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ON THE MIXED PROBLEM FOR HYPERBOLIC EQUATIONS OF SECOND ORDER WITH THE NEUMANN BOUNDARY CONDITION

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1. Introduction

The purpose of this paper is to complete the results of § 3 of [5]. Let S be a sufficiently smooth compact hypersurface in R^n and let Ω be the interior or exterior domain of S.

Consider a hyperbolic equation of second order

$$(1.1)\qquad L[u]=\frac{\partial^2}{\partial t^2}u+a_1(x,t:D)\frac{\partial}{\partial t}u+a_2(x,t:D)u=f$$
$$a_1(x,t:D)=\sum_{j=1}^n 2h_j(x,t)\frac{\partial}{\partial x_j}+h(x,t)$$
$$a_2(x,t:D)=-\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right)+\sum_{j=1}^n b_j(x,t)\frac{\partial}{\partial x_j}+c(x,t)$$

where the coefficients belong to $\mathcal{B}(\Omega\times(0,T))$[1]. We assume that $a_2(x,t:D)$ is an elliptic operator satisfying

$$(1.2)\qquad \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j\geqq d\sum_{j=1}^n\xi_j^2\qquad(d>0)$$
$$a_{ij}(x,t)=a_{ji}(x,t)$$

for all $(x,t)\in\Omega\times(0,T)$ and $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in R^n$, and that $h_j(x,t)$ $(j=1,2,\cdots,n)$ are real-valued. For this equation we consider the following boundary condition

$$(1.3)\qquad B[u(x,t)]=\frac{\partial}{\partial n_t}u(x,t)-\sigma_1(s,t)\frac{\partial u}{\partial t}(x,t)$$
$$+\sigma_2(s,t)u(x,t)=0\qquad\text{on } S,$$

where

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1) $\mathcal{B}(\omega)$, ω being an open set, is the set of all C^∞-functions defined in ω such that their all partial derivatives of any order are bounded.

$$\frac{\partial}{\partial n_t}=\sum_{i,j=1}^{n} a_{ij}(s, t)\nu_i\frac{\partial}{\partial x_j},$$

$\nu=(\nu_1, \nu_2, \cdots, \nu_n)$ is the outer unit normal of S at $s\in S$, $\sigma_i(s, t)$ $(i=1, 2)$ are smooth function defined on $S\times[0, T]$ and $\sigma_1(s, t)$ is real-valued.

Our problem is to obtain $u(x, t)$ satisfying

$$\begin{cases} L[u(x, t)]=f(x, t) & \text{in} \quad \Omega\times(0, T) \\ B[u(x, t)]=0 & \text{on} \quad S\times[0, T] \\ u(x, 0)=u_0(x) & \\ \dfrac{\partial u}{\partial t}(x, 0)=u_1(x) & \end{cases}$$

for any given initial data $\{u_0(x), u_1(x)\}$ and any second member $f(x, t)$. Let us denote this problem by $P(L, B)$.

Since we like to treat this problem in L^2-sense it is necessary to assume that L and B satisfy the inequality

$$\sigma_1(s, t)\leqslant \sum_{j=1}^{n} h_j(s, t)\nu_j \quad \text{on} \quad S\times[0, T]^{2)}, \tag{1.4}$$

which is invariant with a change of variables.

This problem is a generalization of the problem considered in §3 of [5].

We state our theorem:

Theorem 1. *For any initial data* $\{u_0(x), u_1(x)\}\in H^{m+2}(\Omega)\times H^{m+1}(\Omega)$ *and any second member* $f(x, t)\in H^{m+1}(\Omega\times(0, T))$, *if they satisfy the compatibility condition of order* m[3], *there exists a solution* $u(x, t)\in \mathcal{E}_t^0(H^{m+2}(\Omega))\cap\mathcal{E}_t^1(H^{m+1}(\Omega))\cap\cdots\cap\mathcal{E}_t^{m+2}(L^2(\Omega))$[4] *of* $P(L, B)$ *and it is unique in* $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$.

The mixed problem for second order hyperbolic equations with the Neumann type boundary condition is mainly studied under the assumption that the boundary condition does not depend on t (for example Ladyzenskaya [9], Ikawa [5]). The case where the boundary condition varies with t is treated by the author in § 3 of [5]. But there we assumed that $h_j(x, t)$ $(j=1, 2, \cdots, n)$ are identically zero and $b_j(x, t)$ $(j=1, 2, \cdots, n)$ are real-valued on $S\times[0, T]$, moreover to show the existence of the solution, the regularity of $f(x, t)$ in $H^1(\Omega)'$ is required and by that method we could not extend these results to the case where $\sum_{j=1}^{n} h_j(s, t)\nu_j \neq 0$ on $S\times[0, T]$. On the other hand in [6] such restrictions on L are not

2) See Remark of [17], and Theorem 1 of [7].

3) This definition will be given precisely in § 3.

4) $u(x, t)\in\mathcal{E}_t^k(E)$ means that $u(x, t)$ is k-times continuousely differentiable as E-valued function.

posed but the boundary condition treated there does not satisfy the condition (1.4).

It seems to us that the difficulties of this problem due to the following two facts:

(i) $B\left[\frac{\partial u}{\partial t}(x, t)\right] \neq 0$ on S in general since the boundary condition depends on t, and the problem $P(L, B)$ cannot be extend to non-homogeneous boundary condition in the L^2-sense under the condition (1.4)[5]. (ii) We do not know a general theory of integration of an evolution equation

$$\begin{cases} \dfrac{d}{dt} U(t) = A(t)U(t) + F(t) \\ U(0) = U_0 , \end{cases}$$

which is applicable to our problem where the definition domain of $A(t)$ varies with t.

The essential part of this paper is to derive the energy inequality of any order. The necessity of the energy inequalities of any order is caused by the fact that we cannot use, in this case, the method in the proofs of the regularity of the solution of [5] and [6] and still more we have to use the two energy inequalities to show the existence of the solution, for example when $m=0$. To prove the existence of the solution we make an approximation by the solutions satisfying the boundary condition

$$B_\varepsilon = \frac{\partial}{\partial n_t} - (\sigma_1 - \varepsilon)\frac{\partial}{\partial t} + \sigma_2 . \qquad \varepsilon > 0 \tag{1.5}$$

whose existence is already shown in [6].

2. Energy inequalities

In this section we show the following

Theorem 2. *Let m be non-negative integer. There exists a constant C_m and for all $u(x, t) \in H^{m+3}(\Omega \times (0, T))$ the solution of $P(L, B)$ the energy inequality*

$$\begin{aligned} &||u(x, t)||^2_{m+2, L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(x, t)\right\|^2_{m+1, L^2(\Omega)} + \cdots \\ &\quad + \left\|\frac{\partial^{m+2} u}{\partial t^{m+2}}(x, t)\right\|_{L^2(\Omega)} \\ &\leqslant C_m \Big\{ ||u(x, 0)||^2_{m+2, L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(x, 0)\right\|^2_{m+1, L^2(\Omega)} \end{aligned} \tag{2.1}$$

5) See the appendix of [6].

$$+\|f(x, 0)\|^2_{m, L^2(\Omega)}+\left\|\frac{\partial f}{\partial t}(x, 0)\right\|^2_{m-1, L^2(\Omega)}+\cdots+\left\|\frac{\partial^m f}{\partial t^m}(x, 0)\right\|^2_{L^2(\Omega)}$$
$$+\int_0^t\left(\left\|\frac{\partial f}{\partial t}(x, s)\right\|^2_{m, L^2(\Omega)}+\cdots+\left\|\frac{\partial^{m+1} f}{\partial t^{m+1}}(x, s)\right\|^2_{L^2(\Omega)}\right)ds\Big\}$$

holds for all $t\in[0, T]$.

