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<th>A splitting property of oriented homotopy equivalence for a hyperelementary group</th>
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1. Introduction

Let $G$ be a finite group. In this paper a $G$-space means a complex $G$-representation space of finite dimension. For a $G$-space $V$ we denote by $S(V)$ its unit sphere with respect to some $G$-invariant inner product. After tom Dieck [1] and [2] we call two $G$-spaces $V$ and $W$ oriented homotopy equivalent if there exists a $G$-map $f: S(V) \to S(W)$ such that for each subgroup $H$ of $G$ the induced map $f^H: S(V)^H \to S(W)^H$ on the $H$-fixed point sets has degree one with respect to the coherent orientations which are inherited from the complex structures on $V^H$ and $W^H$. Let $R(G)$ be the complex $G$-representation ring, $R_h(G)$ the additive subgroup of $R(G)$ consisting of $x=V-W$ such that $V$ and $W$ are oriented homotopy equivalent, and $R_Q(G)$ the additive subgroup of $R(G)$ consisting of $x=V-W$ such that $\dim V^H = \dim W^H$ for all the subgroups $H$ of $G$. We denote by $j(G)$ the quotient group $R_Q(G)/R_h(G)$.

If $G$ has a normal cyclic subgroup $A$ and a Sylow $p$-subgroup $H$ such that $G$ is the semidirect product of $H$ by $A$, we call $G$ a hyperelementary group. Especially if $G$ is the direct product of $A$ and $H$, we call $G$ an elementary group. tom Dieck showed that for an arbitrary finite group $G$ the restriction homomorphism from $j(G)$ to the direct sum of $j(K)$ is injective, where $K$ runs over the hyperelementary subgroups of $G$ ([1; Proposition 5.1]). Our purpose of this paper is to consider oriented homotopy equivalence for hyperelementary groups and to give a sufficient condition for a hyperelementary group to have a splitting property defined below.

Choose an integer $m$ which is a multiple of the orders of the elements of $G$, and let $Q(m)$ be the field obtained by adjoining the $m$-th roots of unity to $Q$, where $Q$ is the field of rational numbers. The Galois group $\Gamma = \Gamma(m)$ of $Q(m)$ over $Q$ acts on $R(G)$ via its action on character value. Actually $\Gamma$ acts on the set $\text{Irr}(G)$ of isomorphism classes of irreducible $G$-spaces. Let $Z[\Gamma]$ be the integral group ring of $\Gamma$, and $I(\Gamma)$ its augmentation ideal. Then we have $R_0(G) = I(\Gamma)R(G)$. We put $R_i(G) = I(\Gamma)R_i(G)$. According to [3] we have $R_i(G) \subset R_h(G)$. Let us say that $G$ has Property 1 if $R_i(G)$ coincides with $R_h(G)$.
For example the abelian groups and the $p$-groups have Property 1, and some hyperelementary groups do not have Property 1 (see [1] and [6]). In section 3 we obtain other groups which have Property 1.

For each orbit $C \in X(G) = \text{Irr}(G)/\Gamma$, we let $F(C)$ be the free abelian group on elements of $C$. Then we have $R(G) = \bigoplus_{C \in X(G)} F(C)$. Let $f_C$ be the canonical projection from $R(G)$ to $F(C)$. Let us say that $G$ has Property 2 (we called this a splitting property) if for each element $x$ of $R_\delta(G)$ and each element $C$ of $X(G)$ $f_C(x)$ belongs to $R_\delta(G)$. This property is of our interest. If $G$ has Property 1, then $G$ has Property 2; the converse is not true. It is remarkable that $R_\delta(G)$ is determined by oriented homotopy equivalence between the irreducible $G$-spaces if $G$ has Property 2.

Our main results are Theorems 6.11 and 6.12, and the latter indicates the importance of Property 2. Additionally we give a counter example to [1; Proposition 5.2] in section 7.

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2. Preparation

Let $S(G)$ the set of normal subgroups of $G$. If a $G$-space $V$ is given we write $V = \bigoplus_{H \in S(G)} V(H)$, where $V(H)$ collects the faithful irreducible $G/H$-subspaces (see [2; p. 252]).

**Lemma 2.1** ([2]). If $x = V - W \in R_\delta(G)$, then for all $H \in S(G)$ we have $x(H) = V(H) - W(H) \in R_\delta(G)$.

Let $V$ and $W$ be $G$-spaces. If $f$ is an $N_G(H)$-map from $S(V)^H$ to $S(W)^H$ and $g$ is an element of $G$, then there uniquely exists an $N_G(gHg^{-1})$-map $h$ from $S(V)^{gHg^{-1}}$ to $S(W)^{gHg^{-1}}$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
S(V)^H & \xrightarrow{f} & S(W)^H \\
\downarrow{g_*} & & \downarrow{g_*} \\
S(V)^{gHg^{-1}} & \xrightarrow{h} & S(W)^{gHg^{-1}}
\end{array}
\]

where $g_*$ are the maps canonically given by the actions of $g$.

**Proposition 2.2.** Let $V$ and $W$ be $G$-spaces. We have $V^H - W^H \in R_\delta(N_G(H))$ if and only if we have $V^{gHg^{-1}} - W^{gHg^{-1}} \in R_\delta(N_G(gHg^{-1}))$.

Proof. This proposition follows from the fact that each $g_*$ of the above diagram preserves the orientation of the sphere.

Let $V$ and $W$ be $G$-spaces such that $\dim V^H = \dim W^H$ for all subgroups $H$ of $G$ (i.e. $V - W \in R_\delta(G)$). We put $n = \dim V (= \dim W)$. If $g$ is an element of
$G$, $g$ has $n$ eigenvalues $a_1(g), \ldots, a_n(g)$ (resp. $b_1(g), \ldots, b_n(g)$) with respect to its action on $V$ (resp. $W$). We reorder $(a_j(g))$ and $(b_j(g))$ as follows: there is an integer $k$ such that for each $j \leq k$ we have $a_j(g) = b_j(g) = 1$ and for each $j \geq k$ we have $a_j(g) \neq 1$ and $b_j(g) \neq 1$. We get an algebraic integer $z(g)$ defined by

$$z(g) = \prod_{j=1}^{k} \frac{(1 - b_j(g))}{(1 - a_j(g))},$$

where we put $(1 - b_j(g)) / (1 - a_j(g)) = 1$ for $j < k$. Summing up these algebraic integers $z(g)$ over the elements $g$ of $G$ we have an integer $P = P(G; W - V)$, that is,

$$P(G; W - V) = \sum_{g \in G} z(g).$$

**Lemma 2.3** (due to T. Petrie). Let $V$ and $W$ be $G$-spaces as above. $V$ and $W$ are oriented homotopy equivalent if and only if the following two conditions (i) and (ii) are satisfied.

(i) For each non-trivial subgroups $H$ of $G$ (i.e. $H \neq \{1\}$), we have $V^H - W^H \in R_0(N_G(H))$.

