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IGUSA LOCAL ZETA FUNCTIONS AND INTEGRATION FORMULAS ASSOCIATED TO EQUIVARIANT MAPS

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Abstract

We give integration formulas in order to calculate Igusa local zeta functions of prehomogeneous vector spaces. By the integration formula, we determine explicit forms of Igusa local zeta functions of four prehomogeneous vector spaces.

1. Introduction

The purpose of this paper is to give integration formulas associated to equivariant maps in order to calculate Igusa local zeta functions of prehomogeneous vector spaces. Let G be a connected linear algebraic group over a p-adic field K, G_0 an open compact subgroup of $G(K)$, and V, W finite dimensional K-vector spaces. We assume that G acts on V, W rationally, and there exists a G-equivariant polynomial map $\psi: V \to W$. For a point $v \in V$ and a $\mathbb{C}$-valued continuous integrable function F on the orbit $G(K) \cdot \psi(v)$, we give an integration formula which expresses a p-adic integral of $F \circ \psi$ on the orbit $G(K) \cdot v$ by a sum of integrals of F on orbits $G_0 \cdot w_i$ in terms of $i = 1, 2, \ldots$ ($w_i \in W$). The aim of this integration formula is to reduce the calculation of the integral of $F \circ \psi$ to those of F. In particular, we give an explicit form of this integration formula for a $Sp(n)$-invariant map. This explicit form is expressed by Hall polynomials and partitions. By this integration formula, we determine explicit forms of Igusa local zeta functions of four prehomogeneous vector spaces.

For the study of zeta functions of prehomogeneous vector spaces, we have to give explicit forms of p-adic local zeta functions (see, e.g. [2], [8], [22] and [28]). Some p-adic local orbital zeta functions of regular irreducible prehomogeneous vector spaces were given explicitly in [2], [8], [21] and [23]. If the domain of integration is the whole space over integer ring, the p-adic local zeta function is called the Igusa local zeta function. J. Igusa gave explicit forms of the Igusa local zeta functions of twenty four types among twenty nine types of regular irreducible prehomogeneous vector spaces (cf. [9]–[14]). However in the unknown cases of regular irreducible prehomogeneous vector spaces, by their established methods, it is difficult to calculate explicitly p-adic local orbital zeta functions or Igusa local zeta functions. So we try to simplify these calculations by using the integration formula associated to equivariant maps. Actually, if a equivariant map ψ is a $SL(n)$-invariant map, then this formula is

the same form as that of [11, Lemma 8]. J. Igusa applied the formula of [11, Lemma 8] to some complicated calculations of Igusa local zeta functions (cf. [11]–[14]). As a first step, we treat the formula for a $Sp(n)$-invariant map, which is given explicitly in Theorem 4.3. In [27], we used the formula of Theorem 4.3 to determine explicit forms of Igusa local zeta functions of two regular 2-simple prehomogeneous vector spaces $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)})$ which have universally transitive open orbits.

In order to give an explicit form of the integration formula for a map ψ, we have to calculate integrals on the fibers $(\psi|G(K)\cdot v)^{-1}(G_0\cdot w_i)$ and the orbits $G_0\cdot w_i$. In case of the $Sp(n)$-invariant map, we give explicit forms of these integrals by using some results of spherical functions of alternating forms of [5]. As a byproduct of this calculation, we get explicit forms of local densities of alternating forms in a certain special case (Proposition 4.4), because these integrals relate to local densities of alternating forms.

By the integration formula associated to the $Sp(n)$-invariant map, we determine explicit forms of the Igusa local zeta functions of the following prehomogeneous vector spaces:

(a) $(GL(1) \times Sp(n) \times SO(3), \Lambda_1 \otimes \Lambda_1)$ $(n \geq 2)$,

(b) $(GL(1)^3 \times Sp(n), \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ $(n \geq 2)$,

(c) $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$ $(n > m)$,

(d) $(GL(1)^2 \times Sp(n) \times SL(2), \Lambda_1 \otimes (2\Lambda_1) + 1 \otimes \Lambda_1)$ $(n \geq 2)$.

As for the Igusa local zeta function of the space (b), (c) and (d), these explicit forms were unknown. For a p-adic field K, K-forms of the space (a) were classified into two cases in [20]. As for one case, the Igusa local zeta function was already calculated in [7]. In this paper, for these two cases, we calculate uniformly their Igusa local zeta functions. Furthermore our calculation is easier than that of [7]. The space (a) is a non-regular irreducible reduced prehomogeneous vector space (irreducible prehomogeneous vector spaces were classified in [25]), the space (b) is a non-regular simple prehomogeneous vector space (simple prehomogeneous vector spaces were classified in [15]), and the spaces (c), (d) are non-regular 2-simple prehomogeneous vector spaces of type I (2-simple prehomogeneous vector spaces of type I are classified in [16]). By the formula, we reduce calculations of the Igusa local zeta function of the space (a) to that of the quotient space $(GL(1) \times SO(3), \Lambda_1 \otimes \Lambda_1)$, the space (b) to that of the quotient space $(GL(1)^3, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$, the space (c) to that of the quotient space $(GL(1)^4 \times SL(2m+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$, and the space (d) to that of the quotient space $(GL(1)^2 \times SL(2), (2\Lambda_1) \oplus \Lambda_1^*)$ respectively. The Igusa local zeta functions of the quotient spaces $(GL(1) \times SO(3), \Lambda_1 \otimes \Lambda_1)$ and $(GL(1)^4 \times SL(2m+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ were calculated explicitly in [14] and [26] respectively. We can easily calculate the Igusa local zeta functions of the quotient spaces $(GL(1)^3, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ and $(GL(1)^2 \times SL(2), (2\Lambda_1) \oplus \Lambda_1^*)$ by established methods. Therefore we achieve simplifications of these calculations by our formula. Furthermore by our formula, we give an answer to

Remark A of [7]: why the Igusa local zeta function of $(GL(1)\times SO(3), \Lambda_1\otimes\Lambda_1)$ divide that of the space (a).

The plan of this paper is as follows. In Section 2, we define Igusa local zeta functions of prehomogeneous vector spaces, and review some known properties of Hall-Littlewood polynomials and Hall polynomials. In Section 3, we give an integration formula associated to equivariant maps. As an example, we give two explicit forms of this formula for a $SL(n)$-invariant map. In Section 4, we give our main result on an explicit form of the formula for a $Sp(n)$-invariant map. In Section 5 and 6, we prove two lemmas for the proof of our main result. In Section 7, we determine explicit forms of the Igusa local zeta functions of prehomogeneous vector spaces (a), (b), (c) and (d) by using our main result.

NOTATION. Let K be a p-adic field i.e. a finite extension of $\mathbb{Q}_p$, and $\mathcal{O}_K$ the ring of integers in K. We fix a prime element π in $\mathcal{O}_K$, and then $\pi\mathcal{O}_K$ is the ideal of nonunits of $\mathcal{O}_K$. The cardinality of the residue field $\mathcal{O}_K/\pi\mathcal{O}_K$ is denoted by q. We denote by $|\ \ |_K$ the absolute value of K normalized as $|\pi|_K = q^{-1}$. For a commutative ring R, we denote by $M(m,n;R)$ the totality of $m\times n$ matrices over R, and by $\mathrm{Alt}(n;R)$ the totality of $n\times n$ alternating matrices over R $(m,n\in\mathbb{Z}_{>0})$. If $m=n$, we write $M(n;R)$ instead of $M(m,n;R)$. We denote by $\det(x)$ the determinant of $x\in M(n;R)$. For any $x\in M(m,n;R)$, ${}^t x$ is the transpose of x. We denote by $\mathrm{Pf}(y)$ the Pfaffian of $y\in\mathrm{Alt}(2n;R)$. For any positive integer n, $\mathfrak{S}_n$ is the symmetric group in n latters. The cardinality of a set E is denoted by $\sharp(E)$.

2. Preliminaries

2.1. Igusa local zeta functions of prehomogeneous vector spaces. We shall define Igusa local zeta functions of prehomogeneous vector spaces. For details, we refer to [9] and [24].

We denote by $\overline{K}$ the algebraic closure of a p-adic field K. Let G be a connected linear algebraic group defined over K, V a finite dimensional $\overline{K}$-vector space with K-structure, and $\rho: G\to GL(V)$ a rational representation of G on V defined over K. Let the triple (G,ρ,V) be a prehomogeneous vector space i.e. there exists a proper algebraic subset S of V such that $V(\overline{K})\setminus S(\overline{K})$ is a single $G(\overline{K})$-orbit. A point $v\in V\setminus S$ is called a generic point. The set S is called the singular set of (G,ρ,V) and also defined over K. For a K-rational character χ of G, a non-zero K-rational function f on V is called a relative invariant of (G,ρ,V) corresponding to χ if $f(\rho(g)v)=\chi(g)f(v)$ for all $g\in G$ and $v\in V$. Let $S_1,\dots,S_l$ be the K-irreducible hypersurface contained S in V. Take a K-irreducible polynomial function $f_i\in K[V]$ defining S_i for each $i=1,\dots,l$. Then $f_1,\dots,f_l$ are relative invariants of (G,ρ,V) and any relative invariant in $K(V)$ can be written uniquely as $cf_1^{\nu_1}\cdots f_l^{\nu_l}$ with $c\in K^\times$, $\nu_1,\dots,\nu_l\in\mathbb{Z}$. These $f_1,\dots,f_l$ are called the basic relative invariants of (G,ρ,V). Let dv be the Haar measure on $V(K)$ normalized by $\int_{V(\mathcal{O}_K)} dv = 1$,

and $\mathcal{S}(V(K))$ the Schwartz-Bruhat space of $V(K)$. For the basic relative invariants $f_1, \dots, f_l$, and $\Phi \in \mathcal{S}(V(K))$, we put

$$Z(s; \Phi) = \int_{V(K)} \prod_{i=1}^{l} |f_i(v)|_K^{s_i} \Phi(v)\, dv \qquad (s = (s_1, \dots, s_l) \in \mathbb{C}^l,\ \mathrm{Re}(s_i) > 0).$$

It is known that this local zeta function $Z(s; \Phi)$ is a rational function of $q^{-s_1}, \dots, q^{-s_l}$ (see, e.g. [1], [3] and [14]). Let Φ_0 be the characteristic function of $V(\mathcal{O}_K)$. We put $Z(s) = Z(s; \Phi_0)$. This local zeta function $Z(s)$ is called the Igusa local zeta function of (G, ρ, V).

