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DIRECT SUM OF LOCAL MODULES WITH EXTENDING FACTOR MODULES

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1. Introduction

Rings whose cyclic modules are continuous have been studied by Jain and Mohamed [9]. These rings are semiperfect rings. Semiperfect rings whose cyclics are π-injective (extending) are studied by Goel and Jain [6] (Vanaja [14]). We call a module an FE module if every factor module is extending. It was proved in [14] that for a semiperfect ring R, R_R is FE if and only if R_R is extending and every factor module of $R/Soc\,R$ is π-injective. One can easily extend the above result to modules M which are projective and semiperfect in $\sigma[M]$. In this case M is a direct sum of local modules. We extend the above result for any module M which is a direct sum of local modules.

The proof in the case when M is semiperfect and projective in $\sigma[M]$ heavily depends on the fact that M is a direct sum of locals with local endomorphism ring and this decomposition of M complements direct summands. Some sufficient conditions for a decomposition of a module M as a direct sum of locals to complement summands are proved in Section 4.

In Section 5 some important properties of an FE module which is a direct sum of two local modules are obtained. In Section 6 FE modules which are direct sum of local modules are considered. We do not assume that M is projective in $\sigma[M]$ or that the endomorphism ring of these local modules are local. We show that if $M = \bigoplus_{i\in I} M_i$ is an FE module, where each M_i is a local module, then this decomposition complements summands and any factor module of M is isomorphic to $\bigoplus_{i\in I} M_i/X_i$, for some $X_i \subseteq M_i$ (6.2). Our main theorem (6.3) is as follows.

Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module. Then the following are equivalent:
(a) $\bigoplus_{i\in I} M_i/X_i$ is uniform-extending, for all $X_i \subseteq M_i$;
(b) $\bigoplus_{i\in I} M_i/X_i$ is extending, for all $X_i \subseteq M_i$;
(c) every factor module of M is extending;
(d) every factor module of M is uniform-extending;
(e) M is uniform-extending and $\bigoplus_{i\in I}(M_i/Soc\,M_i)/Y_i$ is π-injective, for all $Y_i \subseteq M_i/Soc\,M_i$;

(f) M is extending and every factor module of $M/Soc\,M$ is π-injective.

Suppose M is a direct sum of local modules. We prove that M^2 is FE if and only if M^n is FE, for all $n \in \mathbb{N}$. Also $M^{(\mathbb{N})}$ is FE if and only if $M^{(K)}$ if FE, for any set K. If M is a self-generator also, then M is SFE (i.e. every subfactor module of M is extending) if and only if M is SE (i.e. every submodule of M is extending) and FE. Also, a self-projective self-generator modules is SFE if and only if M is FE.

We also study Fπ modules M (i.e. with every factor module of M is π-injective), where M is either a direct sum of locals, of M is projective in $\sigma[M]$ and is a direct sum of indecomposables.

2. Definitions and notation

All rings considered are associative rings with identity and all modules considered are right unitary modules. A module M is called *extending* (*uniform extending*) if every (uniform) submodule is essential in a summand of M. An extending module M is called *π-injective* if whenever M_1, M_2 are summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M. An extending module is called *continuous* if any submodule isomorphic to a summand is a summand.

Let N be a submodule of M. By $N \trianglelefteq M$ we mean that N is an essential submodule of M and by $N \ll M$ we mean that N is small submodule of M. If every proper submodule of M is $\ll M$, then M is called a *hollow* module. We shall denote the Jacobson radical and the socle of M by $Rad\,M$, $Soc\,M$ respectively. For any module M we define $Top\,M = M/Rad\,M$ and $\overline{M} = M/Soc\,M$.

By a subfactor of M we mean a submodule of a factor module of M or equivalently, a factor of a submodule of M. $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are submodules of M-generated modules. If $N \in \sigma[M]$ we denote by $\widehat{N}$ *the injective hull of N in $\sigma[M]$*. We call a module N in $\sigma[M]$ *semiperfect* (*f-semiperfect*) in $\sigma[M]$ if for every (finitely generated) submodule K of N, N/K has a projective cover in $\sigma[M]$.

A module is called *uniserial* if its submodules are linearly ordered by inclusion. If a module M is a direct sum of uniserial modules, then we say M is *serial.* A module M is called *homo-uniserial* if for any non-zero finitely generated submodules K, L of M, the factor modules $K/Rad\,K$ and $L/Rad\,L$ are simple and isomorphic. A module M is called *homo-serial* if it is a direct sum of homo-uniserial modules. A submodule N of M is said to be a *finitely contained submodule* (denoted briefly by f.c. submodule) with respect to the decomposition $M = \bigoplus_{i\in I} M_i$ of M if N is contained in $\bigoplus_{k\in K} M_k$, where K is a finite subset of I.

For a module M we define the following.

FUE	every factor of M is uniform extending
FE	every factor of M is extending
Fπ	every factor of M is π-injective
FI	every factor of M is injective in $\sigma[M]$
SFE	every subfactor of M is extending
SFπ	every subfactor of M is π-injective
SE	every submodule of M is extending
Sπ	every submodule of M is π-injective

Finally we recall the definition of a quasi-discrete module, which is the dual notion of a π-injective module. A module M is called *lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll M$. A module M is called *quasi-discrete* if it is lifting and if M_1 and M_2 are summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is a summand of M.

For other standard definitions are notations we refer [4], Mohamed and Müller [12] and Wisbauer [15].

3. Preliminaries

We study here conditions under which a module is a direct sum of local modules and the conditions under which every submodule X of a module $M = \bigoplus_{i=1}^{n} M_i$ has a decomposition $X = \bigoplus_{i=1}^{n} X_i$, where each $X_i \subseteq M_i$.

Lemma 3.1. *Let M be a local module such that M is FUE. Then M is uniserial. Hence any FUE module M which is semiperfect and projective in $\sigma[M]$ is serial.*

Proof. For any submodule X of M, M/X is indecomposable and uniform extending. Hence $Soc(M/X)$ is either zero or simple. By Wisbauer [15, 55.1] M is uniserial. □

Proposition 3.2. *Let M be a finitely generated self-projective extending module such that $\overline{M}$ is an FE module. Suppose M satisfies one of the following conditions:*

(i) *M is continuous;*

(ii) *$\widehat{M}$ is projective in $\sigma[M]$;*

(iii) *M is f-semiperfect in $\sigma[M]$,*

Then M is serial.

Proof. By [4, 9.3 (ii)] $\overline{M}$ is a direct sum of uniform modules and by Dung [3, proposition 13] M is a direct sum of uniform modules. Let $M = \bigoplus_{i=1}^{n} M_i$, where each M_i is uniform. Without loss of generality we may assume that each M_i is non-simple.

(i) Suppose M is continuous. Then $End\, M_i$ is local as the endomorphism ring of an indecomposable continuous module is local. Hence each M_i is a local module. By 3.1 each $\overline{M_i}$ is uniserial and therefore M is serial.

(ii) Suppose $\widehat{M}$ is projective in $\sigma[M]$. Then $\widehat{M} = \bigoplus_{i=1}^{n} \widehat{M_i}$. Since $\widehat{M_i}$ is uniform $End\, \widehat{M_i}$ is local. $\widehat{M_i}$ is local as $\widehat{M_i}$ is projective in $\sigma[M]$. Now $\widehat{M_i}$ is isomorphic to a summand of M. By the previous case $\widehat{M_i}$ is serial. Therefore M_i is uniserial.

(iii) Suppose M is f-semiperfect in $\sigma[M]$. $M/Rad\, M$ is FE, as $Soc\, M \subseteq Rad\, M$, and hence is a direct sum of uniform modules [4, 9.3 (ii)]. As $M/Rad\, M$ is regular it is semisimple. Hence M is semiperfect. By 3.1 M is serial. □

For evsy reference we define the following.