(ii) It holds that $P(G; W - V) \equiv 0 \mod |G|.$

Provided (i), then (ii) is equivalent to the condition: $P(G; V - W) \equiv 0 \mod |G|.$

Let $s$ be a positive integer, and $V$ a $G$-space of dimension $n$. We are going to define an element $Q(s; V)$ of $R(G)$. Let $x(1), \ldots, x(n)$ be indeterminates, and $y(i)$ the elementary symmetric polynomial of degree $i$ for each $1 \leq i \leq n$. We define a polynomial $Q$ of $y(1), \ldots, y(n)$ by

$$Q(y(1), \ldots, y(i), \ldots, y(n)) = \prod_{j=1}^{i} (1 + x(j) + \cdots + x(j)^{i-1}).$$

We define $Q(s; V)$ by

$$Q(s; V) = Q(V, \ldots, \Lambda^j V, \ldots, \Lambda^n V),$$

where $\Lambda^j V$ is the $j$-fold exterior power of $V$. By the usual identification we let $Q(s; V)(g)$ stands for trace $(g; Q(s; V))$. Then it holds that

$$(2.4) \quad Q(s; V)(g) = \prod_{j=1}^{s} (1 + a_j(g) + \cdots + a_j(g)^{j-1}),$$

where $a_1(g), \ldots, a_n(g)$ are all the eigenvalues of $g$ on $V$. Since $Q(s; V) \in R(G)$, we have

$$(2.5) \quad \sum_{h \in H} Q(s; V)(h) \equiv 0 \mod |H|$$

for each subgroup $H$ of $G$. 
3. A few remarks about Property 1

Let $L$ be a finite abelian group. We denote the integral group ring of $L$ by $\mathbb{Z}[L]$, the augmentation ideal of $\mathbb{Z}[L]$ by $\mathcal{I}(L)$, i.e.

$$\mathcal{I}(L) = \{\sum x \in \mathbb{Z}[L] : \sum x = 0\},$$

where $\mathbb{Z}$ is the ring of integers.

**Proposition 3.1.** We have the following.

(i) For $x, x' \in L$, it holds that $xx' - x \equiv x' - 1 \mod \mathcal{I}(L)^2$.
(ii) For $x \in L$ and $z \in \mathbb{Z}$, it holds that $zx - z1 \equiv x' - 1 \mod \mathcal{I}(L)^2$.
(iii) $\mathcal{I}(L)/\mathcal{I}(L)^2$ is isomorphic to $L$.

Since the proof is straightforward, we omit it.

Let $G$ be a direct product $H \times K$ as finite group. We denote by $\phi$ the Euler function, that is, for a positive integer $n$ $\phi(n)$ is the number of the units of $\mathbb{Z}_n = \mathbb{Z}/(n)$.

**Proposition 3.2.** Let $V$ be an irreducible $H$-space, and $W$ an irreducible $K$-space. Assume $(\phi(|H|), \dim W) = (\phi(|K|), \dim V) = 1$. Then for an element $x$ belongs to $R_\epsilon(G)$ if and only if $\text{Res}^G_{\epsilon} x \in R_\epsilon(H)$ and $\text{Res}^H_{\epsilon} x \in R_\epsilon(K)$, where $\epsilon(\gamma)$ are integers.

Proof. The only if part is clear. We are going to prove the if part. $\Gamma$ acts on the orbits $\Gamma(V \otimes W)$, $\Gamma V$ and $\Gamma W$ which are subsets of $\text{Irr}(G)$, $\text{Irr}(H)$ and $\text{Irr}(K)$ respectively. Let $\Gamma_{V \otimes W}$, $\Gamma_V$ and $\Gamma_W$ be the isotropy subgroups of $V \otimes W$, $V$ and $W$ respectively. We have $\Gamma_{V \otimes W} = \Gamma_V \cap \Gamma_W$. Put $M = \Gamma/\Gamma_V$ and $N = \Gamma/\Gamma_W$. The order of $M$ (resp. $N$) divides $\phi(|H|)$ (resp. $\phi(|K|)$).

Since $x \in R_\epsilon(G)$, there exists $\mu \in \Gamma$ such that

$$x \equiv (\mu - 1)(V \otimes W) \mod R_\epsilon(G).$$

We put $y = (\mu - 1)(V \otimes W)$. $\text{Res}^G_{\epsilon} x \in R_\epsilon(H)$ and $\text{Res}^H_{\epsilon} x \in R_\epsilon(K)$ are equivalent to $\text{Res}^G_{\epsilon} y \in R_\epsilon(H)$ and $\text{Res}^H_{\epsilon} y \in R_\epsilon(K)$ respectively. We have $\text{Res}^G_{\epsilon} y = (\dim W)(\mu - 1)V$. By Proposition 3.1 (ii) it holds that

$$\text{Res}^G_{\epsilon} y \equiv (\mu \dim W - 1)V \mod R_\epsilon(H).$$

$\text{Res}^G_{\epsilon} y \in R_\epsilon(H)$ implies $\mu \dim W \in \Gamma_V$. Since $(|M|, \dim W) = 1$, we have $\mu \in \Gamma_V$. In the same way we obtain $\mu \in \Gamma_W$. Therefore we have $\mu \in \Gamma_{V \otimes W}$; this means $y = 0$ in $R(G)$. Consequently $x$ belongs to $R_\epsilon(G)$.

For a group $G$ we denote by $C(G)$ its center. Since the dimensions of the
irreducible $G$-spaces divide $|G/C(G)|$, we have the following proposition.

**Proposition 3.3.** If both $H$ and $K$ have Property 1 and if it holds that $(|H/C(H)|, \phi(|K|))= (|K/C(K)|, \phi(|H|))=1$, then $G=H\times K$ has Property 1.

As the abelian groups and the $p$-groups have Property 1, we have the following.

**Corollary 3.4.** Let $H$ be an abelian group, and $K$ a $p$-group. Provided $(\phi(|H|), p)=1$, then $G=H\times K$ has Property 1.

**Corollary 3.5.** Let $H$ be a $p$-group and $K$ a $q$-group. Provided $(p, q)=(p, q-1)=(q, p-1)=1$, then $G=H\times K$ has Property 1.

4. The irreducible spaces of the hyperelementary group

Let $G$ have a normal cyclic subgroup $A$ and a Sylow $p$-subgroup $H$ such that $G$ is the semidirect product of $H$ by $A$, that is, $G$ is a hyperelementary group. The irreducible representations of $G$ can be constructed by the method of little group of Wigner and Mackey (see [7; 8.2]).

