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# PERFECT CATEGORIES I

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Let $R$ be a ring with identity. We assume that an $R$-module $M$ has two decompositions: $M=\sum_{\alpha\in I}\oplus M_\alpha=\sum_{\beta\in J}\oplus N_\beta$, where $M_\alpha$'s and $N_\beta$'s are completely indecomposable. Then it is well known as the Krull-Remak-Schmidt-Azumaya's theorem that $M$ satisfies the following two conditions:

I. *The decompositions are unique up to isomorphism.*

II$'$. *For a given finite set* $\{N_{\beta_1},\cdots,N_{\beta_n}\}$ *we can find a set* $\{M_{\alpha_1},\cdots,M_{\alpha_n}\}$ *such that* $M=N_{\beta_1}\oplus\cdots\oplus N_{\beta_n}\oplus\sum_{\alpha\neq(\alpha_i)}\oplus M_\alpha$ *and* $N_{\beta_i}\approx M_{\alpha_i}$ *for* $i=1,2,\cdots,n$ (*or* $M=M_{\alpha_1}\oplus\cdots\oplus M_{\alpha_n}\oplus\sum_{\beta\neq(\beta_i)}\oplus N_\beta$).

Those facts were generalized in a Grothendieck category $\mathfrak{A}$ by P. Gabriel, [5]. Recently, the author and Y. Sai have treated

II. *The condition* II$'$ *is true for any infinite subset* $\{N_{\beta_i}\}$,

in a case of modules in [7], and shown that Condition II is satisfied for any $M$ in the induced full subcategory $\mathfrak{B}$ from $\{M_\alpha\}$ in the category $\mathfrak{M}_R$ of $R$-modules if and only if $\{M_\alpha\}$ is an elementwise $T$-nilpotent system with respect to a certain ideal $\mathfrak{C}$ of $\mathfrak{B}$. Furthermore, the author and H. Kanbara have shown in [10] and [12] that Condition II is satisfied for a given $M$ if and only if $\{M_\alpha\}$ is an elementwise semi-$T$-nilpotent system with respect to $\mathfrak{C}\cap\mathrm{Hom}_R(M,M)$.

Conditions I and II$'$ are categorical and hence, we can easily generalize the arguments in modules to those in $\mathfrak{A}$ (see [5] and [7]). However, the definition of the elementwise $T$-nilpotency is not categorical. Therefore, we treat, in this paper, a Grothendieck category with a generating set of small objects, e.g. $\mathfrak{M}_R$, locally noteherian categories and functor categories of small additive categories to the category $Ab$ of abelian groups.

We shall show in the section two that almost all of essential properties in [7], [8], [9], [10], [11] and [12] are valid in such a category.

In the final section, making use of such generalized properties, we define perfect (resp. semi-prefect) Grothendieck categories $\mathfrak{A}$ and give a characterization of them with respect to a generating set of $\mathfrak{A}$. This characterization gives us a generalization of [2]. Theorem $P$ for $(\mathfrak{C}, Ab)$, where $\mathfrak{C}$ is an amenable additive small category. Especially, if $\mathfrak{C}$ is a full additive subcategory with finite coproducts of finitely generated abelian groups, we show that $(\mathfrak{C}, Ab)$ is perfect if and only if the complete isomorphic class of indecomposable $p$-torsion groups in $\mathfrak{C}$ is finite for every prime $p$.

## 1. Perliminary results

Let $\mathfrak{A}$ be a Grothendieck category, namely a complete, co-complete $C_3$-abelian category (see [14], Chap. III). We call an object $A$ in $\mathfrak{A}$ *samll* if $[A, \sum\oplus -]\approx\sum\oplus[A, -]$ and call $\mathfrak{A}$ *quasi-small* if every object $A$ in $\mathfrak{A}$ is a union of some small subobjects $A^{\alpha}$ in $A$: $A=\bigcup_{\alpha} A^{\alpha}$.

If $\mathfrak{A}$ has a generating set of small objects, then $\mathfrak{A}$ is quasi-small. For example, the category $\mathfrak{M}_R$ of modules over a ring $R$ is quasi-small and more generally the functor category $(\mathfrak{C}, Ab)$ and its full subcategory $L(\mathfrak{C}, Ab)$ of left exact functors are quasi-small, where $\mathfrak{C}$ is a small additive category and $Ab$ is the category of abelian groups, (cf. [13], p. 109, Theorem 5.3 and p. 99, Proposition 2.3). It is clear that if $\mathfrak{A}$ is locally noetherian (see [4], p. 356), then $\mathfrak{A}$ is quasi-small.

By $J(A)$ we denote the Jacobason radical for any object $A$ in $\mathfrak{A}$, i.e. $J(A)=\cap N$, where $N$ runs through all maximal subobjects in $A$ and $J(A)=A$ if $A$ does not contain any maximal subobjects. $A$ is called *finitely generated* if $A=\bigcup_{\alpha\in I} A_{\alpha}$ for some subobjects $A_{\alpha}$ of $A$, then $A=\bigcup_{\beta\in J} A_{\beta}$ for a finite subset $J$ of $I$.

Let $N$ be a subobject in $M$. $N$ is called *samll in $M$* if $N+T=M$ implies $T=M$ for any subobject $T$ in $M$. Following to [13], we define a semi-perfect (resp. perfect) objest $P$ in $\mathfrak{A}$. $P$ is called *semi-perfect* (resp. *perfect*) if $P$ is projective and every factor object of $P$ has a projectove cover (resp. any coproduct of copies of $P$ is semi-perfect).

From the proof of Lemma in [16], we have

**Lemma 1.** *Let $P$ be a projective object in an abelian category $\mathfrak{C}$. Then* $J([P, P])=\{f\mid\in[P, P], \operatorname{Im} f$ *is samll in* $P\}$.

