Harada, M. and Yukimoto, Y.
Osaka J. Math.
22 (1985), 321-326

ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES IV

To the memory of Professor Takehiko MIYATA

MANABU HARADA AND YOSHITO YUKIMOTO

(Received February 8, 1984)

In the previous papers [1] and [2], we have studied conditions under which every maximal submodule of a finite direct sum D of certain hollow modules over a right artinian ring with 1 contains a non-zero direct summand of D. The present objective is to generalize slightly Theorems 3 and 4 of [2] related to the property mentioned above.

Throughout this paper, R will represent a right artinian ring with identity, and every R-module will be assumed to be a unitary right R-module with finite composition length. We denote the Jacobson radical and the length of a composition series of an R-module M by $J(M)$ and $|M|$, respectively. Occasionally, we write $J=J(R)$. If M has a unique maximal submodule $J(M)$, M is called hollow (local). When this is the case, $M\approx eR/A$ for some primitive idempotent e and a right ideal A in eR.

Let $\{N_i\}_{i=1}^n$ be a family of hollow modules, and $D=\sum_{i=1}^n\oplus N_i$. We are interested in the following condition [1]:

(**) *Every maximal submodule of D contains a non-zero direct summand of D.*

As was claimed in [1], [2], whenever we study the conition (**), we may restrict ourselves to the case where R is basic and $N_i=eR/A_i$ for a fixed primitive idempotent e and a right ideal A_i in eR. Now, let $N=eR/A$ be a hollow module. Put $\Delta=eRe/eJe=\overline{eRe}=\mathrm{End}_R(N/J(N))=\mathrm{End}_R(eR/eJ)$, and $\Delta(A)$ $(=\Delta(N))=\{\bar{x}\,|\,x\in eRe$ and $xA\subset A\}$ (see [2]). We denote by $N^{(m)}$ the direct sum of m copies of N. Then $N^{(m+1)}=N\oplus N^{(m)}$. If M is a maximal submodule of $N^{(m)}$ then $N\oplus M$ is a maximal submodule of $N^{(m+1)}$. Thus we get a mapping $\theta(m)$ of the isomorphism classes of maximal submodules in $N^{(m)}$ into the isomorphism classes of maximal submodules in $N^{(m+1)}$.

Theorem 1 (cf. [3], Corollary 2 to Theorem 3). *Let $N=eR/A$ be a hollow module. Then the following conditions are equivalent*:

1) $[\Delta\colon\Delta(A)]=k$.

2) *If $m>k$, every maximal submodule M in $D=N^{(m)}$ contains a submodule*

isomprphic to $N^{(m-k)}$ *but not to* $N^{(m-k+1)}$. *In this case, such a submodule of* M *is a direct summand of* D.

3) $\theta(i)$ *is not epic for every* $i\leq k-1$, *but* $\theta(j)$ *is epic for every* $j\geq k$.

Proof (cf. [2], the proof of Theorem 3).

1)$\to$2). Put $D=N^{(m)}=D(k)\oplus D'(n)$, where $m=k+n$, $D(k)$ is the direct sum of the first k copies of N and $D'(n)$ the direct sum of the last n copies of N. Let $\{\bar{1}, \bar{\delta}_2, \cdots, \bar{\delta}_k\}$ be a set of linearly independent elements in Δ over $\Delta(A)$. Set $\beta_i=(\tilde{\delta}_i, \tilde{0}, \cdots, \overset{i}{\check{\tilde{e}}}, \tilde{0}, \cdots, \tilde{0})$ in $D(k)$, and $M=\sum_{i=2}^{k}\beta_i R+D'(n)+J(D)$ in D, where $\tilde{x}$ means the residue class of x in eR/A. Then M is a maximal submodule of D. Suppose that $M\supset M_1\oplus M_2\oplus\cdots\oplus M_q\oplus M^*$ and $M_i\approx N$ for all i. Then

