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Osaka University
A SPLITTING PROPERTY OF ORIENTED HOMOTOPY EQUIVALENCE FOR A HYPERELEMENTARY GROUP

MASAHARU MORIMOTO

(Received December 5, 1980)

1. Introduction

Let $G$ be a finite group. In this paper a $G$-space means a complex $G$-representation space of finite dimension. For a $G$-space $V$ we denote by $S(V)$ its unit sphere with respect to some $G$-invariant inner product. After tom Dieck [1] and [2] we call two $G$-spaces $V$ and $W$ oriented homotopy equivalent if there exists a $G$-map $f: S(V) \rightarrow S(W)$ such that for each subgroup $H$ of $G$ the induced map $f^H: S(V)^H \rightarrow S(W)^H$ on the $H$-fixed point sets has degree one with respect to the coherent orientations which are inherited from the complex structures on $V^H$ and $W^H$. Let $R(G)$ be the complex $G$-representation ring, $R_a(G)$ the additive subgroup of $R(G)$ consisting of $x=V-W$ such that $V$ and $W$ are oriented homotopy equivalent, and $R_d(G)$ the additive subgroup of $R(G)$ consisting of $x=V-W$ such that $\dim V^H = \dim W^H$ for all the subgroups $H$ of $G$. We denote by $j(G)$ the quotient group $R_d(G)/R_a(G)$.

If $G$ has a normal cyclic subgroup $A$ and a Sylow $p$-subgroup $H$ such that $G$ is the semidirect product of $H$ by $A$, we call $G$ a hyperelementary group. Especially if $G$ is the direct product of $A$ and $H$, we call $G$ an elementary group. tom Dieck showed that for an arbitrary finite group $G$ the restriction homomorphism from $j(G)$ to the direct sum of $j(K)$ is injective, where $K$ runs over the hyperelementary subgroups of $G$ ([1; Proposition 5.1]). Our purpose of this paper is to consider oriented homotopy equivalence for hyperelementary groups and to give a sufficient condition for a hyperelementary group to have a splitting property defined below.

Choose an integer $m$ which is a multiple of the orders of the elements of $G$, and let $Q(m)$ be the field obtained by adjoining the $m$-th roots of unity to $Q$, where $Q$ is the field of rational numbers. The Galois group $\Gamma=\Gamma(m)$ of $Q(m)$ over $Q$ acts on $R(G)$ via its action on character value. Actually $\Gamma$ acts on the set $\text{Irr}(G)$ of isomorphism classes of irreducible $G$-spaces. Let $Z[\Gamma]$ be the integral group ring of $\Gamma$, and $I(\Gamma)$ its augmentation ideal. Then we have $R_0(G)=I(\Gamma)R(G)$. We put $R_1(G)=I(\Gamma)R_a(G)$. According to [3] we have $R_1(G) \subset R_a(G)$. Let us say that $G$ has Property 1 if $R_1(G)$ coincides with $R_a(G)$. 

For example the abelian groups and the $p$-groups have Property 1, and some hyperelementary groups do not have Property 1 (see [1] and [6]). In section 3 we obtain other groups which have Property 1.

For each orbit $C \in X(G) = \text{Irr}(G)/\Gamma$, we let $F(C)$ be the free abelian group on elements of $C$. Then we have $R(G) = \bigoplus_{C \in X(G)} F(C)$. Let $f_C$ be the canonical projection from $R(G)$ to $F(C)$. Let us say that $G$ has Property 2 (we called this a splitting property) if for each element $x$ of $R_\delta(G)$ and each element $C$ of $X(G)$ $f_C(x)$ belongs to $R_\delta(G)$. This property is of our interest. If $G$ has Property 1, then $G$ has Property 2; the converse is not true. It is remarkable that $R_\delta(G)$ is determined by oriented homotopy equivalence between the irreducible $G$-spaces if $G$ has Property 2.

Our main results are Theorems 6.11 and 6.12, and the latter indicates the importance of Property 2. Additionally we give a counter example to [1; Proposition 5.2] in section 7.

The author wishes to express his hearty thanks to Professor M. Nakaoka and Professor K. Kawakubo for their kind advice.

2. Preparation

Let $S(G)$ the set of normal subgroups of $G$. If a $G$-space $V$ is given we write $V = \bigoplus_{H \in S(G)} V(H)$, where $V(H)$ collects the faithful irreducible $G/H$-subspaces (see [2; p. 252]).

Lemma 2.1 ([2]). If $x = V - W \in R_\delta(G)$, then for all $H \in S(G)$ we have $x(H) = V(H) - W(H) \in R_\delta(G)$.

Let $V$ and $W$ be $G$-spaces. If $f$ is an $N_G(H)$-map from $S(V)^H$ to $S(W)^H$ and $g$ is an element of $G$, then there uniquely exists an $N_G(gHg^{-1})$-map $h$ from $S(V)^{gHg^{-1}}$ to $S(W)^{gHg^{-1}}$ such that the following diagram is commutative:

\[
\begin{array}{c}
S(V)^H & \xrightarrow{f} & S(W)^H \\
\downarrow{g_*} & & \downarrow{g_*} \\
S(V)^{gHg^{-1}} & \xrightarrow{h} & S(W)^{gHg^{-1}}
\end{array}
\]

where $g_*$ are the maps canonically given by the actions of $g$.

Proposition 2.2. Let $V$ and $W$ be $G$-spaces. We have $V^H - W^H \in R_\delta(N_G(H))$ if and only if we have $V^{gHg^{-1}} - W^{gHg^{-1}} \in R_\delta(N_G(gHg^{-1}))$.

Proof. This proposition follows from the fact that each $g_*$ of the above diagram preserves the orientation of the sphere.

Let $V$ and $W$ be $G$-spaces such that $\dim V^H = \dim W^H$ for all subgroups $H$ of $G$ (i.e. $V - W \in R_\delta(G)$). We put $n = \dim V (= \dim W)$. If $g$ is an element of
Let \( V \) and \( W \) be \( G \)-spaces as above. \( V \) and \( W \) are oriented homotopy equivalent if and only if the following two conditions (i) and (ii) are satisfied.

(i) For each non-trivial subgroups \( H \) of \( G \) (i.e. \( H \neq \{1\} \)), we have \( V^H = W^H \in R_*(\mathcal{N}_c(H)) \).

(ii) It holds that \( P(G; W-V) \equiv 0 \mod |G| \).

Provided (i), then (ii) is equivalent to the condition: \( P(G; V-W) \equiv 0 \mod |G| \).

