ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES IV

To the memory of Professor Takehiko MIYATA

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In the previous papers [1] and [2], we have studied conditions under which every maximal submodule of a finite direct sum $D$ of certain hollow modules over a right artinian ring with 1 contains a non-zero direct summand of $D$. The present objective is to generalize slightly Theorems 3 and 4 of [2] related to the property mentioned above.

Throughout this paper, $R$ will represent a right artinian ring with identity, and every $R$-module will be assumed to be a unitary right $R$-module with finite composition length. We denote the Jacobson radical and the length of a composition series of an $R$-module $M$ by $J(M)$ and $|M|$, respectively. Occasionally, we write $J=J(R)$. If $M$ has a unique maximal submodule $J(M)$, $M$ is called hollow (local). When this is the case, $M \cong eR/A$ for some primitive idempotent $e$ and a right ideal $A \in eR$.

Let $\{N_i\}_{i=1}^\infty$ be a family of hollow modules, and $D=\bigoplus_{i=1}^\infty N_i$. We are interested in the following condition [1]:

$$** \text{ Every maximal submodule of } D \text{ contains a non-zero direct summand of } D.$$  

As was claimed in [1], [2], whenever we study the condition (**) , we may restrict ourselves to the case where $R$ is basic and $N_i=eR/A_i$ for a fixed primitive idempotent $e$ and a right ideal $A_i$ in $eR$. Now, let $N=eR/A$ be a hollow module. Put $\Delta=eRe/Ne=\text{End}_R(N/J(N))=\text{End}_R(eR/Ne)$, and $\Delta(A) = \Delta(N) = \{x \mid x \in eRe \text{ and } xA \subseteq A\}$ (see [2]). We denote by $N^{(m)}$ the direct sum of $m$ copies of $N$. Then $N^{(m+1)} = N \oplus N^{(m)}$. If $M$ is a maximal submodule of $N^{(m)}$ then $N \oplus M$ is a maximal submodule of $N^{(m+1)}$. Thus we get a mapping $\theta(m)$ of the isomorphism classes of maximal submodules in $N^{(m)}$ into the isomorphism classes of maximal submodules in $N^{(m+1)}$.

Theorem 1 (cf. [3], Corollary 2 to Theorem 3). Let $N=eR/A$ be a hollow module. Then the following conditions are equivalent:

1) $[\Delta : \Delta(A)] = k$.

2) If $m > k$, every maximal submodule $M$ in $D=N^{(m)}$ contains a submodule
isomphic to $N^{(m-k)}$ but not to $N^{(m-k+1)}$. In this case, such a submodule of $M$ is a direct summand of $D$.

3) $\theta(i)$ is not epic for every $i \leq k-1$, but $\theta(j)$ is epic for every $j \geq k$.

Proof (cf. [2], the proof of Theorem 3).

1) $\rightarrow$ 2). Put $D = N^{(m)} = D(k) \oplus D'(n)$, where $m = k+n$, $D(k)$ is the direct sum of the first $k$ copies of $N$ and $D'(n)$ the direct sum of the last $n$ copies of $N$. Let $\{\bar{1}, \bar{2}, \ldots, \bar{k}\}$ be a set of linearly independent elements in $\Delta$ over $\Delta(A)$. Set $\beta_i = (\delta_i, 0, \ldots, 0, \bar{1}, \bar{2}, \ldots, \bar{k}, 0, \ldots, 0)$ in $D(k)$, and $M = \sum_{i=1}^k \beta_i R + D'(n) + J(D)$ in $D$, where $\bar{x}$ means the residue class of $x$ in $eR/A$. Then $M$ is a maximal submodule of $D$. Suppose that $M \not\supset M_i \oplus M_2 \oplus \cdots \oplus M_s \oplus M^*$ and $M_i \approx N$ for all $i$. Then