DEFINITION 3.3. Let M be a finitely generated self-projective module. We say M is a *module of type $\mathcal{A}$* if M satisfies one of the following conditions:

(i) M is continuous;

(ii) $\widehat{M}$ is projective in $\sigma[M]$;

(iii) M is f-semiperfect in $\sigma[M]$,

Let $M = \bigoplus_{i \in I} M_i$ be an R-module. The following gives a sufficient condition for every $X \subseteq M$ to have decomposition $X = \bigoplus_{i \in I} X_i$, $X_i \subset M_i$, where I is a finite set.

Lemma 3.4. *Let $M = \bigoplus_{i=1}^{k} M_i$ be such that $Hom(A, B) = 0$, where A and B are subfactors of M_i and M_j respectively, $1 \leq i, j \leq k$ and $i \neq j$. If $X \subset M$, then $X = \bigoplus_{i=1}^{k} X_i$, $X_i \subset M_i$.*

Proof. Let $X_i = X \cap M_i$, $Y = \bigoplus_{i=1}^{k} X_i$ and $\eta : M \to M/Y$ be the natural map. For any $K \subset M$, let $\eta(K) = K^*$. Then $M^* = \bigoplus_{i=1}^{k} M_i^*$ and $X^* \cap M_i^* = 0$, for $i = 1, \ldots, k$. The proof is by induction on k.

Let $p_i : M^* \to M_i^*$ be the projection map for $i = 1, 2, \ldots, k$. Suppose $k = 2$. As $X^* \cap M_2^* = 0$, $g = p_1|_{X^*}$ is a monomorphism. But then the map $p_2 g^{-1}$ from $g(X^*)$ to M_2^* is the zero map. Hence p_2 is zero on X^*. Similarly, p_1 is zero on X^*. Hence $X = X_1 \oplus X_2$. Suppose the assertion is true for $n < k$. Suppose $M = \bigoplus_{i=1}^{k} M_i$. Let $q_1 = \bigoplus_{i=2}^{k} p_i$. Then q_1 is one-one on X^*. By induction hypothesis $q_1(X^*) = \bigoplus_{i=2}^{k} A_i$, where $A_i \subset M_i^*$, $i = 2, \ldots, k$. By assumption the map $p_1 q_1^{-1}$ is zero on each A_i and hence is the zero map. This implies that p_1 is zero on X^*. Similarly, p_i is zero on X^*, for any $i = 2, 3, \ldots, k$. Therefore $X^* = 0$ and $X = \bigoplus_{i=1}^{k} X_i$. □

The above Lemma was extended to any arbitrary set I in [13, 2.3] which we state below.

Proposition 3.5. *Let $M = \bigoplus_{i \in I} M_i$. Then the following are equivalent:*
(a) *for distinct k and j in I, no two non-zero subfactors of M_k and M_j are isomorphic;*
(b) *for distinct k and j in I, $Hom(A_k, A_j) = 0$, where A_k and A_j are subfactors of M_k and M_j respectively;*
(c) *for distinct k and j in I, $\sigma[M_k] \cap \sigma[M_j] = 0$;*
(d) *for any $k \in I$, $\sigma[M_k] \cap \sigma[M^k] = 0$, where $M^k = \bigoplus_{i \in I \setminus \{k\}} M_i$;*
(e) *for any $N \in \sigma[M]$ there exists a unique $N_i \in \sigma[M_i]$, $i \in I$, such that $N = \bigoplus_{i \in I} N_i$.*

DEFINITION 3.6. Let M be an R-module. We say $\sigma[M] = \bigoplus_{i \in I} \sigma[M_i]$ if $M = \bigoplus_{i \in I} M_i$ and any one (and hence all) of the equivalent conditions in Proposition 3.5 is satisfied.

Suppose $\sigma[M] = \bigoplus_{i \in I} \sigma[M_i]$. If $X \subseteq M$, then $X = \bigoplus_{i \in I} X_i$, where each $X_i \subseteq M_i$. X is extending (π-injective) if and only if each X_i is extending (π-injective). Thus M is FE (Fπ) if and only if each M_i is FE (Fπ). Whenever we want to prove some result regarding a module M we try to get a decomposition of $\sigma[M]$ and use the above observations.

4. Extending property of a module with a semisimple summand

We are interested in the extending property of a direct sum of local modules where we do not assume that the endomorphism rings of the local modules are local. It has been proved in [12, 2.22] that if M is π-injective and is a direct sum of uniform modules, then this decomposition of M complements summands. We prove here a similar result. Suppose $M = N \bigoplus K$, where $N = \bigoplus_{i \in I} N_i$, each N_i is a hollow module and $K = \bigoplus_{j \in J} S_j$, each S_j is simple and $\overline{N}$-injective. We show that (i) if N is uniform extending, then M is uniform extending, and (ii) if N is π-injective, then M is extending and $M = \bigoplus_{i \in I} N_i \oplus \bigoplus_{j \in J} S_j$ complements direct summands.

We first state some known results regarding extending and π-injectivity of modules which will be used often in the sequel.

The following Result is Theorem 2.13 in Mohamed and Müller [12] which gives a necessary and sufficient condition for a direct sum of π-injective modules to be π-injective.

RESULT 4.1. Let $\{M_i : i \in I\}$ be a family of π-injective modules. Then the following are equivalent:

(a) $M = \bigoplus_{i \in I} M_i$ is π-injective;

(b) $\bigoplus_{j \in I \setminus \{i\}} M_j$ is M_i-injective, for all $i \in I$.

Using 4.1 and Theorem 12 in Harada and Oshiro [7] we get the following.

Corollary 4.2. *Suppose $M = \bigoplus_{i \in I} M_i$ is uniform extending, where each M_i is uniform, $End\, M_i$ is local and M_i is M_j-injective for $i \neq j$, $i, j \in I$. Then M is π-injective.*

Kamal and Müller have proved the following result in [11, Lemma 4].

RESULT 4.3. Let M, N be R-modules, $\phi : E(M) \to E(N)$ an arbitrary homomorphism and $X = \{x \in M : \phi(x) \in N\}$. If there exists a homomorphism $\psi : Y \to N$, $X \subseteq Y \subseteq M$, such that $\psi(x) = \phi(x)$ for all $x \in X$, then $X = Y$. Moreover the submodule $B = \{x + \phi(x) : x \in X\}$ of $M \oplus N$ is closed.

We now give some sufficient conditions for a module M which is a direct sum of hollow modules to be uniform extending. The following Lemma can be easily proved.

Lemma 4.4. *Let $M = A \oplus B$ be a module and $p : M \to B$ be the projection map. Suppose that C is a submodule of M such that $p|_C : C \to B$ is one-one and $p(C)$ is a summand of B. Then $M = A \oplus C \oplus D$, where $B = p(C) \oplus D$.*

Proposition 4.5. *Let $M = N \oplus K$ be an R-module, where K is a semisimple module.*

(1) *If M is extending, then for any simple submodule S of K, S is $\overline{N}$-injective.*

(2) *If N is uniform extending and K is $\overline{N}$-injective, then M is uniform extending.*

(3) *If $N = \bigoplus_{i \in I} N_i$, where each N_i is a hollow module, is uniform extending and any simple submodule of K is $\overline{N}$-injective, then M is uniform extending.*

Proof. (1) Let $f : L/Soc\, N \to S$ be a non-zero map, where $Soc\, N \subseteq L \subseteq N$. Consider $g = f\eta$, where $\eta : L \to L/Soc\, N$ is the natural map. We have $Ke\, g \supseteq Soc\, N$. As N is extending, $L \trianglelefteq T$, a direct summand of N. Let $\overline{g} : E(T) \to E(S)$ be an extension of g and $U = \{x \in T : \overline{g}(x) \in S\}$. We claim that $U = T$.