Let \( s \) be a positive integer, and \( V \) a \( G \)-space of dimension \( n \). We are going to define an element \( Q(s; V) \) of \( R(G) \). Let \( x(1), \ldots, x(n) \) be indeterminates, and \( y(i) \) the elementary symmetric polynomial of degree \( i \) for each \( 1 \leq i \leq n \). We define a polynomial \( Q \) of \( y(1), \ldots, y(n) \) by

\[
Q(y(1), \ldots, y(i), \ldots, y(n)) = \prod_{j=1}^n (1 + x(j) + \cdots + x(j)^{i-1}).
\]

We define \( Q(s; V) \) by

\[
Q(s; V) = Q(V, \ldots, \Lambda^j V, \ldots, \Lambda^n V),
\]

where \( \Lambda^j V \) is the \( j \)-fold exterior power of \( V \). By the usual identification we let \( Q(s; V)(g) \) stands for trace \( (g; Q(s; V)) \). Then it holds that

\[
Q(s; V)(g) = \prod_{j=1}^n (1 + a_j(g) + \cdots + a_j(g)^{s-1}),
\]

where \( a_1(g), \ldots, a_s(g) \) are all the eigenvalues of \( g \) on \( V \). Since \( Q(s; V) \in R(G) \), we have

\[
\sum_{H \subseteq G} Q(s; V)(h) \equiv 0 \mod |H|
\]

for each subgroup \( H \) of \( G \).
3. A few remarks about Property 1

Let $L$ be a finite abelian group. We denote the integral group ring of $L$ by $\mathbb{Z}[L]$, the augmentation ideal of $\mathbb{Z}[L]$ by $I(L)$, i.e.

$$I(L) = \{ \sum_{x \in L} z(x)x: z(x) \in \mathbb{Z}, \text{ and } \sum_{x \in L} z(x) = 0 \} ,$$

where $\mathbb{Z}$ is the ring of integers.

**Proposition 3.1.** We have the following.

(i) For $x, x' \in L$, it holds that $xx' - x \equiv x' - 1 \mod I(L)^2$.

(ii) For $x \in L$ and $z \in \mathbb{Z}$, it holds that $zx - z1 \equiv x' - 1 \mod I(L)^2$.

(iii) $I(L)/I(L)^2$ is isomorphic to $L$.

Since the proof is straightforward, we omit it.

Let $G$ be a direct product $H \times K$ as finite group. We denote by $\phi$ the Euler function, that is, for a positive integer $n$, $\phi(n)$ is the number of the units of $\mathbb{Z} = \mathbb{Z}/(n)$.

**Proposition 3.2.** Let $V$ be an irreducible $H$-space, and $W$ an irreducible $K$-space. Assume $(\phi(|H|), \dim W) = (\phi(|K|), \dim V) = 1$. Then for an element

$$x = \sum_{\gamma \in \Gamma} z(\gamma)\gamma(V \otimes W) \in R_\varphi(G) ,$$

$x$ belongs to $R_\varphi(G)$ if and only if $\text{Res}_H^G x \in R_\varphi(H)$ and $\text{Res}_K^G x \in R_\varphi(K)$, where $z(\gamma)$ are integers.

Proof. The only if part is clear. We are going to prove the if part. $\Gamma$ acts on the orbits $\Gamma(V \otimes W)$, $\Gamma V$ and $\Gamma W$ which are subsets of $\text{Irr}(G)$, $\text{Irr}(H)$ and $\text{Irr}(K)$ respectively. Let $\Gamma_{V \otimes W}$, $\Gamma_V$ and $\Gamma_W$ be the isotropy subgroups of $V \otimes W$, $V$ and $W$ respectively. We have $\Gamma_{V \otimes W} = \Gamma_V \cap \Gamma_W$. Put $M = \Gamma/\Gamma_V$ and $N = \Gamma/\Gamma_W$. The order of $M$ (resp. $N$) divides $\phi(|H|)$ (resp. $\phi(|K|)$). Since $x \in R_\varphi(G)$, there exists $\mu \in \Gamma$ such that

$$x \equiv (\mu - 1)(V \otimes W) \mod R_\varphi(G) .$$

We put $y = (\mu - 1)(V \otimes W)$. $\text{Res}_H^G x \in R_\varphi(H)$ and $\text{Res}_K^G x \in R_\varphi(K)$ are equivalent to $\text{Res}_H^G y \in R_\varphi(H)$ and $\text{Res}_K^G y \in R_\varphi(K)$ respectively. We have $\text{Res}_H^G y = (\dim W)(\mu - 1)V$. By Proposition 3.1 (ii) it holds that

$$\text{Res}_H^G y \equiv (\mu_{\dim W} - 1)V \mod R_\varphi(H) .$$

$\text{Res}_H^G y \in R_\varphi(H)$ implies $\mu_{\dim W} \in \Gamma_V$. Since $(|M|, \dim W) = 1$, we have $\mu \in \Gamma_V$. In the same way we obtain $\mu \in \Gamma_W$. Therefore we have $\mu \in \Gamma_{V \otimes W}$; this means $y = 0$ in $R(G)$. Consequently $x$ belongs to $R_\varphi(G)$.

For a group $G$ we denote by $C(G)$ its center. Since the dimensions of the
irreducible $G$-spaces divide $|G/C(G)|$, we have the following proposition.

**Proposition 3.3.** If both $H$ and $K$ have Property 1 and if it holds that $(|H/C(H)|, \phi(|K|))=(|K/C(K)|, \phi(|H|))=1$, then $G=H \times K$ has Property 1.

As the abelian groups and the $p$-groups have Property 1, we have the following.

**Corollary 3.4.** Let $H$ be an abelian group, and $K$ a $p$-group. Provided $(\phi(|H|), p)=1$, then $G=H \times K$ has Property 1.

**Corollary 3.5.** Let $H$ be a $p$-group and $K$ a $q$-group. Provided $(p, q)=1$, then $G=H \times K$ has Property 1.

### 4. The irreducible spaces of the hyperelementary group

Let $G$ have a normal cyclic subgroup $A$ and a Sylow $p$-subgroup $H$ such that $G$ is the semidirect product of $H$ by $A$, that is, $G$ is a hyperelementary group. The irreducible representations of $G$ can be constructed by the method of little group of Wigner and Mackey (see [7; 8.2]).