Notations and preliminary lemmas

First of all let us remark that it suffices to show (2.1) under the assumption $\sigma_2(s, t)\equiv 0$. Take $a(x, t)$ a sufficiently smooth function defined on $\bar{\Omega}\times[0, T]$ with the following properties

(i) $a(s, t)=1$ on $S\times[0, T]$

(ii) $2>|a(x, t)|>\frac{1}{2}$ for all $(x, t)\in\bar{\Omega}\times[0, T]$

(iii) $\frac{\partial}{\partial n_t}a(x, t)=\sigma_2(s, t)$ on $S\times[0, T]$,

and put $u(x, t)=a(x, t)\, v(x, t)$ then $v(x, t)$ satisfies

$$\begin{cases} L[v]+a(x, t)^{-1}[L, a(x, t)]v=a(x, t)^{-1}f & \text{in } \Omega\times[0, T] \\ \frac{\partial}{\partial n_t}v-\sigma_1\frac{\partial v}{\partial t}=0 & \text{on } S\times[0, T]. \end{cases}$$

There are no difficulties to derive the estimate of $u(x, t)$ from that of $v(x, t)$. Therefore in this section we assume that $\sigma_2(s, t)\equiv 0$.

Let Σ be Ω or $R^n_+=\{(x', x_n);\ x_n>0\}$. Any $u(x, t)\in H^{p+1}(\Sigma\times(0, T))$ ($p\geqslant 0$ integer) belongs to $\mathcal{E}^0_t(H^p(\Sigma))\cap\mathcal{E}^1_t(H^{p-1}(\Sigma))\cap\cdots\cap\mathcal{E}^p_t(L^2(\Sigma))$ by changing its values on a set measure zero of $\Sigma\times(0, T)$ if necessary. Let us denote the space $\mathcal{E}^0_t(H^p(\Sigma))\cap\mathcal{E}^1_t(H^{p-1}(\Sigma))\cap\cdots\cap\mathcal{E}^p_t(L^2(\Sigma))$ by $\mathcal{E}(p, \Sigma)$, and for $u(x, t)\in\mathcal{E}(p, \Sigma)$define $|||u(x, t)|||_{p,\Sigma}$ by

$$|||u(x, t)|||^2_{p,\Sigma}=\sum_{j=0}^p\left\|\left(\frac{\partial}{\partial t}\right)^j u(x, t)\right\|^2_{p-j, L^2(\Sigma)}, \tag{2.2}$$

and for $u(x, t)\in\mathcal{E}(1, \Sigma)$, $\|u(x, t)\|_{\mathcal{H}(t)}$ by

$$\|u(x, t)\|^2_{\mathcal{H}(t)}=\sum_{i,j=1}^n\int_\Sigma a_{ij}(x, t)\frac{\partial u}{\partial x_i}(x, t)\overline{\frac{\partial u}{\partial x_j}(x, t)}\,dx$$
$$+\|u(x, t)\|^2_{L^2(\Sigma)}+\left\|\frac{\partial u}{\partial t}(x, t)\right\|^2_{L^2(\Sigma)}.$$

Then from the condition (1.2) there exists a constant $M>0$ such that

$$\frac{1}{M}|||u(x, t)|||_{1,\Sigma}^2 \leq ||u(x, t)||_{\mathcal{H}(t)}^2 \leq M|||u(x, t)|||_{1,\Sigma}^2$$

for all $t \in [0, T]$ and $u(x, t) \in \mathcal{E}(1, \Sigma)$.

Lemma 2.1. *Let $u(x, t) \in H^2(\Sigma \times (0, T))$ satisfies $L[u]=f$ in $\Sigma \times (0, T)$ we have the estimate*

$$(2.3) \qquad ||u(x, t)||_{\mathcal{H}(t)}^2 \leqslant ||u(x, 0)||_{\mathcal{H}(0)}^2 + c\int_0^t ||u(x, s)||_{\mathcal{H}(s)}^2 ds + \int_0^t ||f(x, s)||^2 ds + 2\,\mathrm{Re}\int_0^t ds\int_{\partial\Sigma}\Big(Bu\overline{\frac{\partial u}{\partial t}}\Big)(x, s)dS$$

holds for all $t \in [0, T]$, where c is a constant determined by L.

Proof. By the integration by parts

$$\int_0^t ds\int_\Sigma\Big(\frac{\partial u}{\partial t}\overline{L[u]} + L[u]\overline{\frac{\partial u}{\partial t}}\Big)(x, s)dx$$
$$= \int_0^t ds\int_\Sigma \frac{\partial}{\partial t}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2 + \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\overline{\frac{\partial u}{\partial x_j}}\Big)(x, s)dx$$
$$-\int_0^t ds\int_{\partial\Sigma}\Big(\frac{\partial u}{\partial n_s}\overline{\frac{\partial u}{\partial t}} + \frac{\partial u}{\partial t}\overline{\frac{\partial u}{\partial n_s}} - 2(\Sigma h_j\nu_j)\Big|\frac{\partial u}{\partial t}\Big|^2\Big)dS$$
$$+\int_0^t ds\int_\Sigma\Big(\text{a quadratic form of } u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_j}\Big)dx\,.$$

Therefore

$$||u(x, t)||_{\mathcal{H}(t)}^2 - ||u(x, 0)||_{\mathcal{H}(0)}^2$$
$$= \int_0^t ds\int_\Sigma 2\mathrm{Re}\,\frac{\partial u}{\partial t}\bar{f}\,dx + \int_0^t ds\int_{\partial\Sigma} 2\mathrm{Re}\Big(\frac{\partial u}{\partial n_s} - \Sigma h_j\nu_j\frac{\partial u}{\partial t}\Big)\overline{\frac{\partial u}{\partial t}}\,dS$$
$$+\int_0^t ds\int_\Sigma\Big(\text{a quadratic form of } u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_j}\Big)dx$$

By taking account of the condition (1.4)

$$2\mathrm{Re}\int_0^t ds\int_{\partial\Sigma}\Big(\frac{\partial u}{\partial n_s} - \Sigma h_j\nu_j\frac{\partial u}{\partial t}\Big)\overline{\frac{\partial u}{\partial t}}\,dS$$
$$\leqslant 2\mathrm{Re}\int_0^t ds\int_{\partial\Sigma} Bu\overline{\frac{\partial u}{\partial t}}\,dS$$

then we have (2.3). Q.E.D.

When $\Sigma = R_+^n$ we denote its point by $x=(x', x_n)$ where $x' \in R^{n-1}$, $x_n > 0$, and omit the notation R_+^n in (2.2).

Lemma 2.2. *Let $p(x', t)$ be a real valued function in $\mathscr{B}(R^{n-1}\times(0, T))$. For any $u(x, t)\in H^3(R^n_+\times(0, T))$ we have the estimates*

$$(2.4)\qquad 2\mathrm{Re}\int_0^t ds\int_{R^{n-1}} p(x', s)\frac{\partial^2 u}{\partial t^2}(x', 0, s)\overline{\frac{\partial u}{\partial x_j}(x', 0, s)}dx'$$
$$\leqslant C\{\varepsilon|||u(x, t)|||_2^2+C(\varepsilon)|||u(x, t)|||_1^2+|||u(x, 0)|||_2^2$$
$$+\int_0^t |||u(x, s)|||_2^2\, ds\}$$

for $t\in[0, T]$ and $j=1, 2, \cdots, n-1$, and

$$(2.5)\qquad 2\mathrm{Re}\int_0^t ds\int_{R^{n-1}} p(x', s)\frac{\partial^2 u}{\partial t^2}(x', 0, s)\overline{\frac{\partial u}{\partial t}(x', 0, s)}dx'$$
$$\leqslant C\{\varepsilon|||u(x, t)|||_2^2+C(\varepsilon)|||u(x, t)|||_1^2$$
$$+|||u(x, 0)|||_2^2+\int_0^t |||u(x, s)|||_2^2\, ds\}\ ,$$

where C is a constant determined by $p(x', t)$, ε is an arbitrary positive number and $C(\varepsilon)$ depends only on ε.