**Proposition 1.** *Let $P$ be a projective object in the Grothendieck category $\mathfrak{A}$. Then the following statements are equivalent.*

1) *$S_P=[P, P]$ is a local ring; $S_P/J(S_P)$ is a division ring.*
2) *Every proper subobject in $P$ is small in $P$.*
3) *$P$ is semi-perfect and directly indecomposable.*

(cf. [8], Theorem 5).

Proof. 1)$\rightarrow$2). Since $S_P$ is local, $J(S_P)$ consists of all non-isomorphisms. Let $N$ be a proper subobject of $P$ and assume $P=T+N$. Since $P/T\overset{\varphi}{\approx}N/N\cap T$, we have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & T\cap N \arrow[r] & N \arrow[r, "\nu'"] & N/N\cap T \arrow[r] & 0 \\
 & & & P/T \arrow[u, "\varphi"'] & \\
 & & & P \arrow[u, "\nu"'] \arrow[uul, dashed, "\alpha"] &
\end{tikzcd},$$

where $\nu$ and $\nu$ are the canonical epimorphisms. Since $P$ is projective, we obtain $\alpha\in[P, N]\subseteq S_P$ such that $\nu'\alpha=\varphi\nu$. Since $N\neq P$, $\alpha\in \mathrm{J}(S_P)$. Hence, $N=\mathrm{Im}\,\alpha+T\cap N$ and $P=\mathrm{Im}\,\alpha+T$. Therefore, $P=T$ by Lemma 1.

2)$\to$1). Let $f$ be not isomorphic. If $\mathrm{Im}\,f=P$, $P=P_0+\mathrm{Ker}\,f$. Since $\mathrm{Ker}\,f$ is proper, $\mathrm{Ker}\,f$ is small in $P$, which is a contradiction. Hence, $\mathrm{Im}\,f\neq P$. Let $g$ be another non-isomorphism. Since $\mathrm{Im}\,f$ and $\mathrm{Im}\,g$ are samll in $P$, $P\neq\mathrm{Im}\,f+\mathrm{Im}\,g\supseteq\mathrm{Im}\,(f+g)$. Hence, $S_P$ is a local ring.
2)$\to$3). It is clear from the definition.
3)$\to$2). Let $T$ be a proper subobject of $P$ and $P'\to P/T$—$0$ a projective cover of $P/T$. Since $P$ is indecomposable, $P\approx P'$. Hence, $T$ is small in $P$.

For the rest of this section, we always assume that the abelian category $\mathfrak{A}$ is quasi-small in the sense given in the beginning of this section.

We shall generalize the notions of summability and elementiwse $T$-nilpotent systems in $\mathfrak{M}_R$ to a case of quasi-small categories, (cf. [7] and [8]).

A set of morphisms $\{f_\beta\}_{\beta\in K}$ of an object $L$ to an object $Q$ is called *summable* if for any small subobject $L^n$ in $L$ $f_\beta|L^n=0$ for almost all $\beta\in K$. Let $M=\sum_I\oplus M_\alpha$ and $N=\sum_J\oplus N_\beta$ be two coproducts in $\mathfrak{A}$, and let $i_\alpha$, $p_\beta$ be an injection $M_\alpha$ to $M$ and a projection of $N$ to $N_\beta$, respectively. Let $f$ be any element in $[M, N]$ and put $f_{\beta\alpha}=p_\beta f i_\alpha$. If $M_\alpha^n$ is a small subobject of $M_\alpha$, $f_{\beta\alpha}|M_\alpha^n=0$ for almost all $\beta$. Therefore, the $\{f_{\beta\alpha}\}_\beta$ is a set of summable morphisms of $M_\alpha$ to $N$. Conversely, let $\{f_{\beta\alpha}\}_{\beta\in J}$ be a set of summable morphisms of $M_\alpha$ to $N$ and $M_\alpha=\cup M_\alpha^n$, where $M_\alpha^n$'s are small subobjects in $M_\alpha$. Since a finite union of small subobjects is again small, we assume $\{M_\alpha^n\}$ forms a directed family and $M_\alpha=\varinjlim M_\alpha^n$. Furthermore, $\sum_{\beta\in I} f_{\beta\alpha}|M_\alpha^n$ gives an element in $[M_\alpha, N]$. Hence, we have a unique element $f$ in $[M, N]$ such that $fi_\alpha^n=\sum f_{\beta\alpha}|M_\alpha^n$. Thus, we have

**Lemma 2.** *Let $M_i=\sum_{\alpha_i\in I_i}\oplus M_{i\alpha_i}$ be objects in the quasi-small category $\mathfrak{A}$ for $i=1, 2$ and 3. Then $[M_1, M_2]$ is isomorphic to the set of row summable matrices with entries $a_{\alpha_j\alpha_i}$. Furthermore, the composition $[M_2, M_3]\,[M_1, M_2]$ corresponds to the product of martices, where $a_{\alpha_j\alpha_i}\in[M_{i\alpha_i}, M_{j\alpha_j}]$.*

**Corollary 1.** *Let $P$ be projective and directly indecomposable object in $\mathfrak{A}$ with a set of small generators. If $S_P=[P, P]$ is a local ring, then $P$ is semi-perfect and $\mathrm{J}(P)$ is a unique maximal subobject of $P$. Hence, $P$ is finitely generated.*

Proof. Let $Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset \cdots$ be a series of proper subobjects in $P$. If $P = \cup Q_j$, we have a diagram

$$\begin{tikzcd} \sum \oplus Q_j \arrow[r, "\nu"] & P \arrow[r] & 0 \\ & P \arrow[ul, dashed, "f"] \arrow[u, equal, "1_P"'] & \end{tikzcd} \qquad \text{(exact)}$$

, where $\nu$ is given naturally by inclusions. We obtain $f \in [P, \sum \oplus Q_j]$ such that $\nu f = 1_P$ and put $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\alpha \\ \vdots \end{pmatrix}$ and $\nu = (i_1, i_2, \cdots, i_\alpha, \cdots)$. Then $1_P = \sum i_\alpha f_\alpha$. However, any of $f_\alpha$'s is not isomorphic, which is a contradiction (cf. [1]). Hence, we have a maximal subobject by the Zorn's lemma. Therefore, $\mathrm{J}(P)$ is a unique maximal, subobject of $P$ by Proposition 1.