Let $V = \{x + \overline{g}(x) : x \in U\}$. Then $V \oplus S = U \oplus S$. As $S \oplus T$ is extending V is a summand of $T \oplus S$ (4.3). Let $S \oplus T = V \oplus W$. Since S has exchange property, either $S \oplus T = V' \oplus W \oplus S$ or $= V \oplus W' \oplus S$, where V' (resp. W') is a summand of V (resp. W).

Suppose $S \oplus T = V \oplus W' \oplus S$. Then $S \oplus T = U \oplus S \oplus W'$. This implies U is a summand of T. As $U \trianglelefteq T$, $U = T$.

Suppose $S \oplus T = S \oplus V' \oplus W$. Let $\phi : V \to U$ be given by $\phi(x + \overline{g}(x)) = x$. Then ϕ is an isomorphism. Let $V = V' \oplus V''$, $U' = \phi(V')$ and $U'' = \phi(V'')$. Then $U = U' \oplus U''$ and U'' is simple. We have $V' \oplus S = U' \oplus S$ and $S \oplus T = S \oplus V' \oplus W = S \oplus U' \oplus W$. So U' is a summand of T. Let $T = U' \oplus T'$. Then $U = U' \oplus (T' \cap U)$ and $\overline{g}$ is zero on $(T' \cap U)$, since $\overline{g}$ is zero on $Soc\, N$. $\overline{g}\,|_U$ can be extended to T by defining $\overline{g}(T') = 0$. By 4.3 we must have $T = U$. It is now easy to prove that f can be extended to $\overline{N}$.

(2) Let U be a uniform submodule of M. Suppose $p : M \to N$ and $q : M \to K$ are the projection maps. As U is uniform either $p\,|_U$ or $q\,|_U$ is one-one. Suppose $q\,|_U$ is one-one. As $q(U)$ is a direct summand of M we get $M = N \oplus U \oplus V$, where $K = q(U) \oplus V$ (4.4). Suppose $q\,|_U$ is not one-one. Then $p\,|_U$ is one-one. Consider $f : p(U) \to K$, where $f = qp^{-1}$. As f is not one-one and K is $\overline{N}$-injective, f can be extended to $g : N \to K$. There exists a summand L of N such that $p(U) \trianglelefteq L$. We have $U = \{x + g(x) : x \in p(U)\}$. Now $W = \{x + g(x) : x \in L\}$ is a summand of M and $U \trianglelefteq W$. Hence M is uniform extending.

(3) Let $N = \bigoplus_{i \in I} N_i$, $K = \bigoplus_{j \in J} S_j$, where each N_i is hollow and each S_j is simple.

Suppose U is a uniform submodule of M. If U is simple, then $U \subseteq L = \bigoplus_{i \in F} N_i \bigoplus T$, where F is a finite subset of I and T is a finitely generated submodule of K. By (2) L and hence M extends U.

Suppose U is not simple. Let $p_i : M \to N_i$, $q_j : M \to S_j$ be the projection maps for each $i \in I$ and each $j \in J$. By [13, 7.5] there exists $i \in I$ such that $p_i\,|_U : U \to N_i$ is one-one. If $q_j(U) = 0$, for all $j \in J$, then $U \subseteq N$. N extends U and so M also extends U.

Suppose $q_j(U) \neq 0$, for some $j \in J$. Let $f = q_j(p_i\,|_U)^{-1}$. As S_j is $\overline{N}$-injective, $f : p_i(U) \to S_j$ can be extended to N_i. Since N_i is hollow $N_i = p_i(U)$. By 4.4 U is a summand of M. □

Since any finite direct sum of uniform modules is extending if and only if it is uniform extending [4, 8.5] we get the following proved in [13, 6.4].

Corollary 4.6. *Let N be a finite direct sum of uniform modules, K a finitely generated semisimple module and $M = N \oplus K$. Then the following are equivalent*:

(a) *M is extending*;

(b) *N is extending and K is $\overline{N}$-injective.*

As a Corollary to 4.5 we get [4, 8.14].

Corollary 4.7. *Let $M = \bigoplus_{i \in I} M_i \oplus \bigoplus_{j \in J} M_j$, where each M_j is simple, each M_i is indecomposable of length 2, for $i \in I$, and M_i and M_k are relatively injective for $i \neq k \in I$. Then M is extending.*

Proof. Let $N = \bigoplus_{i\in I} M_i$. Then N is π-injective by 4.1. By 4.5 M is uniform extending. Hence M is extending [4, 8.13]. □

Theorem 4.8. *Let $M = \bigoplus_{i\in I} N_i \oplus \bigoplus_{j\in J} S_j$, where each N_i is hollow and each S_j, is simple. Suppose $N = \bigoplus_{i\in I} N_i$ is π-injective and any S_j is $\overline{N}$-injective. Then*

(1) *M is extending;*

(2) *The decomposition $M = \bigoplus_{i\in I} N_i \oplus \bigoplus_{j\in J} S_j$ (and hence any decomposition of M into indecomposables) complements direct summands.*

Proof. (1) If N_i, $i \in I$, is simple, then N_i is $\overline{N}$-injective. Hence without loss of generality we can assume that N has no simple summand. Each N_i is uniform and by 4.5 (3) M is uniform extending. Let C be any non-zero closed submodule of M. Then C contains a uniform summand U of M [11, Proposition 6]. Let $\{U_\alpha\}_{\alpha\in\Gamma_1}$ be a maximal local summand of M such that each U_α is non-simple contained in C and let $\{V_\beta\}_{\beta\in\Gamma_2}$ be a maximal local summand of M such that each V_β is a simple summand of M contained in C.

Suppose $A = \bigoplus_{\alpha\in\Gamma_2} U_\alpha$ and $B = \bigoplus_{\beta\in\Gamma_2} V_\beta$. Clearly $A \cap B = 0$. We show $A \oplus B$ is a summand of M.

Let $q : M \to K$ and $p : M \to N$ be the projection maps. As $N \cap B = 0$, $q|_B$ is one-one. Also $q(B)$ is a summand of K. By 4.4 $M = N \oplus B \oplus E$, where $K = q(B) \oplus E$. As the decomposition $K = \bigoplus_{j\in J} S_j$ complements summands $E = \bigoplus_{j\in J_1} S_j$, for some subset J_1 of J.

We next show that $\{p(U_\alpha) \mid \alpha \in \Gamma_1\}$ is a local summand of N and hence a summand of N. Since N is π-injective and $p|_A$ is one-one it is sufficient to prove that $p(U_\alpha)$ is a summand of N, for all $\alpha \in \Gamma_1$.

Fix $\alpha \in \Gamma_1$. If $q(U_\alpha) = 0$, then $p(U_\alpha) = U_\alpha$ is a direct summand of N.

Suppose $q(U_\alpha) \neq 0$. Let $q_j : K \to S_j$ and $p_i : N \to N_i$ be the projection maps, for $j \in J$ and $i \in I$. As $q(U_\alpha) \neq 0$, there exists a $j \in J$ such that the map $q_j : q(U_\alpha) \to S_j$ is non-zero. By [13, 7.5] there exists an $i \in I$ such that the map $p_i p$ is one-one on U_α. Consider the map

$$(q_j q)(p_i p)^{-1} : (p_i p)(U_\alpha) \to S_j.$$

This is not an one-one map and hence has an extension to N_i. Since every proper submodule of N_i is small in N_i we must have $(p_i p)(U_\alpha) = N_i$. As p is onto we have $p_i : p(U_\alpha) \to N_i$ is an isomorphism. Hence $p(U_\alpha)$ is a summand of N. In fact $N = p(U_\alpha) \oplus \bigoplus_{k\in I\setminus\{i\}} N_k$.