Proof. At first remark that for any $v(x)\in H^1(R^n_+)$

$$(2.6)\qquad \int_{R^{n-1}}|v(x', 0)|^2 dx'\leqslant \text{const.}||v(x)||^2_{1, L^2(R^n_+)}$$

$$(2.7)\qquad \int_{R^{n-1}}|v(x', 0)|^2 dx'\leqslant \varepsilon||v(x)||^2_{1, L^2(R^n_+)}+C(\varepsilon)||v(x)||^2_{L^2(R^n_+)}\ .$$

By the integration by parts

$$2\mathrm{Re}\int_0^t ds\int_{R^{n-1}} p(x', s)\frac{\partial^2 u}{\partial t^2}(x', 0, s)\overline{\frac{\partial u}{\partial x_j}(x', 0, s)}dx'$$
$$=2\mathrm{Re}\int_{R^{n-1}}\Big[p\frac{\partial u}{\partial t}\overline{\frac{\partial u}{\partial x_j}}\Big]_0^t dx'-\int_0^t ds\int\frac{\partial}{\partial x_j}\Big(p(x', s)\Big|\frac{\partial u}{\partial t}(x', 0, s)\Big|^2\Big)dx'$$
$$+\int_0^t ds\int_{R^{n-1}}\Big(\frac{\partial p}{\partial x_j}(x', s)\Big|\frac{\partial u}{\partial t}(x', 0, s)\Big|^2-2\mathrm{Re}\frac{\partial p}{\partial s}(x', s)\frac{\partial u}{\partial t}\overline{\frac{\partial u}{\partial x_j}}\Big)dx'$$

since $\int\frac{\partial}{\partial x_j}\Big(p\Big|\frac{\partial u}{\partial t}\Big|^2\Big)dx'=0$

$$\leqslant|p|_0\int_{R^{n-1}}\Big(\Big|\frac{\partial u}{\partial t}(x', 0, t)\Big|^2+\Big|\frac{\partial u}{\partial x_j}(x', 0, t)\Big|^2$$
$$+\Big|\frac{\partial u}{\partial t}(x', 0, 0)\Big|^2+\Big|\frac{\partial u}{\partial x_j}(x', 0, 0)\Big|^2\Big)dx'$$
$$+2|p|_1\int_0^t ds\int_{R^{n-1}}\Big(\Big|\frac{\partial u}{\partial t}(x', 0, s)\Big|^2+\Big|\frac{\partial u}{\partial x_j}(x', 0, s)\Big|^2\Big)dx'$$

by using (2.6) and (2.7)

$$\leqslant |p|_0\Big\{\varepsilon\Big(\Big\|\frac{\partial u}{\partial t}(x,t)\Big\|_1^2+\Big\|\frac{\partial u}{\partial x_j}(x,t)\Big\|_1\Big)+C(\varepsilon)\Big(\Big\|\frac{\partial u}{\partial t}(x,t)\Big\|^2+\Big\|\frac{\partial u}{\partial x_j}(x,t)\Big\|^2\Big)$$
$$+\text{const.}\Big(\Big\|\frac{\partial u}{\partial t}(x,0)\Big\|_1^2+\Big\|\frac{\partial u}{\partial x_j}(x,0)\Big\|_1^2\Big)\Big\}$$
$$+|p|_1\int_0^t \text{const.}\Big(\Big\|\frac{\partial u}{\partial t}(x,s)\Big\|_1^2+\Big\|\frac{\partial u}{\partial x_j}(x,s)\Big\|_1^2\Big)ds.^{6)}$$

Thus (2.4) is proved. (2.5) is seen by the same manner. Q.E.D.

Lemma 2.3. *For any* $u(x,t)\in H^3(R^n_+\times(0,T))$, $v(x,t)\in H^2(R^n_+\times((0,T))$

(2. 8)
$$2\mathrm{Re}\int_0^t ds\int_{R^{n-1}}\frac{\partial^2 u}{\partial t^2}(x',0,s)\,\overline{v(x',0,s)}dx'$$
$$\leqslant C\{\varepsilon|||u(x,t)|||_2^2+C(\varepsilon)|||u(x,t)|||_1^2+|||v(x,t)|||_1^2$$
$$+|||u(x,0)|||_2^2+|||v(x,0)|||_1^2$$
$$+\int_0^t(|||u(x,s)|||_2^2+|||v(x,s)|||_2^2)ds\}$$

holds, where C does not depend on u and v.

Proof.

$$2\mathrm{Re}\int_0^t ds\int_{R^{n-1}}\frac{\partial^2 u}{\partial t^2}(x',0,s)\,\overline{v(x',0,s)}dx'$$
$$=2\mathrm{Re}\int_{R^{n-1}}\Big[\frac{\partial u}{\partial t}(x',0,s)\,\overline{v(x',0,s)}\Big]_0^t dx'$$
$$-\int_0^t ds\int_{R^{n-1}}\frac{\partial u}{\partial t}(x',0,s)\overline{\frac{\partial v}{\partial t}(x',0,s)}dx'$$
$$\leqslant\int_{R^{n-1}}\Big(\Big|\frac{\partial u}{\partial t}(x',0,t)\Big|^2+|v(x',0,t)|^2+\Big|\frac{\partial u}{\partial t}(x',0,0)\Big|^2+|v(x',0,0)|^2\Big)dx'$$
$$+\int_0^t ds\int_{R^{n-1}}\Big(\Big|\frac{\partial u}{\partial t}(x',0,s)\Big|^2+\Big|\frac{\partial v}{\partial t}(x',0,s)\Big|^2\Big)dx'$$

by using (2.6) and (2.7)

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6) For $p(x)\in\mathcal{B}^k(\omega)$ $|p|_k$ denotes its norm, namely

$$|p|_k=\sum_{|\alpha|\leqslant k}\sup_{x\in\omega}\Big|\Big(\frac{\partial}{\partial x}\Big)^\alpha p(x)\Big|.$$

$$\leqslant \varepsilon\left\|\frac{\partial u}{\partial t}(x, t)\right\|_1^2+C(\varepsilon)\left\|\frac{\partial u}{\partial t}(x, t)\right\|^2$$

$$+\text{const.}\left(\|v(x, t)\|_1^2+\|v(x, 0)\|_1^2+\left\|\frac{\partial u}{\partial t}(x, 0)\right\|_1^2\right)$$

$$+\text{const.}\int_0^t\left(\left\|\frac{\partial u}{\partial t}(x, s)\right\|_1^2+\left\|\frac{\partial v}{\partial t}(x, s)\right\|_1^2\right)ds.$$

Thus we get (2.8). Q.E.D.

Lemma 2.4. *Let p be any integer $\geqslant 1$. There exists a constant M_p such that for any solution $u(x, t)\in\mathcal{E}(p+1, \Omega)$ of $P(L, B)$ the estimate*

$$\text{(2. 9)}\qquad |||u(x, t)|||_{p+1,\Omega}^2\leqslant M_p(|||u^{(p)}(x, t)|||_{1,\Omega}^2+|||u(x, t)|||_{p,\Omega}^2 +|||f(x, t)|||_{p-1,\Omega}^2)^{7)}$$

holds for all $t\in[0, T]$.

Proof. Let us remark that the well known a priori estimate concerning an elliptic operator $a_2(x, t: D)$

$$\|w\|_{l+2,L^2(\Omega)}^2\leqslant K_l\left(\|a_2 w\|_{l,L^2(\Omega)}^2+\left\langle\frac{\partial}{\partial n_t}w\right\rangle_{l+1/2,L^2(S)}^2+\|w\|_0^2\right)$$

holds for all $w\in H^{l+2}(\Omega)$.

The differentiation of (1.1) and (1.3) with respect to t of k-times gives

$$L[u^{(k)}]+\sum_{j=1}^k\binom{k}{j}L^{(j)}[u^{(k-j)}]=f^{(k)}$$

$$B[u^{(k)}]+\sum_{j=1}^k\binom{k}{j}B^{(j)}[u^{(k-j)}]=0.$$

Therefore we have for $k=0, 1, 2, \cdots, p-1$

$$a_2u^{(k)}=-a_1u^{(k+1)}-u^{(k+2)}-\sum_{j=1}^k\binom{k}{j}L^{(j)}[u^{(k-j)}]+f^{(k)}$$

$$\frac{\partial}{\partial n_t}u^{(k)}=-\sigma_1u^{(k+1)}-\sum_{j=1}^k\binom{k}{j}B^{(j)}[u^{(k-j)}]$$

and by applying the above apriori estimate by taking $l=p-1-k$ we get

$$\text{(2. 10)}\qquad \|u^{(k)}\|_{p+1-k}^2\leqslant K_{p-1-k}\left\{\left\|-a_1u^{(k+1)}-u^{(k+2)}-\sum_{j=1}^k\binom{k}{j}L^{(j)}[u^{(k-j)}] +f^{(k)}\right\|_{p-1-k}^2+\left\langle-\sigma_1u^{(k+1)}-\sum_{j=1}^k\binom{k}{j}B^{(j)}[u^{(k-j)}]\right\rangle_{p-k-1/2}^2+\|u^{(k)}\|_0^2\right\}$$