**Corollary 2** ([6], Theorem 2.8.) *Let $P$ be projective and artinian, then $P$ is finitely generated, and $S_P$ is right artinian.*

Proof. Since $S_P$ is a semi-primary ring by [5], Proposition 2.7, it is clear from the above corollary. $S_P$ is right artinian from [6], Lemma 2.6.

## 2. Coproducts of completely indecomposable objects

We studied Krull-Remak-Schmidt-Azumaya's theorm for a direct decomposition of a module as completely indecomposable modules in [7], [8], [10] and [12]. We shall generalize many results in a case of modules to a case of Grothendieck abelian categories $\mathfrak{A}$ with a set of small generators.

An object $M$ in $\mathfrak{A}$ is called *completely indecomposable* if $S_M = [M, M]$ is a local ring. The following lemma was given in [7], p. 343, Remark 4 without proof. We shall give here its proof for the sake of completeness.

**Lemma 3.** *Let $M = \sum_{i=1}^{\infty} \oplus M_i$ and $M_i$'s be completely indecomposable objects in a $C_3$-abelian category $\mathfrak{C}$. Let $\{f_i\}_{i=1}^{n}$ be a set of morphisms $f_i \in [M_i, M_{i+1}]$. Put $M_i' = \mathrm{Im}(1_{M_i} + f_i)$. Then $M_t \cap (M'_{i_1} + M'_{i_2} + \cdots + M'_{i_s}) \subseteq \mathrm{Ker}(f_n f_{n-1} \cdots f_t)$ for $1 \leq t \leq n$ and $(i_1, i_2, \cdots, i_s) \subseteq (1, 2, \cdots, n)$ and $M_t \cap (M_i + \sum_{j=1}^{n} M_j') \subseteq \mathrm{Im}(f_{t-1} \cdots f_1) + \mathrm{Ker}(f_n \cdots f_t)$ for $i \leq t \leq n$.*

Proof. We take $\{M_i\}_{i=1}^{n+1}$ and we construct a small full subcategory $\mathfrak{C}_0$ such that $\mathfrak{C}_0$ contains all $M_i'$ and kernels and images in $\mathfrak{C}_0$ are those in $\mathfrak{C}$, (see [14], p. 101, Lemma 2.7). Then there exists an exact covariant imbedding functor of $\mathfrak{C}_0$ to $Ab$ by [14], p. 101, Theorem 2.6. Hence, we may assume that all of $M_i$ are abelian groups. In this case the lemma is clear.

We shall make use of the same condition I. II and III given in [7], p. 331–332, (see the introduction). Condition I is satisfied for any two decompositions as coproducts of completely indecomposable objects in an arbitrary Grothendieck category (see [5] or [8], Theorem 7′). We are now interested in Condition II.

From now on we assume that a Grothendieck category $\mathfrak{A}$ has a generating set of small objects, namely quasi-small in the sense of §1.

First, we shall generalize the notions of elementwise semi-$T$-nilpotent system defined in [7] and [8].

Let $\mathfrak{C}$ be an ideal in $\mathfrak{A}$. We take a set of objects $\{M_\alpha\}$ and consider morphisms $f_{\alpha_i}: M_{\alpha_i}\to M_{\alpha_{i+1}}$, which belong to $\mathfrak{C}$. If for any small subobject $M^n_{\alpha_1}$ of $M_{\alpha_1}$ there exists $m$ such that $f_{\alpha_m}f_{\alpha_{m-1}}\cdots f_{\alpha_1}|M^n_{\alpha_1}=0$, we call $\{f_{\alpha_i}\}$ a *locally right T-nilpotent system* (with respect to $\mathfrak{C}$). If for any subset $\{M_\alpha\}$ and any set $\{f_{\alpha_i}\}$, $\{f_{\alpha_i}\}$ is locally rihgt $T$-nilpotent system, we call $\{M_\alpha\}$ is a *locally right T-nilpotent system.* If $\alpha_i\neq\alpha_j$ for $i\neq j$ in the above, we call $\{f_{\alpha_i}\}$ and $\{M_\alpha\}$ *locally right semi-T-nilpotent systems.* Similarly, if we replace $f_{\alpha_i}$ by $g_{\alpha_i}: M_{\alpha_{i+1}}\to M_{\alpha_i}$ and $g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_m}=0$ for some $m$, we call $\{g_\alpha\}$ left $T$-nilpotent.

If we replace *elementwise* (*semi-*) *T-nilpotent system* by *locally right* (*semi-*) *T-nilpotent systems* in the arguments in [7], [8], [9], [10] and [12], we know that many results in them are valid in $\mathfrak{A}$ without changing proofs. For instance, in order to prove the same result of [7], Lemma 9 for $\mathfrak{A}$, we can replace the relations 2) and 3) in [7], p. 336 by Lemma 3 and elements $x$ by small subobjects, and we use the same argument, taking a projection of $M$ to $M_n$ if necessary.

Let $\{M_\nu\}$ be a set of completely indecomposable objects and $\mathfrak{B}$ be the induced full additive category from $\{M_\alpha\}$: objects of $\mathfrak{B}$ consist of all coproducts of some $M_\alpha$ (and their all isomorphic images). We can express all morphisms in $\mathfrak{B}$ by row summable matrices $(a_{\beta\alpha})$ by Lemma 2. We define an ideal $\mathfrak{C}$ of $\mathfrak{B}$ as follows: $\mathfrak{C}$ consists of all morphisms $(a_{\beta\alpha})$ such that $a_{\gamma\delta}: M_\delta\to M_\gamma$ is not isomorphic for all $\gamma$, $\delta$. Then we have from Theorem 9 in [7].