$$(\alpha)\qquad M_i\not\subset J(D)\,.$$

Actually, if not, $N\approx M_i\subset J(D)=DJ$, which is impossible. Since $M_i\approx eR/A$, $M_i=\rho R$ and $\mathrm{r}_R(\rho)=\{r\in eR\,|\,\rho r=0\}=A$. Now let $\rho=\sum\beta_i y_i+y+j$, where $y_i\in eRe$, $y\in D'(n)$ and $j\in J(D)$. Then $\rho=(\sum_{i\geq 2}\tilde{\delta}_i y_i, \tilde{y}_2, \cdots, \tilde{y}_k, \tilde{0}, \cdots, \tilde{0})+(\tilde{0}, \tilde{0}, \cdots, \tilde{z}_{k+1}, \cdots, \tilde{z}_{k+n})+(\tilde{j}_1, \tilde{j}_2, \cdots, \tilde{j}_{k+n})$, where $z_i\in eRe$ and $j_i\in eJ$. By the structure of D and $r_R(\rho)=A$, $(y_i+j_i)A\subset A$, and $(\sum\delta_i y_i+j_i)A\subseteq A$. Noting that $eA=A$, we see that $\bar{y}_i\in\Delta(A)$ for $2\leq i\leq k$, and $\sum_{i\geq 2}\bar{\delta}_i\bar{y}_i\in\Delta(A)$. Therefore, $\bar{y}_i=0$ for $2\leq i\leq k$, since $\{\bar{1}, \bar{\delta}_2, \cdots, \bar{\delta}_k\}$ is linearly independent. Hence

$$(\beta)\qquad \pi(M_i)\subset J(D(k))\,,$$

where $\pi\colon D=D(k)\oplus D'(n)\to D(k)$ is the projection. Let p_s be the projection on the s-th component of $D=N^{(k+n)}$. Since $M_1\not\subset J(D)$ and $\pi(M_1)\subset J(D(k))$, $p_j\,|\,M_1$ is an epimorphism for some $j>k$, say $j=k+1$, and hence an isomorphism for $M_1\approx N$. Therefore

$$(\gamma)\qquad D=D(k)\oplus M_1\oplus D'(n-1)\,,$$

where $D'(n)=N\oplus D'(n-1)$. Now assume that $D=D(k)\oplus M_1\oplus M_2\oplus\cdots\oplus M_s\oplus D'(n-s)$. Let $\pi_{D'(n-s)}$ be the projection of D onto $D'(n-s)$ in the above decomposition. Suppose $\pi_{D'(n-s)}(M_{s+1})\subset J(D'(n-s))$. Then $\pi_{D(k)\oplus D'(n-s)}(M_{s+1})\subset J(D)$ by (β). On the other hand, $0=M_{s+1}\cap(M_1\oplus M_2\oplus\cdots\oplus M_s)=\ker(\pi_{D(k)\oplus D'(n-s)}\,|\,M_{s+1})$, so M_{s+1} is monomorphic to a submodule in $J(D)$, which is impossible. Hence $\pi_{D'(n-s)}(M_{s+1})\not\subset J(D'(n-s))$, and so $D=D(k)\oplus M_1\oplus M_2\oplus\cdots\oplus M_{s+1}\oplus D'(n-s-1)$ by the above argument. Accordingly, $q\leq n$, and hence M does not contain a submodule of D isomorphic to $N^{(n+1)}$. Let M' be an arbitrary maximal submodule of D. Then, by induction on m and [2], Theorem 2, $M'=N'^{(m-k)}\oplus M^*$, where $N'\approx N$.

2)$\to$1). Take $m=k+1$. By (α) and the argument employed in proving (γ), we see that D contains a direct summand which is isomorphic to N. Hence

$[\Delta: \Delta(A)]=k$ by [2], Theorem 2.

1)$\leftrightarrow$3). In case $\theta(t)$ is epic, every maximal submodule M of $N^{(t+1)}$ contains a direct summand M_1 which is isomorphic to N. Then, by 2), M_1 is also a direct summand of $N^{(t+1)}$. Hence $\theta(t)$ is epic if and only if $N^{(t+1)}$ satisfies (**), and the equivalence of 1) and 3) is clear by [2], Theorem 3 (see Remark below).