7) $w^{(k)}(x, t)$ denotes the k-times derivative with respect to t of a function $w(x, t)$, $L^{(k)}$ and $B^{(k)}$ are differential operators obtained by differentiating the corresponding coefficients of L and B k-times in t.

$$\leqslant K_{p-1-k}K'_k(||u^{(k+1)}||^2_{p-k}+||u^{(k+2)}||^2_{p-k-1}+\langle u^{(k+1)}\rangle^2_{p-k-1/2}$$
$$+|||u(x, t)|||^2_p+|||f(x, t)|||^2_{p-1}),$$

where K'_k depends on L, σ_1 and k. First take $k=p-1$ and it follows

$$||u^{(p-1)}||^2_2\leqslant\text{const.}\ (|||u^{(p)}|||^2_{1,\Omega}+|||u(x, t)|||^2_p+|||f|||^2_{p-1}). \tag{2.11}$$

Next take $k=p-2$, then

$$||u^{(p-2)}||^2_3\leqslant\text{const.}\ (||u^{(p-1)}||^2_2+||u^{(p)}||^2_1+|||u(x, t)|||^2_p+|||f|||^2_{p-1}),$$

substituting (2.11)

$$||u^{(p-2)}||^2_3\leqslant\text{const.}\ (|||u^{(p)}|||^2_1+|||u(x, t)|||^2_p+|||f|||^2_{p-1}).$$

Step by step we get for all $k=0, 1, \cdots, p-1$

$$||u^{(k)}||^2_{p+1-k}\leqslant\text{const.}\ (|||u^{(p)}|||^2_1+|||u(x, t)|||^2_p+|||f|||^2_{p-1}),$$

from which (2.9) follows immediately. Q.E.D.

We state a simple lemma without proof.

Lemma 2.5. *Let $\gamma(t)$ and $\rho(t)$ be two positive functions defined on* $[0, a]$ $(a>0)$. *Suppose that $\gamma(t)$ is summable on $(0, a)$ and that $\rho(t)$ is non-decreasing. Then the inequality*

$$\gamma(t)\leqslant c\int_0^t\gamma(s)ds+\rho(t)\qquad\text{for all}\quad t\in[0, a]$$

implies

$$\gamma(t)\leqslant e^{ct}\rho(t)\qquad\text{for all}\quad t\in[0, a].$$

Proof of Theorem 2

Proposition 2.6. *Let k be a non-negative integer and $\varphi(x)$ be a real-valued function in $C_0^\infty(R^n)$ with a support contained in an open set V. Let $u(x, t)\in H^{k+2}(R^n_+\times(0, T))$ satisfy* (1.1) *in $V\cap R^n_+$ and* (1.3) *in $V\cap R^{n-1}$. Then*

$$\begin{aligned}&||(\varphi u)^{(k)}(x, t)||^2_{\mathcal{H}(t)}\\&\leqslant C_k\Big\{\varepsilon|||u(x, t)|||^2_{k+1,\tilde V}+C(\varepsilon)|||u(x, t)|||^2_{k,\tilde V}+||u(x, 0)||^2_{k+1,L^2(\tilde V)}\\&+\left\|\frac{\partial u}{\partial t}(x, 0)\right\|^2_{k,L^2(\tilde V)}+|||f(x, 0)|||^2_{k-1,\tilde V}+\int_0^t\left\|\frac{\partial f}{\partial t}(x, s)\right\|^2_{k-1,\tilde V}ds\\&+\int_0^t|||u(x, s)|||^2_{k+1,\tilde V}ds\Big\}\end{aligned} \tag{2.12}$$

holds for all $t\in[0, T]$, where C_k depends on L, B, φ and k and $\tilde V=V\cap R^n_+$.

Proof. Put $v(x, t)=\varphi(x)u(x, t)$, then

$$L[v(x, t)] = -([L, \varphi]u)\,(x, t)+\varphi(x)f(x, t) \tag{2.13}$$

$$B[v(x, t)] = -\frac{\partial\varphi}{\partial n_t}u(x', 0, t)\,. \tag{2.14}$$

The differentiation of these two equations k-times with respect to t gives

$$\begin{aligned} L[v^{(k)}(x, t)] &= -\sum_{l=1}^{k}\binom{k}{l}L^{(l)}[v^{(k-l)}]-([L, \varphi]u)^{(k)}(x, t) \\ &\quad +\varphi(x)f^{(k)}(x, t) \\ B[v^{(k)}(x, t)] &= -k\Big(\frac{\partial}{\partial n_t}\Big)' v^{(k-1)}(x, t)+k\sigma_1' v^{(k)}(x, t) \\ &\quad -\sum_{l=2}^{k}\binom{k}{l}B^{(l)}[v^{(k-l)}] \\ &\quad -\frac{\partial\varphi}{\partial n_t}u^{(k)}(x', 0, t)-\sum_{l=1}^{k}\binom{k}{l}\Big(\frac{\partial\varphi}{\partial n_t}\Big)^{(l)}u^{(k-l)}\,. \end{aligned}$$

Then by applying Lemma 2.1 for $v^{(k)}(x, t)$ we have

$$\begin{aligned} ||v^{(k)}(x, t)||^2_{\mathcal{H}(t)} &\leqslant ||v^{(k)}(x, 0)||^2_{\mathcal{H}(0)}+c\int_0^t ||v^{(k)}(x, s)||^2_{\mathcal{H}(s)}ds \\ &\quad +\int_0^t ||-\sum_{l=1}^{k}\binom{k}{l}L^{(l)}[v^{(k-l)}]-([L, \varphi]u)^{(k)}+\varphi f^{(k)}||^2 ds \\ &\quad +2\mathrm{Re}\int_0^t\!\!\int\Big(-k\Big(\frac{\partial}{\partial n_t}\Big)' v^{(k-1)}+k\sigma_1' v^{(k)}-\frac{\partial\varphi}{\partial n_t}\cdot u^{(k)} \\ &\quad -\sum_{l=2}^{k}\binom{k}{l}B^{(l)}[v^{(k-l)}]-\sum_{l=1}^{k}\binom{k}{l}\Big(\frac{\partial\varphi}{\partial n_t}\Big)^{(l)}u^{(k-l)}\Big)\overline{\frac{\partial v^{(k)}}{\partial t}}\,dx'\,ds\,. \end{aligned} \tag{2.15}$$

Evidently we have

$$||v^{(k)}(x, 0)||^2_{\mathcal{H}(0)}\leqslant \text{const.}\ (||u^{(k)}(x, 0)||^2_{1,L^2(\tilde{V})}+||u^{(k+1)}(x, 0)||^2_{L^2(\tilde{V})})$$

$$||-\sum_{l=1}^{k}\binom{k}{l}L^{(l)}[v^{(k-l)}]-([L, \varphi]u)^{(k)}(x, t)||^2\leqslant \text{const.}\ |||u(x, t)|||^2_{k+1,\tilde{V}}\,.$$

Since

$$\Big|\Big|\Big|\Big(\sum_{l=2}^{k}\binom{k}{l}B^{(l)}[v^{(k-l)}]-\sum_{l=1}^{k}\binom{k}{l}\Big(\frac{\partial\varphi}{\partial n_t}\Big)^{(l)}u^{(k-l)}\Big)(x, s)\Big|\Big|\Big|^2_t$$
$$\leqslant \text{const.}\ |||u(x, s)|||^2_{k-1+i,\tilde{V}}\,,$$

by applying Lemma 2.3 we have

$$2\mathrm{Re}\int_0^t\int\Big(-\sum_{l=2}^{k}\binom{k}{l}B^{(l)}[v^{(k-l)}]-\sum_{l=1}^{k}\binom{k}{l}\Big(\frac{\partial\varphi}{\partial n_t}\Big)^{(l)}u^{(k-l)}\Big)\overline{\frac{\partial v^{(k)}}{\partial t}}dx'ds$$
$$\leqslant C\{\varepsilon|||v^{(k-1)}(x,t)|||_2^2+C(\varepsilon)|||v^{(k-1)}(x,t)|||_1^2+|||v^{(k-1)}(x,0)|||_2^2$$
$$+|||u(x,t)|||_{k,\tilde{V}}^2+|||u(x,0)|||_{k,\tilde{V}}^2$$
$$+\int_0^t(|||v^{(k-1)}(x,s)|||_2^2+|||u(x,s)|||_{k+1,\tilde{V}}^2)ds\}$$
$$\leqslant C'\{\varepsilon|||u(x,t)|||_{k+1,\tilde{V}}^2+C(\varepsilon)|||u(x,t)|||_{k,\tilde{V}}^2+|||u(x,0)|||_{k+1,\tilde{V}}^2$$
$$+|||u(x,t)|||_{k,\tilde{V}}^2+|||u(x,0)|||_{k,\tilde{V}}^2+\int_0^t|||u(x,s)|||_{k+1,\tilde{V}}^2\,ds\}\,.$$