**Theorem 1.** *Let $\mathfrak{A}$ be a Grothendieck category with a generating set of small objects, and $\mathfrak{B}$ the induced full subadditive category from a set of completely indecomposable objects $M_\alpha$. Then the following statements are equivalent.*

1) *For any two decompositions $M=\sum_I\oplus Q_\alpha=\sum_J\oplus N_\beta$ of any object $M$ in $\mathfrak{B}$, Condition* II *in* [7] *is satisfied, where $Q_\alpha$, $N_\beta$ are indecomposable.*

2) *The ideal $\mathfrak{C}$ in $\mathfrak{B}$ defined above is the Jacobson radical of $\mathfrak{B}$.*

3) *$\{M_\alpha\}$ is a locally right T-nilpotent system.*

Similarly from [12], Theorem or [10], Lemma 5 we have

**Theorem 2.** *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be as above. Then the following statements are equivalent.*

1) *For given two decompositions $M=\sum_I\oplus Q_\alpha=\sum_J\oplus N_\beta$ of a given object $M$*

*in* $\mathfrak{B}$, *Condition* II *is satisfied, where* $Q_\alpha$, $N_\beta$ *are indecomposable.*

2) $\mathfrak{C}\cap S_M=\mathrm{J}(S_M)$, *where* $S_M=[M,M]$

3) $\{Q_\alpha\}_I$ *is a locally right semi-T-nilpotent system with respect to* $\mathfrak{C}$.

REMARK. Using Lemmas 2 and 3, we can obtain theorems concerned with exchange properties in $\mathfrak{A}$ in [6] and [9] if we replace *semi-T-nilpotent* by *locally right semi-T-nilpotent*.

## 3. Perfect categories

H. Bass defined a perfect or semi-perfect ring in [2]. Recently, M. Weidenfeld and G. Weidenfeld have generalized them to a functor category ($\mathfrak{C}$, $Ab$) of an additive category $\mathfrak{C}$ in [17].

We shall define a perfect or semi-perfect Grothendieck category $\mathfrak{A}$ and study some properties of $\mathfrak{A}$, which are analogous to ones in [2], as an application of §2.

Let $\mathfrak{A}$ be a Grothendieck category. $\mathfrak{A}$ is called *perfect* (resp. *semi-perfect*) if every (resp. finitely generated) object $A$ in $\mathfrak{A}$ has a projective cover (cf. [2]).

Let $\mathfrak{A}'$ be the spectral Grothedieck category given in [7], p. 331, Example 2. Then every object in $\mathfrak{A}'$ has a trivial projetive cover and hence, $\mathfrak{A}'$ is perfect. However, $\mathfrak{A}'$ has completely different properties from ones in $\mathfrak{M}_R$, where $R$ is a right perfect ring.

We are interested, in this section, in perfect categories with similar properties of perfect rings. Hence, in order to exclude such a special perfect category we assume that $\mathfrak{A}$ is quasi-small, namely $\mathfrak{A}$ has a generating set of small objects.

As seen in [2] and [13], the fact $P\neq\mathrm{J}(P)$ for a projective $P$ in $\mathfrak{A}$ is very important to study perfect categories. In the spectral category $\mathfrak{A}'$ above, this fact is not true. On the other hand, that fact was shown in $\mathfrak{M}_R$ and noted in ($\mathfrak{C}$, $Ab$) by [2] and [17], respectively.

We first generalize them as follows:

**Proposition 2.** *Let* $\mathfrak{A}$ *be a Grothendieck category and* $A$ *an object in* $\mathfrak{A}$. *If* $A$ *is a retract of a coproduct of either*

a) *porjective objects* $P$ *such that* $\mathrm{J}(P)$ *is small in* $P$, *or*

b) *noetherian objects,*

*then* $A\neq\mathrm{J}(A)$.

We need two lemmas for the proof, the first of which is well known.

**Lemma 4.** *Let* $P$ *be a small and projective object in* $\mathfrak{A}$. *Then* $P$ *is finitely generated and* $\mathrm{J}(P)$ *is small in* $P$.

See [3], p. 105.

**Lemma 5.** *Let* $\{A_i\}_I$ *be a family of objects in* $\mathfrak{A}$ *such that* $[A_i, \mathrm{J}(A_i)]$ *is* *contained in* $\mathrm{J}([A_i, A_i])$ *for all* $i\in I$. *Put* $A=\sum_{\alpha\in I}\oplus A_i$. *Then for* $f\in[A, A]$ *with* $\mathrm{Ker}(1-f)\neq 0$, $\mathrm{Im} f\neq \mathrm{J}(\mathrm{Im} f)$.