Since $A$ is cyclic, its irreducible representations form a group $Y$. The group $G$ acts on $Y$ by

$$(g\chi)(a) = \chi(g^{-1}ag)$$

for $g \in G$, $\chi \in Y$, $a \in A$. This action induces the action of $G$ on the set $Irr(A)$ of irreducible $A$-spaces. For $V \in Irr(A)$ and $g \in G$, we have an irreducible $A$-space $g_*V$ by this action. Let \{V(i): i \in Y/H\} be a system of representatives for the orbits of $H$. For each $i \in Y/H$, let $H(i)$ be the subgroup of $H$ consisting of those elements $h$ such that $h_*V(i)=V(i)$, and let $G(i)=AH(i)$ be the corresponding subgroup of $G$. We can canonically extend $V(i)$ to the $G(i)$-space, that is, $h \in H(i)$ acts trivially on $V(i)$. Let $W$ be an irreducible $H(i)$-space; $W$ can be extended to $G(i)$-space, too. By taking the tensor product of $V(i)$ and $W$ we obtain an irreducible $G(i)$-space $V(i) \otimes W$. Then $\text{Ind}_{G(i)}^G V(i) \otimes W$ is irreducible, moreover each irreducible $G$-space is obtained in this way ([7; Proposition 25]).

We denote by $C_H(A)$ the centralizer of $A$ in $H$, i.e.

$$C_H(A) = \{g \in H: g^{-1}ag = a \text{ for all } a \in A\}.$$ 

**Proposition 4.1.** If the kernel of $\text{Ind}_C^G V(i) \otimes W$ is $\{1\}$, then the kernel of the $A$-space $V(i)$ is $\{1\}$, and $H(i)=C_H(A)$.

Proof. This comes from the fact that ker $V(i) \subset$ ker $\text{Ind}_C^G \{V(i) \otimes W\}$.

Since $C_H(A)$ is normal in $H$, $H$ acts on $Irr(C_H(A))$ by
(X_{g^kH})(h) = \chi_{\pi}(g^{-1}hg),

where \( g \in H, h \in H \), and \( \chi_{\pi} \) is the corresponding character to \( W \in \text{Irr}(C_H(A)) \).

**Proposition 4.2.** Put \( K = C_H(A) \), and let \( V \) be an irreducible \( A \)-space with the trivial kernel, \( W \) an irreducible \( K \)-space and \( h \) an element of \( H \). Then we have

\[ \text{Ind}_{A}^{G} V \otimes (h \ast W) = \text{Ind}_{A}^{G} (h^{-1} \ast V) \otimes W \]

Proof. If we identify the representation spaces with the corresponding characters, by direct calculation we have

\[ \{\text{Ind}_{A}^{G} V \otimes (h \ast W)\}(g) = \text{Ind}_{A}^{G} \{h^{-1} \ast V\} \otimes W\} \text{ for each } g \in G. \]

**Proposition 4.3.** We have the following.

(i) \( \gamma \text{Ind}_{G(i)}^{G} (\gamma V(i)) \otimes (\gamma W) \text{ for } \gamma \in \Gamma \).

(ii) \( \text{Res}_{H}^{G} \text{Ind}_{G(i)}^{G} V(i) \otimes W = \text{Ind}_{H(i)}^{G} W \).

(iii) \( \text{Res}_{K}^{G} \text{Ind}_{G(i)}^{G} V(i) \otimes W = \dim W \otimes \bigoplus_{h \in H/K(i)} h \ast V(i) \)

(iv) If \( \ker \text{Ind}_{G(i)}^{G} V(i) \otimes W = \ker \text{Ind}_{G(j)}^{G} V(j) \otimes W' \), then we have \( H(i) = H(j) \).

Proof. (i): This holds clearly.

(ii): Since \( H \setminus G/G(i) \) consists of the only one coset, (ii) follows from the Mackey decomposition.

(iii): Since \( A \setminus G/G(i) \) can be identified with \( H/H(i) \), we have (iii) by the Mackey decomposition.

(iv): Put \( U = \text{Ind}_{G(i)}^{G} V(i) \otimes W \) and \( U' = \text{Ind}_{G(j)}^{G} V(j) \otimes W' \). From \( \ker U = \ker U' \) we have \( \ker \text{Res}_{K}^{G} U = \ker \text{Res}_{A}^{G} U' \). By (iii) we have \( \ker V(i) = \ker V(j) \). This implies \( H(i) = H(j) \).

**Proposition 4.4.** Put \( K = C_H(A) \), and let \( V \) be an irreducible \( A \)-space with the trivial kernel, \( U \) and \( W \) irreducible \( K \)-spaces. Set \( M = \text{Ind}_{A}^{G} V \otimes U \) and \( N = \text{Ind}_{K}^{G} V \otimes W \). Provided \( \Gamma M \subseteq \Gamma N \) as subset of \( \text{Irr}(G) \), then we have

\[ \langle \Gamma \text{Res}_{K}^{G} M, \Gamma \text{Res}_{K}^{G} N \rangle_{K} = \{0\}. \]

Proof. For \( \gamma \in \Gamma \) we have

\[ \gamma \text{Res}_{K}^{G} M = \bigoplus_{\lambda \in H/K} \gamma h \ast U \]

by Proposition 4.3 (ii). Proposition 4.2 implies \( \text{Ind}_{A}^{G} V \otimes (h \ast U) \subseteq \Gamma M \). Since \( \Gamma M \subseteq \Gamma N \), we have

\[ \langle \gamma h \ast U, \gamma' h' \ast W \rangle_{K} = 0 \]

for each \( \gamma \in \Gamma, \gamma' \in \Gamma, [h] \in H/K \) and \( [h'] \in H/K \). This relation yields the consequence of Propositions 4.4.
**Proposition 4.5.** Let $L$ be a subgroup of $H$, then we have $N_G(L) = C_A(L)N_H(L)$.

Proof. Let $a$ and $h$ are elements of $A$ and $H$ respectively. If $ah \in N_G(L)$, we have $(ah)^{-1}Lah = L$, consequently $a^{-1}La = hLh^{-1}$. For each $g \in L$, there exists $h' \in H$ such that $a^{-1}ga = h'$. Then we have $a^{-1}(gag^{-1}) = h'g^{-1} \in A \cap H$. This means that $a^{-1}gag^{-1} = 1$ and $h'g^{-1} = 1$. Therefore we have $ga = ag$, that is, we have $a \in C_A(L)$. This yields $L = hLh^{-1}$. We obtain $h \in N_H(L)$. The above argument shows $N_G(L) \subseteq C_A(L)N_H(L)$. On the other hand $N_G(L) \supseteq C_A(L)N_H(L)$ holds obviously. Hence we have $N_G(L) = C_A(L)N_H(L)$.

Let $h$ be an element of $H$, then $h$ acts on the generators $a$ of $A$ by

$$h \cdot a = hah^{-1}.$$ 

Let $L$ be the subset of $H$ consisting of elements $h$ such that

$$T(h) = \prod_{x \in G} b$$

is not equal to the unit element 1 of $G$, where $a$ is a fixed generator of $A$, and $\langle h \rangle \cdot a$ is the orbit of $a$ with respect to the above action of the group $\langle h \rangle$ generated by $h$. $L$ is defined independently of the choice of $a$.