It is now easy to see that $\{p(U_\alpha) \mid \alpha \in \Gamma_1\}$ is a local summand of N and hence $P(A)$ is a summand of N. If $N = D \oplus p(A)$, then by 4.4 $M = D \oplus A \oplus B \oplus E$.

$C = A \oplus B \oplus ((D \oplus E) \cap C)$. Suppose $(D \oplus E) \cap C \neq 0$. Then $(D \oplus E) \cap C$ is a closed submodule of M and hence must contain a uniform summand of M which is a contradiction to the maximality of A or B. Therefore $C = A \oplus B$ and hence M is extending.

(2) If C is a summand of M, then $M = D \oplus A \oplus B \oplus E$, where $C = A \oplus B$, $D \subseteq N$ and $E \subseteq K$ (from the proof of (1)). Since the decomposition $N = \bigoplus_{i \in I} N_i$ complements summands $D = \bigoplus_{i \in I_1} N_i$, where I_1 is a subset of I. Hence

$$M = \bigoplus_{i \in I_1} N_i \oplus A \oplus B \oplus \bigoplus_{j \in J_1} S_j.$$

Thus (2) follows. By [1, 12.5] any decomposition of M into indecomposables complements direct summands. $\square$

Suppose the modules A and B are π-injective. Then $A \oplus B$ is π-injective, if A and B are relatively injective. In the case when $A = \bigoplus_{i \in I} A_i$ and $B = \bigoplus_{j \in J} B_j$, where the A_i's and B_j's are uniserial, it is enough to assume that the A_i and B_j are relatively injective, for all $i \in I$ and $j \in J$.

Lemma 4.9. *Let $A = \bigoplus_{i \in I} A_i$ and $B = \bigoplus_{j \in J} B_j$, where all A_i's and B_j's are uniserial modules. Suppose A, B are π-injective and A_i, B_j are relatively injective, for all $i \in I$ and all $j \in J$. Then $A \oplus B$ is π-injective.*

Proof. Let $f : X \to B$ be a non-zero map. $X \subseteq A_i$. Then $f(X)$ is essential in a direct summand C of B. Since $B = \bigoplus_{j \in J} B_j$ complements direct summands [12, 2.22] and C is uniform, $C \simeq B_j$, for some $j \in J$. Hence f can be extended to A_i. Therefore B is A-injective. Similarly A is B-injective. So $A \oplus B$ is π-injective. $\square$

5. Basic properties

Our main object is to study an FE module M which is a direct sum of local modules. As a prelude we take up the case when M is a direct sum of two local modules.

Proposition 5.1. *Let $M = A \oplus B$, where A, B are cyclic uniserial module. Suppose $A \oplus Top\,B$ is extending. Then $Top\,B \simeq Top\,X$, $X \subset A$ implies $X = Soc\,A$ or A.*

Proof. Suppose $Top\,X \simeq Top\,B$, $X \subseteq A$. If $Rad\,X = 0$, then $X = Soc\,A$. Suppose $Rad\,X \neq 0$. Then the obvious map $f : X \to Top\,B$ is not one-one. By 4.5 f has an extension to A, which gives us $X = A$. $\square$

For easy reference we define the following condition on a decomposition of a module.

DEFINITION 5.2. Let M be an R-module. The *decomposition* $M = \oplus B$ is said to *satisfy* $(*)$ if,
(1) A and B are cyclic uniserial,
(2) $Top\,X \simeq Top\,B$, $X \subset A$ implies $X = Soc\,A$ or A and
(3) $Top\,Y \simeq Top\,A$, $Y \subset B$ implies $Y = Soc\,B$ or B.

Lemma 5.3. *Let $M = A \oplus B$ be a decomposition of an R-module M satisfying $(*)$. If $Top\,A \simeq Top\,B$, then for all $X \subset A$, A/X is continuous and hence $End\,A/X$ is local.*

Proof. Let $Y/X \simeq A/X$, $X \subseteq Y \subseteq A$. Then $Top\,Y \simeq Top\,B$. Therefore $Y = Soc\,A$ or A. Hence A/X is continuous. □

Proposition 5.4. *Let $M = A \oplus B$ be a uniform-extending R-module such that A, B are local modules. Suppose $\overline{A}/X_0 \oplus \overline{B}/Y_0$ is π-injective for all $X_0 \subseteq \overline{A}$ and $Y_0 \subseteq \overline{B}$. Then $M = A \oplus B$ satisfies $(*)$.*

Proof. By 3.1 $\overline{A}$ and $\overline{B}$ are uniserial. As A and B are local and uniform-extending, they are uniserial. If both A and B are not simple, then by 4.6 and 5.1 the decomposition $A \oplus B$ satisfies $(*)$. If both A and B are simple then the Proposition is trivial. Suppose B is simple and A is not simple. Then $A \oplus Top\,B = A \oplus B$ is extending. Again by 5.1 we get the Proposition. □

We prove below an important property of an extending module M with a decomposition satisfying $(*)$.

Proposition 5.5. *Let $M = A \oplus B$ be a decomposition of an extending R-module M satisfying $(*)$. If $B \not\simeq Soc\,A$, then B is A-injective.*

Proof. If A is simple, then the proof is trivial. We assume that A is not simple. Let $f : L \to B$ be a non-zero homomorphism, where $L \subseteq A$. Consider the extension $g : E(A) \to E(B)$ of f and let $U = \{x \in A : g(x) \in B\}$ and $V = \{x + g(x) : x \in U\}$. By 4.3 V is closed in M. Let $M = V \oplus W$. Since $\theta : V \to U$ given by $\theta(x + g(x)) = x$ is an isomorphism, V is uniserial. As the uniform dimension of M is 2, W is indecomposable and hence uniform. By [13, 7.5] W is uniserial. Let $\pi_1 : V \oplus W \to V$ be the projection map.

Case (i): Let L be not simple. Now $Top\,V \simeq Top\,A$ or $Top\,B$.

If $Top\,V \simeq Top\,B$, then $Top\,U \simeq Top\,B$. As $U \supseteq L \neq Soc\,A$, $U = A$ by $(*)$.
Suppose $Top\,V \simeq Top\,A$. Then $Top\,U \simeq Top\,A$. $Top\,g(U) \simeq Top\,A$. By $(*)$ $g(U) = B$ or $Soc\,B$. If $g(U) = B$, then $Top\,U \simeq Top\,B$ and hence $U = A$ by $(*)$.

Suppose $g(U) = Soc\,B$. Then g is not one-one and so $A \cap W = 0$. We note that $\theta\pi_1|_A : A \to U$ is one-one and $\theta\pi_1$ is identity on $Ke\,f$. Also $Ke\,f = Ke\,g$. As $Ke\,f$ is a proper submodule of L, $\theta\pi_1(Ke\,f) = Ke\,f$ is a proper submodule of $\theta\pi_1(L)$. A is uniserial and $g\theta\pi_1(L) = g\theta\pi_1(A) = Soc\,B$ imply $A = L$.

Case (ii): Let $L = Soc\,A$. Then B is not simple. As before we have $M = V \oplus W$, $V \simeq U$, $g : U \to B$ is an extension of f, and g does not have any proper extension to any submodule of A (4.3). If V is simple, then V has exchange property and so either A or B is simple. Hence V and therefore U is not simple. But by case (i), g has an extension to A and hence $U = A$. □

Corollary 5.6. *Let $M = A \oplus B$ be a decomposition of an extending R-module M satisfying $(*)$. Then:*

(1) *if both A and B are not simple, then $A \oplus B$ is π-injective;*

(2) *B is $\overline{A}$-injective and hence $\overline{A}/X_0$-injective, for all $X_0 \subseteq \overline{A}$.*

Proof. (1) follows easily from (5.5).