Proof. Put $B=\mathrm{Im} f$ and assume $B=\mathrm{J}(B)$. Since $\mathrm{J}(B)\subset \mathrm{J}(A)$, $f\in[A, \mathrm{J}(A)]$. $\mathrm{Ker}(1-f)\neq 0$ from the assumption and hence, $\mathrm{Ker}(1-f)\cap\sum_{i=1}^{n}\oplus A_{\alpha_1}\neq 0$ for some finite indeces $\alpha_i\in I$. Let $e_1$ be the projection of $A$ to $A_{\alpha_i}$. Since $f\in[A, \mathrm{J}(A)]$, $e_1fe_1|A_{\alpha_1}\in[A_{\alpha_1}, \mathrm{J}(A_{\alpha_1})]\subset \mathrm{J}(S_{A_{\alpha_1}})$. Hence, $e_1(1-f)e_1|A_{\alpha_1}=(e_1-e_1fe_1)|A_{\alpha_1}$ is automorphic. Therefore, $A=(1-f)(A_{\alpha_1})\oplus\mathrm{Ker}\, e_1=(1-f)(A_{\alpha_1})\oplus\sum_{\beta\neq\alpha_1}\oplus A_\beta$ and $A_{\alpha_1}\overset{1-f}{\approx}(1-f)(A_{\alpha_1})$. Let $e_2$ be the projection of $A$ to $A_{\alpha_2}$ in the above decompositon. Then we obtain $A=(1-f)(A_{\alpha_1})\oplus(1-f)(A_{\alpha_2})\oplus\sum_{\beta\neq\alpha_1,\alpha_2}\oplus A_\beta$ and $A_{\alpha_2}\overset{1-f}{\approx}(1-f)(A_{\alpha_2})$. Repearting this argument, we know that $(1-f)|\sum_{i=1}^{n}\oplus A_{\alpha_i}$ is isomorphic, which is a contraadiction, (this argument is due to [1]).

Proof of Proposition 2. It is clear for the case a) from Lemmas 4 and 5 and [10], Proposition 1. Let $A$ be a noetherian object. Then $A\neq \mathrm{J}(A)$ and $\mathrm{J}(A)$ is small in $A$. Hence, $1-f$ is epimorphic for any $f$ in $[A, \mathrm{J}(A)]$. Therefore, $1-f$ is unit, since $A$ is noetherian. Thus, $[A, \mathrm{J}(A)]\subset \mathrm{J}(S_A)$.

**Corollary 1** ([2] and [17]). *Let* $\mathfrak{A}$ *be a Grothendieck category which is one of the following types:*

a) $\mathfrak{M}_R$ *for some ring* $R$,

b) $(\mathfrak{C}, Ab)$, *where* $\mathfrak{C}$ *is a small additive category,*

c) *Locally noetherian.*

*Then* $P\neq \mathrm{J}(P)$ *for every non-zero projective obejcet* $P$.

**Corollary 2.** *Let* $\mathfrak{C}$ *be an artinian abelian category and* $\mathrm{L}(\mathfrak{C}, Ab)$ *the left exact functor category. Then* $Q\neq \mathrm{J}(Q)$ *for every retract* $Q$ *of any coproduct of generators* $\{\mathrm{H}^A\}_{A\in\mathfrak{C}}$, *where* $\mathrm{H}^A(-)=[A, -]$.

Proof. $\mathrm{L}(\mathfrak{C}, Ab)$ is locally noetherian by [4], Proposition 7 in p. 356.

For the study of perfect categories, we recall an induced category. Let $\{M_\alpha\}_I$ be a given set of some objects in a Grothendieck category $\mathfrak{A}$. By $\mathfrak{C}_f$ we denote the full subadditive category of $\mathfrak{A}$, whose objects consist of all finite coproducts of $M_\alpha'$ which is isomorphic to some $M_\beta$ in $\{M_\alpha\}_I$. We call $\mathfrak{C}_f$ the *finitely induced additive category from* $\{M_\alpha\}$, (see [7]). If all $M_\alpha$ are completely indecomposable, $\mathfrak{C}_f$ is amenable (see [3], p. 119) by [7], Theorem 7′.

Let $A$ be an object in $\mathfrak{A}$. By $\mathrm{S}(A)$ we denote the socle of $A$, namely $\mathrm{S}(A)$ $=$the union of all minimal subobjects in $A$.

Following to [15], we call $\mathfrak{A}$ *semi-artinian* if $\mathbf{S}(A)\neq 0$ for all non-zero object $A$ in $\mathfrak{A}$.

If $\mathfrak{A}$ is a Grothendieck category with a generating set of small projective, then $\mathfrak{A}$ is equivalent to $(\mathfrak{C}, Ab)$ by Freyd's theorem (see [14], p. 109, Theorem 5.2), where $\mathfrak{C}$ is a small additive category. In this case, $\mathfrak{A}$ is also equivalent to a subcategory of modules by [4]. We give here categorical proofs in the following for some properties in $\mathfrak{A}$, however we note that we can prove them ring-theoretical (see Remark below).

First, we generalize [15], Proposition 3.2.

**Proposition 3.** ([15]). *Let $\mathfrak{A}$ be a Grothendieck category with a generating set $\{P_\alpha\}$ of small projective. Then $\mathfrak{A}$ is semi-artinian if and only if* 1) *$\{P_\alpha\}$ is a left $T$-nilpotent system with respect to* $\mathrm{J}(\mathfrak{A})$ *and* 2) $\mathrm{S}(A)\neq 0$ *for every non-zero quotient object $A$ of $P_\alpha/J(P_\alpha)$ for all $\alpha$.*