Since $A$ is cyclic, its irreducible representations form a group $Y$. The group $G$ acts on $Y$ by

$$(g\chi)(a) = \chi(g^{-1}ag)$$

for $g \in G$, $\chi \in Y$, $a \in A$. This action induces the action of $G$ on the set $Irr(A)$ of irreducible $A$-spaces. For $V \in Irr(A)$ and $g \in G$, we have an irreducible $A$-space $g_*V$ by this action. Let $\{V(i): i \in Y/H\}$ be a system of representatives for the orbits of $H$. For each $i \in Y/H$, let $H(i)$ be the subgroup of $H$ consisting of those elements $h$ such that $hV(i)=V(i)$, and let $G(i)=AH(i)$ be the corresponding subgroup of $G$. We can canonically extend $V(i)$ to the $G(i)$-space, that is, $h \in H(i)$ acts trivially on $V(i)$. Let $W$ be an irreducible $H(i)$-space; $W$ can be extended to $G(i)$-space, too. By taking the tensor product of $V(i)$ and $W$ we obtain an irreducible $G(i)$-space $V(i) \otimes W$. Then $\text{Ind}_G^{G(i)}V(i) \otimes W$ is irreducible, moreover each irreducible $G$-space is obtained in this way ([7; Proposition 25]).

We denote by $C_H(A)$ the centralizer of $A$ in $H$, i.e.

$$C_H(A) = \{g \in H: g^{-1}ag = a\ \text{for all}\ a \in A\} .$$

**Proposition 4.1.** If the kernel of $\text{Ind}_G^{G(i)}V(i) \otimes W$ is $\{1\}$, then the kernel of the $A$-space $V(i)$ is $\{1\}$, and $H(i)=C_H(A)$.

**Proof.** This comes from the fact that $\ker V(i) \subset \ker \text{Ind}_G^{G(i)} \{V(i) \otimes W\}$.

Since $C_H(A)$ is normal in $H$, $H$ acts on $Irr(C_H(A))$ by
Proposition 4.2. Put $K=C_H(A)$, and let $V$ be an irreducible $A$-space with the trivial kernel, $W$ an irreducible $K$-space and $h$ an element of $H$. Then we have

$$\text{Ind}^G_K V \otimes (h \ast W) = \text{Ind}^G_K (h^{-1} \ast V) \otimes W$$

Proof. If we identify the representation spaces with the corresponding characters, by direct calculation we have

$$\{\text{Ind}^G_K V \otimes (h \ast W)\}(g) = \text{Ind}^G_K \{h^{-1} \ast V\} \otimes W\}(g) \quad \text{for each } g \in G.$$

Proposition 4.3. We have the following.

(i) $\gamma \text{Ind}^G_{G(i)} V(i) \otimes W = \text{Ind}^G_{G(i)} (\gamma V(i)) \otimes (\gamma W)$ for $\gamma \in \Gamma$.

(ii) $\text{Res}^G_H \text{Ind}^G_{G(i)} V(i) \otimes W = \text{Ind}^H_{H(i)} W$.

(iii) $\text{Res}^G_H \text{Ind}^G_{G(i)} V(i) \otimes W = \dim W \bigoplus_{[h] \in H/H(k)} h \ast V(i)$

(iv) If $\ker \text{Ind}^G_{G(i)} V(i) \otimes W = \ker \text{Ind}^G_{G(j)} V(j) \otimes W'$, then we have $H(i) = H(j)$.

Proof. (i): This holds clearly.

(ii): Since $H \backslash G/G(i)$ consists of the only one coset, (ii) follows from the Mackey decomposition.

(iii): Since $A \backslash G/G(i)$ can be identified with $H/H(i)$, we have (iii) by the Mackey decomposition.

(iv): Put $U = \text{Ind}^G_{G(i)} V(i) \otimes W$ and $U' = \text{Ind}^G_{G(j)} V(j) \otimes W'$. From $\ker U = \ker U'$ we have $\ker \text{Res}^G_A U = \ker \text{Res}^G_A U'$. By (iii) we have $\ker V(i) = \ker V(j)$. This implies $H(i) = H(j)$.

Proposition 4.4. Put $K=C_H(A)$, and let $V$ be an irreducible $A$-space with the trivial kernel, $U$ and $W$ irreducible $K$-spaces. Set $M=\text{Ind}^G_K V \otimes U$ and $N=\text{Ind}^G_K V \otimes W$. Provided $\Gamma M \equiv \Gamma N$ as subset of $\text{Irr}(G)$, then we have

$$\langle \Gamma \text{Res}^G_K M, \Gamma \text{Res}^G_K N \rangle_K = \{0\}.$$

Proof. For $\gamma \in \Gamma$ we have

$$\gamma \text{Res}^G_K M = \bigoplus_{[h] \in H/K} \gamma h \ast U$$

by Proposition 4.3 (ii). Proposition 4.2 implies $\text{Ind}^G_K V \otimes (h \ast U) \in \Gamma M$. Since $\Gamma M \equiv \Gamma N$, we have

$$\langle \gamma h \ast U, \gamma' h' \ast W \rangle_K = 0$$

for each $\gamma \in \Gamma$, $\gamma' \in \Gamma$, $[h] \in H/K$ and $[h'] \in H/K$. This relation yields the consequence of Propositions 4.4.
Proposition 4.5. Let $L$ be a subgroup of $H$, then we have $N_G(L) = C_A(L)N_H(L)$.

Proof. Let $a$ and $h$ are elements of $A$ and $H$ respectively. If $ah \in N_G(L)$, we have $(ah)^{-1}Lah = L$, consequently $a^{-1}La = hLh^{-1}$. For each $g \in L$, there exists $h' \in H$ such that $a^{-1}ga = h'$. Then we have $a^{-1}(gag^{-1}) = h'g^{-1} \in A \cap H$. This means that $a^{-1}gag^{-1} = 1$ and $h'g^{-1} = 1$. Therefore we have $ga = ag$, that is, we have $a \in C_A(L)$. This yields $L = hLh^{-1}$. We obtain $h \in N_H(L)$. The above argument shows $N_G(L) \subseteq C_A(L)N_H(L)$. On the other hand $N_G(L) \supseteq C_A(L)N_H(L)$ holds obviously. Hence we have $N_G(L) = C_A(L)N_H(L)$.

Let $h$ be an element of $H$, then $h$ acts on the generators $a$ of $A$ by

$$h \cdot a = hah^{-1}.$$ 

Let $L$ be the subset of $H$ consisting of elements $h$ such that

$$T(h) = \prod_{a \in \langle h \rangle \cdot a} b$$

is not equal to the unit element $1$ of $G$, where $a$ is a fixed generator of $A$, and $\langle h \rangle \cdot a$ is the orbit of $a$ with respect to the above action of the group $\langle h \rangle$ generated by $h$. $L$ is defined independently of the choice of $a$.

Proposition 4.6. The above $L$ is a subgroup of $H$.

Proof. If $h \in K = C_H(A)$, we have $\langle h \rangle \cdot a = \{a\}$. This implies $T(h) \neq 1$. We get $L \supseteq K$, moreover we see that $L$ is the union of several cosets of $H/K$. We remark that $H/K$ is a cyclic $p$-group. If we can show that $h \in L$ implies $h^m \in L$ for $1 \leq m \leq p$, we see that $L$ is a subgroup of $H$.