In Theorem 1, we have studied a direct sum of isomorphic copies of a fixed hollow module. Next, let $N_1=eR/A_1$ and $N_2=eR/A_2$. If there exists an epimorphism φ of N_1 to N_2 then we write $N_1>N_2$. Since φ is given by the left-sided multiplication of a unit element x in eRe, we have $xA_1\subset A_2$, and furthermore $N_1\approx eR/xA_1$. Hence, when we study the direct sum $N_1\oplus N_2$ with $N_1>N_2$, we may assume that $A_1\subset A_2$.

Theorem 2. *Let $\{N_i=eR/A_i\}_{i=1}^n$ be a family of hollow modules ($n\geq 2$). Assume that $|A_1|\geq|A_2|\geq\cdots\geq|A_n|$. Then $D=\sum_{i=1}^n\oplus N_i$ satisfies (**) if and only if, for any sequence $\{\bar\delta_2,\cdots,\bar\delta_n\}$ of $n-1$ elements in Δ, there exist an integer t ($2\leq t\leq n$) and $\bar y_i\in\Delta(A_t, A_i)$ ($2\leq i\leq t-1$) such that*

$$(\#)\qquad \sum_{i=2}^{t-1}\bar\delta_i\bar y_i+\bar\delta_t\in\Delta(A_t, A_1)\,,$$

where $\Delta(A_t, A_i)=\{\bar x\mid x\in eRe \text{ and } xA_t\subset A_i\}$.

Proof. We may assume that R is basic. Take the maximal submodule M in D generated by $\beta_i=(\tilde\delta_i, \tilde 0, \cdots, \overset{\overset{i}{\vee}}{\tilde e}, \tilde 0, \cdots, \tilde 0)$ ($i=2, 3, \cdots, n$). Then M contains a direct summand M_1 of D, i.e., $D=M_1\oplus D_1$ and $M_1\approx N_p$ for some p; M_1 is generated by $\alpha=\sum_{i\geq 2}\beta_iy_i+j$, where $y_i\in eRe$ ($y_q\notin eJe$ for some q) and $j\in J(D)$. Now, $\alpha=(\sum_{i\geq 2}\tilde\delta_iy_i+\tilde j_1, \tilde y_2+\tilde j_2, \cdots, \tilde y_n+\tilde j_n)$. Assume that $\bar y_n=\bar y_{n-1}=\cdots=\bar y_{t+1}=0$ and $\bar y_t\neq 0$. Let π_t be the projection of $D=\sum\oplus N_i$ onto N_t. Then $\pi_t|M_1$ is an epimorphism, so $M_1>N_t$. On the other hand, let π be the projection of $D=M_1\oplus D_1$ onto M_1. We shall show that $\pi_t|M_1$ is an isomorphism. Suppose, to the contrary, that $|M_1|>|N_t|$. Then, since $|N_k|\leq|N_t|$, $\pi(N_k)\subset J(M_1)$ for $k\leq t$, and $\alpha=\pi(\alpha)=\pi(\sum_{i\geq 2}\tilde\delta_iy_i+\tilde j_1, \tilde 0, \cdots, \tilde 0)+\pi(\tilde 0, \tilde y_2+\tilde j_2, \tilde 0, \cdots, \tilde 0)+\cdots+\pi(\tilde 0, \cdots, \tilde y_t+\tilde j_t, \tilde 0, \cdots, \tilde 0)+\pi(\tilde 0, \cdots, \tilde y_{t+1}+\tilde j_{t+1}, \tilde 0, \cdots, \tilde 0)+\cdots+\pi(\tilde 0, \cdots, \tilde y_n+\tilde j_n)\in J(M_1)\subset J(D)$, which is a contradiction. Hence $M_1\approx N_t$. Now, let φ: $eR\to M_1$ be a homomorphism given by setting $\varphi(er)=\alpha er$. Then, since $y_t'(\ker\varphi)\subset A_t$ and $|M_1|=|N_t|$, we have $\ker\varphi=y_t'^{-1}A_t$, where $y_t'=y_t+j_te$. Hence $(\sum_{i=2}^t\delta_iy_i+j_1e)\cdot y_t'^{-1}A_t\subset A_1$ and $(y_i+j_ie)y_t'^{-1}A_t\subset A_i$ ($2\leq i\leq t-1$). Conversely, assume the above property. Let M be a maximal submodule of D, and put $\bar D=D/J(D)\supset\bar M=M/J(D)$. If $\bar M$ contains some $\overline{eR/A_i}$ then $M\supset eR/A_i$. Hence we may assume that $\bar{M}=\sum \bar{\beta}_i R$, where $\beta_i=(\tilde{\delta}_i, \tilde{0}, \cdots, \overset{i}{\check{\tilde{e}}}, \tilde{0}, \cdots, \tilde{0})\in M$. By assumption, there exists $\{y_i\}_{i=2}^{t-1}$ such that $\sum_{i=2}^{t-1} \bar{\delta}_i y_i+\bar{\delta}_t \in \Delta(A_t, A_1)$ and $\bar{y}_i\in\Delta(A_t, A_i)$ $(i\geq 2)$. We define a homomorphism $\theta\colon N_t\to\sum_{j=i}^{t-1}\oplus N_j$ by setting $\theta(x)=((\sum_{i=2}^{t-1}\tilde{\delta}_i y_i+\tilde{\delta}_t+\tilde{j})x, \tilde{y}_2 x, \cdots, \tilde{y}_{t-1}x)$, where $j\in eJe$ and $(\sum \delta_i y_i+\delta_t+j)A_t\subset A_1$. Then $\sum_{i=1}^{t}\oplus N_i=\sum_{i=1}^{t-1}\oplus N_i\oplus N_t(\theta)$ and $N_t(\theta)=(\theta+1_{N_t})N_t=(\theta+1_{N_t})\tilde{e}R=(\sum \beta_i y_i+\tilde{j}e)R\subset M$.