Proof. If $\mathfrak{A}$ is semi-artinian, 2) is clear. The following agrument is similar to one in [2], p. 470. Let $\{f_i\}$ be a set in $\mathrm{J}(\mathfrak{A})$ and $f_i\colon P_{i+1}\to P_i$. We define inductively a series of subobjects $K_\alpha$ of $P_{\alpha 1}$ as follows: $K_0=0$, $K_1=\mathrm{S}(P_1)$, $K_2/K_1=S(P_1)$, $\cdots$. If $\alpha$ is a limit, $K_\alpha=\bigcup_{\beta<\alpha} K_\beta$. Since $\mathfrak{A}$ is a Grothendieck category, $P_1=K_\gamma$ for some $\gamma$. Put $I_i=\mathrm{Im}\, f_1f_2\cdots f_i$. Then $I_i$ is finitely generated, since so is $P_{i+1}$. Let $\alpha_i$ be the least number such that $K_{\alpha_i}\supset I_i$. If $\alpha_i$ is a limit, then $I_i=\bigcup_{\beta<\alpha_i}(K_\beta\cap I_i)$ and hence, $I_i\subset K_\beta$ for some $\beta<\alpha_i$. Therefore, we can express $\alpha_i=\delta_i+1$. Since $K_{\alpha_i}/K_{\delta_i}$ is semi-simple, $\mathrm{J}(K_{\alpha_i}/K_{\delta_i})=0$ and $\mathrm{Im}\, f_{i+1}\subset \mathrm{J}(P_i)$ by Lemma 1. Hence, $\mathrm{Im}\, f_1f_2\cdots f_{i+1}=\mathrm{Im}\,((f_1f_2\cdots f_i)f_{i+1})\subset K_{\delta_i}$. Therefore, $\alpha_i>\alpha_{i+1}$ which means that $\{f_i\}$ is a left $T$-nilpotent. Conversely, we assume 1) and 2). We show that for any non-zero object $A$, there exists $P_1$ and $f\in[P_1, A]$ such that $f(\mathrm{J}(P_1))=0$ and $f\neq 0$. If it were not true, we would have some $P_1$ and $f\in[P_1, A]$ such that $f(\mathrm{J}(P_1))\neq 0$. If we consider an exact sequence,

$$\begin{tikzcd}
\mathrm{J}(P_1) \arrow[r, "f"] & f(\mathrm{J}(P_1)) \arrow[r] & 0 \\
& P_2 \arrow[ul, dashed, "f_1"] \arrow[u, dashed, "f_1'"'] &
\end{tikzcd}$$

we have some $P_2$, $f_1'\in[P_2, f(\mathrm{J}(P_1)]$ and $f_1\in[P_2, \mathrm{J}(P_1)]$ such that $f_1'=ff_1$. Since $f_1'(\mathrm{J}(P_2))\neq 0$, we can find $P_3$ and $f_2\in[P_3, \mathrm{J}(P_2)]$ such that $f_2'=ff_1f_2\in[P_2, A]$ and $f_2'(\mathrm{J}(P_2))\neq 0$. Repeating this argument we have $f_n'=ff_1\cdots f_n\neq 0$ and $f_i\in[P_{i+1}, \mathrm{J}(P_i)]\subset \mathrm{J}(\mathfrak{A})\cap[P_{i+1}, P_i]$ for all $n$ by Lemma 1, which contradicts to 1). Hence, $\mathfrak{A}$ is semi-artinian from 2).

In order to characterize some perfect Grothendieck categories, we give some notes here. For a project object $P$ such that $P\neq \mathrm{J}(P)$ we obtain from [13], Theorem 5.2 that $P=\sum\oplus P_\alpha$ is semi-perfect if and only if $P_\alpha$'s are semi-perfect of $\mathrm{J}(P_\alpha)\neq P_\alpha$ and $\mathrm{J}(P)$ is small in $P$. Further if $P$ is semi-perfect, $P=\sum\oplus Q_\alpha$

by [13], Corollary 4.4, where $Q_\alpha$'s are completely indecomposable. Similarly from Lemma 5 and [10], Proposition 1 and Corollary 1 to Theorem 3 we obtain

**Lemma 6.** *Let $\mathfrak{A}$ be a quasi-samll Grothendieck category and $\{P_\alpha\}_I$ a family of semi-perfect objects in $\mathfrak{A}$. Then $P=\sum_I \oplus P_\alpha$ is semi-perfect (resp. perfect) and $P \neq \mathrm{J}(P)$ if and only if $\{P_\alpha\}_I$ is a locally right semi-T-nilpotent (resp. T-nilpotent) system with respect to $\mathrm{J}([P, P])$ and $P_\alpha \neq \mathrm{J}(P_\alpha)$ for all $\alpha$.*

**Theorem 3.** *An abelian category $\mathfrak{A}$ is a Grothendieck category with a generating set of finitely generated objects and is semi-perfect if and only if $\mathfrak{A}$ is equivalent to $(\mathfrak{C}_f^0, Ab)$, where $\mathfrak{C}_f$ is the finitely induced sub-additive category from $\{P_\alpha\}_I$, where $P_\alpha$'s are completely indecomposable objects in $\mathfrak{A}$.*

Proof. Let $\{G_\alpha\}$ be a generating set of finitely generated objects. If $\mathfrak{A}$ is semi-perfect, we have a projective cover $P_\alpha$ of $G_\alpha$; $0 \to K_\alpha \to P_\alpha \xrightarrow{f} G_\alpha \to 0$ is exact and $K_\alpha$ is small in $P_\alpha$. Furthermore, $P_\alpha$ contains a finitely generated subobject $P'$ such that $f(P')=G_\alpha$. Hence, $P_\alpha=K+P'$ implies that $P_\alpha$ is also finitely generated. Therefore, $\mathfrak{A}$ has a generating set of projective small $P_\alpha$. We have $P \neq \mathrm{J}(P)$ for every projective object $P$ by Proposition 2. Thus $P_\alpha=\sum_{i=1}^{n_\alpha} \oplus P_{\alpha_i}$ by [13], Corollary 4.4, where $P_{\alpha_i}$'s are completely indecomposable. Let $\mathfrak{C}_f$ be the induced subadditive category from $\{P_{\alpha_i}\}$. Then $\mathfrak{A}$ is equivalent to $(\mathfrak{C}^0, Ab)$ by Freyd's Theorem. Conversely, if $\mathfrak{A} \approx (\mathfrak{C}^0, Ab)$, $\{H_C(-)=[-, C]\}$ is a generating set of finitely generated projective objects by Lemma 4. Further $\mathfrak{A}$ is semi-perfect by Proposition 1 and [14], Corollary 5.3.