$$M_i \subset J(D).$$

Actually, if not, $N \approx M_i \subset J(D) = DJ$, which is impossible. Since $M_i \approx eR/A$, $M_i = pR$ and $r(e) = \{r \in eR | pr = 0\} = A$. Now let $\rho = \sum \beta_i y_i + y_j$, where $y_i \in eR$, $y_j \in D'(n)$ and $j \in J(D)$. Then $\rho = (\sum_{i=2}^k \delta_i y_i, 0, \ldots, 0) + (0, 0, \ldots, 0, \bar{1}, \bar{2}, \ldots, \bar{k}) \oplus (\bar{1}, \bar{2}, \ldots, \bar{k})$, where $z_i \in eR$ and $j \in eJ$. By the structure of $D$ and $r(\rho) = A$, $(y_i + j) A \subset A$, and $r(\delta_i y_i + j) A \subset A$. Noting that $eA = A$, we see that $\bar{y}_i \in \Delta(A)$ for $2 \leq i \leq k$, and $\sum_{i=2}^k \delta_i \bar{y}_i \in \Delta(A)$. Therefore, $\bar{y}_i = 0$ for $2 \leq i \leq k$, since $\{\bar{1}, \bar{2}, \ldots, \bar{k}\}$ is linearly independent. Hence

$$\pi(M_i) \subset J(D).$$

where $\pi : D = D(k) \oplus D'(n) \rightarrow D(k)$ is the projection. Let $p_s$ be the projection on the $s$-th component of $D = N^{(k+n)}$. Since $M_i \subset J(D)$ and $\pi(M_i) \subset J(D(k))$, $p_j | M_i$ is an epimorphism for some $j > k$, say $j = k+1$, and hence an isomorphism for $M_j \approx N$. Therefore

$$D = D(k) \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_s \oplus M^* \oplus D'(n-s),$$

where $D'(n) = N \oplus D'(n-1)$. Now assume that $D = D(k) \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_s \oplus D'(n-s)$. Let $\pi_{D'(n-s)}$ be the projection of $D$ onto $D'(n-s)$ in the above decomposition. Suppose $\pi_{D'(n-s)}(M_{s+1}) \subset J(D(n-s))$. Then $\rho = \pi_{D'(n-s)}(M_{s+1}) \subset J(D)$ by (2). On the other hand, $0 = M_{s+1} \cap (M_1 \oplus M_2 \oplus \cdots \oplus M_s) = \ker(\pi_{D(k) \oplus D'(n-s)} | M_{s+1})$, so $M_{s+1}$ is monomorphic to a submodule in $J(D)$, which is impossible. Hence $\pi_{D'(n-s)}(M_{s+1}) \subset J(D(n-s))$, and so $D = D(k) \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_{s+1} \oplus D'(n-s-1)$ by the above argument. Accordingly, $q \leq n$, and hence $M$ does not contain a submodule of $D$ isomorphic to $N^{(s+1)}$. Let $M'$ be an arbitrary maximal submodule of $D$. Then, by induction on $m$ and [2], Theorem 2, $M' = N^{(m-k)} \oplus M^*$, where $N^* \approx N$.

2) $\rightarrow$ 1). Take $m = k+1$. By (a) and the argument employed in proving (γ), we see that $D$ contains a direct summand which is isomorphic to $N$. Hence
FINITE DIRECT SUM OF HOLLOW MODULES

IV 323

[Δ: Δ(A)] = k by [2], Theorem 2.

1) ↔ 3). In case θ(t) is epic, every maximal submodule M of \( N(t+1) \) contains a direct summand \( M_i \) which is isomorphic to \( N \). Then, by 2), \( M_i \) is also a direct summand of \( N(t+1) \). Hence \( θ(t) \) is epic if and only if \( N(t+1) \) satisfies (**), and the equivalence of 1) and 3) is clear by [2], Theorem 3 (see Remark below).

In Theorem 1, we have studied a direct sum of isomorphic copies of a fixed hollow module. Next, let \( N_1 = eR/A_1 \) and \( N_2 = eR/A_2 \). If there exists an epimorphism \( φ \) of \( N \) to \( N_2 \) then we write \( N \supset N_2 \). Since \( φ \) is given by the left-sided multiplication of a unit element \( x \) in \( eR \), we have \( xA_1 \subset A_2 \), and furthermore \( N_1 \approx eR/xA_1 \). Hence, when we study the direct sum \( N_1 \oplus N_2 \) with \( N_1 \supset N_2 \), we may assume that \( A_1 \subset A_2 \).