Suppose $1 \leq m < p$. Since $\langle h \rangle \cdot a = \langle h^m \rangle \cdot a$, $h \in L$ implies $h^m \in L$. Let $h$ be an element of $H - K$, then we have the disjoint sum such that

$$\langle h \rangle \cdot a = \sum_{j=0}^{p-1} h^j \langle h^p \rangle \cdot a.$$ 

If $T(h^p) = 1$, we have

$$T(h) = \prod_{j=0}^{p-1} h^j T(h^p) h^{-j} = 1.$$ 

Therefore $h^p \in L$ implies $h \in L$; this means that $h \in L$ implies $h^p \in L$. This completes the proof of Proposition 4.6.

Proposition 4.7. Put $K = C_K(A)$, and let $V$ be an irreducible $A$ space with the trivial kernel, $W$ a $K$-space, $a$ a generator of $A$ and $h$ an element of $H$. We have the following.

(i) Provided $h \in H - L$, the all eigenvalues of $ah$ on $\operatorname{Ind}_{K}^{A} V \otimes W$ are determined independently of the choice of the generator $a$ of $A$. 

(ii) Provided \( h \in L \), \( ah \) does not have 1 as its eigenvalue on \( \text{Ind}_{L}^{G} V \otimes W \). Here \( L \) is the group defined above.

As we can prove this by direct calculation, we omit the proof.

5. On the case: \( G \) is generated by two elements

In this section \( G = AH \) will be a hyperelementary group such that \( H \) is cyclic.

**Remark 5.1.** Let \( K \) be a subgroup of \( H \), then \( K \) is normal in \( H \). If \( W \) is a \( K \)-space, then for any \( h \in H \) we have \( h_{*}W = W \).

**Proposition 5.2.** We have the following.

(i) Let \( U = \text{Ind}_{H(i)}^{G} V(i) \otimes W \) be an irreducible \( G \)-space. Then \( \ker U = (\ker V(i))(\ker W) \) holds, where \( \ker V(i) \subset A \) and \( \ker W \subset H(i) \).

(ii) If irreducible \( G \)-spaces \( U \) and \( U' \) have the same kernel, \( \Gamma U = \Gamma U' \) holds.

(iii) \( G \) has Property 2.

**Proof.** (i): By the definition of the induced representation and Remark 5.1 we obtain \( \ker U = (\ker V(i))(\ker W) \).

(ii): Suppose \( U = \text{Ind}_{H(i)}^{G} V(i) \otimes W \) and \( U' = \text{Ind}_{H(j)}^{G} V(j) \otimes W' \), then by (i) we have \( \ker V(i) = \ker V(j) \) and \( \ker W = \ker W' \) (see Proposition 4.3 (iv)).

Since both \( A \) and \( H(i) = H(j) \) are cyclic, we have \( \Gamma V(i) = \Gamma V(j) \) and \( \Gamma W = \Gamma W' \).

From Proposition 4.3 (i) we obtain \( \Gamma U = \Gamma U' \).

(iii): Lemma 2.1 and above (ii) imply (iii).

**Proposition 5.3.** Let \( V(i) \) be an irreducible \( A \)-space as before, \( W \) an \( H(i) \)-space and \( \gamma \) an element of \( \Gamma \). Put \( x = \text{Ind}_{H(i)}^{G} \{ (\gamma V(i)) \otimes W - V(i) \otimes W \} \). Then \( x \) belongs to \( R_{s}(G) \) if and only if \( \text{Res}^{G}_{H(i)} x \) belongs to \( R_{s}(G(i)) \).

**Proof.** The only if part is clear. We will prove the if part by induction on \( |G| \). If \( |A| = 1 \) or \( |H| = 1 \) then Proposition 5.3 is trivial. Make the inductive hypothesis: for each hyperelementary group of the same type as \( G \) has and of smaller order than \( |G| \) Proposition 5.3 is valid.

We assume that \( \text{Res}^{G}_{H(i)} x \) belongs to \( R_{s}(G(i)) \). By Lemma 2.1 and the inductive hypothesis it is sufficient to prove the proposition in the case: \( V(i)(\{1\}) = V(i) \) and \( W(\{1\}) = W \). In this case we have \( x^{L} = 0 \) in \( R(N_{c}(L)) \) for each non-trivial subgroup \( L \) of \( G \). By Lemma 2.3 we complete the proof if we show \( P = P(G; x) \equiv 0 \mod |G| \).

Choose a positive integer \( s \) such that

\[ \gamma(\exp(2\pi \sqrt{-1}/|A|)) = \exp(2\pi s \sqrt{-1}/|A|) \quad \text{and} \quad s \equiv 1 \mod |H| \text{.} \]

By (2.4) and (2.5) we have
\[ P \equiv \sum_{g \in G} \{ z(g) - Q(s; \text{Ind}^{G}_{G(i)} V(i) \otimes W)(g) \} \mod |G| \]
\[ = 1 - s^n, \]
where \( n = \dim \text{Ind}^{G}_{G(i)} V(i) \otimes W \). Since \( s \equiv 1 \mod |H| \), we have \( P \equiv 0 \mod |H| \).
On the other hand, \( \text{Res}_{G(i)}^G x \in R_{G(i)}(G(i)) \implies \text{Res}_{G}^G x \in R_{G}(A) \); we have \( P(A; \text{Res}_{G}^G x) \equiv 0 \mod |A| \). From (2.5) we obtain
\[ \sum_{g \in A} \{ z(g) - Q(s; \text{Ind}^{G}_{G(i)} V(i) \otimes W)(g) \} \equiv 0 \mod |A|. \]
The left hand side of the above relation is equal to \( 1 - s^n \). This means that \( P \equiv 0 \mod |A| \). Consequently we have \( P \equiv 0 \mod |G| \). This completes the proof.

**Proposition 5.4.** Let \( V(i) \) be an irreducible \( A \)-space as before, and \( U \) and \( W \) \( H(i) \)-spaces. Put \( x = \text{Ind}^{G}_{G(i)} (V(i) \otimes U - V(i) \otimes W) \). Then \( x \) belongs to \( R_{G}(G) \) if and only if \( \text{Res}_{G}^G x \) belongs to \( R_{G}(H) \).

**Proof.** The only if part is clear. We will prove the if part by induction on \( |G| \). If \( |A| = 1 \) or \( |H| = 1 \) then Proposition 5.4 is trivial. Make the inductive hypothesis: for each hyperelementary group of the same type as \( G \) and of smaller order than \( |G| \) Proposition 5.4 is valid.