REMARK. If we put all $A_i=A$ in Theorem 2, then we obtain [2], Theorem 2. Next, in [2], Theorem 3, we can take a set of linearly independent elements $\{\bar{\delta}_{i1}, \cdots, \bar{\delta}_{is_i}\}$ in Δ over $\Delta(N_i)$. Apply Theorem 2 for the set $\{\bar{\delta}_{ij}\}_{i=1}^{t}$. Then we obtain [2], Theorem 3, because $\Delta(N_i, N_j)\neq 0$ implies $N_i\approx N_j$.

The next is a dual to [3], Corollary to Theorem 4.

Corollary 1. *Let N_1 and N_2 be hollow modules. Assume that* $[\Delta\colon \Delta(N_2)]=k<\infty$. *Then $N_1\oplus N_2^{(k)}$ satisfies* (**) *if and only if $N_1>N_2$ or $N_1<N_2$.*

Proof. Apply Theorem 2 to a basis $\{\bar{e}, \bar{\delta}_2, \cdots, \bar{\delta}_k\}$ of Δ over $\Delta(N_2)$.

For two hollow modules N_1 and N_2, we put $N_1\sim N_2$ when $N_1>N_2$ or $N_1<N_2$. Given a family $\{eR/A_i\}_{i=1}^{n}$ of hollow modules, we set

$$D=\sum_{i=1}^{n}\oplus eR/A_i=\sum_{j=1}^{n_1}\oplus eR/A_{1j}\oplus\sum_{j=2}^{n_2}\oplus eR/A_{2j}\oplus\cdots\oplus\sum_{j=1}^{n_m}\oplus eR/A_{mj}\,,$$

where ($eR/A_{ik}\sim eR/A_{ij}$ forsome k and j, and) $eR/A_{ik}\not\sim eR/A_{i'j}$ for all k and j provided $i\neq i'$.