To estimate the remained terms remark that from (2.14)

$$\frac{\partial v}{\partial x_n}=-\frac{1}{a_{nn}}\Big(\sum_{j=1}^{n-1}a_{nj}\frac{\partial v}{\partial x_j}+\sigma_1\frac{\partial v}{\partial t}-\frac{\partial\varphi}{\partial n_t}u\Big),$$

then

$$\frac{\partial v^{(k-1)}}{\partial x_n}=-\frac{1}{a_{nn}}\Big(\sum_{j=1}^{n-1}a_{nj}\frac{\partial v^{(k-1)}}{\partial x_j}+\sigma_1 v^{(k)}-\frac{\partial\varphi}{\partial n_t}u^{(k-1)}\Big)+B_{k-1}u\,,$$

where B_{k-1} is a boundary operator of the order $\leqslant k-1$. Then

$$2\mathrm{Re}\int_0^t ds\int_{R^{n-1}}\frac{\partial t}{\partial v^{(k)}}\overline{\Big(-\Big(\frac{\partial}{\partial n_t}\Big)'v^{(k-1)}\Big)}dx'$$
$$=2\mathrm{Re}\int_0^t ds\int\frac{\partial v^{(k)}}{\partial t}\overline{\Big(-\sum_{j=1}^{n-1}\Big(a'_{nj}+a'_{nn}\frac{a_{nj}}{a_{nn}}\Big)\frac{\partial v^{(k-1)}}{\partial x_j}\Big)}dx'$$
$$+2\mathrm{Re}\int_0^t ds\int\frac{\partial^2}{\partial t^2}u^{(k-1)}\varphi(x)\overline{B_{k-1}u}\,dx'$$

by applying Lemma 2.2 and 2.3

$$\leqslant C\Big\{\varepsilon|||v^{(k-1)}(x,t)|||_{2,\tilde{V}}^2+C(\varepsilon)|||v^{(k-1)}(x,t)|||_{1,\tilde{V}}^2+\varepsilon|||u(x,t)|||_{k+1,\tilde{V}}^2$$
$$+C(\varepsilon)|||u(x,t)|||_{k,\tilde{V}}^2+|||u(x,0)|||_{k+1,\tilde{V}}^2$$
$$+|||v^{(k-1)}(x,0)|||_{2,\tilde{V}}^2+\int_0^t(|||v^{(k-1)}(x,s)|||_{2,\tilde{V}}^2+|||u(x,s)|||_{k+1,\tilde{V}}^2)ds\Big\}$$
$$\leqslant C'\Big(\varepsilon|||u(x,t)|||_{k+1,\tilde{V}}^2+C(\varepsilon)|||u(x,t)|||_{k,\tilde{V}}^2$$
$$+|||u(x,0)|||_{k+1,\tilde{V}}^2+\int_0^t|||u(x,s)|||_{k+1,\tilde{V}}^2ds\Big)\,.$$

And by applying Lemma 2.3 we have

$$2\mathrm{Re}\int_0^t ds\int\frac{\partial v^{(k)}}{\partial t}\sigma_1\overline{v^{(k)}}dx'$$

$$\leqslant C\Big(\varepsilon|||u(x,t)|||^2_{k+1,\tilde V}+C(\varepsilon)|||u(x,t)|||^2_{k,\tilde V}+|||u(x,0)|||^2_{k+1,\tilde V}$$

$$+\int_0^t|||u(x,s)|||^2_{k+1,\tilde V}ds\Big)\,.$$

$$2\mathrm{Re}\int_0^t ds\int\frac{\partial v^{(k)}}{\partial t}\overline{\left(\frac{\partial\varphi}{\partial n_t}\right)u^{(k)}}dx'$$

$$=\int_0^t ds\int\left(\varphi(x)\cdot\frac{\partial\varphi}{\partial n_t}(x)\right)u^{(k+1)}u^{(k)}dx'$$

from Lemma 2.3

$$\leqslant C\Big(\varepsilon|||u(x,t)|||^2_{k+1,\tilde V}+C(\varepsilon)|||u(x,t)|||^2_{k,\tilde V}+|||u(x,0)|||^2_{k+1,\tilde V}$$

$$+\int_0^t|||u(x,s)|||^2_{k+1,\tilde V}ds\Big)\,.$$

Therefore inserting these estimates into (2.15), we get for some C'

$$||v^{(k)}(x,t)||^2_{\mathcal{H}(t)}\leqslant C'\Big\{\varepsilon|||u(x,t)|||^2_{k+1,\tilde V}+C(\varepsilon)|||u(x,t)|||^2_{k,\tilde V}+|||u(x,0)|||^2_{k+1,\tilde V}$$

$$+\int_0^t|||u(x,s)|||^2_{k+1,\tilde V}ds+\int_0^t|||f(x,s)|||^2_{k-1,\tilde V}ds\Big\}\,,$$

from this inequality (2.12) follows by using only

$$|||u(x,0)|||^2_{k+1,\tilde V}\leqslant\text{const.}\left(||u(x,0)||^2_{k+1,L^2(\tilde V)}+\left\|\frac{\partial u}{\partial t}(x,0)\right\|^2_{k,L^2(\tilde V)}\right.$$

$$\left.+|||f(x,0)|||^2_{k-1,\tilde V}\right)$$

which is derived from $Lu=f$, and

$$\int_0^t|||f(x,s)|||^2_{k-1,\tilde V}ds\leqslant\text{const.}\left(|||f(x,0)|||^2_{k-1,\tilde V}+\int_0^t\left\|\frac{\partial f}{\partial t}(x,s)\right\|^2_{k-1}ds\right)$$

Q.E.D.

Now we prove Theorem 2. Let $\{\varphi_j(x)\}_{j=1}^N$ be a partition of unity in a neighborhood of S, namely $\varphi_j(x)\in C_0^\infty(R^n)$ such that

$$\sum_{j=1}^N\varphi_j(x)^2=1\qquad\text{in a neighborhood of } S.$$

Assume that the support of φ_j is contained in a sufficiently small neighborhood U_j of some $s_j\in S$ and there exists a smooth transformation $\Psi_j=(\psi_{j1}(x),\cdots,\psi_{jn}(x))$ from U_j onto V_j in R^n such that

$$\Psi_j(U_j\cap\Omega)=V_j\cap R^n_+$$
$$\Psi_j(U_j\cap S)=V_j\cap R^{n-1}$$
$$\Psi_j(s_j)=0\,.$$

For the function $w(x)$ defined in a domain containing some $U_j\cap\Omega$ we denote by $\tilde{w}_j(y)$ the function defined in $V_j\cap R^n_+$ by $\tilde{w}_j(y)=\tilde{w}_j(\Psi_j(x))=w(x)$. Then

$$L_j[\tilde{u}_j(y,t)]=\tilde{f}_j(y,t)\quad \text{in}\quad (V_j\cap R^n_+)\times(0,T) \tag{2.16}$$
$$B_j[\tilde{u}_j(y,t)]=0\quad \text{in}\quad (V_j\cap R^{n-1})\times[0,T]\,, \tag{2.17}$$

where

$$L_j=\frac{\partial^2}{\partial t^2}+2\sum_{k=1}^n\Big(\sum_{l=1}^n h_l\frac{\partial\psi_{jk}}{\partial x_l}\Big)(y,t)\frac{\partial^2}{\partial y_k\partial t}$$
$$-\sum_{i,k=1}^n\Big(\sum_{p,q=1}^n a_{pq}\frac{\partial\psi_{ji}}{\partial x_p}\frac{\partial\psi_{jk}}{\partial x_q}\Big)(y,t)\frac{\partial^2}{\partial y_i\partial y_k}$$
$$+(\text{first order})$$
$$B_j=-\sum_{i=1}^n\Big(\sum_{p,q=1}^n a_{pq}\frac{\partial\psi_{ji}}{\partial x_p}\frac{\varphi\psi_{jn}}{\partial x_q}\Big)\frac{\partial}{\partial y_i}-\tilde{\sigma}_1(y',t)\frac{\partial}{\partial t}\,.$$