We assume that \( \text{Res}_{G}^G x \) belongs to \( R_{G}(H) \). By Lemma 2.1 and the inductive hypothesis it is sufficient to prove the proposition in the case: \( V(i) (\{1\}) = V(i) \), \( U(\{1\}) = U \) and \( W(\{1\}) = W \). Since \( K = C_H(A) \) is cyclic, those conditions imply
\[ U - W \equiv \gamma W_0 - W_0 \mod R_1(K), \]
where \( W_0 \) is some irreducible \( K \)-space with the trivial kernel and \( \gamma \) is some element of \( T \). Without loss of generality we may assume that \( W = W_0 \) and \( U = \gamma W_0 \). By this assumption we have \( x^L = 0 \) for each non-trivial subgroup \( L \) of \( G \). If we show that \( P = P(G; x) \equiv 0 \mod |G| \), by Lemma 2.4 we obtain Proposition 5.4. Choose a positive integer \( s \) such that
\[ \gamma (\exp(2\pi \sqrt{-1} |H|)) = \exp(2\pi s \sqrt{-1} |H|) \] and \( s \equiv 1 \mod |A| \).
By (2.4) and (2.5) we have
\[ P \equiv \sum_{g \in G} \{ z(g) - Q(s; \text{Ind}^{G}_{G(i)} V(i) \otimes W)(g) \} \mod |G| \]
\[ = 1 - s^n, \]
where \( n = \dim \text{Ind}^{G}_{G(i)} V(i) \otimes W \). Since \( s \equiv 1 \mod |A| \), we have \( P \equiv 0 \mod |A| \).
On the other hand, \( \text{Res}_{G}^G x \in R_{G}(H) \) implies \( P(H; \text{Res}_{G}^G x) \equiv 0 \mod |H| \). From (2.5) we obtain
\[ \sum_{g \in G} \{ z(g) - Q(s; \text{Ind}^{G}_{G(i)} V(i) \otimes W)(g) \} \equiv 0 \mod |H|. \]
The left hand side of the above relation is equal to $1-s^\pi$. This means that $P=0 \mod |H|$. Consequently we have $P=0 \mod |G|$. 

**Proposition 5.5.** Let $V(i)$ be an irreducible $A$-space as before, $W$ an irreducible $H(i)$-space, and $\gamma$ and $\gamma'$ elements of $\Gamma$. Put $x=\text{Ind}_{G(i)}^G \{\gamma(V) \otimes (\gamma'W) - V \otimes W\}$. Then $x$ belongs to $R_a(G)$ if and only if $\text{Res}_{G(i)}^G x \in R_a(G(i))$ and $\text{Res}_{G}^G x \in R_a(H)$.

**Proof.** The only if part is clear. We prove the if part. Put 
\[ y = \text{Ind}_{G(i)}^G \{(\gamma V(i)) \otimes (\gamma' W) - (\gamma V(i)) \otimes W\} \]
\[ z = \text{Ind}_{G(i)}^G \{(\gamma V(i)) \otimes W - V(i) \otimes W\} \]
We have $x=y+z$; we have $\text{Res}_{H}^G x = \text{Res}_{H} y$. $\text{Res}_{G(i)}^G x \in R_a(H)$ means that $\text{Res}_{H} y \in R_a(H)$. By Proposition 5.4 we have $y \in R_a(G)$. This and $\text{Res}_{G(i)}^G x \in R_a(G(i))$ imply $\text{Res}_{G(i)}^G z \in R_a(G(i))$. By Proposition 5.3 we have $z \in R_a(G)$. Consequently we have $x=y+z \in R_a(G)$.

6. **Hyerelementary groups and Property 2**

In this section $G=AH$ will be a hyperelementary group such that all the elementary subgroups of the quotient groups of the subgroups of $G$ have Property 2.

**Remark.** If an elementary group $K=A \times H$ satisfies one of the conditions: 
(i) $(\phi(|A|), p)=1$, (ii) $|H| \leq p^4$ and (iii) $H$ is metacyclic, then $K$ has Property 2.

Let $R(G, f)$ be the subgroup of $R(G)$ built from the irreducible $G$-spaces which yield faithful $A$-spaces when they are restricted to $A$. Put $R_a(G, f)=R(G, f) \cap R_a(G)$, and $R_b(G, f)=R(G, f) \cap R_b(G)$.

**Proposition 6.1.** Let $x$ be an element of $R_a(G, f)$, $B$ a subgroup of $A$ and $K$ a subgroup of $C_b(B)$. Then for each $C \in X(G)=\text{Irr}(G)/\Gamma$ we have $\text{Res}_{K}^G f_C(x) \in R_a(BK)$.

**Proof.** It is sufficient to prove the proposition in the case that $K=C_b(B)$. In this case we have $K \subset C_b(A)$. Put $L=C_b(A)$. Let $V$ be an irreducible $A$-space with the trivial kernel, and $U$ and $W$ irreducible $L$-spaces. If $\Gamma \text{Ind}_{L}^G V \otimes U = \Gamma \text{Ind}_{L}^G V \otimes W$, we have \[ \langle \Gamma \text{Res}_{K}^G \text{Ind}_{L}^G V \otimes U, \Gamma \text{Res}_{K}^G \text{Ind}_{L}^G V \otimes W \rangle_{BK} = \{0\} \]
by Proposition 4.4. Since $BK$ has Property 2 by the assumption, we have $\text{Res}_{K}^G f_C(x) \in R_a(BK)$ for each $C \in X(G)$.

**Proposition 6.2.** Put $K=C_b(A)$, and let $V$ be an irreducible $A$-space with
the trivial kernel, \( W \) a \( K \)-space and \( \gamma \) an element of \( \Gamma \). Put \( x = \text{Ind}^G_K \{ (\gamma V) \otimes W - V \otimes W \} \). Then \( x \) belongs to \( R_\delta(G) \) if and only if for each subgroup \( B \) of \( A \) and \( L = C_H(B) \) we have \( \text{Res}^\delta_B x \in R_\delta(BL) \).

Proof. The only if part is clear. We will prove the if part by induction on \( |G| \). If \( |A| = 1 \) or \( |H| = 1 \), then Proposition 6.2 is trivial. Make the inductive hypothesis: for each hyperelementary group which satisfies the same condition as \( G \) satisfies and whose order is smaller than \( |G| \) Proposition 6.2 is valid.

Assume that for each \( B \subset A \) and \( L = C_H(B) \) we have \( \text{Res}^\delta_B x \in R_\delta(BL) \). Firstly we get \( x \in R_\delta(G) \). By Propositions 3.1, 4.3, 4.4 and 6.1 it is sufficient to prove the proposition in the case that there exist a positive integer \( r \), an irreducible \( K \)-space \( U \) and elements \( h(m) \) of \( H \), \( 1 \leq m \leq r \), such that

\[
W = \bigoplus_{m=1}^{r} h(m)_* U.
\]

By Propositions 3.1 and 4.2 we have

\[
\text{Ind}^G_K \{ (\gamma V) \otimes h(m)_* U - V \otimes h(m)_* U \} \equiv \text{Ind}^G_K \{ (\gamma V) \otimes U - V \otimes U \} \mod R_\delta(G).
\]

This enables us to assume that \( W \) itself is irreducible.