2.2. Hall-Littlewood polynomials and Hall polynomials. We shall review some known properties of Hall-Littlewood polynomials and Hall polynomials. For details, we refer to [18]

For a positive integer m, we put

$$\Lambda_m^+ = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m;\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0\},$$
$$|\lambda| = \sum_{i=1}^{m} \lambda_i, \quad n(\lambda) = \sum_{i=1}^{m} (i-1)\lambda_i.$$

For $\lambda, \mu \in \Lambda_m^+$, we write $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for all $i \geq 1$. For a non-negative integer i and $\lambda \in \Lambda_m^+$, the number $m_i(\lambda)$ of λ_j's which are equal to i is called the multiplicity of i in λ. For a non-negative integer m, we put

$$w_m(t) = \prod_{j=1}^{m} (1 - t^j),$$

($w_0(t) = 1$). For $\lambda \in \Lambda_m^+$, we put

$$w_\lambda^{(m)}(t) = \prod_{i=0}^{+\infty} w_{m_i(\lambda)}(t).$$

The Hall-Littlewood polynomial $P_\lambda(x; t)$ is defined by

$$P_\lambda(x; t) = P_\lambda(x_1, x_2, \dots, x_m; t)$$
$$= \frac{(1-t)^m}{w_\lambda^{(m)}(t)} \cdot \sum_{\sigma \in S_m} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(m)}^{\lambda_m} \prod_{1 \leq i < j \leq m} \frac{x_{\sigma(i)} - t x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}}$$

for each $\lambda \in \Lambda_m^+$. For $\lambda \in \Lambda_m^+$, $P_\lambda(x; t)$ is a polynomial in $x_1, \dots, x_m$ and t, and the set $\{P_\lambda(x; t);\ \lambda \in \Lambda_m^+\}$ forms a $\mathbb{Z}[t]$-basis of the ring $\mathbb{Z}[t][x_1, \dots, x_m]^{\mathfrak{S}_m}$ of symmetric polynomials in $x_1, \dots, x_m$ with coefficients in $\mathbb{Z}[t]$. We denote by $f_{\mu\nu}^\lambda(t)$ the structure

constants of the ring $\mathbb{Z}[t][x_1, \ldots, x_m]^{\mathfrak{S}_m}$ with respect to the basis $\{P_\lambda(x; t); \lambda \in \Lambda_m^+\}$:

$$P_\mu(x; t) \cdot P_\nu(x; t) = \sum_\lambda f_{\mu\nu}^\lambda(t) \cdot P_\lambda(x; t) \qquad (f_{\mu\nu}^\lambda(t) \in \mathbb{Z}[t]).$$

Unless $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \subset \lambda$, we have $f_{\mu\nu}^\lambda(t) = 0$. If we put

$$g_{\mu\nu}^\lambda(t) = t^{2n(\lambda)-2n(\mu)-2n(\nu)} \cdot f_{\mu\nu}^\lambda(t^{-1}) \in \mathbb{Z}[t],$$

then this polynomial $g_{\mu\nu}^\lambda(t)$ is called the Hall polynomial corresponding to λ, μ, ν (cf. [18, Chapter III, Section 3]). We use the following property.

Lemma 2.1 ([18, Chapter 3, Section 4, Example 1]).

$$\sum_{\lambda \in \Lambda_m^+} t^{n(\lambda)} P_\lambda(x_1, x_2, \ldots, x_m; t) = \prod_{i=1}^m (1 - x_i)^{-1}.$$

3. Integration formula

In this section, we give an integration formula associated to equivariant maps (Proposition 3.1). As an example, we give two explicit forms of the formula for a $SL(n)$-invariant map.

Let G be a connected linear algebraic groups defined over K. Let V, W be finite dimensional vector spaces with K-structure, $\rho: G \to GL(V)$ a rational representation of G on V defined over K, and $\rho': G \to GL(W)$ a rational representation of G on W defined over K. We assume that there exists a G-equivariant K-polynomial map $\psi: V \to W$ such that

$$\psi(\rho(g)v) = \rho'(g)\psi(v) \quad \text{for all } g \in G, \ v \in V.$$

We fix a point $v_0 \in V(K)$. Let H be the fixer of v_0 in $G(K)$, and H' the fixer of $\psi(v_0)$ in $G(K)$. We see that $H \subset H'$ and H, H' are closed subgroups of $G(K)$. Since $G(K)$ is countable at ∞, we have $\rho(G(K))v_0 \cong G(K)/H$, $\rho'(G(K))\psi(v_0) \cong G(K)/H'$ by a theorem of L.S. Pontrjagin. Let dv be a measure on $G(K)/H$ satisfying $d(\rho(g)v) = \chi(g)\,dv$ for every $g \in G(K)$, where χ is a element of $\mathrm{Hom}(G(K), \mathbb{C}^\times)$. Let dw be a measure on $G(K)/H'$ satisfying $d(\rho'(g)w) = \chi'(g)\,dw$ for every $g \in G(K)$, where χ' is a element of $\mathrm{Hom}(G(K), \mathbb{C}^\times)$. We assume that $dv \neq 0$ and $dw \neq 0$. Since χ and χ' are locally constant homomorphisms, there exists a open compact subgroup G_0 of $G(K)$ satisfying $\chi(g) = \chi'(g) = 1$ for every $g \in G_0$. Furthermore since $G(K)$ is countable at ∞, there exists a sequence $\{g_i'\}_{i=1}^\infty$ in $G(K)$ such that $G(K) = \bigcup_{i=1}^\infty G_0 g_i'$. Hence we can take a subsequence $\{g_i\}_{i=1}^\infty$ $(\subset \{g_i'\}_{i=1}^\infty)$ such that

$$\rho'(G(K))\psi(v_0) = \bigcup_{i=1}^\infty \rho'(G_0 g_i)\psi(v_0) \qquad \text{(disjoint union)}.$$

The set $\rho'(G_0 g_i)\psi(v_0)$ is open compact in $\rho'(G(K))\psi(v_0)$ for every $i = 1, 2, \ldots$. If we put $U_i = (\psi_0)^{-1}(\rho'(G_0 g_i)\psi(v_0))$ for $\psi_0 = \psi|\rho(G(K))v_0$ and every i, then we have

$$\rho(G(K))v_0 = \bigcup_{i=1}^{\infty} U_i \qquad \text{(disjoint union)}.$$

Proposition 3.1. *Let F be any $\mathbb{C}$-valued continuous function on $\rho'(G(K))\psi(v_0)$, and Φ a $\mathbb{C}$-valued function on $\rho(G(K))v_0$ satisfying $\Phi(\rho(g)v) = \Phi(v)$ for every $g \in G_0$, $v \in V(K)$. We assume that $F(\psi(v))\Phi(v)$ and $\Phi(v)$ are integrable on $\rho(G(K))v_0$ for dv. Then we have*

$$\int_{\rho(G(K))v_0} F(\psi(v))\Phi(v)\,dv = \sum_{i=1}^{\infty} \frac{\int_{U_i} \Phi(v)\,dv}{\int_{\rho'(G_0 g_i)\psi(v_0)} dw} \cdot \int_{\rho'(G_0 g_i)\psi(v_0)} F(w)\,dw.$$

Proof. Let dg be the Haar measure on $G(K)$ normalized by $\int_{G_0} dg = 1$. Then we have

$$\begin{aligned}
\int_{U_i} F(\psi(v))\Phi(v)\,dv &= \int_{G_0}\int_{U_i} F(\psi(\rho(g)v))\Phi(\rho(g)v)\,dv\,dg \\
&= \int_{U_i} \Phi(v)\left\{\int_{G_0} F(\rho'(g)\psi(v))\,dg\right\} dv \\
&= \int_{U_i} \Phi(v)\,dv \cdot \int_{G_0} F(\rho'(gg_i)\psi(v_0))\,dg.
\end{aligned}$$

We get the following equation similarly.

$$\int_{\rho'(G_0 g_i)\psi(v_0)} F(w)\,dw = \int_{\rho'(G_0 g_i)\psi(v_0)} dw \cdot \int_{G_0} F(\rho'(gg_i)\psi(v_0))\,dg.$$

By [9, Lemma 1], we have $\int_{\rho'(G_0 g_i)\psi(v_0)} dw \neq 0$. Hence we get the above formula from

$$\int_{\rho(G(K))v_0} F(\psi(v))\Phi(v)\,dv = \sum_{i=1}^{\infty} \int_{U_i} F(\psi(v))\Phi(v)\,dv. \qquad \square$$

As an example, we shall give two explicit forms of the above integration formula for a $SL(m)$-invariant map. We take positive integers m, n such that $m \leq n$. The group $G = GL(n) \times SL(m)$ acts on $V = M(n, m)$ by $(g, h)\cdot v = gx^t h$ for $(g, h) \in G$ and $v \in V$. We denote by I the set of all $i = (i_1, \ldots, i_m) \in \mathbb{Z}^m$ where $1 \leq i_1 < \cdots < i_m \leq n$, and by $\eta_i(v)$ the the determinant of the $m \times m$ submatrix of v obtained by crossing out its k-th rows for $k \neq i_1, \ldots, i_m$. Put $W = M(\sharp(I), 1)$. We define the $SL(m)$-invariant polynomial map $\eta\colon V \to W$ as $\eta(v) = {}^t(\eta_i(v))_{i\in I}$. Put $v_0 = {}^t(1_m, 0)$ and $G_0 = GL(n; \mathcal{O}_K) \times SL(m; \mathcal{O}_K)$. Let dv be the Haar measure on $V(K)$ normalized by

