{(M)} \quad \text{on} \quad \{t < \tau_m\}.$$

Thus, we conclude $Y_{t\wedge\tau_m} = Y(M)_{t\wedge\tau_m}$. For a more detailed discussion on construction of solutions to SDEs without drift we refer to [8]. $\square$

We formulate now the two dimensional marginal laws of a solution to the martingale problem related to L.

Proposition 5.10. *Let $X = X^{x_0}$ be a solution to the martingale problem related to L with initial condition x_0 such that* (TA) *is realized. The joint law of* $(X_s^{x_0}, X_T^{x_0})$,

$0 < s < T$, has a density given by

$$(x_1, x_2) \mapsto p_s(x_0, x_1) p_{T-s}(x_1, x_2).$$

Proof. Let $f \in C^0(\mathbb{R}^2)$ with compact support. We have to evaluate

$$\mathbb{E}(f(X_s, X_T)) = \mathbb{E}(\mathbb{E}(f(X_s, X_T)|X_s))$$

for $0 < s < T$. In order to calculate the previous conditional expectation we need some preliminary results. The first one is an adaptation of Theorem 3.21. $\square$

Lemma 5.11. *Let $u \in \mathcal{U}_L$ such that $(\partial_t + L)u = 0$ holds in the C^0-generalized sense. Then we have*

$$\mathbb{E}(u(T, X_T)|\mathcal{F}_s^X) = u(s, X_s)$$

a.s. In particular, $(u(t, X_t))$ is a $\mathcal{F}^X$-martingale.

Proof. The same as for Theorem 3.21, but we take conditional expectations instead of expectations on X^{x_0} starting from zero instead of s. $\square$

We focuse now on the case that L is of divergence type. If $p_t(x, y)$ is the fundamental solution associated with L, we set

$$u(x_1, t, z) := \begin{cases} \int dx \, p_{T-t}(x, z) f(x_1, x) & : t < T; \\ f(x_1, z) & : t = T. \end{cases}$$

We already know $u(x_1, \cdot) \in \mathcal{U}_L$ by Proposition 5.3 and $(\partial_t + L)u(x_1, t, \cdot) = 0$ in the C^0-generalized sense. Using the above lemma, we now have

$$\begin{aligned} \mathbb{E}(f(X_s, X_T)) &= \mathbb{E}(\mathbb{E}(f(X_s, X_T)|X_s)) \\ &= \mathbb{E}(u(X_s, s, X_s)) \\ &= \int dx_1 \, p_s(z, x_1) u(x_1, s, x_1) \\ &= \int dx_1 \, p_s(z, x_1) \int dx_2 \, p_{T-s}(x_2, x_1) f(x_1, x_2). \end{aligned}$$

This proves the result if L is of divergence type.

In the general case, we set again $Z := k(X)$, where $k \in C^1(\mathbb{R})$ and L^1 are defined in Lemma 2.16 and recalled before Lemma 5.6. If $f \in C^0(\mathbb{R}^2)$ with compact support then we have

$$\mathbb{E}(f(X_s, X_T)) = \mathbb{E}(f(k^{-1}(Z_s), k^{-1}(Z_T)))$$

$$= \int dz_1\, dz_2\, r_s(z_0, z_1) r_{T-s}(z_1, z_2) \tag{5.15}$$
$$f(k^{-1}(z_1)) f(k^{-1}(z_2)),$$

where $(r_t(z, \cdot))$ is the law density of Z_t^z which solves the martingale problem related to L^1 which is of divergence type.

According to (5.14), (5.15) equals

$$\int dz_1\, dz_2\, p_s(x_0, k^{-1}(z_1)) p_{T-s}(k^{-1}(z_1), k^{-1}(z_2)) k'(k^{-1}(z_1)) k'(k^{-1}(z_2)).$$

Using the change of variables $x_i = k^{-1}(z_i)$, $i = 1, 2$, we complete the proof. □

We conclude the paper with the following theorem on the Dirichlet-Fukushima structure of $f(X)$, where X is the solution to the martingale problem related to L.