Assertion 6.3. Let \( M \neq \{1\} \) be a subgroup of \( G \). We have \( x^M \in R_\delta(N_\delta(M)) \).

Proof. If \( A \cap M \neq \{1\} \), then we have \( x^M = 0 \) in \( R(N_\delta(M)) \). We assume \( A \cap M = \{1\} \). In this case \( M \) is conjugate to a subgroup of \( H \). By Proposition 2.2 we may assume \( M \subset H \). By Proposition 4.5 we have \( N_\delta(M) = C_\delta(M)N_\delta^H(M) \). The proof is divided into the following three cases.

Case 1. \( C_\delta(M) = A \)

Put \( B = C_\delta(M) \), \( L = C_H(B) \) and \( y = \text{Res}^\delta_H x \). We have

\[
y = \text{Ind}^G_K \{ (\gamma \text{ Res}^A_H V) \otimes W -(\text{Res}^A_H V) \otimes W \}.
\]

By Proposition 25 of \([7; 8.2]\) we have \( y \) in another form as follows:

\[
y = \text{Ind}^G_K \{ (\gamma \text{ Res}^A_H V) \otimes U -(\text{Res}^A_H V) \otimes U \},
\]

where \( U \) is an \( L \)-space. For a subgroup \( C \) of \( B \), we put \( N = C_\delta(H) \); we have \( \text{Res}^N_H y = \text{Res}^C_N x \in R_\delta(CN) \) by the assumption. By the inductive hypothesis \( y \) belongs to \( R_\delta(BH) \). This implies \( x^M = y^M \in R_\delta(N_\delta(M)) \).

Case 2. \( C_\delta(M) = A \) and \( N_\delta^H(M) = H \)

Put \( N = N_\delta(M) \), \( D = H \cap N \), \( E = K \cap N \) and \( y = \text{Res}^E_H x \), then we have

\[
y = \sum_{(h_k) \in K} \text{Ind}^G_K \{ (\gamma h_k V) \otimes (\text{Res}^E_H h_k W) -(h_k V) \otimes (\text{Res}^E_H h_k W) \}.
\]

By Proposition 3.1 we have
\[ y \equiv \sum_{\{a\in H, C\}} \text{Ind}^H_{\{a\}} \{ (\gamma V) \otimes (\text{Res}^H_{\{a\}} h_\ast W) - V \otimes (\text{Res}^H_{\{a\}} h_\ast W) \mod R_4(N) \}
\]
\[ = \text{Ind}^H \{ (\gamma V) \otimes U - V \otimes U \}, \]
where
\[ U = \bigoplus_{\{a\in H, C\}} \text{Res}^H_{\{a\}} h_\ast W. \]

For a subgroup \( B \) of \( A \) and \( L = C_\phi(B) \) we have \( \text{Res}^N_{BL} y = \text{Res}^N_{BL} x \in R_4(\phi(N)) \). We have \( y = R_4(N_\phi(M)) \) by the inductive hypothesis. This implies \( x^M = y^M \in R_4(N_\phi(M)). \)

Case 3. \( N_\phi(M) = G \)

We have reduced the problem to the case that \( W \) is irreducible. In this case \( \text{Ind}^A_{\{a\}} (\gamma V) \otimes W \) and \( \text{Ind}^A_{\{a\}} V \otimes W \) are irreducible. If \( (\text{Ind}^A_{\{a\}} V \otimes W)^M \neq \{0\} \), then we have \( (\text{Ind}^A_{\{a\}} V \otimes W)^M = \text{Ind}^A_{\{a\}} V \otimes W. \) We get \( \ker \text{Ind}^A_{\{a\}} V \otimes W \supseteq M. \) By the inductive hypothesis we have \( x \in R_4(G) \). This completes the proof of Assertion 6.3.

If we show \( P = P(G; x) \equiv 0 \mod |G| \), we complete the proof of Proposition 6.2. Choose a positive integer \( s \) such that
\[ \gamma(\exp(2\pi \sqrt{-1}/|A|)) = (2\pi s \sqrt{-1}/|A|) \] and \( s \equiv 1 \mod |H| \).

By (2.4) and (2.5) we have
\[ P \equiv \sum_{g \in G} \{ \gamma(g) - Q(s; \text{Ind}^A_{\{a\}} V \otimes W(g)) \} \mod |G|. \]

Since \( s \equiv 1 \mod |H| \), we have \( P \equiv 0 \mod |H| \). On the other hand there exist integers \( n_c \) for the cyclic subgroups \( C \) of \( H \) such that
\[ P = \sum_{C \in \text{cyclic}} n_c P(G; \text{Res}^A_{\{a\}} x). \]

If we can show \( P(G; \text{Res}^A_{\{a\}} x) \equiv 0 \mod |A|, \) we see that \( P \equiv 0 \mod |A| \); consequently we obtain \( P \equiv 0 \mod |G| \). \( P(G; \text{Res}^A_{\{a\}} x) \equiv 0 \mod |A|, \) follows from the following assertion.

**Assertion 6.4.** For each cyclic subgroup \( C \) of \( H \), we have \( \text{Res}^A_{\{a\}} x \in R_4(AC). \)

Proof. Put \( y = \text{Res}^A_{\{a\}} x \) and \( M = C \cap K \). We have
\[ y = \sum_{\{a\} \in H/C} \text{Ind}^A_{\{a\}} \{ (\gamma h_\ast V) \otimes \text{Res}^H_{\{a\}} h_\ast W \} \mod R_4(AC) \]
\[ = \sum_{\{a\} \in H/C} \text{Ind}^A_{\{a\}} \{ (\gamma V) \otimes (\text{Res}^H_{\{a\}} h_\ast W) - V \otimes (\text{Res}^H_{\{a\}} h_\ast W) \} \mod R_4(AC) \]
\[ = \text{Ind}^A_{\{a\}} \{ (\gamma V) \otimes U - V \otimes U \}, \]
where
\[ U = \bigoplus_{\{a\} \in H/C} \text{Res}^A_{\{a\}} h_\ast W. \]
Since we have $\text{Res}^g_H y = \text{Res}^g_M x \in R_h(AM)$ by the assumption, we have $\text{Res}^g_A x = y \in R_h(AC)$ by Proposition 5.3. This completes the proof of Assertion 6.4.