**Proposition 4.6.** The above $L$ is a subgroup of $H$.

Proof. If $h \in K = C_H(A)$, we have $\langle h \rangle \cdot a = \{a\}$. This implies $T(h) \neq 1$. We get $L \supseteq K$, moreover we see that $L$ is the union of several cosets of $H/K$. We remark that $H/K$ is a cyclic $p$-group. If we can show that $h \in L$ implies $h^m \in L$ for $1 \leq m \leq p$, we see that $L$ is a subgroup of $H$.

Suppose $1 \leq m < p$. Since $\langle h \rangle \cdot a = \langle h^m \rangle \cdot a$, $h \in L$ implies $h^m \in L$.

Let $h$ be an element of $H - K$, then we have the disjoint sum such that

$$\langle h \rangle \cdot a = \sum_{j=0}^{p-1} h^j \langle h^p \rangle \cdot a.$$ 

If $T(h^p) = 1$, we have

$$T(h) = \prod_{j=0}^{p-1} h^j T(h^p) h^{-j} = 1.$$ 

Therefore $h^p \in L$ implies $h \in L$; this means that $h \in L$ implies $h^p \in L$. This completes the proof of Proposition 4.6.

**Proposition 4.7.** Put $K = C_H(A)$, and let $V$ be an irreducible $A$ space with the trivial kernel, $W$ a $K$-space, $a$ a generator of $A$ and $h$ an element of $H$. We have the following.

(i) Provided $h \in H - L$, the all eigenvalues of $ah$ on $\text{Ind}^A_K V \otimes W$ are determined independently of the choice of the generator $a$ of $A$. 


(ii) Provided $h \in L$, $ah$ does not have 1 as its eigenvalue on $\text{Ind}_{\beta \chi}^\alpha V \otimes W$. Here $L$ is the group defined above.

As we can prove this by direct calculation, we omit the proof.

5. On the case: $G$ is generated by two elements

In this section $G = AH$ will be a hyperelementary group such that $H$ is cyclic.

REMARK 5.1. Let $K$ be a subgroup of $H$, then $K$ is normal in $H$. If $W$ is a $K$-space, then for any $h \in H$ we have $h_k W = W$.

**Proposition 5.2.** We have the following.

(i) Let $U = \text{Ind}_{\delta \varepsilon \iota}^\delta V(i) \otimes W$ be an irreducible $G$-space. Then $\ker U = (\ker V(i))(\ker W)$ holds, where $\ker V(i) \subset A$ and $\ker W \subset H(i)$.

(ii) If irreducible $G$-spaces $U$ and $U'$ have the same kernel, $\Gamma U = \Gamma U'$ holds.

(iii) $G$ has Property 2.

Proof. (i): By the definition of the induced representation and Remark 5.1 we obtain $\ker U = (\ker V(i))(\ker W)$.

(ii): Suppose $U = \text{Ind}_{\delta \varepsilon \iota}^\delta V(i) \otimes W$ and $U' = \text{Ind}_{\delta \varepsilon \iota}^\delta V(j) \otimes W'$, then by (i) we have $\ker V(i) = \ker V(j)$ and $\ker W = \ker W'$ (see Proposition 4.3 (iv)). Since both $A$ and $H(i) = H(j)$ are cyclic, we have $\Gamma V(i) = \Gamma V(j)$ and $\Gamma W = \Gamma W'$. From Proposition 4.3 (i) we obtain $\Gamma U = \Gamma U'$.

(iii): Lemma 2.1 and above (ii) imply (iii).

**Proposition 5.3.** Let $V(i)$ be an irreducible $A$-space as before, $W$ an $H(i)$-space and $\gamma$ an element of $\Gamma$. Put $x = \text{Ind}_{\delta \varepsilon \iota}^\delta \{ (\gamma V(i)) \otimes W - V(i) \otimes W \}$. Then $x$ belongs to $R_\delta(G)$ if and only if $\text{Res}_{\gamma \iota}^\delta x$ belongs to $R_\delta(G(i))$.

Proof. The only if part is clear. We will prove the if part by induction on $|G|$. If $|A| = 1$ or $|H| = 1$ then Proposition 5.3 is trivial. Make the inductive hypothesis: for each hyperelementary group of the same type as $G$ has and of smaller order than $|G|$ Proposition 5.3 is valid.

We assume that $\text{Res}_{\gamma \iota}^\delta x$ belongs to $R_\delta(G(i))$. By Lemma 2.1 and the inductive hypothesis it is sufficient to prove the proposition in the case: $V(i)\{1\} = V(i)$ and $W(\{1\}) = W$. In this case we have $x^L = 0$ in $R(N_\delta(L))$ for each non-trivial subgroup $L$ of $G$. By Lemma 2.3 we complete the proof if we show $P = P(G; x) \equiv 0 \pmod{|G|}$. Choose a positive integer $s$ such that

$$\gamma(\exp(2\pi \sqrt{-1}/|A|)) = \exp(2\pi s \sqrt{-1}/|A|) \text{ and } s \equiv 1 \pmod{|H|}.$$ 

By (2.4) and (2.5) we have
\[ P \equiv \sum_{s \in \mathbb{A}} \{ z(g) - Q(s); \text{Ind}_G^G (V(i) \otimes W)(g) \} \mod |G| \\
= 1 - s^a, \]

where \( n = \text{dim} \text{Ind}_G^G (V(i) \otimes W) \). Since \( s \equiv 1 \mod |H| \), we have \( P \equiv 0 \mod |H| \).

On the other hand, \( \text{Res}_{G}^G x \in R_H(G(i)) \) implies \( \text{Res}_{G}^G x \in R_H(A) \); we have \( P(A; \text{Res}_{G}^G x) \equiv 0 \mod |A| \). From (2.5) we obtain

\[ \sum_{s \in \mathbb{A}} \{ z(g) - Q(s); \text{Ind}_G^G (V(i) \otimes W)(g) \} \equiv 0 \mod |A|. \]

The left hand side of the above relation is equal to \( 1 - s^a \). This means that \( P \equiv 0 \mod |A| \). Consequently we have \( P \equiv 0 \mod |G| \). This completes the proof.

**Proposition 5.4.** Let \( V(i) \) be an irreducible \( A \)-space as before, and \( U \) and \( W \) \( H(i) \)-spaces. Put \( x = \text{Ind}_G^G (V(i) \otimes U - V(i) \otimes W) \). Then \( x \) belongs to \( R_H(G) \) if and only if \( \text{Res}_{H}^G x \) belongs to \( R_H(H) \).

**Proof.** The only if part is clear. We will prove the if part by induction on \( |G| \). If \( |A| = 1 \) or \( |H| = 1 \) then Proposition 5.4 is trivial. Make the inductive hypothesis: for each hyperelementary group of the same type as \( G \) and of smaller order than \( |G| \) Proposition 5.4 is valid.