Theorem 5.12. *Let $X = X^{x_0}$ be a solution to the martingale problem related to L with initial condition x_0 and $f \in W^{1,2}_{\mathrm{loc}}$. Then $f(X)$ is an $(\mathcal{F}_t)$-Dirichlet process with martingale part*

$$M_t^f = \int_0^t f'(X_s)\sigma(X_s)\, dW_s.$$

Proof. We recall that X is $(\mathcal{F}_t)$-Dirichlet with martingale part $M_t = \int_0^t \sigma(X_s)\, dW_s$. First we will assume that the technical assumption (TA) is verified. We aim to apply Theorem 4.3. For this we need to check conditions H1), H2), H3) of previous section.

Proposition 5.10 says that

$$q_{s,t}(x_1, x_2) = p_s(x_0, x_1) p_{t-s}(x_1, x_2),$$

so H1) is verified.

Let $T, M > 0$. Let $x_1, x_2 \in \mathbb{R}$, $\delta > 0$. Using local Aronson estimates it is immediate to show that the quantity

$$\int_0^T q_{s,s+\delta}(x_1, x_2)\, ds = p_\delta(x_1, x_2) \int_0^T p_s(x_0, x_1)\, ds$$

is well defined and

$$\sup_{|x_1| \le M} \int_0^T p_s(x_0, x_1)\, ds < \infty;$$

also there is a constant $C > 0$ such that, for $\delta > 0$

$$\sqrt{\delta}\, p_\delta(x_1, x_2) \le C \exp\left(-\frac{(x_2 - x_1)^2}{C\delta}\right) \le C$$

for $|x_1|, |x_2| \leq M$. Consequently $f(X)$ is an $(\mathcal{F}_t)$-Dirichlet process if assumption (TA) is verified.

Suppose now that (TA) is not verified, and let $M > 0$ with $|x_0| < M$. We define Ω_M, τ^M, $L(M)$ as before Lemma 5.9. According to the same lemma, on Ω_M, the stopped process X^{τ^M} coincides with $X(M)^{\tau^M}$ where $X(M)$ solves a martingale problem related to $L(M)$. Using local Aronson estimates we easily get that

$$\int_0^T f'(X_s)^2\, ds = \int_0^T f'(X(M)_s)^2\, ds < \infty$$

almost surely. Therefore

$$M_t^f = \int_o^t (f'\sigma)(X_s)\, dW_s$$

is well-defined. We define

$$A_t^f = f(X_t) - f(x_0) - M_t^f$$

Now A^f is a zero quadratic variation process since, on Ω_M, it coincides with

$$A_t^{M,f} := f(X(M)_t) - f(x_0) - \int_0^t (f'\sigma)(X(M)_s)\, dW_s$$

and assumption (TA) is fulfilled for $L(M)$. $\square$

Corollary 5.13. *Let X be a solution to the martingale problem related to L.*
(i) *The map $\mathcal{A}\colon \mathcal{D}_L \longrightarrow \mathcal{C}$ defined by $\mathcal{A}(f) = \int_0^{\cdot}(Lf)(X_s)\, ds$ can be extended continuously to $W_{\mathrm{loc}}^{1,2}$.*
(ii) *If L is close to divergence type, then $A\colon l \longrightarrow \int_0^{\cdot} l'(X_s)\, ds$ can be continuously extended from C^1 to L_{loc}^2.*

REMARK 5.14. Point (ii) means that when L is close to divergence type, the term $\int l'(a) L_t^X(da)$ can even be extended to L_{loc}^2.

Proof of the Corollary. Point (i) is an immediate consequence of Theorem 5.12. In fact the extension will be given by

$$\mathcal{A}(f) = f(X) - f(X_0) - \int_0^{\cdot} f'(X_s)\sigma(X_s)\, dW_s.$$

Concerning point (ii), Proposition 2.17 implies that for $l \in L_{\mathrm{loc}}^2$

$$A(l) = \mathcal{A}(\tilde{T} l),$$

where $\tilde{T}$ is the inverse of the extension of the operator $\mathcal{L}$ to $W^{1,2}_{\mathrm{loc}}$. Since $\mathcal{A}$ and $\tilde{T}$ are continuous, the result follows. □

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