**Theorem 2.** Let \( \{ N_i = eR/A_i \}_{i=1}^n \) be a family of hollow modules \( (n \geq 2) \). Assume that \( |A_1| \geq |A_2| \geq \cdots \geq |A_n| \). Then \( D = \sum_{i=1}^n N_i \) satisfies (**) if and only if, for any sequence \( \{ δ_2, \ldots, δ_n \} \) of \( n-1 \) elements in \( Δ \), there exist an integer \( t (2 \leq t \leq n) \) and \( \gamma_i \in Δ(A_i, A_i) \) \( (2 \leq i \leq t-1) \) such that

\[
\sum_{i=2}^{t-1} δ_i \gamma_i + \gamma_t \in Δ(A_t, A_t),
\]

where \( Δ(A_i, A_i) = \{ x \in eR \text{ and } xA_i \subset A_i \} \).

**Proof.** We may assume that \( R \) is basic. Take the maximal submodule \( M \) in \( D \) generated by \( β_i = (δ_i, 0, \ldots, 0, 0, \ldots, 0) \) \( (i=2, 3, \ldots, n) \). Then \( M \) contains a direct summand \( M_i \) of \( D \), i.e., \( D = M_1 \oplus D_1 \) and \( M_i \approx N_i \) for some \( \rho \); \( M_i \) is generated by \( \alpha = \sum \beta_i y_i + j, \) where \( y_i \in eR \) \( (y_i \in eRe \text{ for some } q) \) and \( j \in J(D) \).

Now, \( α = \sum \delta_i y_i + j, 3_j + j_2, \ldots, 3_n + j_n \). Assume that \( 3_j = 3_j = \cdots = 3_{t-1} = 0 \) and \( 3_t \neq 0 \). Let \( π \) be the projection of \( D = \sum \oplus N_i \) onto \( N_t \). Then \( π| M_1 \) is an epimorphism, so \( M_1 \supset N_t \). On the other hand, let \( π \) be the projection of \( D = M_1 \oplus D_1 \) onto \( M_1 \). We shall show that \( π|M_1 \) is an isomorphism. Suppose, to the contrary, that \( |M_1| > |N_t| \). Then, since \( |N_k| < |N_1| \), \( π(N_k) \subset J(M_1) \) for \( k \leq t \), and \( α = π(α) = π(\sum δ_j y_j + j, 0, \ldots, 0) + π(0, 3_j + j_2, 0, \ldots, 0) + \cdots + π(0, 0, \ldots, 3_n + j_n) \) \( \subset J(M_1) \subset J(D) \), which is a contradiction. Hence \( M_1 \approx N_t \). Now, let \( ϕ: eR \to M_1 \) be a homomorphism given by setting \( ϕ(ex) = exr \). Then, since \( y_i(\ker ϕ) \subset A_i \) and \( |M_1| = |N_t| \), we have \( \ker ϕ = y^{t-1}_i A_i \), where \( y_i = y_i + j_i e \). Hence \( (\sum_{i=2}^{t-1} δ_i y_i + j_i e) \cdot y^{t-1}_i A_i \subset A_i \) and \( (y_i + j_i e) y^{t-1}_i A_i \subset A_i \) \( (2 \leq i \leq t-1) \). Conversely, assume the above property. Let \( M \) be a maximal submodule of \( D \), and put \( D = D/J(D) \supset M = M/J(D) \). If \( M \) contains some \( eR/A_i \) then \( M \supset eR/A_i \). Hence we may
assume that $\bar{M}=\sum \beta_i R$, where $\beta_i=({\delta_i, 0, \cdots, \delta_0, \cdots, 0}) \in M$. By assumption, there exists $\{y_i\}_{i=1}^t$ such that $\sum_{i=1}^t \delta_i y_i + \delta_t$ belongs to $\Delta(A_i, A_t)$ and $\bar{y}_i \in \Delta(A_t, A_i)$ ($i \geq 2$).

We define a homomorphism $\theta: N_i \to \sum_{j=1}^t \oplus N_j$ by setting $\theta(x) = (\sum_{i=1}^t \delta_i y_i + \delta_t + j)x$, $\bar{y}_i x, \cdots, \bar{y}_t x$, where $j \in e_j e$ and $(\sum \delta_i y_i + \delta_t + j)A_i \subset A_i$. Then $\sum_{i=1}^t \oplus N_i = \sum_{i=1}^{t-1} \oplus N_i \oplus N_i(\theta)$ and $N_i(\theta) = (\theta + 1_{N_i})N_i = (\theta + 1_{N_i})\theta R = (\sum \beta_i y_i + \delta e)R \subset M$.