(2) If B is not simple, then (2) follows from 5.5. Suppose B is simple and L a proper submodule of $\overline{A}$. As the decomposition $A \oplus B$ satisfies $(*)$ there exists no non-zero map from L to B. Hence B is $\overline{A}$-injective. □

Let M be as in Proposition 5.5. We give a sufficient condition for $\overline{A}/X_0$ to be B-injective.

Proposition 5.7. *Assume that the decomposition of an extending module $M = A \oplus B$ satisfies $(*)$. Suppose $\overline{B} \oplus \overline{A}/X_0$, where $X_0 \subseteq \overline{A}$, is π-injective. If $\overline{A}/X_0 \not\simeq Soc\,B$, then $\overline{A}/X_0$ is B-injective.*

Proof. If $Soc\,B = 0$, then $B = \overline{B}$ and trivially $\overline{A}/X_0$ is B-injective. Assume $Soc\,B \neq 0$ and let $f : L \to \overline{A}/X_0$, where $L \subset B$, be a non-zero homomorphism.

If f is not one-one, then f induces a map $g : L/Soc\,B \to \overline{A}/X_0$ which has an extension $h : \overline{B} \to \overline{A}/X_0$. Then ηh is an extension of f, where $\eta : B \to \overline{B}$ is the natural map.

Suppose f is one-one. Then $f(Soc\,B) \neq \overline{A}/X_0$ as $Soc\,B \not\simeq \overline{A}/X_0$. Let $f(L) = T/X_0$. Consider $f^{-1} : T/X_0 \to L$. By 5.6 and f^{-1} can be extended to $\theta : \overline{A}/X_0 \to B$. $\operatorname{Im}\theta \neq Soc\,B$ and the decomposition $B \oplus A$ satisfies $(*)$ implies $\operatorname{Im}\theta = B$. Then θ^{-1} is an extension of f. Hence $\overline{A}/X_0$ is B-injective. □

Proposition 5.8. *Let $M = A \oplus B$, where A and B are local modules. Suppose $A/X \oplus B/Y$ is uniform extending for all $X \subseteq A$ and $Y \subseteq B$. Then $\overline{A}/X_0 \oplus \overline{B}/Y_0$ is π-injective, for all $X_0 \subset \overline{A}$ and $Y_0 \subset \overline{B}$.*

Proof. By 3.1 and 5.1 the decomposition $A \oplus B$ satisfies $(*)$. If both $\overline{A}/X_0$, $\overline{B}/Y_0$ are not simple, then $\overline{A}/X_0 \oplus \overline{B}/Y_0$ is π-injective (5.6). Suppose $\overline{A}/X_0$ is simple. Then $\overline{A}/X_0 \oplus B$ is extending implies $\overline{A}/X_0 \oplus \overline{B}$ is π-injective (4.5). By 4.1 $\overline{A}/X_0 \oplus \overline{B}/Y_0$ is π-injective. The proof is similar in the case when $\overline{B}/Y_0$ is simple. □

Theorem 5.9. *Let $M = A \oplus B$ be an extending R-module such that A, B are local and $\overline{A}/X_0 \oplus \overline{B}/Y_0$ is π-injective for all $X_0 \subseteq \overline{A}$ and $Y_0 \subseteq \overline{B}$. Then:*

(1) *if B/Y and A are not simple, then $B/Y \oplus A$ is π-injective.*

(2) *if $Top\, A \not\simeq Top\, B$, then $\sigma[\overline{M}] = \sigma[\overline{A}] \oplus \sigma[\overline{B}]$ and $\overline{M}$ is $S\pi$ and $F\pi$.*

(3) *if both A and B are not continuous, then $\sigma[A] \cap \sigma[B] = 0$.*

Proof. By Proposition 5.4 the decomposition $M = A \oplus B$ satisfies $(*)$.

(1) follows from 5.6 and 5.7.

(2) By 3.5 $\sigma[\overline{A}] \cap \sigma[\overline{B}] = 0$ if and only if there is no non-zero isomorphism between subfactors of $\overline{A}$ and subfactors of $\overline{B}$. Let $\theta : A_2/A_1 \to B_2/B_1$ be a non-zero isomorphism, where $Soc\, A \subset A_1 \subset A_2 \subset A$ and $Soc\, B \subset B_1 \subset B_2 \subset B$. θ has an extension $g : A/A_1 \to B/B_1$. Then $Top\, A \simeq Top\, B$, a contradiction.

Let $X \subset \overline{M}$. Then $X = X_1 \oplus X_2$, where $X_1 \subset \overline{A}$ and $X_2 \subset \overline{B}$ (3.4). So $\overline{M}$ is Sπ and Fπ.

(3) By 5.3 $Top\, A \not\simeq Top\, B$ and by 5.6 (1) $A \oplus B$ is π-injective. Suppose $X \subseteq A$. We claim that $Top\, X \not\simeq Top\, B$. If $Top\, X \simeq Top\, B$, then $N = Soc\, A$. Since B is not continuous there exists a proper submodule B' of B such that $B' \simeq B$. As A is B-injective the obvious map $f : B' \to X \subseteq A$ can be extended to B. This contradicts that the decomposition $A \oplus B$ satisfies $(*)$. Similarly for any submodule Y of B, $Top\, Y \not\simeq Top\, A$.

Suppose $f : X \to B/Y$ is a non-zero map, where $X \subseteq A$. From the above observation it follows that B/Y is not simple. By (1) f has an extension g to A. Then $Top\, g(A) \simeq Top\, A$, a contradiction. Thus (3) follows. □

6. FE modules which are direct sum of local modules

In this Section we first derive some properties of the module $M = \bigoplus_{i \in I} M_i$, where each M_i is local and $\bigoplus_{i \in I} M_i/X_i$ is uniform extending, for all $X_i \subseteq M_i$. We use these to prove our main Theorem. Suppose $M = \bigoplus_{i \in I} M_i$, where each M_i is local. We show that M^2 is FE if and only if $M^{(n)}$ is FE, for all $n \in \mathbb{N}$. Also $M^{(\mathbb{N})}$ is FE if and only if $M^{(K)}$ if FE, for any set K.

Lemma 6.1. *Let $M = \bigoplus_{i \in I} M_i$ be a uniform extending R-module, where each M_i is local and non-simple. Suppose that $\overline{M_i}/Y_i \oplus \overline{M_j}/Y_j$ is π-injective, for all $Y_i \subseteq \overline{M_i}$, $Y_j \subseteq \overline{M_j}$. Then M is π-injective.*

Proof. It is easy to see that each M_i is cyclic uniserial and the decomposition $M_i \oplus M_j$ satisfies $(*)$, for $i \neq j \in I$ (5.4). Let

$$I_1 = \{i \in I \mid M_i \text{ is continuous}\} \text{ and}$$
$$I_2 = \{i \in I \mid M_i \text{ is not continuous}\}.$$

By 5.6 (1) and 4.9 it is enough to show that $A = \bigoplus_{i \in I_1} M_i$ and $B = \bigoplus_{i \in I_2} M_i$ are π-injective.

For each $i \in I_1$, $End M_i$ is local and hence by 5.6 (1) and 4.2, A is π-injective. 5.9 implies that $\sigma[M_k] \cap \sigma[M_j] = 0$, for $j \neq k \in I_2$. Hence $\sigma[B] = \bigoplus_{i \in I_2} \sigma[M_i]$ (3.5). It is clear that B is π-injective. □

The next Proposition is an important step in proving our main Theorem.