We assume that \( \text{Res}_{H}^G x \) belongs to \( R_H(H) \). By Lemma 2.1 and the inductive hypothesis it is sufficient to prove the proposition in the case: \( V(i) \{1\} = V(i), U\{1\} = U \) and \( W\{1\} = W \). Since \( K = C_H(A) \) is cyclic, those conditions imply

\[ U - W \equiv \gamma W_0 - W_0 \mod R_1(K), \]

where \( W_0 \) is some irreducible \( K \)-space with the trivial kernel and \( \gamma \) is some element of \( T \). Without loss of generality we may assume that \( W = W_0 \) and \( U = \gamma W_0 \). By this assumption we have \( x^L = 0 \) for each non-trivial subgroup \( L \) of \( G \). If we show that \( P = P(G; x) \equiv 0 \mod |G| \), by Lemma 2.4 we obtain Proposition 5.4. Choose a positive integer \( s \) such that

\[ \gamma (\exp(2\pi \sqrt{-1}/|H|)) = \exp(2\pi s \sqrt{-1}/|H|) \]

and \( s \equiv 1 \mod |A| \).

By (2.4) and (2.5) we have

\[ P \equiv \sum_{s \in \mathbb{A}} \{ z(g) - Q(s); \text{Ind}_G^G (V(i) \otimes W)(g) \} \mod |G| \\
= 1 - s^a, \]

where \( n = \text{dim} \text{Ind}_G^G (V(i) \otimes W) \). Since \( s \equiv 1 \mod |A| \), we have \( P \equiv 0 \mod |A| \). On the other hand, \( \text{Res}_{H}^G x \in R_H(H) \) implies \( P(H; \text{Res}_{H}^G x) \equiv 0 \mod |H| \). From (2.5) we obtain

\[ \sum_{s \in \mathbb{A}} \{ z(g) - Q(s); \text{Ind}_G^G (V(i) \otimes W)(g) \} \equiv 0 \mod |H|. \]
The left hand side of the above relation is equal to $1 - s^\gamma$. This means that $P = 0 \mod |H|$. Consequently we have $P = 0 \mod |G|$.  

**Proposition 5.5.** Let $V(i)$ be an irreducible $A$-space as before, $W$ an irreducible $H(i)$-space, and $\gamma$ and $\gamma'$ elements of $\Gamma$. Put $x = \text{Ind}_{G(i)}^G \{ \gamma(V) \otimes (\gamma'W) - V \otimes W \}$. Then $x$ belongs to $R_a(G)$ if and only if $\text{Res}_{G(i)}^G x \in R_a(G(i))$ and $\text{Res}_G^G x \in R_a(H)$.  

Proof. The only if part is clear. We prove the if part. Put $y = \text{Ind}_{G(i)}^G \{ (\gamma V(i)) \otimes (\gamma'W) - (\gamma V(i)) \otimes W \}$ and $z = \text{Ind}_{G(i)}^G \{ (\gamma V(i)) \otimes W - V(i) \otimes W \}$. We have $x = y + z$; we have $\text{Res}_H^G x = \text{Res}_H^G y$. $\text{Res}_H^G x \in R_a(H)$ means that $\text{Res}_H^G y \in R_a(H)$. By Proposition 5.4 we have $y \in R_a(G)$. This and $\text{Res}_{G(i)}^G x \in R_a(G(i))$ imply $\text{Res}_{G(i)}^G z \in R_a(G(i))$. By Proposition 5.3 we have $z \in R_a(G)$. Consequently we have $x = y + z \in R_a(G)$.  

**6. Hyperelementary groups and Property 2**  

In this section $G = AH$ will be a hyperelementary group such that all the elementary subgroups of the quotient groups of the subgroups of $G$ have Property 2.  

**Remark.** If an elementary group $K = A \times H$ satisfies one of the conditions: (i) $(\phi(|A|), p) = 1$, (ii) $|H| \leq p^4$ and (iii) $H$ is metacyclic, then $K$ has Property 2.  

Let $R(G, f)$ be the subgroup of $R(G)$ built from the irreducible $G$-spaces which yield faithful $A$-spaces when they are restricted to $A$. Put $R_a(G, f) = R(G, f) \cap R_a(G)$, and $R_a(G, f) = R(G, f) \cap R_0(G)$.  

**Proposition 6.1.** Let $x$ be an element of $R_a(G, f)$, $B$ a subgroup of $A$ and $K$ a subgroup of $C_H(B)$. Then for each $C \in X(G) = \text{Irr}(G)/\Gamma$ we have $\text{Res}_{BK}^G f_c(x) \in R_a(BK)$.  

Proof. It is sufficient to prove the proposition in the case that $K = C_H(B)$. In this case we have $K \subseteq C_H(A)$. Put $L = C_H(A)$. Let $V$ be an irreducible $A$-space with the trivial kernel, and $U$ and $W$ irreducible $L$-spaces. If $\Gamma \text{Ind}_{KL}^A V \otimes U = \Gamma \text{Ind}_{KL}^A V \otimes W$, we have

$$\langle \Gamma \text{Res}_{BK}^G \text{Ind}_{KL}^A V \otimes U, \Gamma \text{Res}_{BK}^G \text{Ind}_{KL}^A V \otimes W \rangle_{BK} = \{0\}$$

by Proposition 4.4. Since $BK$ has Property 2 by the assumption, we have $\text{Res}_{BK}^G f_c(x) \in R_a(BK)$ for each $C \in X(G)$.  

**Proposition 6.2.** Put $K = C_H(A)$, and let $V$ be an irreducible $A$-space with
the trivial kernel, \( W \) a \( K \)-space and \( \gamma \) an element of \( \Gamma \). Put \( x = \text{Ind}_{G, K}^G \{(\gamma V) \otimes W - V \otimes W\} \). Then \( x \) belongs to \( R_k(G) \) if and only if for each subgroup \( B \) of \( A \) and \( L = C_H(B) \) we have \( \text{Res}_{B, L}^G \) \( x \in R_k(BL) \).

**Proof.** The only if part is clear. We will prove the if part by induction on \( |G| \). If \( |A| = 1 \) or \( |H| = 1 \), then Proposition 6.2 is trivial. Make the inductive hypothesis: for each hyperelementary group which satisfies the same condition as \( G \) satisfies and whose order is smaller than \( |G| \) Proposition 6.2 is valid.

Assume that for each \( B \subset A \) and \( L = C_H(B) \) we have \( \text{Res}_{B, L}^G \) \( x \in R_k(BL) \). Firstly we get \( x \in R_k(G) \). By Propositions 3.1, 4.3, 4.4 and 6.1 it is sufficient to prove the proposition in the case that there exist a positive integer \( r \), an irreducible \( K \)-space \( U \) and elements \( h(m) \) of \( H \), \( 1 \leq m \leq r \), such that

\[
W = \bigoplus_{m=1}^{\ell} h(m)_\ast U.
\]

By Propositions 3.1 and 4.2 we have

\[
\text{Ind}_{G, K}^G \{(\gamma V) \otimes h(m)_\ast U - V \otimes h(m)_\ast U\} \equiv \text{Ind}_{G, K}^G \{(\gamma V) \otimes U - V \otimes U\} \mod R_k(G).
\]

This enables us to assume that \( W \) itself is irreducible.