$\int_{V(\mathcal{O}_K)} dv = 1$. By [11] or [14], we see that there exist a measure dw on $\eta(G(K)\cdot v_0)$ such that $d(g\cdot w) = |\det(g)|_K^m\, dw$ for every $g \in GL(n, K)$. We have

$$(G(K)\cdot v_0)\bigcap M(n,m;\mathcal{O}_K) = \bigcup_{\lambda\in\Lambda_m^+} G_0\cdot(v_0 a_\lambda) \qquad \text{(disjoint union)}$$

where $a_\lambda = \mathrm{diag}(\pi^{\lambda_1},\dots,\pi^{\lambda_m}) \in GL(m;K)$ for $\lambda \in \Lambda_m^+$, and

$$\eta\left((G(K)\cdot v_0)\bigcap M(n,m;\mathcal{O}_K)\right) = \bigcup_{i\in\mathbb{N}} \pi^i \eta(G_0\cdot v_0) \qquad \text{(disjoint union).}$$

If we put $U_i = (\eta|V(\mathcal{O}_K))^{-1}\left(\pi^i\eta(G_0\cdot v_0)\right)$, then we have

$$U_i = \bigcup_{|\lambda|=i} G_0\cdot(v_0 a_\lambda)\,.$$

By [14], we have

$$\int_{U_i} dv = \frac{w_n(q^{-1})}{w_{n-m}(q^{-1})} \sum_{e_1+\cdots+e_m=i} \prod_{j=1}^{m} q^{-(n-j+1)e_j},$$

where $(e_1,\dots,e_m)\in\mathbb{N}^m$. We normalize the measure dw by

$$\int_{\eta(G_0\cdot v_0)} dw = \frac{w_n(q^{-1})}{w_{n-m}(q^{-1})}.$$

Then we have

$$\int_{\pi^i\eta(G_0\cdot v_0)} dw = q^{-ni}\frac{w_n(q^{-1})}{w_{n-m}(q^{-1})}$$

(cf. [14, p.192]). Therefore by Proposition 3.1 we have the following formula.

Proposition 3.2 (J. Igusa). *Let F denote any $\mathbb{C}$-valued continuous function on $\psi(M(n,m;\mathcal{O}_K))$. Then we have*

$$\int_{M(n,m;\mathcal{O}_K)} F(\eta(v))\,dv = \sum_{e_1,\dots,e_m=0}^{\infty}\left(\prod_{j=1}^{m} q^{-(n-j+1)e_j}\right)\int_{\eta(G_0\cdot v_0)} F(\pi^{|e|}w)\,dw$$

where $|e| = e_1+\cdots+e_m$.

We shall remark on the measure dw. Put $V' = \{v\in V;\ \eta_{12\cdots m}(v)\neq 0\}$, $W' = \psi(V')$, and write every $v\in V'$ as

$$v = \begin{pmatrix} 1_m \\ z \end{pmatrix} y, \qquad t = \det(y).$$

In [11, Lemma 8], the measure dw was expressed by

$$dw|W' = \frac{(1-q^{-1})}{w_m(q^{-1})} \cdot t^{n-1}\, dt \wedge dz,$$

since t and z form coordinates on W'. In [14, Proposition 10.6.1], the measure dw was expressed by the image measure $dw = (\eta|G_0 \cdot v_0) \cdot dv$ on $\eta(G_0 \cdot v_0)$.

We shall give another form of this formula. By the method of Section 5, we have

$$\int_{G_0 \cdot (v_0 a_\lambda)} dv = \frac{w_n(q^{-1}) w_m(q^{-1})}{w_{n-m}(q^{-1}) w_\lambda^{(m)}(q^{-1})} \cdot q^{-(n-m+1)|\lambda|-2n(\lambda)}.$$

Therefore we have the following formula.

Proposition 3.3. *Let F denote any $\mathbb{C}$-valued continuous function on $\psi(M(n,m;\mathcal{O}_K))$. Then we have*

$$\int_{M(n,m;\mathcal{O}_K)} F(\eta(v))\, dv = \sum_{\lambda \in \Lambda_m^+} \frac{w_m(q^{-1})}{w_\lambda^{(m)}(q^{-1})} q^{-(n-m+1)|\lambda|-2n(\lambda)} \int_{\eta(G_0 \cdot v_0)} F(\pi^{|\lambda|} w)\, dw.$$

4. Main result

In this section, we give an explicit form of the formula of Proposition 3.1 for a $Sp(n)$-invariant map. In Section 7, we apply this formula to calculations of some Igusa local zeta functions. We also remark local densities of alternating forms.

We take positive integers l, n such that $l \leq 2n$, and a positive integer m such that $l = 2m$ or $2m+1$. Put

$$J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{Alt}(2n),$$

and define

$$Sp(n) = \{g \in GL(2n);\ g J_n {}^t g = J_n\}.$$

The group $Sp(n) \times GL(l)$ acts on $M(2n,l)$ by $(g,h) \cdot x = g x {}^t h$ for $(g,h) \in Sp(n) \times GL(l)$ and $x \in M(2n,l)$. The group $GL(l)$ act on $\mathrm{Alt}(l)$ by $h \cdot y = h y {}^t h$ for $h \in GL(l)$ and $y \in \mathrm{Alt}(l)$. We define the $Sp(n)$-invariant map $\varphi\colon M(2n,l) \to \mathrm{Alt}(l)$ as $\varphi(x) = {}^t x J_n x$ for $x \in M(2n,l)$. The map φ satisfies $\varphi((g,h) \cdot x) = h \cdot \varphi(x)$ for $(g,h) \in Sp(n) \times GL(l)$ and $x \in M(2n,l)$. It is well-known that an algebra homomorphism $\varphi^*\colon \mathbb{C}[\mathrm{Alt}(l)] \to \mathbb{C}[M(2n,l)]^{Sp(n;\mathbb{C})}$ is surjective (see, e.g. [4, Theorem 4.2.2]). This $Sp(n)$-invariant map φ defines $\mathrm{Alt}(l)$ as the $GL(l)$-equivariant quotient of $M(2n,l)$ by $Sp(n)$. Hence we see that the prehomogeneous vector space $(GL(l), \Lambda_2, \mathrm{Alt}(l))$ is the quotient of the prehomogeneous vector space $(Sp(n) \times GL(l), \Lambda_1 \otimes \Lambda_1, M(2n,1))$ (cf. [19]).

Put

$$Y_{2m} = \{y \in \mathrm{Alt}(2m;\, \mathcal{O}_K);\ \mathrm{Pf}(y) \neq 0\},$$
$$Y_{2m+1} = \{y \in \mathrm{Alt}(2m+1;\, \mathcal{O}_K);\ \mathrm{rank}(y) = 2m\}.$$

For $\lambda \in \Lambda_m^+$, we put

$$(\pi^\lambda)_{2m} = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_m} \\ -\pi^{\lambda_m} & 0 \end{pmatrix} \in Y_{2m},$$
$$(\pi^\lambda)_{2m+1} = \begin{pmatrix} (\pi^\lambda)_{2m} & 0 \\ 0 & 0 \end{pmatrix} \in Y_{2m+1}.$$

Then the set Y_l is expressed by the intersection of $\mathrm{Alt}(l;\, \mathcal{O}_K)$ and the $GL(l;\, K)$-orbit of the point $(\pi^{(0)})_l$. By the theory of elementary divisors, the $GL(l;\, \mathcal{O}_K)$-orbit decompositions of Y_l are given by

$$Y_l = \bigcup_{\lambda \in \Lambda_m^+} GL(l;\, \mathcal{O}_K) \cdot (\pi^\lambda)_l \qquad \text{(disjoint union)}.$$

Let dx be the Haar measure on $M(2n, l;\, K)$ normalized by $\int_{M(2n,l;\,\mathcal{O}_K)} dx = 1$, and dy the Haar measure on $\mathrm{Alt}(l;\, K)$ normalized by $\int_{\mathrm{Alt}(l;\,\mathcal{O}_K)} dy = 1$. Put

$$U_{\lambda,l} = \{x \in M(2n, l;\, \mathcal{O}_K);\ \varphi(x) \in GL(l;\, \mathcal{O}_K) \cdot (\pi^\lambda)_l\}.$$

We prove the following in Section 5 and 6.

Lemma 4.1. *For $\lambda \in \Lambda_m^+$, we have*

$$\int_{GL(2m;\,\mathcal{O}_K)\cdot(\pi^\lambda)_{2m}} dy = q^{-4n(\lambda)-|\lambda|} \cdot w_{2m}(q^{-1}) \cdot (w_\lambda^{(m)}(q^{-2}))^{-1},$$
$$\int_{GL(2m+1;\,\mathcal{O}_K)\cdot(\pi^\lambda)_{2m+1}} dy = q^{-4n(\lambda)-3|\lambda|} \cdot (1-q^{-1})^{-1} \cdot w_{2m+1}(q^{-1}) \cdot (w_\lambda^{(m)}(q^{-2}))^{-1}.$$

Lemma 4.2. *For $\lambda \in \Lambda_m^+$, we have*

$$\int_{U_{\lambda,2m}} dx = w_{2m}(q^{-1}) w_n(q^{-2}) (w_{n-m}(q^{-2}))^{-1} \times \sum_{\mu,\nu \in \Lambda_m^+} q^{-4n(\lambda)-|\lambda|-(2n-2m+1)|\mu|} \cdot (w_\lambda(q^{-2}))^{-1} \cdot g_{\mu\nu}^\lambda(q^{-2}),$$
$$\int_{U_{\lambda,2m+1}} dx = w_{2m+1}(q^{-1}) w_n(q^{-2}) ((1-q^{-1}) \cdot w_{n-m}(q^{-2}))^{-1} \times \sum_{\mu,\nu \in \Lambda_m^+} q^{-4n(\lambda)-|\lambda|-(2n-2m+1)|\mu|-2|\nu|} \cdot (w_\lambda(q^{-2}))^{-1} \cdot g_{\mu\nu}^\lambda(q^{-2}).$$

If we put $G_0 = Sp(n; \mathcal{O}_K) \times GL(l; \mathcal{O}_K)$, then we have the following formula by Proposition 3.1 and the above lemmas.