Corollary 2. *Let D be as above. Then D satisfies* (**) *if and only if so does some $\sum_j\oplus eR/A_{ij}$.*

Proof. If some $D_i=\sum_{j=1}^{n_i}\oplus eR/A_{ij}$ satisfies (**), then so does D by [2], Lemma 1. Next, we shall show that D does not satisfy (**) if none of D_i does. We may assume that $|A_{i1}|\geqslant A_{i2}\geqslant\cdots\geqslant|A_{in_i}|$. Then there exists $\{\bar{\delta}_{i2}, \bar{\delta}_{i3}, \cdots, \bar{\delta}_{in_i}\}\subset\Delta$ for which (#) never holds if $n_i\geq 2$. If D satisfies (**) then there exist B_t and $\bar{y}_h\in\Delta(B_t, B_1)$ such that

$$(\delta)\qquad \sum_{h=2}^{t-1}\bar{\varepsilon}_h\bar{y}_h+\bar{\varepsilon}_t\in\Delta(B_t, B_1)\,,$$

where B_p is equal to some A_{ij}, $|B_p|\geq|B_{p+1}|$ for all p, ε_p is equal to some δ_{ij}, and $\delta_{i1}=e$ for all i. First, assume that $B_t=A_{ik}$ and $B_1=A_{i1}$. Since $\Delta(A_{ij}, A_{i'j'})=0$ for $i\neq i'$, (δ) becomes

$$\bar{\delta}_{i2}\bar{y}_{i_2}+\cdots+\bar{\delta}_{ik-1}\bar{y}_{i_{k-1}}+\bar{\delta}_{ik}\in\Delta(A_{ik}, A_{i1})$$

and $\bar{y}_{i_p}\in\Delta(A_{ik}, A_{ip})$, which is a contradiction. Next, assume that $B_t=A_{ik}$ and $B_1=A_{i'1}$ for $i\neq i'$. Then (δ) becomes

$$\bar{e}\bar{y}_{i_1}+\bar{\delta}_{i2}\bar{y}_{i_2}+\cdots+\bar{\delta}_{ik-1}\bar{y}_{i_{k-1}}+\bar{\delta}_{ik}=0$$

and $\bar{y}_{i_p}\in\Delta(A_{ik}, A_{ip})$. But, $\bar{e}\bar{y}_{i_1}$ being in $\Delta(A_{ip}, A_{i1})$, we have a contradiction. Therefore D does not satisfy (**).

Corollary 3. *Let $\{N_i=eR/A_i\}_{i=1}^{m+1}$ be a family of hollow modules ($m\geq 1$). Assume that $[\Delta\colon\Delta(A_i)]=n$ for all i and $A_i\supset A_j$ for $i<j$. If $n\leq 3$ then $\sum_{i=1}^{n+1}\oplus N_i$ satisfies* (**).

Proof. If $n=1$, this is clear by [2], Theorem 1. Assume $n=2$. If $\Delta(A_3, A_1)\supsetneqq\Delta(A_3)$ then (#) holds trivially. So, we assume that $\Delta(A_3, A_1)=\Delta(A_3)$. Since $\Delta(A_3)=\Delta(A_3, A_1)\supset\Delta(A_2, A_1)\supset\Delta(A_2)$, we get $\Delta(A_3)=\Delta(A_2)=\Delta(A_2, A_1)$. In view of $[\Delta\colon\Delta(A_3)]=2$, for any $\bar{\delta}_2, \bar{\delta}_3\in\Delta$ we can find $\bar{z}_2, \bar{z}_3\in\Delta(A_3)$ such that $\bar{\delta}_2\bar{z}_2+\bar{\delta}_3\bar{z}_3\in\Delta(A_3)=\Delta(A_3, A_1)$ and $\{\bar{z}_2, \bar{z}_3\}\neq 0$. This shows that $\{\bar{z}_2, \bar{z}_3\}$ satisfies (#). Finally, assume that $n=3$. Let $\bar{\delta}_2$, $\bar{\delta}_3$ and $\bar{\delta}_4$ be elements in Δ. First assume that $\Delta(A_3)\subsetneqq\Delta(A_3, A_1)$. Then $[\Delta/\Delta(A_3, A_1)\colon\Delta(A_3)]\leq 1$. If $\bar{\delta}_3$ is in $\Delta(A_3, A_1)$ then (#) holds trivially. So, assume that $\bar{\delta}_3\notin\Delta(A_3, A_1)$. Then there exist $\bar{y}_3, \bar{y}_4\in\Delta(A_3)$ such that $\bar{\delta}_3\bar{y}_3+\bar{\delta}_4\bar{y}_4\in\Delta(A_3, A_1)$ and $\{\bar{y}_3, \bar{y}_4\}\neq 0$. Since $\bar{y}_4\neq 0$ by $\bar{\delta}_3\notin\Delta(A_3, A_1)$, $\bar{\delta}_3\bar{y}_3\bar{y}_4^{-1}+\bar{\delta}_4\in\Delta(A_3, A_1)\subset\Delta(A_4, A_1)$, and $\bar{y}_3\bar{y}_4^{-1}\in\Delta(A_3)\subset\Delta(A_4, A_3)$. Hence (#) holds. Next, assume that $\Delta(A_3)=\Delta(A_3, A_1)$. Then $\Delta(A_3)=\Delta(A_2, A_1)=\Delta(A_2)$, as in the case $n=2$. There exist $\bar{y}_2, \bar{y}_3, \bar{y}_4\in\Delta(A_3)$ such that $\bar{\delta}_2\bar{y}_2+\bar{\delta}_3\bar{y}_3+\bar{\delta}_4\bar{y}_4\in\Delta(A_3)\subset\Delta(A_3, A_1)\subset\Delta(A_4, A_1)$ and $\{\bar{y}_i\}\neq 0$. Now, by making use of a similar argument as above, we can easily see that $\{\bar{y}_i\}$ satisfies (#).