From (2.16) and (2.17), Proposition 2.6 shows

$$\|\tilde{\varphi}_j(y)\tilde{u}_j^{(m+1)}(y,t)\|^2_{\mathcal{H}(t)}\leqslant C_{jm}\Big(\|\tilde{u}_j(y,0)\|^2_{m+2,L^2(\tilde{V}_j)}$$
$$+\Big\|\frac{\partial\tilde{u}_j}{\partial t}(y,0)\Big\|^2_{m+1,L^2(\tilde{V}_j)}+\varepsilon|||\tilde{u}_j(y,t)|||^2_{m+2,\tilde{V}_j}$$
$$+C(\varepsilon)|||\tilde{u}_j(y,t)|||^2_{m+1,\tilde{V}_j}+\int_0^t|||\tilde{u}_j(y,t)|||^2_{m+2,\tilde{V}_j}dt$$
$$+|||\tilde{f}_j(y,0)|||^2_{m,\tilde{V}_j}+\int_0^t\Big\|\frac{\partial\tilde{f}_j}{\partial t}(y,s)\Big\|^2_{m,\tilde{V}_j}ds\Big)\,,$$

therefore we have

$$|||\varphi_j(x)u^{(m+1)}(x,t)|||^2_{1,\Omega}\leqslant C_{jm}\Big(\|u(x,0)\|^2_{m+2,L^2(\Omega)} \tag{2.18}$$
$$+\Big\|\frac{\partial u}{\partial t}(x,0)\Big\|^2_{m+1,L^2(\Omega)}+\varepsilon|||u(x,t)|||^2_{m+2,\Omega}$$
$$+C(\varepsilon)|||u(x,t)|||^2_{m+1,\Omega}+|||f(x,0)|||^2_{m,\Omega}$$
$$+\int_0^t\Big\|\frac{\partial f}{\partial t}(x,s)\Big\|^2_{m,\Omega}ds+\int_0^t|||u(x,s)|||^2_{m+2,\Omega}ds\Big)\,.$$

And

$$(2.19)\qquad |||(1-\sum_{j=1}^{N}\varphi_j(x)^2)^{1/2}u^{(m+1)}|||_{1,\Omega}^2 \leqslant c_m\Big(||u(x,0)||_{m+2,L^2(\Omega)}^2 + \Big\|\frac{\partial u}{\partial t}(x,0)\Big\|_{m+1,L^2(\Omega)}^2 + |||f(x,0)|||_{m,\Omega}^2 + \int_0^t \Big\|\frac{\partial f}{\partial t}(x,s)\Big\|_{m,\Omega}^2 ds + \int_0^t |||u(x,s)|||_{m+2,\Omega}^2 ds\Big).$$

Since it holds for some constant c_1

$$|||u^{(m+1)}(x,t)|||_{1,\Omega}^2 \leqslant \sum_{j=1}^{N} |||\varphi_j(x)u^{(m+1)}(x,t)|||_{1,\Omega}^2 + |||(1-\sum_{j=1}^{N}\varphi_j(x)^2)^{1/2}u^{(m+1)}|||_{1,\Omega}^2 + c_1||u^{(m+1)}(x,t)||_{L^2(\Omega)}^2,$$

by summing up (2.18) and (2.19) and by applying Lemma 2.4 we get for some constant C'_m

$$|||u(x,t)|||_{m+2,\Omega}^2 \leqslant C'_m\Big(\varepsilon|||u(x,t)|||_{m+2,\Omega}^2 + C(\varepsilon)|||u(x,t)|||_{m+1,\Omega}^2 + ||u(x,0)||_{m+2,L^2(\Omega)}^2 + \Big\|\frac{\partial u}{\partial t}(x,0)\Big\|_{m+1,L^2(\Omega)}^2 + |||f(x,0)|||_{m,\Omega}^2 + \int_0^t \Big\|\frac{\partial f}{\partial t}(x,s)\Big\|_{m,\Omega}^2 ds + \int_0^t |||u(x,s)|||_{m+2,\Omega}^2 ds\Big).$$

Fix ε such that $C'_m\varepsilon<1$. Then we have for some constant C''_m

$$|||u(x,t)|||_{m+2,\Omega}^2 \leqslant C''_m\Big(||u(x,0)||_{m+2,L^2(\Omega)}^2 + \Big\|\frac{\partial u}{\partial t}(x,0)\Big\|_{m+1,L^2(\Omega)}^2 + |||f(x,0)|||_{m,\Omega}^2 + \int_0^t \Big\|\frac{\partial f}{\partial t}(x,s)\Big\|_{m,\Omega}^2 ds + \int_0^t |||u(x,s)|||_{m+2,\Omega}^2 ds\Big)$$

here we used

$$|||u(x,t)|||_{m+1,\Omega}^2 \leqslant \text{const.}\Big(|||u(x,0)|||_{m+1,\Omega}^2 + \int_0^t |||u(x,s)|||_{m+2,\Omega}^2 ds\Big).$$

From this (2.1) follows by applying Lemma 2.5 by taking

$$\gamma(t) = |||u(x, t)|||^2_{m+2,\Omega}$$
$$\rho(t) = C''_m(||u(x, 0)||^2_{m+2,L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(x, 0)\right\|^2_{m+1,L^2(\Omega)}$$
$$+|||f(x, 0)|||^2_{m,\Omega} + \int_0^t \left\|\frac{\partial f}{\partial t}(x, s)\right\|^2_{m,\Omega} ds\Big) .$$

Q.E.D.

3. Existence and regularity of the solution (Proof of Theorem 1)

At first we explain the compatibility condition of general order. Let m be an integer $\geqslant 0$ and $\{u_0(x), u_1(x)\} \in H^{m+2}(\Omega) \times H^{m+1}(\Omega)$ and $f(x, t) \in \mathcal{E}(m, \Omega)$ $\frac{\partial f}{\partial t}(x, t) \in H^m(\Omega \times (0, T))$. Define $u_p(x) \in H^{m+2-p}(\Omega)$ ($p=2, 3, \cdots, m+1$) successively by the formula

$$(3.1) \qquad u_p(x) = -\sum_{k=0}^{p-2} \binom{p-2}{k} \{a_2^{(k)}(x, 0: D) u_{p-k-2} + a_1^{(k)}(x, 0: D) u_{p-k-1}\}$$
$$+ f^{(p-2)}(x, 0) .$$

DEFINITION 3.1. Given data $u_0(x)$, $u_1(x)$, $f(x, t)$ such that $u_0(x) \in H^{m+2}(\Omega)$, $u_1(x) \in H^{m+1}(\Omega)$, $f(x, t)$, $\frac{\partial f}{\partial t}(x, t) \in H^m(\Omega \times (0, T))$ are said to satisfy the compatibility condition of order m when

$$\sum_{k=0}^{p} \binom{p}{k} \left(\left(\frac{\partial}{\partial n_t} \right)^{(k)} u_{p-k} - (\sigma_1)^{(k)} u_{p-k+1} + (\sigma_2)^{(k)} u_{p-k} \right)_{t=0} = 0$$

holds on S for $p=0, 1, \cdots, m$.

DEFINITION 3.2. $S^m(L, B)$ is a space of all data $\Phi = (u_0, u_1, f)$ satisfying the compatibility condition of order m equipped with the following norm

$$\| \Phi \|^2_{m,\Omega} = ||u_0||^2_{m+2,L^2(\Omega)} + ||u_1||^2_{m+1,L^2(\Omega)}$$
$$+ |||f(x, 0)|||^2_{m,\Omega} + \int_0^t \left\|\frac{\partial f}{\partial t}(x, s)\right\|^2_{m,\Omega} ds .$$

REMARK. $S^m(L, B)$ is a Hilbert space and $S^{m+1}(L, B) \subset S^m(L, B)$.