Theorem 4.3. *For any $\mathbb{C}$-valued continuous function F on $\mathrm{Alt}(l; \mathcal{O}_K)$, we have*

$$\int_{M(2n,l;\mathcal{O}_K)} F(\varphi(x))\,dx = \sum_{\lambda \in \Lambda_m^+} A_{\lambda,l} \cdot \int_{GL(l;\mathcal{O}_K)\cdot(\pi^\lambda)_l} F(y)\,dy,$$

where

$$A_{\lambda,l} = \frac{w_n(q^{-2})}{w_{n-m}(q^{-2})} \sum_{\mu,\nu \in \Lambda_m^+} g_{\mu\nu}^{\lambda}(q^2) \times \begin{cases} q^{-(2n-2m+1)|\mu|} & (l = 2m) \\ q^{-(2n-2m-1)|\mu|} & (l = 2m+1) \end{cases}.$$

We shall remark on local densities of alternating forms. For $A \in Y_{2n}$ and $B \in Y_l$, we denote by $N_i(B, A)$ the number of solutions T in $M(2n, l; \mathcal{O}_K/\pi^i\mathcal{O}_K)$ of the congruence ${}^tTAT \equiv B \bmod \pi^i$. Then the density $\mu(B, A)$ of integral representation of B by A are defined by

$$\mu(B, A) = \lim_{i \to \infty} q^{-2^{-1}il(4n-l+1)} \cdot N_i(B, A).$$

Since $\mu(B, A)$ depend only on the $GL(2n; \mathcal{O}_K)$-orbit containing A and the $GL(l; \mathcal{O}_K)$-orbit containing B, we may consider only $\mu((\pi^\lambda)_l, (\pi^\xi)_{2n})$ for $\lambda \in \Lambda_m^+$ and $\xi \in \Lambda_n^+$. In case of $l = 2m$, the local density $\mu((\pi^\lambda)_{2m}, (\pi^\xi)_{2n})$ was given explicitly in [5] and [6]. Here we treat local densities $\mu((\pi^\lambda)_l, J_n)$. From [5, Lemma 3.2] we have

$$\int_{U_{\lambda,2m}} dx = q^{-4n(\lambda)-|\lambda|} \cdot w_{2m}(q^{-1}) \cdot (w_\lambda^{(m)}(q^{-2}))^{-1} \cdot \mu((\pi^\lambda)_{2m}, J_n).$$

By an argument similar to the proof of [5, Lemma 3.2], we have

$$\int_{U_{\lambda,2m+1}} dx = q^{-4n(\lambda)-3|\lambda|} \cdot (1 - q^{-1})^{-1} \cdot w_{2m+1}(q^{-1}) \cdot (w_\lambda^{(m)}(q^{-2}))^{-1} \cdot \mu((\pi^\lambda)_{2m+1}, J_n).$$

Hence by Lemma 4.2 we observe that $\mu((\pi^\lambda)_l, J_n)$ is equal to the coefficient of $\int_{GL(l;\mathcal{O}_K)\cdot(\pi^\lambda)_l} f(y)\,dy$ in the formula of Theorem 4.3.

Proposition 4.4. *For $\lambda \in \Lambda_m^+$, we have*

$$\mu((\pi^\lambda)_l, J_n) = A_{\lambda,l}.$$

5. Proof of Lemma 4.1

In this section, we shall prove Lemma 4.1. We can prove Lemma 4.1 by [5, Corollary of Lemma 2.7], but we give an alternative proof of Lemma 4.1 by using

only $GL(l;\mathbb{F}_q)$-orbits on $\mathrm{Alt}(l;\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. We also applied this method to the case of Proposition 3.3.

For $\lambda \in \Lambda_m^+$, we express the partition λ as pairs of the sets $\{m_i\}_{1\le i\le r}$ and $\{k_i\}_{1\le i\le r}$ by

$$\lambda = (\overbrace{k_r,\dots,k_r}^{m_r}, k_{r-1},\dots,k_3,\overbrace{k_2,\dots,k_2}^{m_2},\overbrace{k_1,\dots,k_1}^{m_1}),$$

where $m_1+\cdots+m_r = m$ and $0 \le k_1 \le \cdots \le k_r$ for some r. We put

$$\pi(\{m_i\},\{k_i\})_{2m} = \begin{pmatrix} \pi^{k_1}J_{m_1} & & & \\ & \pi^{k_2}J_{m_2} & & \\ & & \cdots & \\ & & & \pi^{k_r}J_{m_r} \end{pmatrix} \in Y_{2m}(\mathcal{O}_K),$$

$$\pi(\{m_i\},\{k_i\})_{2m+1} = \begin{pmatrix} \pi(\{m_i\},\{k_i\})_{2m} & \\ & 0 \end{pmatrix} \in Y_{2m+1}(\mathcal{O}_K).$$

Then we identify $(\pi^\lambda)_l$ as $\pi(\{m_i\},\{k_i\})_l$. For $l \ge 2k$, we put

$$J_{l,k} = \begin{pmatrix} J_k & \\ & 0 \end{pmatrix} \in \mathrm{Alt}(l;K).$$

We see the following lemma easily.

Lemma 5.1. *The $GL(l;\mathbb{F}_q)$-orbit decompositions of $\mathrm{Alt}(l;\mathbb{F}_q)$ are given by*

$$\mathrm{Alt}(l;\mathbb{F}_q) = \bigcup_{0\le 2k\le l} GL(l;\mathbb{F}_q)\cdot J_{l,k} \qquad (\textit{disjoint union}),$$

and we have

$$\sharp(GL(2m;\mathbb{F}_q)\cdot J_{2m,k}) = q^{4mk-2k^2-k}\frac{w_{2m}(q^{-1})}{w_k(q^{-2})\cdot w_{2m-2k}(q^{-1})},$$

$$\sharp(GL(2m+1;\mathbb{F}_q)\cdot J_{2m+1,k}) = q^{4mk-2k^2+k}\frac{w_{2m+1}(q^{-1})}{w_k(q^{-2})\cdot w_{2m-2k+1}(q^{-1})}.$$

For convenience, we put

$$\sharp(J_{l,k}) = \sharp(GL(l;\mathbb{F}_q)\cdot J_{l,k}).$$

By this lemma we have

$$\int_{GL(l;\mathcal{O}_K)\cdot(J_{l,k}+\pi\,\mathrm{Alt}(l;\mathcal{O}_K))} dy_l = \sharp(J_{l,k})\int_{J_{l,k}+\pi\,\mathrm{Alt}(l;\mathcal{O}_K)} dy_l = q^{-l(l-1)/2}\cdot\sharp(J_{l,k}),$$

where dy_l is the Haar measure on $\mathrm{Alt}(l;K)$ normalized by $\int_{\mathrm{Alt}(l;\mathcal{O}_K)} dy_l = 1$. This calculation is same as in the proof of Igusa's key lemma (cf. [14, Theorem 10.3.1]). Hence we see

$$\begin{aligned}\int_{GL(l;\mathcal{O}_K)\cdot\pi(\{m_i\},\{k_i\})_l} dy_l &= q^{-(l(l-1)/2)k_1} \int_{GL(l;\mathcal{O}_K)\cdot\pi(\{m_i\},\{k_i-k_1\})_l} dy_l \\ &= q^{-(l(l-1)/2)(k_1+1)} \sharp(J_{l,m_1}) \\ &\quad \times \int_{GL(l-2m_1;\mathcal{O}_K)\cdot\pi(\{m_i\}_{i\geq 2},\{k_i-k_1-1\}_{i\geq 2})_{l-2m_1}} dy_{l-2m_1}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_{GL(l;\mathcal{O}_K)\cdot\pi(\{m_i\},\{k_i\})_l} dy_l &= q^{-(l(l-1)/2)(k_1+1)} \cdot \sharp(J_{l,m_1}) \\ &\quad \times q^{-((l-2m_1-1)(l-2m_1)/2)(k_2-k_1)} \cdot \sharp(J_{l-2m_1,m_2}) \\ &\quad \times q^{-((l-2m_1-2m_2-1)(l-2m_1-2m_2)/2)(k_3-k_2)} \cdot \sharp(J_{l-2m_1-2m_2,m_3}) \\ &\quad \times \cdots \\ &\quad \times q^{-\left(\left(l-1-2\sum_{i=1}^{r-1} m_i\right)\left(l-2\sum_{i=1}^{r-1} m_i\right)\Big/2\right)(k_r-k_{r-1})} \cdot \sharp(J_{l-m_1-\cdots-m_{r-1},m_r}).\end{aligned}$$