Proposition 6.2. *Let $M = \bigoplus_{i \in I} M_i$, where each M_i is a local module, be such that if $Y_i \subseteq M_i$, for all $i \in I$, then $\bigoplus_{i \in I} M_i/Y_i$ is uniform extending. Suppose for each $i \in I$, $X_i \subseteq M_i$. Then*

(1) *$\bigoplus_{i \in I_1} M_i/X_i$, where $I_1 = \{i \in I \mid M_i/X_i$ is non-simple$\}$, is π-injective.*
(2) *$\bigoplus_{i \in I} M_i/X_i$ is extending;*
(3) *the decomposition $\bigoplus_{i \in I} M_i/X_i$ complements summands;*
(4) *any uniform submodule of $\bigoplus_{i \in I} M_i/X_i$ is a f.c. submodule;*
(5) *if I is finite, then for any $X \subseteq M$, $M/X \simeq \bigoplus_{i \in I} M_i/Y_i$, for some $Y_i \subseteq M_i$;*
(6) *if $f : \overline{M} \to L$ is an onto map, then $f(\overline{M_i})$ is a summand of L, for all $i \in I$;*
(7) *if $Y \subseteq M$, then there exists $Y_i \subseteq M_i$, for all $i \in I$, such that $M/Y \simeq \bigoplus_{i \in I} M_i/Y_i$;*
(8) *for all $Y \subseteq M$, any decomposition of M/Y into indecomposable modules complements summands.*

Proof. We first note that if $Y_i \subseteq \overline{M_i}$ and $Y_j \subseteq \overline{M_j}$, where $i \neq j \in I$, then $\overline{M_j}/Y_j \oplus \overline{M_j}/Y_j$ is π-injective (5.8). Let $I_2 = I \setminus I_1$, where I_1 is as in (1). For $j \in I_1$ and $k \in I_2$, $M_j/X_j \oplus M_k/X_k$ is extending. Hence M_k/X_k is $\overline{M_j/X_j}$-injective (4.5) and hence is $\overline{N}$-injective.

(1) can be easily derived from 6.1 and 5.1.

(2) and (3) follow from 4.6 and 4.5.

(4) Let U be a uniform submodule of $\bigoplus_{i \in I} M_i/X_i$. Then U is essential in a uniform direct summand V of $\bigoplus_{i \in I} M_i/X_i$. By (3) $V \simeq M_i/X_i$, for some $i \in I$ and hence V is cyclic. Therefore U is a f.c. submodule of $\bigoplus_{i \in I} M_i/X_i$.

(5) We use induction on $|I|$, the cardinality of I. If $|I| = 1$, then the result is obvious. Assume that the result is true for all I such that $|I| < n$. Suppose $|I| = n$ and $X \subseteq M$.

If X is not essential in M, then $M = B \oplus C$, where $X \trianglelefteq B$. By (3) the given decomposition of M complements summands and hence $B \simeq \bigoplus_{i \in I'} M_i$, where I'

is a proper subset of I. By induction hypothesis we get the result.

Suppose $X \trianglelefteq M$. Let $A_i = X \cap M_i$, for each $i \in I$, $D = \bigoplus_{i\in I} M_i/A_i$ and $\phi : M \to D$ be the obvious map. Then $\phi(X) \cap M_j/A_j = 0$, for any $j \in I$, and so $\phi(X)$ is not essential in D. By applying the previous case to D we get that $D/\phi(X) \simeq \bigoplus_{i\in I} M_i/Y_i$. As $M/X \simeq D/\phi(X)$ (5) follows.

(6) Let $N_i = \overline{M_i}$ and $N = \bigoplus_{i\in I} N_i$. Fix $i \in I$. Let $A = \sum_{j\in I\setminus\{i\}} f(N_j)$. If $A\cap f(N_i) = 0$, then $f(N_i)$ is a sumand of L. Suppose $0 \neq x \in A\cap f(N_i)$. There exists a finite subset J of $I\setminus\{i\}$ such that $xR \subseteq \sum_{j\in J} f(N_j)$. By (5) $\sum_{j\in J} f(N_j) \simeq \bigoplus_{j\in J} N_j/Y_j$. Since xR is uniform there exists $j \in J$ such that xR is isomorphic to a submodule of N_j/Y_j. As $f(N_i) \oplus N_j/Y_j$ is π-injective and $xR \subseteq f(N_i)$, $f(N_i)$ is isomorphic to a submodule of N_j/Y_j and hence $f(N_i)$-injective. If $j \neq i$, $j \in I$, then $f(N_i)$ is $f(N_j)$-injective and so $f(N_i)$ is A-injective. Therefore $f(N_i)$ is L-injective and a direct summand of L.

(7) As M is extending and the given decomposition of M complements summands ((2) and (3)) we can assume that $Y \trianglelefteq M$. Suppose $N_i = \overline{M_i}$ and $N = \bigoplus_{i\in I} N_i$. It is enough to prove that for $Y \subseteq N$, $N/Y \simeq \bigoplus_{i\in I} N_i/Y_i$, for some $Y_i \subseteq N_i$.

Let X be a proper submodule of N. Consider the natural map $f : N \to N/X$. Consider the collection $\{A_j\}_{j\in J}$ of non-zero submodules of N/X satisfying the following properties:

(i) $J \subseteq I$ and for each $j \in J$, $A_j \simeq N_j/X_j$, a factor module of N_j;

(ii) $\{A_j\}_{j\in J}$ is a local direct summand of N/X;

(iii) $\sum_{j\in J} f(N_j) = A$, where $A = \bigoplus_{j\in J} A_j$.

The collection of such submodules is non-empty as $f(N_i) \neq 0$ for at least one $i \in I$, and for this i, $\{f(N_i)\}$ satisfies the above conditions by (6). By Zorn's lemma we choose a maximal family $\{A_j\}_{j\in J}$ satisfying the above properties. Let $A = \bigoplus_{j\in J} A_j$. We claim that each $f(N_i) \subseteq A$ and hence $A = N/X$. Suppose $f(N_i) \cap A = 0$, for $i \notin J$. Then $\{A_j\}_{j\in J} \cup \{f(N_i)\}$ is a family of submodules of N/X satisfying conditions (i), (ii) and (iii) (using (6)). This contradicts the maximality of $\{A_j\}_{j\in J}$. Let $i \notin J$ and $0 \neq Y = f(N_i) \cap A$. Then $Y \trianglelefteq V$, a direct summand of A. By (4) $V \subseteq \bigoplus_{k\in K} A_k$, where K is a finite subset of J. Hence V is a summand of N/X also. Let $N/X = V \oplus T$. Then $f(N_i) + V = V \oplus L$, where $L = T \cap (f(N_i) + V)$. It is easy to check that $L \cap A = 0$.

Let $p : N/X \to T$ be the projection map along V. Then $pf : N \to T$ is onto and $pf(N_i) = L$. By (6) L is a summand of T and hence a summand of N/X. $L \simeq (f(N_i) + V)/V \simeq f(N_i)/(V \cap f(N_i))$. Hence $L \simeq N_i/X_i$, for some $X_i \subseteq N_i$. Also $f(N_i) \subseteq (A \oplus L)$. If $L \neq 0$, then $\{A_j\}_{j\in J} \cup \{L\}$ is a family subsets satisfying conditions (i), (ii) and (iii), which contradicts the maximality of $\{A_j\}_{j\in J}$. Hence $L = 0$. So $f(N_i) \subseteq A$, for all $i \in I$.

(8) This follows from (7), (3) and [1, 12.5]. □

Theorem 6.3. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module. Then the following are equivalent*:

(a) $\bigoplus_{i\in I} M_i/X_i$ *is uniform-extending, for all* $X_i \subseteq M_i$;

(b) $\bigoplus_{i\in I} M_i/X_i$ *is extending, for all* $X_i \subseteq M_i$;

(c) M *is FE*;

(d) M *is FUE*;

(e) M *is uniform-extending and* $\bigoplus_{i\in I} \overline{M_i}/Y_i$ *is* π*-injective, for all* $Y_i \subseteq \overline{M_i}$;

(f) M *is extending and* $\overline{M}$ *is F*π.