By making use of the above argument and Corollary 1, we can prove the following corollary.

Corollary 4. *Let $\{N_i=eR/A_i\}_{i=1}^{m}$ be a family of hollow modules. Assume that $\Delta(A_i)=\Delta(A_1)$ for all i and $[\Delta\colon\Delta(A_1)]=n$. Then all the direct sums $\sum_{i=1}^{n+1}\oplus T_i$ with T_i isomorphic to some one in $\{N_i\}$ satisfy* (**) *if and only if $\{N_i\}$ is linearly ordered with respect to $<$.*

EXAMPLE. Let k be a field and x an indeterminate. Let $L=k(x)$, and $K=k(x^5)$. Consider the ring

$$R=\begin{pmatrix} L & L \\ 0 & K \end{pmatrix}.$$

Put $A_{4-i}=(0,\ K+Kx+\cdots+Kx^i)\subset e_{11}R$ $(0\leq i\leq 3)$, and $N_i=e_{11}R/A_i$. Then $A_1\supset A_2\supset A_3\supset A_4$ and $\Delta(A_i)=K$. We can show directly the following facts: Both $N_1\oplus N_3\oplus N_4$ and $N_1\oplus N_2\oplus N_4$ satisfy (**). But, no direct sum $N_i\oplus N_j$ $(i\neq j)$ satisfies (**) and neither $N_1\oplus N_2\oplus N_3$ nor $N_2\oplus N_3\oplus N_4$ does. (Note that

$\Delta(A_4, A_1)=K+Kx+Kx^2+Kx^3$ and $\Delta(A_3, A_1)=\Delta(A_4, A_2)=K+Kx+Kx^2$.) $N_1\oplus N_2\oplus N_3\oplus N_4$ satisfies (**), but neither $N_i^{(4)}$ nor $N_i^{(5)}$ does. If $m\geq 6$ then $\sum_{i=1}^{m}\oplus N_i'$ with N_i' isomorphic to some one in $\{N_i\}$ satisfies (**). If we replace $K=k(x^5)$ by $k(x^7)$, none of $N_1'\oplus N_2'\oplus N_3'$ satisfies (**).

References

[1] M. Harada: *On maximal submodules of a finite direct sum of hollow modules* I, Osaka J. Math. **21** (1984), 649–670.

[2] ————: *On maximal submodules of a finite direct sum of hollow modules* II, Osaka J. Math. **21** (1984), 671–677.

[3] ————: *Simple submodules in a finite direct sum of uniform modules*, Methods in Ring Theory, ed. by F. van Oystaeyen, NATO ASI ser. C 129 (1984), 145–159.

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558
Japan