Lemma 3.1. *Any element of $S^m(L, B)$ can be approximated by smooth elements of $S^m(L, B)$.*

Proof. Let $\Phi = (u_0, u_1, f) \in S^m(L, B)$. Take sequences of sufficiently smooth functions $v_{j0} \in H^{m+2}(\Omega)$, $v_{j1} \in H^{m+1}(\Omega)$, $g_j \in H^m(\Omega \times (0, T))$ such that

$$\begin{aligned}
v_{j0} &\to u_0 && \text{in} \quad H^{m+2}(\Omega)\\
v_{j1} &\to u_1 && \text{in} \quad H^{m+1}(\Omega)\\
g_j &\to f && \text{in} \quad \mathcal{E}(m, \Omega)\\
\frac{\partial g_j}{\partial t} &\to \frac{\partial f}{\partial t} && \text{in} \quad H^m(\Omega\times(0, T)) .
\end{aligned}$$

Define v_{jp} for $p=2, 3, \cdots, m+1$ by (3. 1) from v_{j0}, v_{j1} and g_j, and set

$$\gamma_{jl}(s) = \sum_{k=0}^{l} \binom{l}{k}\left(\left(\frac{\partial}{\partial n_t}\right)^{(k)} v_{jl-k} - (\sigma_1)^{(k)} v_{jl-k+1} + (\sigma_2)^{(k)} v_{jl-k}\right).$$

Then $\gamma_{jl}(s)$ $(l=0, 1, 2, \cdots, m)$ are sufficiently smooth function defined on S. Since $v_{jp}\to u_p$ in $H^{m+2-p}(\Omega)$ and $\Phi\in S^m(L, B)$ we have

$$\gamma_{jl}(s) \to 0 \quad \text{in} \quad H^{m-1/2+l}(S) . \tag{3. 2}$$

Let Ω be the interior domain of S and consider the following boundary value problem of a system of elliptic operators

$$\left\{\begin{aligned}
&(\lambda-\Delta)w_p = q_p(x)\in H^{m-p}(\Omega)\\
&\sum_{k=0}^{p} \binom{p}{k}\left(\left(\frac{\partial}{\partial n_t}\right)^{(k)} w_{p-k} - (\sigma_1)^{(k)} w_{p-k+1} + \sigma_2^{(k)} w_{p-k}\right)\\
&\qquad = r_p(s)\in H^{m+1/2-p}(S)\\
&\qquad\qquad (p = 0, 1, 2, \cdots, m) .
\end{aligned}\right. \tag{3. 3}$$

It can be easily seen that for sufficiently large $\lambda>0$ (3.3) has a unique solution in $w_p\in H^{m+2-p}(\Omega)$ and the estimate

$$\sum_{p=0}^{m} \|w_p\|_{m+2-p}^2 \leqslant K \sum_{p=0}^{m} (\|q_p\|_{m-p}^2 + \langle r_p\rangle_{m+1/2-p}^2) \tag{3. 4}$$

holds[8].

Let w_{jp} be the solution of (3.3) for $q_p\equiv 0$, $r_p=\gamma_{jp}(s)$. Then from (3.4)

$$\sum_{p=0}^{m} \|w_{jp}\|_{m+2-p}^2 \leqslant K\Big(\sum_{p=0}^{m} \langle\gamma_{jp}\rangle_{m+1/2-p}^2\Big) \to 0 .$$

Now we take $\{u_{j0}, u_{j1}, f_j\}$ as

$$\begin{aligned}
u_{j0} &= v_{j0}-w_{j0}\\
u_{j1} &= v_{j1}-w_{j1}\\
f_j &= g_j - \sum_{l=2}^{m} \left\{w_{jl} + \sum_{k=0}^{l-2} \binom{l-2}{k}(a_2^{(k)} w_{jl-k-2} + a_1^{(k)} w_{jl-k-1})\right\}\frac{t^{l-2}}{(l-2)!} .
\end{aligned}$$

8) The problem (3.3) satisfies the coerciveness condition by taking $s_i=i-m$, $t_j=m+2-j$, $r_h=1-m+h$, of Agmon-Douglis-Nirenberg (Comm. Pure and Appl. Math., XVII, 35–92).

Then u_{jp} $(p=2, \cdots, m+1)$ constructed from $\Phi_j=(u_{j0}, u_{j1}, f_j)$ are $v_{jp}-w_{jp}$, therefore smooth data Φ_j are in $S^m(L, B)$ and evidently

$$\|\Phi_j-\Phi\|_{m,\Omega} \to 0$$

when $j\to\infty$.

When Ω is the exterior domain of S, the existence of an approximating sequence is deduced to a case with a compact domain by introducing a sphere S_1 containing S[9]. Q.E.D.

Let B_ε be the boundary operator defined by

$$B_\varepsilon = \frac{\partial}{\partial n_t} - (\sigma_1-\varepsilon)\frac{\partial}{\partial t} + \sigma_2 \tag{3. 5}$$

where ε is any positive constant.

Lemma 3.2. *For any element* $\Phi=(u_0, u_1, f)\in S^m(L, B)$ *there exists a sequence* $\Phi_j=(u_{j0}, u_{j1}, f_j)\in S^m(L, B_{1/j})$ $(j=1, 2, \cdots)$ *such that* $\|\Phi_j-\Phi\|_{m,\Omega}\to 0$.

Proof. u_p $(p=2, 3, \cdots, m+1)$ is derived from Φ by (3.1).

$$\begin{aligned}\gamma_{jp}(s) &= \sum_{k=0}^{p}\left\{\left(\frac{\partial}{\partial n_t}\right)^{(k)} u_{p-k} - \left(\sigma_1 - \frac{1}{j}\right)^{(k)} u_{p-k+1} + \sigma_2^{(k)} u_{p-k}\right\} \\ &= \frac{1}{j}\, u_{p+1}(s)\in H^{m+1/2-p}(S)\,.\end{aligned}$$

Ω be the interior domain of S and w_{jp} be the solution of (3.3) for $q_p=0$, $r_p(s) =\gamma_{jp}$, then we have

$$\sum_{p=0}^{m} \|w_{jp}\|^2_{m+2-p} \longrightarrow 0 \qquad (\text{when } j\to\infty)\,.$$

Take u_{j0}, u_{j1}, f_j as

$$\begin{aligned} u_{j0} &= u_0-w_{j0} \\ u_{j1} &= u_1-w_{j1} \\ f_j &= f-\sum_{l=2}^{m}\left\{w_{jl}+\sum_{k=0}^{l-2}\binom{l-2}{k}(a_2^{(k)}W_{jl-k-2}+a_1^{(k)}W_{jl-k-1})\right\}\frac{t^{l-2}}{(l-2)!}\,,\end{aligned}$$

then $\Phi_j=(u_{j0}, u_{j1}, f_j)\in S^m(L, B_{1/j})$ and $\|\Phi_j-\Phi\|_{m,\Omega}\to 0$ when $j\to\infty$.
Q.E.D.

Lemma 3.3. $S^{m+1}(L, B)$ *is dense in* $S^m(L, B)$.

9) See the proof of Proposition 4.1 of [6].

Proof. From Lemma 3.1 for any $\Phi\in S^m(L, B)$ there exists a smooth data $\Phi_j\in S^m(L, B)$ which tends to Φ. We can define u_{jm+2} by the formula (3.1) by taking $p=m+2$ for Φ_j. Since for a smooth function $\gamma(s)$ defined on S there exists a sequence of $v_k(x)\in H^2(\Omega)\cap\mathcal{D}^1_{L^2}(\Omega)$ such that

$$\frac{\partial}{\partial n_0}v_k(x)=\gamma(s) \qquad \text{on } S$$

$$\|v_k\|_{1,L^2(\Omega)}\leqslant\frac{1}{k},$$

we take $w_j(x)\in H^2(\Omega)\cap\mathcal{D}^1_{L^2}(\Omega)$ as $\|w_j(x)\|_{1,L^2(\Omega)}\leqslant\frac{1}{j}$ and

$$\frac{\partial}{\partial n_0}w_j(x)=\sum_{p=0}^{m+1}\binom{m+1}{p}\left\{\left(\frac{\partial}{\partial n}\right)^{(p)}u_{jm+1-p}-(\sigma_1)^{(k)}u_{jm+2-p}\right.$$
$$\left.+(\sigma_2)^{(k)}u_{jm+1-p}\right\},$$

then put

$$\widetilde{\Phi}_j=(u_{j0},\ u_{j1}-w_j,\ f_j)$$

when $m=0$, and

$$\widetilde{\Phi}_j=\left(u_{j0},\ u_{j1},\ f_j-\frac{t^{m-1}}{(m-1)!}w_j\right)$$

when $m\geqslant1$. Then $\widetilde{\Phi}_j\in S^{m+1}(L, B)$ and converges to Φ in $S^m(L, B)$ when j increases infinitely. Q.E.D.