Proof. (a) $\Rightarrow$ (b) follows from 6.2 (2).

(b) $\Rightarrow$ (c) follows from 6.2 (7).

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (e). Let $N_i = \overline{M_i}$. for all $i \in I$ and $A = \bigoplus_{i\in I} N_i/Y_i$. By 6.2 (1) we get that the direct sum of all non-simple N_i/Y_i's is π-injective. By 5.8, for $i \neq j \in I$, $N_i/X_i \oplus N_j/X_j$ is π-injective for all $X_i \subseteq N_i$ and $X_j \subseteq N_j$. By 4.9 A is π-injective.

(e) $\Rightarrow$ (f). By 6.2 (7) applied to $\overline{M}$ we get that any factor module of $\overline{M}$ is π-injective. It remains to prove that M is extending. By 6.1 A, the direct sum of all non-simple M_i's, is π-injective. If M_i is non-simple and M_j is simple, where $i, j \in I$, then $M_i \oplus M_j$ is extending implies that M_j is $\overline{M_i}$-injective (4.5 (1)). Hence M_j is $\overline{A}$-injective. By 4.8 M is extending.

(f) $\Rightarrow$ (a). Let $A = \bigoplus_{i\in I} M_i/X_i$. Define

$$
\begin{aligned}
I_1 &= \{i \in I \mid X_i \neq 0 \text{ and } M_i/X_i \text{ is non-simple}\}.\\
I_2 &= \{i \in I \mid X_i = 0 \text{ and } M_i/X_i \text{ is non-simple}\}.\\
I_3 &= \{i \in I \mid X_i \neq 0 \text{ and } M_i/X_i \text{ is simple}\}.\\
I_4 &= \{i \in I \mid X_i = 0 \text{ and } M_i/X_i \text{ is simple}\}.
\end{aligned}
$$

Let $A_j = \bigoplus_{i\in I_j} M_i/X_i$, for $j = 1, 2, 3$ and 4. Clearly A_1 is π-injective. Also A_2 is π-injective by 6.1. By 5.9, for $i \in I_1$ and $j \in I_2$, M_i/X_i and M_j/X_j are relatively injective. Using 4.9 we get that $A_1 \oplus A_2$ is π-injective.

Let $k \in I_3$. As $A_1 \oplus \overline{A_2} \oplus M_k/X_k$ is π-injective, M_k/X_k is $A_1 \oplus \overline{A_2}$-injective and hence $\overline{A_1} \oplus \overline{A_2}$-injective. Let $k \in I_4$. Since $\bigoplus_{i\in I_1} M_i \bigoplus_{i\in I_2} M_i \oplus M_k/X_k$ is extending M_k/X_k is $(\bigoplus_{i\in I_1} \overline{M_i} \bigoplus_{i\in I_2} \overline{M_i})$-injective (4.5) and hence $\overline{A_1} \oplus \overline{A_2}$-injective. By 4.8 A is extending and hence uniform extending. □

Corollary 6.4. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module. Then the following are equivalent*:

(a) M^2 *is FE*;

(b) M^n *is FE, for all* $n \in \mathbb{N}$.

Proof. (a) $\Rightarrow$ (b). Let

$$K = \{i \in I \mid M_i \text{ is non-simple}\}.$$

Define $N = \bigoplus_{k\in K} M_k$ and $L = \bigoplus_{i\in I\setminus K} M_i$. As M^2 is FE, N^2 is π-injective and hence self-injective. This implies N^n is π-injective (in fact self-injective). Also any simple submodule of L^n is $\overline{N}$-injective and hence $\overline{N^n}$-injective. By 4.8 M^n is extending.

Let $M^n = \bigoplus_{j=1}^n(\bigoplus_{i\in I} M_{ij})$, where $M_{ij} \simeq M_i$, for each $j = i,\dots,n$. By 6.3 (e) it is enough to show that $A = \bigoplus_{j=1}^n(\bigoplus_{i\in I} \overline{M_{ij}}/X_{ij})$ is π-injective, for each $X_{ij} \subseteq \overline{M_{ij}}$. We can write $A = \bigoplus_{j=1}^n A_j$, where each A_j is a factor module of $\bigoplus_{i\in I} \overline{M_{ij}}$. Since M^2 is FE, $A_j \oplus A_k$ is π-injective, for $1 \le j,\ k \le n$. Therefore A is π-injective by (4.1). □

Corollary 6.5. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module. Then the following are equivalent*:

(a) $M^{(\mathbb{N})}$ *is FE*;

(b) $M^{(K)}$ *is FE, for any set K.*

Proof. (a) $\Rightarrow$ (b) Let $N = M^{(k)} = \bigoplus_{j\in J} N_j$, where each $N_j \simeq M_i$, for some $i \in I$. As $N_j/X_j \oplus N_j/X_j$ is FE, $End(N_j/X_j)$ is local (5.3). For any countable subset L of J, $\bigoplus_{i\in L} N_l/X_l$ is extending as it is a factor module of $M^{(\mathbb{N})}$. Hence $\bigoplus_{j\in J} N_j/X_j$ is extending [2, Theorem 2.4]. By 6.3 (b) $M^{(K)}$ is FE.

(b) $\Rightarrow$ (a) is obvious. □

Let M be an R-module. Every simple module in $\sigma[M]$ is isomorphic to a subfactor of M. Hence if M is a self-generator, then M generates every simple module in $\sigma[M]$. For a projective module M in $\sigma[M]$, M generates every simple module in $\sigma[M]$ if and only if M generates every module in $\sigma[M]$.

Lemma 6.6. *Suppose M is an FUE module which is a direct sum of local modules. If M generates every simple module in $\sigma[M]$, then $\overline{M}$ is a homo-serial module.*

Proof. Let $M = \bigoplus_{i\in I} M_i$, where each M_i is local. As M is FUE each M_i is uniserial. Let X be a cyclic proper submodule of M_i. By the hypothesis $Top\,X \simeq Top\,M_j$, for some $j \in I$. For $i \neq j$ and $i,j \in I$, the decomposition $M_i \oplus M_j$ satisfies $(*)$ (5.1). Hence either $X = Soc\,M_i$ or $Top\,X \simeq Top\,M_i$. Thus $\overline{M_i}$ is homo-uniserial. □

Corollary 6.7. *Let M be a direct sum of local modules such that M generates every simple module in $\sigma[M]$. Then M is FUE if and only if M is uniform extending* *and $\overline{M}$ is SFπ.*

Proof. By 6.3 it is enough to prove that if M is FUE, then $\overline{M}$ is SFπ. By 6.6 $M = \bigoplus_{i\in I} M_i$, where each M_i is uniserial and each $\overline{M_i}$ is homo-uniserial. Let $J = \{i \in I \mid \ell(M_i) \geq 3\}$. Suppose $j \in J$, $i \in I$ and $i \neq j$. Then $Top\, M_j \not\simeq Top\, M_i$ as $M_j \oplus Top\, M_i$ is extending. Hence $\sigma[\overline{M_j}] \cap \sigma[\overline{M_i}] = 0$. Thus

$$\sigma[\overline{M}] = \bigoplus_{i\in J} \sigma[\overline{M_i}] \oplus \sigma[\overline{B}],$$

where B is the direct sum of those M_i's which are of length 2. It is easy to see that $\overline{M}$ is SFπ. □

Next we consider the case when M is self-projective.

We recall that a module M is FI if M/X is M-injective, for all $X \subseteq M$. If M is an R-module, then any injective module in $\sigma[M]$ is an epimorphic image of $M^{(I)}$, for some set I. Hence we get the following.