Hence we have

$$\int_{GL(2m;\mathcal{O}_K)\cdot\pi(\{m_i\},\{k_i\})_{2m}} dy_{2m} = q^{P_{2m}(\{m_i\},\{k_i\})} \cdot w_{2m}(q^{-1}) \cdot \prod_{i=1}^{r} (w_{m_i}(q^{-2}))^{-1},$$

$$\int_{GL(2m+1;\mathcal{O}_K)\cdot\pi(\{m_i\},\{k_i\})_{2m+1}} dy_{2m+1} = q^{P_{2m+1}(\{m_i\},\{k_i\})} \cdot \frac{w_{2m+1}(q^{-1})}{1-q^{-1}} \cdot \prod_{i=1}^{r} (w_{m_i}(q^{-2}))^{-1},$$

where we put

$$P_{2m}(\{m_i\},\{k_i\}) = -2\sum_{i=1}^{r} k_i m_i^2 - 4\sum_{i<j} k_i m_i m_j + \sum_{i=1}^{r} k_i m_i,$$

$$P_{2m+1}(\{m_i\},\{k_i\}) = -2\sum_{i=1}^{r} k_i m_i^2 - 4\sum_{i<j} k_i m_i m_j - \sum_{i=1}^{r} k_i m_i.$$

By $2n(\lambda) + |\lambda| = \sum_{i=1}^{r} k_i m_i^2 + 2\sum_{i<j} k_i m_i m_j$, we have

$$\int_{GL(2m;\mathcal{O}_K)\cdot(\pi^{\lambda})_{2m}} dy = q^{-4n(\lambda)-|\lambda|} \cdot w_{2m}(q^{-1}) \cdot (w_{\lambda}^{(m)}(q^{-2}))^{-1},$$

$$\int_{GL(2m+1;\mathcal{O}_K)\cdot(\pi^{\lambda})_{2m+1}} dx = q^{-4n(\lambda)-3|\lambda|} \cdot (1-q^{-1})^{-1} \cdot w_{2m+1}(q^{-1}) \cdot (w_{\lambda}^{(m)}(q^{-2}))^{-1}.$$

Hence we obtain Lemma 4.1.

6. Proof of Lemma 4.2

6.1. Some *p*-adic integrals. In this subsection, we review some results of [5] to prove Lemma 4.2.

For an $y \in \mathrm{Alt}(l; K)$, we denote by $\mathrm{Pf}_i(y)$ $(1 \le 2i \le l)$ the Pfaffian of the upper left $2i$ by $2i$ block of y. For $i = m$ and $l = 2m$, $\mathrm{Pf}_m(y)$ is the Pfaffian of y. We choose the sign of the Pfaffian so that $\mathrm{Pf}(J_m) = 1$. For $y \in Y_l$, we put

$$H_{l,y} = \{h \in GL(l; \mathcal{O}_K);\ \mathrm{Pf}_i(h \cdot y) \neq 0\ (1 \le 2i \le l)\}.$$

For $l = 2m$ or $l = 2m + 1$, $s \in \mathbb{C}^m$, we put

$$\zeta_l(y; s) = \zeta(y; s_1, \ldots, s_m) = \int_{H_{l,y}} \prod_{i=1}^{m} |\mathrm{Pf}_i(h \cdot y)|_K^{s_i}\, dh$$

where dh is the Haar measure on $GL(l; K)$ normalized by $\int_{GL(l;\mathcal{O}_K)} dh = 1$. When $\mathrm{Re}(s_1), \ldots, \mathrm{Re}(s_{m-1}) \ge 0$, the integrals $\zeta_l(y; s)$ is absolutely convergent and has an analytic continuation to a rational function in $q^{-s_1}, \ldots, q^{-s_m}$ by the theory of complex powers of polynomial functions. Set

$$\Psi_z(y) = \frac{\zeta_{2m}(y; s)}{\zeta_{2m}(J_m; s)} \qquad (y \in Y_{2m}),$$

where z is a variables in $\mathbb{C}^m$ which is related with the variable s by

$$\begin{cases} s_i = z_{i+1} - z_i - 2 & (1 \le i \le m-1) \\ s_m = (m+1) - z_m - 2 \end{cases}.$$

The function $\Psi_z(y)$ is called the spherical function on Y_{2m} (cf. [5, Section 2]).

Lemma 6.1 ([5, Theorem 3]). *For any $\lambda \in \Lambda_m^+$, we have*

$$\Psi_z((\pi^\lambda)_{2m}) = q^{2n(\lambda)-(m-1)|\lambda|} \cdot \frac{w_\lambda^{(m)}(q^{-2})}{w_m(q^{-2})} \cdot P_\lambda(q^{z_1}, \ldots, q^{z_m}; q^{-2}).$$

Lemma 6.2 ([5, Theorem 6]).

$$\zeta_{2m}(J_m; s_1, \ldots, s_m) = \prod_{k=1}^{m-1} \frac{1 - q^{-1}}{1 - q^{-2k-1}} \cdot \prod_{1 \le i < j \le m} \frac{1 - q^{z_i - z_j - 1}}{1 - q^{z_i - z_j + 1}}.$$

By Lemma 6.2, we have the following lemma.

Lemma 6.3. *For $m \leq n$, we have*

$$\zeta_{2n}(J_n; s_1, \dots, s_m, \overbrace{0, \dots, 0}^{n-m}) = \frac{w_n(q^{-2}) \cdot w_{2n-2m}(q^{-1}) \cdot w_{2m}(q^{-1})}{w_m(q^{-2}) \cdot w_{n-m}(q^{-2}) \cdot w_{2n}(q^{-1})} \times \zeta_{2m}(J_m; s) \cdot \prod_{i=1}^{m} \frac{1 - q^{-(s_i + \dots + s_m + 2n - 2i + 1)}}{1 - q^{-(s_i + \dots + s_m + 2m - 2i + 1)}}.$$

We denote by X_l^0 the subset of $M(2n, l; \mathcal{O}_K)$ consisting of the elements which can be extended to a unimodular matrix by complementing $2n - l$ column vectors. Put

$$X_l' = \{x \in M(2n, l; \mathcal{O}_K);\ \mathrm{Pf}_i(\varphi(x)) \neq 0\ (1 \leq 2i \leq l)\},$$

and $X_l^{0\prime} = X_l' \cap X_l^0$. Then for $l = 2m$ or $l = 2m + 1$ we set

$$\Phi_l(s) = \Phi_l(s_1, \dots, s_m) = \int_{X_l'} \prod_{i=1}^{m} |\mathrm{Pf}_i(\varphi(x))|_K^{s_i}\, dx,$$

$$\Phi_l^0(s) = \Phi_l^0(s_1, \dots, s_m) = \int_{X_l^{0\prime}} \prod_{i=1}^{m} |\mathrm{Pf}_i(\varphi(x))|_K^{s_i}\, dx.$$

These $\Phi_l(s)$ and $\Phi_l^0(s)$ are absolutely convergent for $\mathrm{Re}(s_1), \dots, \mathrm{Re}(s_{m-1}) \geq 0$, and have analytic continuation to rational functions in $q^{-s_1}, \dots, q^{-s_m}$. We can easily see that $\Phi_{2m}(s) = \Phi_{2m+1}(s)$. We have the following properties of $\Phi_{2m}(s)$ and $\Phi_{2m}^0(s)$.

Lemma 6.4 ([5, Proof of Theorem 5]).

$$\zeta_{2n}(J_n; s_1, \dots, s_m, \overbrace{0, \dots, 0}^{n-m}) = \frac{w_{2n-2m}(q^{-1})}{w_{2n}(q^{-1})} \cdot \Phi_{2m}^0\ (s_1, \dots, s_m).$$

Lemma 6.5 ([5, Lemma 3.1 (i)]).

$$\Phi_{2m}^0(s) = \prod_{i=1}^{m} (1 - q^{-(s_i + \dots + s_m + 2n - 2i + 1)})(1 - q^{-(s_i + \dots + s_m + 2n - 2i + 2)}) \times \Phi_{2m}(s).$$

By Lemma 6.3, 6.4 and 6.5, we have the following lemma.

Lemma 6.6.

$$\Phi_{2m}(s_1, \dots, s_m) = \prod_{i=1}^{m} (1 - q^{-2i+1})(1 - q^{-2n+2m-2i}) \times \zeta_{2m}(J_m; s_1, \dots, s_m) \times \prod_{i=1}^{m} (1 - q^{-(s_i + \dots + s_m + 2m - 2i + 1)})^{-1} (1 - q^{-(s_i + \dots + s_m + 2n - 2i + 2)})^{-1}.$$

6.2. Proof of Lemma 4.2. In this subsection, we prove Lemma 4.2 by using some results of [5]. First we give an explicit form of the integral $\int_{U_{\lambda,2m}} dx$.