**Assertion 6.3.** Let \( M \neq \{1\} \) be a subgroup of \( G \). We have \( x^M \in R_k(N_G(M)) \).

**Proof.** If \( A \cap M \neq \{1\} \), then we have \( x^M = 0 \) in \( R_k(N_G(M)) \). We assume \( A \cap M = \{1\} \). In this case \( M \) is conjugate to a subgroup of \( H \). By Proposition 2.2 we may assume \( M \subset H \). By Proposition 4.5 we have \( N_G(M) = C_A(M)N_H(M) \). The proof is divided into the following three cases.

Case 1. \( C_A(M) \neq A \)

Put \( B = C_A(M) \), \( L = C_H(B) \) and \( y = \text{Res}_{B, H}^A x \). We have

\[
y = \text{Ind}_{B, H}^A \{(\gamma \text{ Res}_{B, H}^A V) \otimes W - (\text{Res}_{B, H}^A V) \otimes W\}.
\]

By Proposition 25 of [7; 8.2] we have \( y \) in another form as follows:

\[
y = \text{Ind}_{B, H}^A \{(\gamma \text{ Res}_{B, H}^A V) \otimes U - (\text{Res}_{B, H}^A V) \otimes U\},
\]

where \( U \) is an \( L \)-space. For a subgroup \( C \) of \( B \), we put \( N = C_H(C) \); we have \( \text{Res}_{B, H}^{CN} y = \text{Res}_{N, H}^C x \in R_k(CN) \) by the assumption. By the inductive hypothesis \( y \) belongs to \( R_k(BH) \). This implies \( x^M = y^M \in R_k(N_G(M)) \).

Case 2. \( C_A(M) = A \) and \( N_H(M) \neq H \)

Put \( N = N_G(M) \), \( D = H \cap N \), \( E = K \cap N \) and \( y = \text{Res}_N^E x \), then we have

\[
y = \sum_{(\ell, \ell') \in H/DK} \text{Ind}_{E, E}^D \{(\gamma h_\ast V) \otimes (\text{Res}_N^E h_\ast W) - (h_\ast V) \otimes (\text{Res}_N^E h_\ast W)\}.
\]

By Proposition 3.1 we have
\[ y = \sum_{\theta \in H \cap K} \text{Ind}_{A \cap K}^{N_{\theta}} \{(\gamma V) \otimes (\text{Res}_{K}^{\theta} h_{\theta} W) - V \otimes (\text{Res}_{K}^{\theta} h_{\theta} W)\} \mod R_{\theta}(N) \]

\[ = \text{Ind}_{A \cap K}^{N_{\theta}} \{(\gamma V) \otimes U - V \otimes U\}, \]

where

\[ U = \bigoplus_{\theta \in H \cap K} \text{Res}_{K}^{\theta} h_{\theta} W. \]

For a subgroup \( B \) of \( A \) and \( L = C_{\theta}(B) \) we have \( \text{Res}_{B \cap L}^{N_{\theta}} y = \text{Res}_{B \cap L}^{N_{\theta}} x \in R_{\theta}(BL) \). We have \( y \in R_{\theta}(N_{\theta}(M)) \) by the inductive hypothesis. This implies \( x^{M} = y^{M} \in R_{\theta}(N_{\theta}(M)) \).

\textbf{Case 3.} \( N_{\theta}(M) = G \)

We have reduced the problem to the case that \( W \) is irreducible. In this case \( \text{Ind}_{A \cap K}^{\theta} (\gamma V) \otimes W \) and \( \text{Ind}_{K}^{\theta} V \otimes W \) are irreducible. If \( (\text{Ind}_{K}^{\theta} V \otimes W)^{M} \neq \{0\} \), then we have \( (\text{Ind}_{K}^{\theta} V \otimes W)^{M} = \text{Ind}_{K}^{\theta} V \otimes W \). We get \( \ker \text{Ind}_{K}^{\theta} V \otimes W \supset M \). By the inductive hypothesis we have \( x \in R_{\theta}(G) \). This completes the proof of Assertion 6.3.

If we show \( P = P(G; x) \equiv 0 \mod |G| \), we complete the proof of Proposition 6.2. Choose a positive integer \( s \) such that

\[ \gamma(\exp(2\pi \sqrt{-1}/|A|)) = \exp(2\pi s \sqrt{-1}/|A|) \] and \( s \equiv 1 \mod |H| \).

By (2.4) and (2.5) we have

\[ P \equiv \sum_{\theta \in H \cap K} \{P(g) - Q(s; \text{Ind}_{K}^{\theta} V \otimes W)(g)\} \mod |G|. \]

Since \( s \equiv 1 \mod |H| \), we have \( P \equiv 0 \mod |H| \). On the other hand there exist integers \( n_{c} \) for the cyclic subgroups \( C \) of \( H \) such that

\[ P = \sum_{\theta \in H \cap K \text{ cyclic}} n_{c} P(G; \text{Res}_{AC}^{C} x). \]

If we can show \( P(G; \text{Res}_{AC}^{C} x) \equiv 0 \mod |A| \), we see that \( P \equiv 0 \mod |A| \); consequently we obtain \( P \equiv 0 \mod |G| \). \( P(G; \text{Res}_{AC}^{C} x) \equiv 0 \mod |A| \), follows from the following assertion.

\textbf{Assertion 6.4.} \textit{For each cyclic subgroup} \( C \) \textit{of} \( H \), \textit{we have} \( \text{Res}_{AC}^{C} x \in R_{\theta}(AC) \).

\textbf{Proof.} \textit{Put} \( y = \text{Res}_{AC}^{C} x \) \textit{and} \( M = C \cap K \). \textit{We have}

\[ y = \sum_{\theta \in H \cap K \text{ cyclic}} \text{Ind}_{A \cap K}^{\theta} \{(\gamma h_{\theta} V) \otimes (\text{Res}_{K}^{\theta} h_{\theta} W) - (h_{\theta} V) \otimes (\text{Res}_{K}^{\theta} h_{\theta} W)\} \]

\[ = \sum_{\theta \in H \cap K \text{ cyclic}} \text{Ind}_{A \cap K}^{\theta} \{(\gamma V) \otimes (\text{Res}_{K}^{\theta} h_{\theta} W) - V \otimes (\text{Res}_{K}^{\theta} h_{\theta} W)\} \mod R_{\theta}(AC) \]

\[ = \text{Ind}_{A \cap K}^{\theta} \{(\gamma V) \otimes U - V \otimes U\}, \]

where

\[ U = \bigoplus_{\theta \in H \cap K} \text{Res}_{K}^{\theta} h_{\theta} W. \]
Since we have $\text{Res}^A_M y = \text{Res}^A_M x \in R_\delta(AM)$ by the assumption, we have $\text{Res}^A_C x = y \in R_\delta(AC)$ by Proposition 5.3. This completes the proof of Assertion 6.4.