**Remark.** If we put all $A_i = A$ in Theorem 2, then we obtain [2], Theorem 2. Next, in [2], Theorem 3, we can take a set of linearly independent elements $\{\delta_{i_1}, \cdots, \delta_{i_n}\}$ in $\Delta$ over $\Delta(N_i)$. Apply Theorem 2 for the set $\{\delta_{i_j}\}_{j=1}^t$. Then we obtain [2], Theorem 3, because $\Delta(N_i, N_j) \neq 0$ implies $N_i \approx N_j$.

The next is a dual to [3], Corollary to Theorem 4.

**Corollary 1.** Let $N_1$ and $N_2$ be hollow modules. Assume that $[\Delta: \Delta(N_2)] = k < \infty$. Then $N_1 \oplus N_2$ satisfies (***) if and only if $N_1 > N_2$ or $N_1 < N_2$.

**Proof.** Apply Theorem 2 to a basis $\{e, \delta_2, \cdots, \delta_k\}$ of $\Delta$ over $\Delta(N_2)$. For two hollow modules $N_1$ and $N_2$, we put $N_1 \approx N_2$ when $N_1 > N_2$ or $N_1 < N_2$. Given a family $\{eR/A_i\}_{i=1}^n$ of hollow modules, we set

$$D = \sum_{i=1}^n \oplus eR/A_i = \sum_{j=1}^{n_1} \oplus eR/A_{i_1} \oplus \sum_{j=2}^{n_2} \oplus eR/A_{i_2} \oplus \cdots \sum_{j=1}^{n_n} \oplus eR/A_{i_m},$$

where $(eR/A_{ik} \approx eR/A_{ij}$ for some $k$ and $j$, and) $eR/A_{ik} \approx eR/A_{ij}$ for all $k$ and $j$ provided $i \neq j$.

**Corollary 2.** Let $D$ be as above. Then $D$ satisfies (***) if and only if so does some $\sum_{i} \oplus eR/A_{ij}$.

**Proof.** If some $D_i = \sum_{i=1}^n \oplus eR/A_{ij}$ satisfies (***) then so does $D$ by [2], Lemma 1. Next, we shall show that $D$ does not satisfy (***) if none of $D_i$ does. We may assume that $|A_{i_1}| \geq |A_{i_2}| \geq \cdots \geq |A_{i_{n_1}}|$. Then there exists $\{\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_{n_1}}\}$, $\subset \Delta$ for which (**) never holds if $n_i \geq 2$. If $D$ satisfies (***) then there exist $B_i$ and $\bar{y}_i \in \Delta(B_i, B_i)$ such that

$$\sum_{k=1}^{t-1} \delta_k \bar{y}_k + \bar{e}_i \in \Delta(B_i, B_i),$$

where $B_p$ is equal to some $A_{ij}$, $|B_p| \geq |B_{p+1}|$ for all $p$, $e_p$ is equal to some $\delta_{ij}$, and $\delta_i = e$ for all $i$. First, assume that $B_i = A_{ik}$ and $B_i = A_{ij}$. Since $\Delta(A_{ik}, A_{ij}) = 0$ for $i \neq i'$, (**) becomes

$$\delta_{i_2} \bar{y}_{i_2} + \cdots + \delta_{i_{k-1}} \bar{y}_{i_{k-1}} + \delta_{i_k} \in \Delta(A_{ik}, A_{ij}),$$

and $\bar{y}_p \in \Delta(A_{ik}, A_{ij})$, which is a contradiction. Next, assume that $B_i = A_{ik}$ and $B_i = A_{ij}$ for $i \neq i'$. Then (**) becomes
and \( \bar{y}_{i_1} + \bar{y}_{i_2} + \cdots + \bar{y}_{i_k} + g_{i_{k-1}} = 0 \)

\( \bar{y}_{i_1} \in \Delta(A_{i_k}, A_{i_1}) \). But, \( \bar{y}_{i_1} \) being in \( \Delta(A_{i_j}, A_{i_1}) \), we have a contradiction. Therefore \( D \) does not satisfy (**).

**Corollary 3.** Let \( \{ N_i = e_{i}R/A_i \} \) be a family of hollow modules \((m \geq 1) \). Assume that \([ \Delta : \Delta(A_{i_j}) ] = n \) for all \( i \) and \( A_i \supset A_j \) for \( i < j \). If \( n \leq 3 \) then \( \sum_{i=1}^{n+1} N_i \) satisfies (**).