If a ring $R$ is right artinian, then $\mathfrak{M}_R$ is right (semi-) perfect. Similarly, we have

**Proposition 4.** *Let $\mathfrak{A}$ be a Grothendieck category with a generating set $\{P_\alpha\}_I$ of projective objects with finite length. Then $\mathfrak{A}$ is semi-perfect. $\mathfrak{A}$ is perfect if and only if $\sum_I \oplus P_\alpha$ is semi-perfect,* (cf. Remark 2 below)

Proof. We may assume that $\mathfrak{A}$ has a generating set of completely indecomposable and small projective objects $P_\alpha$. Then $P_\alpha$ is semi-perfect by Proposition 1 and hence, $\mathfrak{A}$ is semi-perfect. If $\sum_I \oplus P_\alpha$ is semi-perfect, then $\sum \oplus P_\alpha$ is perfect by Lemma 6 and [6], Proposition 2.4.

Analogously to Theorem 3, we have

**Theorem 4.** *An abelian category $\mathfrak{A}$ is a Grothendieck category with a generating set of finitely generated objects and is perfect if and only if $\mathfrak{A}$ is equivalent to $(\mathfrak{C}_f^0, Ab)$, where $\mathfrak{C}_f$ is the finitely induced additive category from a set of some completely indecomposable objects $P_\alpha$ such that $\{P_\alpha\}$ is a right T-nilpotent system* *with respect to* $\mathrm{J}(\mathfrak{C}_f)$.

Proof. If $\mathfrak{A}$ is a perfect Grothendieck category as above, then $\mathfrak{A}\approx(\mathfrak{C}_f^0, Ab)$ by Theorem 3. It is clear from Lemma 6 that $\{P_\alpha\}$ is a right $T$-nilpotent system with respect to $\mathrm{J}(\mathfrak{C}_f)$, since $P_\alpha$ is small. Conversely, if $\mathfrak{A}\approx(\mathfrak{C}_f^0, Ab)$, $\mathfrak{A}$ is a perfect category as in the theorem by Lemmas 4 and 6.

We have immediately from Corollary to Lemma 2, Proposition 3 and Theorems 3 and 4

**Corollary 1.** *Let* $\mathfrak{A}$ *be a Grothendieck category with a generating set of finitely generated. Then* $\mathfrak{A}$ *is semi-perfect if and only if* $\mathfrak{A}$ *has a generating set* $\{P_\alpha\}$ *of completely indecomposable projective objects. In this case* $\{P_\alpha\}$ *is right (resp. left) T-nilpotent if and only if* $\mathfrak{A}$ *is perfect (resp. semi-artinian).*

Let $\mathfrak{A}$ be a Grothendieck category as in the above. Then the induced category from $\{P_{\alpha'}/\mathrm{J}(P_{\alpha'})\}_J$ is equivalent to $\sum_J\oplus\mathfrak{M}_{\Delta_{\alpha'}}$, where $\Delta_{\alpha'}=[P_{\alpha'}/\mathrm{J}(P_{\alpha'}), P_{\alpha'}/\mathrm{J}(P_{\alpha'})]$, where $\{P_{\alpha'}/\mathrm{J}(P_{\alpha'})\}$ is a complete isomorphic representative of $\{P_\alpha/\mathrm{J}(P_\alpha)\}$. Hence, we have

**Corollary 2.** *A (semi-) perfect Grothendieck category with a generating set of finitely generated is equivalent to* $\mathfrak{M}_R$ *with R (semi-) perfect if and only J is finite.*

From Theorems 3 and 4, we may restrict ourselves to a case of functor categories $(\mathfrak{C}, Ab)$, if we are interested in perfect Grothendieck categories. First, we note

**Proposition 5** ([17]). *Let* $\mathfrak{C}$ *be an amenable additive and small category. Then* $(\mathfrak{C}, Ab)$ *is semi-perfect if and only if every object in* $\mathfrak{C}$ *is finite coproduct of completely indecomposable objects.*

Proof. It is clear from Theorem 3 and [3], p. 119.

For a ring $R$, ${}_R\mathfrak{M}$ (resp. $\mathfrak{M}_R$) is naturally equivalent to $(R, Ab)$ (resp. $(R^0, Ab)$). Hence, an analogy of [2], Theorem 2.1 is

**Corollary.** *Let* $\mathfrak{C}$ *be as above. Then* $(\mathfrak{C}, Ab)$ *is semi-perfect if and only if* $(\mathfrak{C}^0, Ab)$ *is semi-perfect.*

Our next aim is to generalize Theorem $P$ of [2] to a case of $(\mathfrak{C}_f, Ab)$. First we shall recall the idea given in [4], Chapter II. Put $R=\sum_{X,Y\in\mathfrak{C}_f}\oplus[X, Y]$ and we can make $R$ a ring by the compositions of morphisms in $\mathfrak{C}$. If we denote the indentity morphism of $X$ by $I_X$, $I_X$ is idemoptent and $I_XI_Y=I_YI_X=0$ if $X\neq Y$. Hence, $R=\sum_{X\in\mathfrak{C}}\oplus RI_X=\sum_{X\in\mathfrak{C}}\oplus I_XR$. In general, $R$ does not contain the identity. We know by [4], Proposition 2 in p. 347 that the covariant functor category $(\mathfrak{C}, Ab)$ is equivalent to the full subcategory of ${}_R\mathfrak{M}$ whose objects consist of all left $R$-modules $A$ such that $RA=A$. Similarly, we know the contravariant functor category $(\mathfrak{C}^0, Ab)$ is equivalent to the full sbucategory of $\mathfrak{M}_R$ with $AR=A$.

**Lemma 7.** *Let* $\mathfrak{C}_f$ *and* $R=\sum\oplus[X, Y]$ *be as above. Then* $\mathrm{J}(R)=\sum\oplus([X, Y]\cap \mathrm{J}(\mathfrak{C}_f))$.