Proposition 6.7. *For $\lambda \in \Lambda_m^+$, we have*

$$\int_{U_{\lambda,2m}} dx = w_{2m}(q^{-1})w_n(q^{-2})(w_{n-m}(q^{-2}))^{-1} \times \sum_{\mu,\nu\in\Lambda_m^+} q^{-2n(\lambda)-2n(\mu)-2n(\nu)-|\lambda|-(2n-2m+1)|\mu|} \cdot (w_\lambda(q^{-2}))^{-1} \cdot f_{\mu\nu}^{\lambda}(q^{-2}).$$

Proof. We shall prove this proposition by imitating the proof of [5, Theorem 8]. By Lemma 2.1 and Lemma 6.6, we have

$$\Phi_{2m}(s) = \zeta_{2m}(J_m\,;\,s)\cdot w_{2m}(q^{-1})w_n(q^{-2})\left(w_m(q^{-2})w_{n-m}(q^{-2})\right)^{-1} \times \sum_{\lambda,\mu,\nu\in\Lambda_m^+} q^{-2n(\mu)-2n(\nu)-(2n-m+1)|\mu|-m|\nu|} \cdot f_{\mu\nu}^{\lambda}(q^{-2}) \cdot P_\lambda(q^{z_1},\dots,q^{z_m}\,;\,q^{-2}).$$

By Lemma 6.1, we have

$$\begin{aligned}
\Phi_{2m}(s) &= \int_{GL(2m;\,\mathcal{O}_K)} \int_{X'_{2m}} \prod_{i=1}^{m} |\mathrm{Pf}_i(h\cdot\varphi(x))|_K^{s_i}\, dx\, dh \\
&= \int_{X'_{2m}} \zeta_{2m}(\varphi(x)\,;\,s)\, dx \\
&= \sum_{\lambda\in\Lambda_m^+} \int_{U_{\lambda,2m}} dx \cdot \zeta_{2m}((\pi^\lambda)_{2m}\,;\,s) \\
&= \zeta_{2m}(J_m\,;\,s) \sum_{\lambda\in\Lambda_m^+} \int_{U_{\lambda,2m}} dx \\
&\quad \times q^{2n(\lambda)-(m-1)|\lambda|} \cdot \frac{w_\lambda^{(m)}(q^{-2})}{w_m(q^{-2})} \cdot P_\lambda(q^{z_1},\dots,q^{z_m}\,;\,q^{-2}).
\end{aligned}$$

Therefore we obtain the formula for $\int_{U_{\lambda,2m}} dx$ by comparing the terms involving $P_\lambda(q^{z_1},\dots,q^{z_m}\,;\,q^{-2})$. □

Next we give an explicit form of the integral $\int_{U_{\lambda,2m+1}} dx$. In order to calculate the integral $\int_{U_{\lambda,2m+1}} dx$, we need the following lemma.

Lemma 6.8. *For $\lambda \in \Lambda_m^+$, we have*

$$\zeta_{2m+1}((\pi^\lambda)_{2m+1}\,;\,s) = \frac{1-q^{-1}}{1-q^{-2m-1}} \cdot \prod_{i=1}^{m} \frac{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}}{1-q^{-(s_i+\cdots+s_m+2m-2i+1)}} \cdot \zeta_{2m}((\pi^\lambda)_{2m}\,;\,s).$$

Proof. For $l = 2m$ or $l = 2m+1$, $\lambda \in \Lambda_m^+$, we put

$$L_{\lambda,l} = \{T \in M(l;\mathcal{O}_K) \cap GL(l;K);\ \mathrm{Pf}_i({}^tT(\pi^\lambda)_l T) \neq 0\ (1 \leq 2i \leq l)\},$$
$$L^0_{\lambda,l} = \{T \in GL(l;\mathcal{O}_K);\ \mathrm{Pf}_i({}^tT(\pi^\lambda)_l T) \neq 0\ (1 \leq 2i \leq l)\}.$$

Let dT be the Haar measure on $M(2m+1;K)$ normalized by $\int_{M(2m+1;\mathcal{O}_K)} dT = 1$, and dT' the Haar measure on $M(2m;K)$ normalized by $\int_{M(2m;\mathcal{O}_K)} dT' = 1$. By [5, Proof of Lemma 3.1], we decompose $L_{\lambda,l}$ as follows:

$$L_{\lambda,l} = \bigcup_{a_1,\ldots,a_l} \bigcup_{b_{ij}} L^0_{\lambda,l} \begin{pmatrix} \pi^{a_1} & & b_{ij} \\ & \ddots & \\ 0 & & \pi^{a_l} \end{pmatrix} \qquad \text{(disjoint union).}$$

Here $a_1 \ldots, a_l$ run through all negative integers and b_{ij} $(1 \leq i < j \leq l)$ is taken from a complete system of representatives of $\mathcal{O}_K/\pi^{a_j}\mathcal{O}_K$. Hence we have

$$\begin{aligned}
&\int_{L_{\lambda,2m+1}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT(\pi^\lambda)_{2m+1}T)|_K^{s_i}\, dT = \int_{L^0_{\lambda,2m+1}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT(\pi^\lambda)_{2m+1}T)|_K^{s_i}\, dT \\
&\times \sum_{a_1,\ldots,a_{2m+1}=1}^{+\infty} q^{-(2m+1)(a_1+\cdots+a_{2m+1})} \cdot \prod_{i=1}^m q^{-(a_1+\cdots+a_{2i})s_i} \cdot q^{\sum_{i=1}^{2m+1}(i-1)a_i} \\
&= \int_{L^0_{\lambda,2m+1}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT(\pi^\lambda)_{2m+1}T)|_K^{s_i}\, dT \\
&\times (1-q^{-1})^{-1} \prod_{i=1}^m \{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}\}^{-1}\{1-q^{-(s_i+\cdots+s_m+2m-2i+2)}\}^{-1}.
\end{aligned}$$

By an argument similar to the above calculation, we have

$$\begin{aligned}
&\int_{L_{\lambda,2m}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT'(\pi^\lambda)_{2m}T')|_K^{s_i}\, dT' \\
&= \int_{L^0_{\lambda,2m}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT'(\pi^\lambda)_{2m}T')|_K^{s_i}\, dT' \\
&\times \prod_{i=1}^m \{1-q^{-(s_i+\cdots+s_m+2m-2i+1)}\}^{-1}\{1-q^{-(s_i+\cdots+s_m+2m-2i+2)}\}^{-1}.
\end{aligned}$$

We see

$$\int_{L_{\lambda,2m+1}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT(\pi^\lambda)_{2m+1}T)|_K^{s_i}\, dT = \int_{L_{\lambda,2m}} \prod_{i=1}^m |\mathrm{Pf}_i({}^tT'(\pi^\lambda)_{2m}T')|_K^{s_i}\, dT'.$$

Therefore we have

$$\begin{aligned}
\zeta_{2m+1}(\pi^{\lambda};\, s) &= (w_{2m+1}(q^{-1}))^{-1} \int_{L^0_{\lambda,2m+1}} \prod_{i=1}^{m} |\mathrm{Pf}_i({}^tT(\pi^{\lambda})_{2m+1}T)|_K^{s_i}\, dT \\
&= \frac{1-q^{-1}}{w_{2m+1}(q^{-1})} \cdot \prod_{i=1}^{m} \{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}\}\{1-q^{-(s_i+\cdots+s_m+2m-2i+2)}\} \\
&\quad \times \int_{L_{\lambda,2m+1}} \prod_{i=1}^{m} |\mathrm{Pf}_i({}^tT(\pi^{\lambda})_{2m+1}T)|_K^{s_i}\, dT \\
&= \frac{1-q^{-1}}{w_{2m+1}(q^{-1})} \cdot \prod_{i=1}^{m} \{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}\}\{1-q^{-(s_i+\cdots+s_m+2m-2i+2)}\} \\
&\quad \times \int_{L_{\lambda,2m}} \prod_{i=1}^{m} |\mathrm{Pf}_i({}^tT'(\pi^{\lambda})_{2m}T')|_K^{s_i}\, dT' \\
&= \frac{1-q^{-1}}{1-q^{-2m-1}} \cdot \prod_{i=1}^{m} \frac{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}}{1-q^{-(s_i+\cdots+s_m+2m-2i+1)}} \cdot \zeta_{2m}((\pi^{\lambda})_{2m};\, s).
\end{aligned}$$

Hence we obtain the formula for $\zeta_{2m+1}(\pi^{\lambda};\, s)$ and $\zeta_{2m}(\pi^{\lambda};\, s)$. □

Proposition 6.9. *For $\lambda \in \Lambda_m^+$, we have*

$$\begin{aligned}
\int_{U_{\lambda,2m+1}} dx &= w_{2m+1}(q^{-1}) w_n(q^{-2})((1-q^{-1}) \cdot w_{n-m}(q^{-2}))^{-1} \\
&\quad \times \sum_{\mu,\nu \in \Lambda_m^+} q^{-2n(\lambda)-2n(\mu)-2n(\nu)-|\lambda|-(2n-2m+1)|\mu|-2|\nu|} \cdot (w_{\lambda}(q^{-2}))^{-1} \cdot f^{\lambda}_{\mu\nu}(q^{-2}).
\end{aligned}$$

Proof. We shall prove this formula by Lemma 6.8 and an argument similar to the proof of Proposition 6.7. By Lemma 6.8, we have

$$\begin{aligned}
\Phi_{2m+1}(s) &= \int_{GL(2m+1;\,\mathcal{O}_K)} \int_{X'_{2m+1}} \prod_{i=1}^{m} |\mathrm{Pf}_i(k \cdot \varphi(x))|_K^{s_i}\, dx\, dh \\
&= \int_{X'_{2m+1}} \zeta_{2m+1}(\varphi(x);\, s)\, dx \\
&= \sum_{\lambda \in \Lambda_m^+} \int_{U_{\lambda,2m+1}} dx \cdot \zeta_{2m+1}((\pi^{\lambda})_{2m+1};\, s) \\
&= \frac{1-q^{-1}}{1-q^{-2m-1}} \cdot \prod_{i=1}^{m} \frac{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}}{1-q^{-(s_i+\cdots+s_m+2m-2i+1)}} \cdot \zeta_{2m}(J_m;\, s) \\
&\quad \times \sum_{\lambda \in \Lambda_m^+} \int_{U_{\lambda,2m+1}} dx \cdot q^{2n(\lambda)-(m-1)|\lambda|} \cdot \frac{w_{\lambda}^{(m)}(q^{-2})}{w_m(q^{-2})} \cdot P_{\lambda}(q^{z_1}, \ldots, q^{z_m};\, q^{-2}).
\end{aligned}$$

By the proof of Proposition 6.7 and $\Phi_{2m+1}(s) = \Phi_{2m}(s)$, we have

$$\left(\prod_{i=1}^{m} \frac{1-q^{-(s_i+\cdots+s_m+2m-2i+3)}}{1-q^{-(s_i+\cdots+s_m+2m-2i+1)}}\right)^{-1} \Phi_{2m+1}(s)$$
$$= \zeta_{2m}(J_m; s) \cdot w_{2m}(q^{-1}) w_n(q^{-2}) (w_m(q^{-2}) w_{n-m}(q^{-2}))^{-1}$$
$$\times \sum_{\lambda,\mu,\nu \in \Lambda_m^+} q^{-2n(\mu)-2n(\nu)-(2n-m+1)|\mu|-(m+2)|\nu|} \cdot f_{\mu\nu}^{\lambda}(q^{-2}) \cdot P_\lambda(q^{z_1}, \ldots, q^{z_m}; q^{-2}).$$

Therefore we obtain the formula for $\int_{U_{\lambda,2m+1}} dx$ by comparing the terms involving $P_\lambda(q^{z_1}, \ldots, q^{z_m}; q^{-2})$. □

By Proposition 6.7 and 6.9, we obtain Lemma 4.2.