Proof. If \( n = 1 \), this is clear by [2], Theorem 1. Assume \( n = 2 \). If \( \Delta(A_3, A_1) = \Delta(A_3, A_2) \), then (#) holds trivially. So, we assume that \( \Delta(A_3, A_1) \supset \Delta(A_3, A_2) \). Since \( \Delta(A_3) = \Delta(A_3, A_1) \supset \Delta(A_3, A_2) \), we get \( \Delta(A_3) = \Delta(A_3, A_2) \). In view of \([ \Delta : \Delta(A_2) ] = 2 \), for any \( \bar{y}_3, \bar{y}_4 \in \Delta(A_2) \) we can find \( \bar{z}_2, \bar{z}_3 \in \Delta(A_3) \) such that \( \bar{y}_3 = \bar{z}_2 + \bar{y}_4 = \bar{z}_3 \) and \( \{ \bar{z}_2, \bar{z}_3 \} \equiv 0 \). This shows that \( \{ \bar{z}_2, \bar{z}_3 \} \) satisfies (#). Finally, assume that \( n = 3 \). Let \( \bar{y}_1, \bar{y}_2, \bar{y}_3 \) be elements in \( \Delta \). First assume that \( \Delta(A_3) = \Delta(A_3, A_1) \). Then \([ \Delta(A_3, A_1) : \Delta(A_2) ] \leq 2 \). If \( \bar{y}_1 \) is in \( \Delta(A_3, A_1) \) then (#) holds trivially. So, we assume that \( \bar{y}_1 \notin \Delta(A_3, A_1) \). Then there exist \( \bar{z}_2, \bar{z}_3 \in \Delta(A_2) \) such that \( \bar{y}_2 = \bar{z}_2 + \bar{y}_3 = \bar{z}_3 \) and \( \{ \bar{z}_2, \bar{z}_3 \} \equiv 0 \). Since \( \bar{y}_1 = 0 \) by \( \bar{y}_1 \notin \Delta(A_3, A_1) \), \( \bar{y}_2 = \bar{y}_3 = \bar{z}_2 + \bar{z}_3 \in \Delta(A_3, A_1) \subset \Delta(A_3, A_1) \), and \( \bar{y}_2 = \bar{z}_2 + \bar{z}_3 \) is in \( \Delta(A_3, A_1) \). Hence (#) holds. Next, assume that \( \Delta(A_3) = \Delta(A_2, A_1) \). Then \( \Delta(A_3) = \Delta(A_3, A_1) \equiv \Delta(A_2) \), as in the case \( n = 2 \). There exist \( \bar{z}_2, \bar{z}_3, \bar{z}_4 \in \Delta(A_2) \) such that \( \bar{y}_2 = \bar{z}_2 + \bar{z}_3 = \bar{z}_4 \) and \( \{ \bar{z}_2, \bar{z}_3 \} \equiv 0 \). Now, by making use of a similar argument as above, we can easily see that \( \{ \bar{y}_1 \} \) satisfies (#).

By making use of the above argument and Corollary 1, we can prove the following corollary.

**Corollary 4.** Let \( \{ N_i = e_{i}R/A_i \} \) be a family of hollow modules. Assume that \( \Delta(A_i) = \Delta(A_1) \) for all \( i \) and \([ \Delta : \Delta(A_1) ] = n \). Then all the direct sums \( \sum_{i=1}^{n+1} T_i \) with \( T_i \) isomorphic to some one in \( \{ N_i \} \) satisfy (**). But, no direct sum \( \sum_{i=1}^{n+1} N_i \) satisfies (**).
\[ \Delta(A_3, A_1) = K + Kx + Kx^2 + Kx^3 \text{ and } \Delta(A_3, A_1) = \Delta(A_4, A_2) = K + Kx + Kx^2. \]

\( N_1 \oplus N_2 \oplus N_3 \oplus N_4 \) satisfies (**), but neither \( N_1^{(i)} \) nor \( N_1^{(j)} \) does. If \( m \geq 6 \) then \( \sum_{i=1}^m N_i' \) with \( N_i' \) isomorphic to some one in \( \{N_i\} \) satisfies (**). If we replace \( K = k(x^5) \) by \( k(x^7) \), none of \( N_1' \oplus N_2' \oplus N_3' \) satisfies (**).

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**References**


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