Let $\Phi\in S^{m+1}(L, B)$ and take $\Phi_j\in S^{m+1}(L, B_{1/j})$ such that Φ_j converges to Φ. For each Φ_j there exists a unique solution $u_j(x, t)\in\mathcal{E}(m+3, \Omega)$ of $P(L, B_{1/j})$. Therefore from Theorem 2 we have

$$|||u_j(x, t)|||^2_{m+2,\Omega}\leqslant C_m\Big(\|u_{j0}\|^2_{m+2,L^2(\Omega)}+\|u_{j1}\|^2_{m+1,L^2(\Omega)}$$
$$+|||f_j(x, 0)|||^2_{m,\Omega}+\int_0^t\left\|\frac{\partial f_j}{\partial t}(x, s)\right\|^2_{m,\Omega}ds\Big)$$

where C_m does not depend on j,[10] which shows $\{u_j(x, t)\}_{j=1,2,\dots}$ is a bounded set in $H^{m+2}(\Omega\times(0, T))$, therefore weakly compact. Thus for some subsequence $\{u_{j_p}(x, t)\}_{p=1,2,3,\dots}$ converges weakly to some $u(x, t)\in H^{m+2}(\Omega\times(0, T))$. It is easy to see that $u(x, t)$ is the solution of $P(L, B)$ for the data Φ. Indeed evidently $u(x, t)$ satisfies $L[u]=f$, on the other hand

10) When L and m are fixed, C_m depends on $\left|\frac{\partial\sigma_1}{\partial t}\right|_{m+1}$. Therefore C does not depend on j.

$$B[u_{j_p}(x, t)] = -\frac{1}{j_p}\frac{\partial u_{j_p}}{\partial t}(x, t)$$

holds and the left-hand side converges to $B[u(x, t)]$ weakly and the right-hand side tends to zero therefore $B[u]=0$. Similarily $u(x, 0)=u_0$, $\frac{\partial u}{\partial t}(x, 0)=u_1(x)$ is assured. Then we get

Proposition 3.4. *For any $\Phi\in S^{m+1}(L, B)$ there exists a solution of $P(L, B)$ in $H^{m+2}(\Omega\times(0, T))$.*

With the aid of these facts we get immediately Theorem 1. Let $\Phi\in S^m(L, B)$, since Lemma 3.2 shows $S^{m+2}(L, B)$ is also dense in $S^m(L, B)$ there exists a sequence of $\Phi_j\in S^{m+2}(L, B)$ converging to Φ. Proposition 3.4 assures that the existence of the solution $u_j(x, t)\in H^{m+3}(\Omega\times(0, T))$ of $P(L, B)$ for Φ_j, then $u_j(x, t)\in \mathcal{E}(m+2, \Omega)$.

By applying Theorem 2 for u_k-u_j

$$\sup_{t\in[0,T]} |||u_j(x, t)-u_k(x, t)|||^2_{m+2,\Omega}\leqslant C_m \| \Phi_j-\Phi_k \|^2_{m,\Omega}\,.$$

This shows the convergence of u_j in $\mathcal{E}(m+2, \Omega)$. Denote its limit by $u(x, t)$, then the passage to the limit of

$$\begin{aligned} L[u_j] &= f_j \\ B[u_j] &= 0 \\ u_j(x, 0) &= u_{j0}(x) \\ \frac{\partial u_j}{\partial t}(x, 0) &= u_{j1}(x) \end{aligned}$$

when $j\to\infty$ shows that $u(x, t)\in\mathcal{E}(m+2, \Omega)$ is the required solution. And we also see the energy inequality

$$(3.5)\qquad |||u(x, t)|||^2_{m+2,\Omega}\leqslant C_m\Big(||u_0||^2_{m+2,L^2(\Omega)}+||u_1||^2_{m+1,L^2(\Omega)} + |||f(x, 0)|||^2_{m,\Omega}+\int_0^t\Big|\Big|\Big|\frac{\partial f}{\partial t}(x, s)\Big|\Big|\Big|^2_{m,\Omega} ds\Big)$$

follows from the passage to the limit of the estimates

$$|||u_j(x, t)|||^2_{m+2,\Omega}\leqslant C_m\Big(||u_{j0}||^2_{m+2,L^2(\Omega)}+||u_{j1}||^2_{m+1,L^2(\Omega)} + |||f_j(x, 0)|||^2_{m,\Omega}+\int_0^t\Big|\Big|\Big|\frac{\partial f_j}{\partial t}(x, s)\Big|\Big|\Big|^2_{m,\Omega} ds\Big).$$

Uniquencess of the solution is derived from the facts that for any solution $u(x, t)\in\mathcal{E}(2, \Omega)$ of $P(L, B)$ the energy inequality

$$|||u(x, t)|||_{1,\Omega}^2 \leqslant c\left(||u_0||_{1,L^2(\Omega)}^2 + ||u_1||_{L^2(\Omega)}^2 + \int_0^t ||f(x, s)||_{L^2(\Omega)}^2\, ds\right)$$

holds, which follows from Lemma 2.1 and Lemma 2.5.

REMARK 3.5. If we combine Theorem 1 and 2 the following holds: *For any solution $u(x, t)$ of $P(L, B)$ in $\mathcal{E}(m+2, \Omega)$, if $\frac{\partial f}{\partial t}(x, t) \in H^m(\Omega \times (0, T))$, the energy inequality*

$$(3.6) \qquad |||u(x, t)|||_{m+2,\Omega}^2 \leqslant C_m\left(||u(x, 0)||_{m+2,L^2(\Omega)}^2 + \left\|\frac{\partial u}{\partial t}(x, 0)\right\|_{m+1,L^2(\Omega)}^2 + |||f(x, 0)|||_{m,\Omega}^2 + \int_0^t \left\|\frac{\partial f}{\partial t}(x, s)\right\|_{m,\Omega}^2 ds\right)$$

holds.

Proof. Since $\Phi = \left(u(x, 0), \frac{\partial u}{\partial t}(x, 0), f(x, t)\right) \in S^m(L, B)$, from Theorem 1 we have a solution $\tilde{u}(x, t) \in \mathcal{E}(m+2, \Omega)$ of $P(L, B)$ for Φ and for $\tilde{u}(x, t)$ the energy inequality (3.6) holds. On the other hand, from the uniqueness of the solution, $\tilde{u}(x, t)$ is nothing but $u(x, t)$. Thus (3.6) holds for $u(x, t)$.

Q.E.D.

REMARK 3.6. Our problem $P(L, B)$ has a finite velocity. Let $\lambda_1(x, t: \xi)$, $\lambda_2(x, t: \xi)$ be the roots of the characteristic equation of L

$$\lambda^2 + 2\sum_{j=1}^n h_j(x, t)\xi_j\lambda - \sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j = 0$$

for $(x, t) \in \Omega \times [0, T]$ and $\xi \in R^n$. Denote

$$(3.7) \qquad \lambda_{\max} = \sup_{\substack{|\xi|=1,\, j=1,2 \\ (x,t)\in\Omega\times[0,T]}} |\lambda_j(x, t: \xi)|$$

and $\Lambda(x_0, t_0) = \{(x, t);\ |x - x_0| \leqslant \lambda_{\max}(t_0 - t)\}$, then we have the following:

Let $u(x, t)$ be C^2-function defined in $\Lambda(x_0, t_0) \cap (\Omega \times [0, T])$ satisfying $L[u] = 0$ in $\Lambda(x_0, t_0) \cap (\Omega \times (0, T))$ and $B[u] = 0$ in $\Lambda(x_0, t_0) \cap (S \times [0, T])$. If $u_0(x)$, $u_1(x)$ are zero in $\Lambda(x_0, t_0) \cap \{\bar{\Omega}, t=0\}$, $u(x, t)$ is identically zero in $\Lambda(x_0, t_0) \cap (\Omega \times (0, T))$. Since the proof is essentially same as that of [16], we omit it.[11]

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11) See §5 of [16].

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