7. Application to Igusa local zeta function

In this section, we shall give explicit forms of the Igusa local zeta functions of prehomogeneous vector spaces (a), (b), (c) and (d) by using the formula ($l = 2m + 1$) of Theorem 4.3.

7.1. $(GL(1) \times Sp(n) \times SO(3), \Lambda_1 \otimes \Lambda_1)$. The group $G = GL(1) \times Sp(n) \times SO(3)$ acts on $V = M(2n, 3)$ by $x \mapsto \alpha g x {}^t h$ for $x \in V$ and $(\alpha, g, h) \in G$. We define the map ω: Alt(3) $\to M(3, 1)$ as $y = (y_{ij})_{1 \le i,j \le 3} \mapsto {}^t(y_{23}, -y_{13}, y_{12})$, where $y_{ij} = -y_{ji}$. We see that $\omega(h \cdot y) = \det(h)^t h^{-1} \omega(y)$ for all $h \in GL(3)$, $y \in$ Alt(3). Hence we identify Alt(3) as $M(3, 1)$. Let φ be the $Sp(n)$-invariant map $M(2n, 3) \to$ Alt(3) defined by $\varphi(x) = {}^t x J_n x$ for $x \in M(2n, 3)$. In [20], K-forms of this space were classified into two cases. For one case, the basic relative invariant is given by $f(\varphi(x))$ where $f(y) = y_{12}^2 + y_{13}^2 + y_{23}^2$ (cf. [25]), and this Igusa local zeta function was calculated in [7]. For another case, the basic relative invariant is given by $f(\varphi(x))$ where $f(y) = y_{12}^2 + y_{13} y_{23}$. We shall treat uniformly these two cases. Let f be a quadratic form in $\mathcal{O}_K[y_{12}, y_{13}, y_{23}]$. By the following lemma, we reduce the calculation of the Igusa local zeta function of $f(\varphi(x))$ to that of f.

Lemma 7.1.

$$Z(s) = \int_{M(2n,3;\mathcal{O}_K)} |f(\varphi(x))|_K^s \, dx = \frac{1-q^{-2n}}{1-q^{-2n-2s}} \int_{M(3,1;\mathcal{O}_K)} |f(y)|_K^s \, dy.$$

Proof. By Theorem 4.3 we have

$$Z(s) = (1-q^{-2n}) \sum_{\mu_1,\nu_1=1}^{+\infty} q^{-(2n-3)\mu_1} \int_{GL(3;\mathcal{O}_K)\cdot(\pi^{\mu_1+\nu_1})_3} |f(y)|_K^s \, dy$$

$$= (1-q^{-2n}) \sum_{\mu_1,\nu_1=1}^{+\infty} q^{-(2n-3)\mu_1} \cdot q^{(-2s-3)\mu_1} \int_{GL(3;\mathcal{O}_K)\cdot(\pi^{\nu_1})_3} |f(y)|_K^s \, dy$$

$$= \frac{1-q^{-2n}}{1-q^{-2n-2s}} \int_{M(3,1;\mathcal{O}_K)} |f(y)|_K^s \, dy,$$

because

$$g_{\mu_1\nu_1}^{\lambda_1} = \begin{cases} 1 & (\lambda_1 = \mu_1 + \nu_1) \\ 0 & (\text{otherwise}) \end{cases}.$$

Hence we have this lemma. □

We put $\overline{f}(y) = f(y) \bmod \pi$, $\overline{Y} = \mathbb{F}_q^3 = Y \bmod \pi$, and $\overline{f}(y, y') = \overline{f}(y + y') - \overline{f}(y) - \overline{f}(y')$. Let $\overline{Y}^{\perp}$ be the set of all y in $\overline{Y}$ satisfying $\overline{f}(y, y') = 0$ for every $y' \in \overline{Y}$. If $\overline{Y}^{\perp} \cap \overline{f}^{-1}(0) = 0$, then $\overline{f}(y)$ is called reduced on $\overline{Y}$ (cf. [14, Chapter 9]).

Lemma 7.2 ([14, Corollary 10.2.1]). *If $\overline{f}(y)$ is reduced on $\overline{Y}$, then the Igusa local zeta function of $f(y)$ is given by*

$$\int_{M(3,1;\mathcal{O}_K)} |f(y)|_K^s \, dy = \frac{(1-q^{-1})(1-q^{-3-s})}{(1-q^{-1-s})(1-q^{-3-2s})}.$$

Hence we obtain $Z(s)$ by Lemma 7.1 and 7.2.

Proposition 7.3. *Let f be a quadratic form in $\mathcal{O}_K[y_{12}, y_{13}, y_{23}]$. We assume that $\overline{f}(y)$ is reduced in $\mathbb{F}_q^3$. Then the Igusa local zeta function of $f(\varphi(x))$ is given by*

$$Z(s) = \frac{(1-q^{-1})(1-q^{-3-s})(1-q^{-2n})}{(1-q^{-1-s})(1-q^{-3-2s})(1-q^{-2n-s})}.$$

By this proposition, we get the Igusa local zeta function of $(GL(1) \times Sp(n) \times SO(3), \Lambda_1 \otimes \Lambda_1)$.

7.2. $(GL(1)^3 \times Sp(n), \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$. The group $G = GL(1)^3 \times Sp(n)$ acts on $V = M(2n, 3)$ by $x \mapsto (\alpha g x_1, \beta g x_2, \gamma g x_3)$ for $x = (x_1, x_2, x_3) \in V$ and $(\alpha, \beta, \gamma, g) \in G$. Let φ be the $Sp(n)$-invariant map $M(2n, 3) \to \mathrm{Alt}(3)$ defined by $\varphi(x) = {}^t x J_n x$ for $x \in M(2n, 3)$. This space has three basic relative invariants $f_1(\varphi(x))$, $f_2(\varphi(x))$, $f_3(\varphi(x))$ for $f_1(y) = y_{12}$, $f_2(y) = y_{13}$, $f_3(y) = y_{23}$, $y = (y_{ij}) = \varphi(x) \in \mathrm{Alt}(3)$ and $x \in V$ (cf. [15]). These $f_1(y)$, $f_2(y)$, $f_3(y)$ are the basic relative invariants of $(GL(1)^3, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$. By an argument similar to the proof of Lemma 7.1, we have the following lemma.

Lemma 7.4.

$$Z(s) = \int_{M(2n,3;\mathcal{O}_K)} \prod_{i=1}^{3} |f_i(\varphi(x))|_K^{s_i}\, dx = \frac{1-q^{-2n}}{1-q^{-2n-s_1-s_2-s_3}} \int_{M(3,1;\mathcal{O}_K)} \prod_{i=1}^{3} |f_i(y)|_K^{s_i}\, dy.$$

By this lemma, we get an explicit form of $Z(s)$.

Proposition 7.5. *The Igusa local zeta function of* $(GL(1)^3 \times Sp(n), \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ *is given by*

$$Z(s) = \prod_{i=1}^{3} \frac{1-q^{-1}}{1-q^{-1-s_i}} \times \frac{1-q^{-2n}}{1-q^{-2n-s_1-s_2-s_3}}.$$

7.3. $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$. The group $G = GL(1)^4 \times Sp(n) \times SL(2m+1)$ acts on $V = M(2n, 2m+1) \oplus M(2m+1, 1) \oplus M(2m+1, 1) \oplus M(2m+1, 1)$ by $(x, z_1, z_2, z_3) \mapsto (\alpha g x {}^t h, \beta h z_1, \gamma h z_2, \delta h z_3)$ for $(x, z_1, z_2, z_3) \in V$ and $(\alpha, \beta, \gamma, \delta, g, h) \in G$. Let φ be the $Sp(n)$-invariant map $M(2n, 2m+1) \to \mathrm{Alt}(2m+1)$ defined by $\varphi(x) = {}^t x J_n x$ for $x \in M(2n, 2m+1)$. This space has four basic relative invariants $f_i(\varphi(x), z)$ for $f_i(y, z) = \mathrm{Pf}(A_i(y, z))$ $(i = 1, 2, 3, 4)$, $z = (z_1, z_2, z_3)$, $y = (y_{ij}) = \varphi(x) \in \mathrm{Alt}(2m+1)$, where we put

$$A_i(y,z) = \begin{pmatrix} y & z_i \\ -{}^t z_i & 0 \end{pmatrix} \quad (i = 1, 2, 3), \qquad A_4(y,z) = \begin{pmatrix} y & z_1 & z_2 & z_3 \\ -{}^t z_1 & 0 & 0 & 0 \\ -{}^t z_2 & 0 & 0 & 0 \\ -{}^t z_3 & 0 & 0 & 0 \end{pmatrix}$$

(cf. [17]). These $f_1(y, z), \ldots, f_4(y, z)$ are the basic relative invariants of $(GL(1)^4 \times SL(2m+1), \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ (cf. [15]). Let dz be the Haar measure on $M(2m+1, 3; K)$ normalized by $\int_{M(2m+1,3;\mathcal{O}_K)} dz = 1$. By Theorem 4.3, we have the following lemma.