**Proposition 6.5.** Put $K = C_H(A)$, and let $V$ be an irreducible $A$-space with the trivial kernel, $W$ a $K$-space and $\gamma$ an element of $\Gamma$. Put $x = \text{Ind}^A_K \{V \otimes (\gamma W) - V \otimes W\}$. Then $x$ belongs to $R_\delta(G)$ if and only if $\text{Res}^A_H x$ belongs to $R_\delta(H)$.

**Proof.** The only if part is clear. We will prove the if part by induction on $|G|$. If $|A| = 1$ or $|H| = 1$, then the proposition is trivial. Make the inductive hypothesis: for each hyperelementary group which satisfies the condition stated at the beginning of this section and whose order is smaller than $|G|$ Proposition 6.5 is valid.

We assume $\text{Res}^A_H x \in R_\delta(H)$ and $|A| \neq 1$. First we have $x \in R_\delta(G)$. By Propositions 3.1, 4.3, 4.4 and 6.1 it is sufficient to prove the proposition in the case that there exist a positive integer $r$, an irreducible $K$-space $U$ and elements $h(m)$ of $H$, $1 \leq m \leq r$, such that

$$W = \bigoplus_{m=1}^r h(m)_*_U.$$

By Propositions 3.1 and 4.2 we have

$$\text{Ind}^A_K \{V \otimes (\gamma h(m)_*_U) - V \otimes (h(m)_*_U)\} \equiv \text{Ind}^A_K \{V \otimes (\gamma U) - V \otimes U\} \mod R_\delta(G).$$

This enables us to assume that $W$ itself is irreducible.

**Assertion 6.6.** Let $L$ be a non-trivial subgroup of $G$. We have $x^L \in R_\delta(N_\delta(L))$.

**Proof.** Since $A$ acts freely on $\text{Ind}^A_K V \otimes \gamma W$ and on $\text{Ind}^A_K V \otimes W$ except the origins, it is sufficient to prove the assertion in the case that $L \cap A = \{1\}$. In this case $L$ is conjugate to a subgroup of $H$. By Proposition 2.2 we may assume $L \subset H$. Then we have $N_\delta(L) = C_A(L)N_H(L)$ by Proposition 4.5. We divide the proof into the following three cases.

Case 1. $C_A(L) \neq A$

We put $B = C_A(L)$ and $y = \text{Res}^A_B x$. We have

$$y = \text{Ind}^A_B \{(\text{Res}^A_B V \otimes (\gamma W) - (\text{Res}^A_B V) \otimes W)\}.$$

Put $M = C_H(B)$, then we have

$$y = \text{Ind}^A_M \{(\text{Res}^A_B V \otimes (\gamma \text{Ind}^A_K W) - (\text{Res}^A_B V) \otimes (\text{Ind}^A_K W))\}.$$

On the other hand we have $\text{Res}^A_H y = \text{Res}^A_H x \in R_\delta(H)$. By the inductive hypothesis we have $y \in R_\delta(BH)$. This implies $x^L = y^L \in R_\delta(N_\delta(L))$. 

Case 2. \( C_A(L) = A \) and \( N_H(L) \neq H \)

Put \( M = N_H(L) \), \( N = N_G(L) \), \( D = K \cap M \) and \( y = \text{Res}_H^G x \). We have \( N = AM \) and

\[
\begin{align*}
y &= \sum_{\{a\} \in H/K} \text{Ind}_{AD}^N \{(h_a V) \otimes (\gamma \text{Res}_D^K h_a W) - (h_a V) \otimes (\text{Res}_D^K h_a W)\} \\
&\equiv \sum_{\{a\} \in H/K} \text{Ind}_{AD}^N \{V \otimes (\gamma \text{Res}_D^K h_a W) - V \otimes (\text{Res}_D^K h_a W)\} \mod R_i(N) \\
&= \text{Ind}_{AD}^N \{V \otimes (\gamma U) - V \otimes U\},
\end{align*}
\]

where

\[
U = \bigoplus_{\{a\} \in H/K} \text{Res}_D^K h_a W.
\]

Since we have \( \text{Res}_M^N y = \text{Res}_M^G x \in R_4(M) \), by the inductive hypothesis we get \( y \in R_k(N) \). This implies \( x^L = y^L \in R_k(N_G(L)) \).

Case 3. \( N_G(L) = G \)

When \( W \) is irreducible, \( \text{Ind}_A^K V \otimes W \) and \( \text{Ind}_A^K V \otimes \gamma W \) are irreducible. This implies that \( x^L = x \) or \( 0 \) in \( R(G) \). If \( x^L = 0 \), Assertion 6.6 is clearly valid. If \( x^L = x \), then \( L \) is included in the kernel of \( x \). By the inductive hypothesis we obtain \( x \in R_d(G) \). This completes the proof of Assertion 6.6.

If we show \( P = P(G; x) \equiv 0 \mod |G| \), we complete the Proof of Proposition 6.5. As usual choose a positive integer \( s \) such that

\[
\gamma(\exp(2\pi\sqrt{-1}/|H|)) = \exp(2\pi s\sqrt{-1}/|H|) \quad \text{and} \quad s \equiv 1 \mod |A|.
\]

By (2.5) we have

\[
P \equiv \sum_{g \in G} \{z(g) - Q(s; \text{Ind}_A^K V \otimes W)(g)\} \mod |G|.
\]

By the inductive hypothesis, for each proper subgroup \( B \) of \( A \) we have \( \text{Res}_B^G x \in R_d(BH) \). This implies \( P(BH; \text{Res}_B^G x) \equiv 0 \mod |BH| \). Therefore we have

\[
P \equiv \sum_{ah \in A_B; \langle a \rangle = A} \{z(ah) - Q(s; \text{Ind}_A^K V \otimes W)(ah)\} \mod |H|.
\]

By Propositions 4.6 and 4.7 we have

\[
P \equiv \sum_{ah \in A_B; \langle a \rangle = A} \{z(ah) - Q(s; \text{Ind}_A^K V \otimes W)(ah)\} \mod |H| \\
\equiv \phi(|A|) \sum_{h \in H \setminus L} \{z(h) - Q(s; \text{Ind}_A^K V \otimes W)(h)\} \mod |H|,
\]

where \( L \) is the group given in Proposition 4.6, \( \phi \) is the Euler function. \( \text{Res}_L^G x \in R_k(L) \) and (2.5) imply

\[
\sum_{h \in L} \{z(h) - Q(s; \text{Ind}_A^K V \otimes W)(h)\} \equiv 0 \mod |L|.
\]
Since $\phi(|A|)$ is a multiple of $|H/K|$ and $|L|$ a multiple of $|K|$, we have

$$P \equiv \phi(|A|) \sum_{h \in H} \{z(h) - \mathcal{Q}(s; \text{Ind}_{\mathcal{A}_K}^\mathcal{K} V \otimes W)(h)\} \mod |H|.$$ 

From $\text{Res}_H^\mathcal{K} x \in R_\mathcal{A}(H)$, we have $P \equiv 0 \mod |H|$. On the other hand for the cyclic subgroups $C$ of $H$ there exist integers $n_C$ such that

$$P = \sum_{C \leq H \text{ cyclic}} n_C \sum_{g \in \mathcal{A}_U} z(g).$$

We obtain $P \equiv 0 \mod |A|$ from the following assertion; consequently we get $P \equiv 0 \mod |G|$.