**Proposition 6.5.** Put $K = C_H(A)$, and let $V$ be an irreducible $A$-space with the trivial kernel, $W$ a $K$-space and $\gamma$ an element of $\Gamma$. Put $x = \text{Ind}^g_A \{V \otimes (\gamma W) - V \otimes W\}$. Then $x$ belongs to $R_h(G)$ if and only if $\text{Res}^g_H x$ belongs to $R_h(H)$.

**Proof.** The only if part is clear. We will prove the if part by induction on $|G|$. If $|A| = 1$ or $|H| = 1$, then the proposition is trivial. Make the inductive hypothesis: for each hyperelementary group which satisfies the condition stated at the beginning of this section and whose order is smaller than $|G|$ Proposition 6.5 is valid.

We assume $\text{Res}^g_H x \in R_h(H)$ and $|A| \neq 1$. Firstly we have $x \in R_h(G)$. By Propositions 3.1, 4.3, 4.4 and 6.1 it is sufficient to prove the proposition in the case that there exist a positive integer $r$, an irreducible $K$-space $U$ and elements $h(m)$ of $H$, $1 \leq m \leq r$, such that

$$W = \bigoplus_{n=1}^{r} h(m) \ast U.$$ 

By Propositions 3.1 and 4.2 we have

$$\text{Ind}^g_A \{V \otimes (\gamma h(m) \ast U) - V \otimes (h(m) \ast U)\} = \text{Ind}^g_A \{V \otimes (\gamma U) - V \otimes U\} \mod R_i(G).$$

This enables us to assume that $W$ itself is irreducible.

**Assertion 6.6.** Let $L$ be a non-trivial subgroup of $G$. We have $x^L \in R_h(N_G(L))$.

**Proof.** Since $A$ acts freely on $\text{Ind}^g_A V \otimes \gamma W$ and on $\text{Ind}^g_A V \otimes W$ except the origins, it is sufficient to prove the assertion in the case that $L \cap A = \{1\}$. In this case $L$ is conjugate to a subgroup of $H$. By Proposition 2.2 we may assume $L \subseteq H$. Then we have $N_G(L) = C_A(L)N_H(L)$ by Proposition 4.5. We divide the proof into the following three cases.

Case 1. $C_A(L) \neq A$

We put $B = C_A(L)$ and $y = \text{Res}^g_B x$. We have

$$y = \text{Ind}^g_B \{(\text{Res}^g_B V) \otimes (\gamma W) - (\text{Res}^g_B V) \otimes W\}.$$ 

Put $M = C_H(B)$, then we have

$$y = \text{Ind}^g_M \{(\text{Res}^g_B V) \otimes (\gamma \text{Ind}^g_M W) - (\text{Res}^g_B V) \otimes (\text{Ind}^g_M W)\}.$$ 

On the other hand we have $\text{Res}^g_H y = \text{Res}^g_H x \in R_h(H)$. By the inductive hypothesis we have $y \in R_h(BH)$. This implies $x^L = y^L \in R_d(N_G(L))$. 


Case 2. \( C_A(L) = A \) and \( N_H(L) \neq H \)

Put \( M = N_H(L) \), \( N = N_G(L) \), \( D = K \cap M \) and \( y = \text{Res}^M_H x \). We have \( N = AM \) and

\[
y = \sum_{(a) \in H/K} \text{Ind}^V_A \{(h_a V) \otimes (\gamma \text{Res}_H^V h_a W) - (h_a V) \otimes (\text{Res}_H^V h_a W)\}
\equiv \sum_{(a) \in H/K} \text{Ind}^V_A \{V \otimes (\gamma \text{Res}_H^V h_a W) - V \otimes (\text{Res}_H^V h_a W)\} \mod R_i(N)
= \text{Ind}^V_A \{V \otimes (\gamma U) - V \otimes U\},
\]

where

\[
U = \bigoplus_{(a) \in H/K} \text{Res}_H^V h_a W.
\]

Since we have \( \text{Res}_M^V y = \text{Res}_M^V x \in R_i(M) \), by the inductive hypothesis we get \( y \in R_i(N) \). This implies \( x^L = y^L = \text{Res}_H^L x \in R_i(N_G(L)) \).

Case 3. \( N_G(L) = G \)

When \( W \) is irreducible, \( \text{Ind}^G_H V \otimes W \) and \( \text{Ind}^G_H V \otimes \gamma W \) are irreducible. This implies that \( x^L = x \) or \( 0 \) in \( R(G) \). If \( x^L = 0 \), Assertion 6.6 is clearly valid. If \( x^L = x \), then \( L \) is included in the kernel of \( x \). By the inductive hypothesis we obtain \( x \in R_i(G) \). This completes the proof of Assertion 6.6.

If we show \( P = P(G; x) \equiv 0 \mod |G| \), we complete the Proof of Proposition 6.5. As usual choose a positive integer \( s \) such that

\[
\gamma(\exp(2\pi \sqrt{-1}/|H|)) = \exp(2\pi s \sqrt{-1}/|H|) \quad \text{and} \quad s \equiv 1 \mod |A|.
\]

By (2.5) we have

\[
P = \sum_{g \in G} \{z(g) - Q(s; \text{Ind}^G_H V \otimes W)(g)\} \mod |G|.
\]

By the inductive hypothesis, for each proper subgroup \( B \) of \( A \) we have \( \text{Res}_{BH}^G x \in R_i(BH) \). This implies \( P(BH; \text{Res}_{BH}^G x) \equiv 0 \mod |BH| \). Therefore we have

\[
P = \sum_{ah \in A; \langle h \rangle = A} \{z(ah) - Q(s; \text{Ind}^G_H V \otimes W)(ah)\} \mod |H|.
\]

By Propositions 4.6 and 4.7 we have

\[
P \equiv \sum_{ah \in A; \langle h \rangle = A} \{z(ah) - Q(s; \text{Ind}^G_H V \otimes W)(ah)\} \mod |H|
\equiv \phi(|A|) \sum_{h \in H - L} \{z(h) - Q(s; \text{Ind}^G_H V \otimes W)(h)\} \mod |H|,
\]

where \( L \) is the group given in Proposition 4.6, \( \phi \) is the Euler function. \( \text{Res}_L^G x \in R_i(L) \) and (2.5) imply

\[
\sum_{h \in L} \{z(h) - Q(s; \text{Ind}^G_H V \otimes W)(h)\} \equiv 0 \mod |L|.
\]
Since \( \phi(|A|) \) is a multiple of \( |H/K| \) and \( |L| \) a multiple of \( |K| \), we have

\[
P \equiv \phi(|A|) \sum_{h \in H} \{z(h) - \phi(s; \text{Ind}_{K}^{H} V \otimes W)(h)\} \text{ mod } |H|.
\]

From \( \text{Res}_{H}^{G} x \in R_{a}(H) \), we have \( P \equiv 0 \mod |H| \). On the other hand for the cyclic subgroups \( C \) of \( H \) there exist integers \( n_{C} \) such that

\[
P = \sum_{C < H \text{ cyclic}} n_{C} \sum_{g \in A_{C}} z(g).
\]

We obtain \( P \equiv 0 \mod |A| \) from the following assertion; consequently we get \( P \equiv 0 \mod |G| \).