Lemma 7.6.

$$\begin{aligned} Z(s) &= \int_{M(2n,2m+1;\mathcal{O}_K) \oplus M(2m+1,3;\mathcal{O}_K)} \prod_{i=1}^{4} |f_i(\varphi(x), z)|_K^{s_i}\, dx\, dz \\ &= \frac{w_{2n}(q^{-2})}{w_{2n-2m}(q^{-2})} \sum_{\lambda,\mu,\nu \in \Lambda_m^+} g_{\mu\nu}^{\lambda}(q^2) \cdot q^{-(2n-2m-1)|\mu|} \\ &\quad \times \int_{GL(2m+1;\mathcal{O}_K) \cdot (\pi^\lambda)' \oplus M(2m+1,3;\mathcal{O}_K)} \prod_{i=1}^{4} |f_i(y, z)|_K^{s_i}\, dy\, dz. \end{aligned}$$

From [26, Section 3], we have the following lemma.

Lemma 7.7.

$$\int_{GL(2m+1;\,\mathcal{O}_K)\cdot(\pi^{\lambda})_{2m+1}\oplus M(2m+1,3;\,\mathcal{O}_K)} \prod_{i=1}^{4} |f_i(y,z)|_K^{s_i}\,dy\,dz$$
$$= \prod_{i=1}^{4} \frac{1-q^{-1}}{1-q^{-s_i-1}} \times \frac{1-q^{-2m}}{1-q^{-s_4-2m}} \times \frac{1-q^{-s_1-s_2-s_3-3}}{1-q^{-s_1-s_2-s_3-s_4-3}} \times \frac{w_{2m+1}(q^{-1})}{(1-q^{-1})w_m(q^{-2})}$$
$$\times q^{-2n(\lambda)} P_{\lambda}(q^{-s_1-s_2-s_3-3}, q^{-s_1-s_2-s_3-s_4-5}, \ldots, q^{-s_1-s_2-s_3-s_4-2m-1};\, q^{-2}).$$

Therefore we have the following result by Lemma 2.1, 7.6 and 7.7.

Proposition 7.8. *The Igusa local zeta function of* $(GL(1)^4 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$ *is given by*

$$Z(s) = \prod_{i=1}^{4} \frac{1-q^{-1}}{1-q^{-1-s_i}} \times \frac{1-q^{-2n+2m-2}}{1-q^{-2n+2m-2-s_1-s_2-s_3}} \times \frac{1-q^{-2m}}{1-q^{-2m-s_4}}$$
$$\times \prod_{j=1}^{m} \frac{1-q^{-2j-1}}{1-q^{-2j-1-s_1-s_2-s_3-s_4}}$$
$$\times \prod_{j=1}^{m-1} \frac{1-q^{-2n+2j-2}}{1-q^{-2n+2j-2-s_1-s_2-s_3-s_4}}.$$

7.4. $\boldsymbol{(GL(1)^2 \times Sp(n) \times SL(2), \Lambda_1 \otimes (2\Lambda_1) + 1 \otimes \Lambda_1)}$**.** The group $G = GL(1)^2 \times Sp(n) \times SL(2)$ acts on $V = M(2n,3) \oplus M(2,1)$ by $(x,z) \mapsto (\alpha g x\, {}^t(2\Lambda)(h), \beta h z)$ for $(x,z) \in V$ and $(\alpha, \beta, g, h) \in G$, where

$$(2\Lambda_1)(h) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \quad \text{for} \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2).$$

For a commutative ring R, we denote by $\mathrm{Sym}(2;R)$ the totality of 2×2 symmetric matrices over R. We define the map $\omega'\colon \mathrm{Sym}(2) \to M(3,1)$ as $y = (y_{ij})_{1\le i,j\le 2} \mapsto {}^t(y_{11}, y_{12}, y_{22})$, where $y_{12} = y_{21}$. We see that $\omega'(hy\,{}^t h) = (2\Lambda_1)(h)\omega'(y)$ for all $h \in GL(2)$, $y \in \mathrm{Sym}(2)$. Hence we identify $\mathrm{Sym}(2)$ as $\mathrm{Alt}(3)$ by the maps ω and ω'. Let φ be the $Sp(n)$-invariant map $M(2n,3) \to \mathrm{Alt}(3)$ defined by $\varphi(x) = {}^t x J_n x$ for $x \in M(2n,3)$. This space has two basic relative invariants $f_1(\varphi(x))$, $f_2(\varphi(x), z)$ for $f_1(y) = \det(y)$, $f_2(y,z) = {}^t z y z$, $y = \varphi(x) \in \mathrm{Sym}(2)$, $z = {}^t(z_1, z_2) \in M(2,1)$ (cf. [17]). These $f_1(y)$ and $f_2(y,z)$ are the basic relative invariants of $(GL(1)^2 \times SL(2), (2\Lambda_1) \oplus \Lambda_1^*)$. Let dz be the Haar measure on $M(2,1;K)$ normalized by $\int_{M(2,1;\,\mathcal{O}_K)} dz = 1$. By an argument similar to the proof of Lemma 7.1, we have the following lemma.

Lemma 7.9.

$$Z(s) = \int_{M(2n,3;\mathcal{O}_K)\oplus M(2,1;\mathcal{O}_K)} |f_1(\varphi(x))|_K^{s_1} |f_2(\varphi(x),z)|_K^{s_2}\, dx\, dz$$
$$= \frac{1-q^{-2n}}{1-q^{-2n-2s_1-s_2}} \int_{\mathrm{Sym}(2;\mathcal{O}_K)\oplus M(2,1;\mathcal{O}_K)} |f_1(y)|_K^{s_1} |f_2(y,z)|_K^{s_2}\, dy\, dz.$$

We shall prove the following lemma.

Lemma 7.10.

$$\int_{\mathrm{Sym}(2;\mathcal{O}_K)\oplus M(2,1;\mathcal{O}_K)} |f_1(y)|_K^{s_1} |f_2(y,z)|_K^{s_2}\, dy\, dz$$
$$= \frac{(1-q^{-1})^2(1-q^{-2})(1-q^{-3-s_1-2s_2})}{(1-q^{-1-s_1})(1-q^{-1-s_2})(1-q^{-2-2s_2})(1-q^{-3-2s_1-2s_2})}.$$

Proof. We decompose $M(2,1;\mathcal{O}_K)$ as

$$M(2,1;\mathcal{O}_K) = \bigcup_{k=0}^{+\infty} GL(2;\mathcal{O}_K)\cdot \begin{pmatrix} \pi^i \\ 0 \end{pmatrix} \quad \text{(disjoint union)}.$$

Then by imitating [14, Section 10.1] we have

$$\int_{\mathrm{Sym}(2;\mathcal{O}_K)\oplus M(2,1;\mathcal{O}_K)} |f_1(y)|_K^{s_1} |f_2(y,z)|_K^{s_2}\, dy\, dz$$
$$= \sum_{i=0}^{+\infty} \int_{\mathrm{Sym}(2;\mathcal{O}_K)\oplus GL(2;\mathcal{O}_K)\cdot {}^t(\pi^i,0)} |f_1(y)|_K^{s_1} |f_2(y,z)|_K^{s_2}\, dy\, dz$$
$$= (1-q^{-2})\cdot \sum_{i=0}^{+\infty} q^{-2i-2is_2} \int_{\mathrm{Sym}(2;\mathcal{O}_K)} |f_1(y)|_K^{s_1} |f_2(y,{}^t(1,0))|_K^{s_2}\, dy$$
$$= \frac{1-q^{-2}}{1-q^{-2-2s_2}} \int_{\mathrm{Sym}(2;\mathcal{O}_K)} |y_{11}y_{22}-y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy.$$

If we split the domain $\mathcal{O}_K$ of y_{22} into the union of $\mathcal{O}_K \setminus \pi\mathcal{O}_K$ and $\pi\mathcal{O}_K$, then we have

$$\int_{\mathrm{Sym}(2;\mathcal{O}_K)} |y_{11}y_{22}-y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy$$
$$= \frac{(1-q^{-1})^2}{1-q^{-1-s_1}} + q^{-1-s_2} \int_{\mathrm{Sym}(2;\mathcal{O}_K)} |\pi y_{11}y_{22}-y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy.$$

By repeating this process, we have

$$\int_{\mathrm{Sym}(2;\mathcal{O}_K)} |\pi y_{11}y_{22}-y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy$$

$$= \frac{(1-q^{-1})^2}{1-q^{-1-s_2}} + q^{-1-s_1} \int_{\mathrm{Sym}(2;\mathcal{O}_K)} |y_{11}y_{22} - \pi y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy,$$

and

$$\int_{\mathrm{Sym}(2;\mathcal{O}_K)} |y_{11}y_{22} - \pi y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy$$
$$= \frac{(1-q^{-1})^2}{1-q^{-1-s_1}} + q^{-1-s_2} \int_{\mathrm{Sym}(2;\mathcal{O}_K)} |y_{11}y_{22} - y_{12}^2|_K^{s_1} |y_{22}|_K^{s_2}\, dy.$$

If we put together the above results, then we get this lemma. □

Therefore by Lemma 7.9 and 7.10, we have the following result.

Proposition 7.11. *The Igusa local zeta function of* $(GL(1)^2 \times Sp(n) \times SL(2), \Lambda_1 \otimes (2\Lambda_1) + 1 \otimes \Lambda_1)$ *is given by*

$$Z(s) = \frac{(1-q^{-1})^2(1-q^{-2})(1-q^{-3-s_1-2s_2})(1-q^{-2n})}{(1-q^{-1-s_1})(1-q^{-1-s_2})(1-q^{-2-2s_2})(1-q^{-3-2s_1-2s_2})(1-q^{-2n-2s_1-s_2})}.$$

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