**Assertion 6.7.** For each cyclic subgroup $C$ of $H$, we have $\text{Res}_C^\mathcal{A} x \in R_\mathcal{A}(AC)$.

**Proof.** Put $y = \text{Res}_C^\mathcal{A} x$ and $D = C \cap K$, then we have

$$y = \sum_{\{\lambda\} \in H/\mathcal{A}_K} \text{Ind}_{\mathcal{A}_C}^\mathcal{A} \{((\mathcal{A}_C \otimes (\gamma \text{ Res}_{\mathcal{D}_K}^\mathcal{K} h_* W) - (h_* V) \otimes (\text{Res}_{\mathcal{D}_K}^\mathcal{K} h_* W)) \mod \mathcal{A}_C(AC)$$

$$= \text{Ind}_{\mathcal{A}_C}^\mathcal{A} (V \otimes \gamma U - V \otimes U),$$

where

$$U = \bigoplus_{\{\lambda\} \in H/\mathcal{A}_K} \text{Res}_{\mathcal{D}_K}^\mathcal{K} h_* W.$$

Moreover we have $\text{Res}_C^\mathcal{A} y = \text{Res}_C^\mathcal{A} x \in R_\mathcal{A}(C)$. By Proposition 5.4 we have $y \in R_\mathcal{A}(AC)$. This completes the proof of Assertion 6.9 consequently completes the proof of Proposition 6.5.

**Proposition 6.10.** Put $K = \mathcal{A}_H(A)$, and let $V$ be an irreducible $A$-space with the trivial kernel, $W$ a $K$-space and $\gamma$ an element of $\Gamma$. Put $x = \text{Ind}_{\mathcal{A}_K}^\mathcal{A} \{\gamma(V \otimes W) - V \otimes W\}$. Then $x$ belongs to $R_\mathcal{A}(G)$ if and only if for each subgroup $B$ of $A$ and $L = \mathcal{A}_H(B)$ we have $\text{Res}_{\mathcal{A}_L}^\mathcal{A} x \in R_\mathcal{A}(BL)$.

**Proof.** The only if part is clear. We prove the if part. Put $y = \text{Ind}_{\mathcal{A}_K}^\mathcal{A} \{\gamma(V \otimes W) - (\gamma V) \otimes W\}$ and $z = \text{Ind}_{\mathcal{A}_K}^\mathcal{A} \{\gamma(V) \otimes W - V \otimes W\}$, then we have $x = y + z$. Since $\text{Res}_{\mathcal{A}_K}^\mathcal{A} z = 0$, we have $\text{Res}_{\mathcal{A}_K}^\mathcal{A} y \in R_\mathcal{A}(H)$ by the assumption. From Proposition 6.5 we obtain $y \in R_\mathcal{A}(G)$. This yields that

$$\text{Res}_{\mathcal{A}_L}^\mathcal{A} z = \text{Res}_{\mathcal{A}_L}^\mathcal{A} x - \text{Res}_{\mathcal{A}_L}^\mathcal{A} y \in R_\mathcal{A}(BL).$$

Proposition 6.2 implies $z \in R_\mathcal{A}(G)$. Hence we conclude that $x \in R_\mathcal{A}(G)$.

**Theorem 6.11.** Let $G$ be a hyperelementary group such that all the elementary subgroups of the quotient groups of the subgroups of $G$ have Property 2. Then $G$ has Property 2.
Proof. We prove it by induction on \(|G|\). If \(|A|=1\) or \(|H|\leq p\), we are aware that \(G\) has Property 2. Make the inductive hypothesis: each hyper-elementary group which satisfies the same condition as \(G\) satisfies and whose order is smaller than \(|G|\) has Property 2.

Let \(x\) be an element of \(R_a(G)\). By Lemma 2.1 and the inductive hypothesis we may assume \(x(\{1\})=x\). This implies \(x\in R_a(G,f)\). Put \(K=C_B(A)\). For a fixed element \(C\) of \(X(G)\), there exist \(\gamma\in\Gamma\), an irreducible \(A\)-space \(V\) and an irreducible \(K\)-space \(W\) such that

\[
f_c(x) \equiv \text{Ind}_{A^{K}}^{\Gamma} \{\gamma(V \otimes W) - V \otimes W}\ \text{mod} \ R_0(G).
\]

By Propositions 6.1 and 6.10 we get \(f_c(x)\in R_a(G)\).

For a subgroup \(B\) of \(A\), we get an elementary subgroup \(BC_B(B)\) of \(G\). Varying \(B\), we obtain several elementary groups. Let \(E(G)\) be the set of all those elementary groups. Lemma 2.1 and Propositions 6.1 and 6.10 yield the following theorem.

**Theorem 6.12.** In the same situation as in Theorem 6.11

\[
\text{Res}: \ R_a(G,f)/R_a(G,f) \to \bigoplus_{K\in\pi(G)} j(K)
\]

is injective. Therefore we obtain a naturally defined injection

\[
j(G) \to \bigoplus_{B} \bigoplus_{K\in\pi(G)} j(K)
\]

where \(B\) runs over the subgroups of \(A\).

7. A closing example

Let \(A\) (resp. \(H\)) be the cyclic group of order 7 (resp. 5) which consists of the 7-th (resp. 5-th) roots of unity, and \(G\) the direct product of \(A\) and \(H\). For each integer \(i\) (resp. \(j\)) with \(0\leq i\leq 6\) (resp. \(0\leq j\leq 4\)) define the \(A\)-(resp. \(H\)-) representation \(v_i\) (resp. \(w_j\)) by

\[
v_i(z) = z^i \text{ for } z\in A
\]

(resp. \(w_j(z) = z^j \text{ for } z\in H\)).

We denote by \(V_i\) (resp. \(W_j\)) the corresponding representation space to \(v_i\) (resp. \(w_j\)). Define an element \(x\) of \(R(G)\) by

\[
x = V_2 \otimes W_1 + V_2 \otimes W_0 + V_2 \otimes W_0 - V_1 \otimes W_1 - V_1 \otimes W_0 - V_1 \otimes W_0.
\]

Then we have \(x\in R_a(G)\cap R(G,f)\); moreover we have \(\text{Res}_H^G x\in R_a(A)\) and \(\text{Res}_A^H x\in R_a(H)\). The \(x\) does not, however, belong to \(R_a(G)\). This is a counter example to [1; Proposition 5.2].
References


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