**Assertion 6.7.** For each cyclic subgroup \( C \) of \( H \), we have \( \text{Res}_{H}^{C} x \in R_{a}(AC) \).

**Proof.** Put \( y = \text{Res}_{H}^{C} x \) and \( D = C \cap K \), then we have

\[
y = \sum_{\{k\} \subseteq H \cap C_{K}} \text{Ind}_{H}^{A} \{h_{*} V \otimes (\gamma \text{ Res}_{D}^{K} h_{k} W) - (h_{*} V) \otimes (\text{Res}_{D}^{K} h_{k} W)\}
\]

\[
\equiv \sum_{\{k\} \subseteq H \cap C_{K}} \text{Ind}_{H}^{A} \{V \otimes (\gamma \text{ Res}_{D}^{K} h_{k} W) - V \otimes (\text{Res}_{D}^{K} h_{k} W)\} \text{ mod } R_{1}(AC)
\]

\[
= \text{Ind}_{H}^{A} (V \otimes \gamma U - V \otimes U),
\]

where

\[
U = \bigoplus_{\{k\} \subseteq H \cap C_{K}} \text{Res}_{D}^{K} h_{*} W.
\]

Moreover we have \( \text{Res}_{C}^{A} y = \text{Res}_{C}^{G} x \in R_{a}(C) \). By Proposition 5.4 we have \( y \in R_{a}(AC) \). This completes the proof of Assertion 6.9 consequently completes the proof of Proposition 6.5.

**Proposition 6.10.** Put \( K = C_{H}^{N}(A) \), and let \( V \) be an irreducible \( A \)-space with the trivial kernel, \( W \) a \( K \)-space and \( \gamma \) an element of \( \Gamma \). Put \( x = \text{Ind}_{A_{K}}^{G} \{\gamma (V \otimes W) - V \otimes W\} \). Then \( x \) belongs to \( R_{a}(G) \) if and only if for each subgroup \( B \) of \( A \) and \( L = C_{H}^{N}(B) \) we have \( \text{Res}_{L}^{G} x \in R_{a}(BL) \).

**Proof.** The only if part is clear. We prove the if part. Put \( y = \text{Ind}_{A_{K}}^{G} \{\gamma (V \otimes W) - (\gamma V) \otimes W\} \) and \( z = \text{Ind}_{A_{K}}^{G} \{\gamma V \otimes W - V \otimes W\} \), then we have \( x = y + z \). Since \( \text{Res}_{L}^{G} z = 0 \), we have \( \text{Res}_{L}^{G} y \in R_{a}(H) \) by the assumption. From Proposition 6.5 we obtain \( y \in R_{a}(G) \). This yields that

\[
\text{Res}_{L}^{G} z = \text{Res}_{L}^{G} x - \text{Res}_{L}^{G} y \in R_{a}(BL).
\]

Proposition 6.2 implies \( z \in R_{a}(G) \). Hence we conclude that \( x \in R_{a}(G) \).

**Theorem 6.11.** Let \( G \) be a hyperelementary group such that all the elementary subgroups of the quotient groups of the subgroups of \( G \) have Property 2. Then \( G \) has Property 2.
Proof. We prove it by induction on \(|G|\). If \(|A|=1\) or \(|H|\leq p\), we are aware that \(G\) has Property 2. Make the inductive hypothesis: each hyper-elementary group which satisfies the same condition as \(G\) satisfies and whose order is smaller than \(|G|\) has Property 2.

Let \(x\) be an element of \(R_\delta(G)\). By Lemma 2.1 and the inductive hypothesis we may assume \(x(\{1\})=x\). This implies \(x\in R_\delta(G, f)\). Put \(K=C_B(A)\). For a fixed element \(C\) of \(X(G)\), there exist \(\gamma\in \Gamma\), an irreducible \(A\)-space \(V\) and an irreducible \(K\)-space \(W\) such that

\[ f_\gamma(x)\equiv \text{Ind}_{A^\kappa}^{A^\delta} \{\gamma(V \otimes W)-V \otimes W\} \mod R_\delta(G). \]

By Propositions 6.1 and 6.10 we get \(f_\gamma(x)\in R_\delta(G)\).

For a subgroup \(B\) of \(A\), we get an elementary subgroup \(BC_B(B)\) of \(G\). Varying \(B\), we obtain several elementary groups. Let \(E(G)\) be the set of all those elementary groups. Lemma 2.1 and Propositions 6.1 and 6.10 yield the following theorem.

**Theorem 6.12.** In the same situation as in Theorem 6.11

\[ \text{Res}: R_\delta(G, f)/R_\delta(G, f) \rightarrow \bigoplus_{K\in \Delta(G)} j(K) \]

is injective. Therefore we obtain a naturally defined injection

\[ j(G) \rightarrow \bigoplus_B \bigoplus_{K\in \Delta(G)} j(K) \]

where \(B\) runs over the subgroups of \(A\).

7. A closing example

Let \(A\) (resp. \(H\)) be the cyclic group of order 7 (resp. 5) which consists of the 7-th (resp. 5-th) roots of unity, and \(G\) the direct product of \(A\) and \(H\). For each integer \(i\) (resp. \(j\)) with \(0\leq i\leq 6\) (resp. \(0\leq j\leq 4\)) define the \(A\)-(resp. \(H\)-) representation \(v_i\) (resp. \(w_j\)) by

\[ v_i(z) = z^i \text{ for } z\in A \]

\[ (\text{resp. } w_j(z) = z^j \text{ for } z\in H). \]

We denote by \(V_i\) (resp. \(W_j\)) the corresponding representation space to \(v_i\) (resp. \(w_j\)). Define an element \(x\) of \(R(G)\) by

\[ x = V_2 \otimes W_1 + V_2 \otimes W_0 + V_2 \otimes W_0 - V_1 \otimes W_1 - V_1 \otimes W_0 - V_1 \otimes W_0. \]

Then we have \(x\in R_\delta(G)\cap R(G, f)\); moreover we have \(\text{Res}^G_A x\in R_\delta(A)\) and \(\text{Res}^H_B x\in R_\delta(H)\). The \(x\) does not, however, belong to \(R_\delta(G)\). This is a counter example to [1; Proposition 5.2].
References


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