Proof. Let $x$ be in $\mathrm{J}(R)$. Then there exists a finite number of objects $X_i$ such that $x=(\sum I_{X_i})x(\sum I_{X_i})\in(\sum I_{X_i})\mathrm{J}(R)(\sum I_{X_i})=\mathrm{J}((\sum I_{X_i})R(\sum I_{X_i}))$. On the other hand $(\sum I_{X_i})R(\sum I_{X_i})\approx[\sum\oplus X_i, \sum\oplus X_i]$. Hence, $x\in\sum([X, Y]\cap \mathrm{J}(\mathfrak{C}_f))$ by [7]., Lemma 8. The converse is clear from the above argument.

We can prove the following theorem by the same method given in [2], Part 1 even though $R$ does not contain the identity (see Remark 1 below). However, we shall give here the proof rather directly (without homological method).

**Theorem 5** (cf. [2]. Theorem P). *Let* $\mathfrak{A}$ *be an arbitrary Grothendieck category,* $\{M_\alpha\}_I$ *a set of completely indecomposable objects in* $\mathfrak{A}$ *and* $\mathfrak{C}_f$ *the finitly induced additive subcategory from* $\{M_\alpha\}$. *Put* $R=\sum_{\mathfrak{C}_f}\oplus[X, Y]$ *as above. Then the following conditions are equivalent.*

1) $(\mathfrak{C}_f, Ab)$, *is perfect.*
2) $\{M_\alpha\}$ *is a left T-nilpotent system with respect to* $\mathrm{J}(\mathfrak{C}_f)$.
3) $\mathrm{J}(R)$ *is left T-nilpotent.*
4) *R satisfies the descending chain condition on principal right ideals in* $\mathrm{J}(R)$.
5) *Every object in* $(\mathfrak{C}_f^0, Ab)$ *contains minimal subobjects.*

*We have the similar result for* $(\mathfrak{C}_f^0\ Ab)$.

Proof. 1)$\leftrightarrow$2) is nothing but Lemma 6.
2)$\rightarrow$3). Let $x_n$ be in $\mathrm{J}(R)$. Then $x_n=\sum x_{n\,j(n)}$, $x_{n\,j(n)}\in[X_{j(n)}, Y_{j(n)}]\cap \mathrm{J}(\mathfrak{C}_f)$. by Lemma 7, where we may assume that $X$, $Y$ are isomorphic to ones in $\{M_\alpha\}$. Hence, $\{x_n\}$ is left $T$-nilpotent by König Graph Theorem.
3)$\rightarrow$4)$\rightarrow$2) is clear.
2)$\leftrightarrow$5) is given by Proposition 3.

REMARK 1. We can prove Theorem 5 by making use of idea in [2], Part 1. For instance, let $\{a_i\}$ be a sequence of elements in $R$. There exist indempotents $I_i$ such that $I_ia_i=a_i$, $a_{i-1}I_i=a_{i-1}$. Then we denote by $[F, \{a_n\}, G]$

1) $F=\sum_{i=1}^{\infty}\oplus RI_ix_i$, 2) The subgroup $G$ of $F$ generated by $\{I_ix_i-a_iI_{i+1}x_{i+1}\}$, where $x_i$ is a base. Then this $[F, \{a_i\}, G]$ takes the place of $[F, \{a_n\}, G]$ given in [2], p. 468, even though $R$ does not contain the identity. From those

arguments we can shown that we may take out the assumption "*in* J$(R)$" in 4), (cf. [17], Proposition in p. 1571).

REMARK 2. Let $\{R_i\}_I$ be a set of perfect rigns. Then $\mathfrak{M}_{R_i}$ is perfect and $\prod_I \mathfrak{M}_{R_i}$ is also perfect, however $\prod_I R_i$ is not a perfect ring if $I$ is infinite.

If a ring $R$ is right artinian, then $\mathfrak{M}_R$ has a generator $R$ of finite length and $\mathfrak{M}_R$ is perfect. However, in gneral categories with a generating set of projective and finite length need not be perfect. For instance, let $K$ a be field and $I$ the set of natural numbers. We define an abelian category $[I, \mathfrak{M}_K]$ of commutative diagrams as follows; the objects of $[I, \mathfrak{M}_K]$ consist of all form $(A_1, A_2, \cdots, A_j, \cdots)$ with arrow $d_{kj}$: $A_j \to A_k$ such that $d_{kj}=0$ for $k>j$, where $A_i \in \mathfrak{M}_K$. Then $[I, \mathfrak{M}_K]$ is an abelian category with a generating set of projective objects $(K, K, \cdots, K, 0, \cdots)=U_i$ of finite length (see [11], Proposition 2.1 and [14], p. 227). We have natural monomorphisms $f_i$: $U_i \to U_{i+1}$. Hence, $[I, \mathfrak{M}_K]$ is not perfect, however $[I, \mathfrak{M}_K]$ is semi-artinian by Proposition 3.

Finally, we shall give the following corollary as an example.

**Corollary.** *Let $\mathfrak{C}$ be a full additive amenable subcategory with finite coproduct in the category of finitely generated torsion abelian groups. Then the following statements are equivalent.*

1) $(\mathfrak{C}, Ab)$ *is perfect.*

2) $(\mathfrak{C}^0, Ab)$ *is perfect.*

3) *Every object in* $(\mathfrak{C}, Ab)$ *contains minimal subobjects.*

4) *Every object in* $(\mathfrak{C}^0, Ab)$ *contains minimal subobjects.*

5) *The completely isomorphic representative class of indecomposable p-torsion objects in* $\mathfrak{C}$ *is finite for all p.*

6) $(\mathfrak{C}_f, Ab)$ *is equivalent to* $\prod \mathfrak{M}_{R_\alpha}$, *where* $R_\alpha$*'s are right artinian rings.*

Proof. The indecomposable objects are left (or right) $T$-nilpotent with respect to J$(\mathfrak{C})$ if and only if 5) is satisfied.

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