Lemma 6.8. *Let A be a local FI module. Then any uniform injective module in $\sigma[A]$ is a factor module of A and uniserial.*

Proposition 6.9. *Let $M = A^{(I)}$ be an FUE module, where A is local and I is an infinite set. Then A is noetherian.*

Proof. We have $\sigma[M] = \sigma[A]$. By 5.8 $\overline{A}$ is an FI module. Assume $V = \bigoplus_{n\in\mathbb{N}} V_n$ is such that each $V_n \in \sigma[\overline{A}]$ is a uniform $\overline{A}$-injective module. Consider $W = \overline{A} \oplus V$. By 6.8 W is a factor module of M and hence uniform extending. By [4, 8.10] W is self-injective and therefore V is $\overline{A}$-injective. By Wisbauer [15, 27.3] $\overline{A}$ and hence A is noetherian. □

Proposition 6.10. *Let $M = \bigoplus_{i\in I} M_i$ be a self-projective module, where each M_i is local. M is FUE if and only if M is uniform extending and every M-generated subfactor of $\overline{M}$ is π-injective.*

Proof. As M is a direct sum of finitely generated modules and is self-projective M is projective in $\sigma[M]$.

Suppose M is FUE. Then M_i is uniserial for all $i \in I$. As M is projective in $\sigma[M]$, $Top\, M_i \simeq Top\, M_j$ implies that $M_i \simeq M_j$, for $i, j \in I$. Hence $M = \bigoplus_{j\in J} M_j^{(K_j)}$, where $Top\, M_k \not\simeq Top\, M_j$, for $k \neq j$ in J. By 5.9 (2) and 3.5

$$\sigma[\overline{M}] = \bigoplus_{j\in J} \sigma[\overline{M}_j^{K_j}].$$

Let T be an M-generated subfactor of $\overline{M}$. Then $T = Y/X$, $X \subseteq Y \subseteq M$. There exists an onto map $f : M^{(K)} \to T$, where K is a set, and this map can be lifted to $g : M^{(K)} \to Y$. If $Z = \operatorname{Im} g$, then $Y/X \simeq Z/(Z \cap X)$. So without loss of generality we can assume that Y is generated by M.

Now $Y = \bigoplus_{j \in J} Y_j$, where $Y_j \subseteq \overline{M_j^{(K_j)}}$. Let $k, j \in J$ and $k \neq j$. By 5.8 and 5.4 the decomposition $M_k \oplus M_j$ satisfies $(*)$ and hence it is easy to see $Hom(M_k, \overline{M_j}) = 0$. Therefore Y_j is M_j-generated for all $j \in J$. Thus it is enough to prove the case where $M = N^{(K)}$, where N is local and $0 \neq Y$ is M-generated.

If $|K| = 1$, then the result is obvious. Suppose $|K| \geq 2$. In this case $\overline{N}$ is injective and projective in $\sigma[\overline{M}]$. We claim that $Y \simeq \overline{N}^{(K')}$, where $K' \subseteq K$. If $|K|$ is infinite, N and hence $\overline{N}$ is noetherian (6.9). It is enough to show that Y contains a summand isomorphic to $\overline{N}$.

As Y is a non-zero submodule of $\overline{M}$ there exists $0 \neq Z \subseteq \overline{N}$ such that Z is an homomorphic image of Y. As $N \oplus N$ satisfies $(*)$ (5.8 and 5.4), any map from N to $\overline{N}$ is onto. Since Z is M-generated, $Z = \overline{N}$. Since $\overline{N}$ is projective in $\sigma[M]$, Y has a summand isomorphic to $\overline{N}$.

The converse follows from 6.3. □

Corollary 6.11. *Let M be a projective semiperfect module in $\sigma[M]$. Then the following are equivalent*:

(a) *M is FUE*;

(b) *M is uniform extending and every M-generated subfactor of $\overline{M}$ is π-injective.*

Corollary 6.12. *Let M be module of type $\mathcal{A}$* (3.3). *Then the following are equivalent*:

(a) *M is FE*;

(b) *M is extending and every M-generated subfactor of $\overline{M}$ is π-injective*;

(c) *M is exteding and $\overline{M}$ is $F\pi$.*

Proof. (a) $\Rightarrow$ (b) follows from 3.2 and 6.11.
(b) $\Rightarrow$ (c) is trivial and (c) $\Rightarrow$ (a) follows from 3.2 and 6.3. □

Taking $M = R$ we get the following [14, 3.5].

Corollary 6.13. *Let R be a ring. Suppose R_R is of type $\mathcal{A}$* (3.3). *Then the following are equivalent*:

(a) *R_R is FE*;

(b) *R_R is extending and $R/Soc\,R$ is $F\pi$*;

(c) *R_R is extending and $R/Soc\,R$ is $SF\pi$*;

(d) *R_R is extending and $R/Soc\,R$ is a ring direct sum of right uniserial rings* *and a semisimple ring.*

In the next Section we show that a module M of type $\mathcal{A}$ is an FE module if and only if it is an SFE module.

7. SFE modules

Suppose M is an FUE module which is a direct sum of local modules. If every simple module in $\sigma[M]$ is generated by M, then $\overline{M}$ is homo-serial (6.6). In this Section we consider FE modules M which are direct sum of local modules and for which $\overline{M}$ is homo-serial. In this case we show that M is SE if and only if M is SFE, if and only M_i and M_j are relatively projective, for all $i \neq j \in I$ with $\ell(M_i) = \ell(M_j) = 2$. Hence any self-projective, self-generator, FE module which is a direct sum of local modules is SFE.

First we consider some properties of the indecomposable summands of an FE module M which is a direct sum of local modules and $\overline{M}$ is homo-serial.

Lemma 7.1. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is local and $\overline{M_i}$ is homo-uniserial, be an FUE module. Let $k \in I$ be such that $\ell(M_k) \geq 3$. Suppose $T_i = Top\, M_i$ and $S_i = Soc\, M_i$ for all $i \in I$. Then*
(1) $\sigma[\overline{M_k}] \cap \sigma[M_j] = 0$, *for $j \neq k \in I$;*
(2) $\sigma[M_k] \cap \sigma[M_j] = 0$, *if $\ell(M_j) \geq 3$ and $j \neq k \in I$;*
(3)

$$\sigma[M] = \bigoplus_{i\in I_1} \sigma[M_i] \oplus \sigma\left[\bigoplus_{i\in I_2} M_i\right],$$

where $I_1 = \{i \in I \mid \ell(M_i) \geq 3$ and M_i is homo-uniserial$\}$ and $I_2 = I \setminus I_1$;
(4) *M_k and M_j are relatively projective, if $\ell(M_j) \geq 2$ and $j \neq k \in I$.*

Proof. (1) Let M_j be not simple. Then $M_k \oplus T_j$ is extending implies that $T_k \not\simeq T_j$. $M_k \oplus M_j$ is π-injective (6.2 (1)) and $T_k \not\simeq T_j$ gives us $T_k \not\simeq S_j$. As $\overline{M_k}$ and $\overline{M_j}$ are homo-uniserial we get (1).

(2) Now $S_k \not\simeq S_j$ as $M_k \oplus M_j$ is π-injective and $T_k \not\simeq T_j$. So (2) follows from (1),

(3) is an easy consequence of (1).

(4) This follows trivially by (2), if $\ell(M_j) \geq 3$. In the case when $\ell(M_j) = 2$, (4) can be easily proved using (1). □

Proposition 7.2. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is local and $\overline{M_i}$ is homo-uniserial, be an FUE module. Suppose L is any serial submodule (subfactor) of M. Then*

(1) *$L \simeq \bigoplus_{i\in I} L_i$, where each $L_i \subseteq M_i$;*
($L \simeq \bigoplus_{i\in I} X_i/Y_i$, where each $Y_i \subseteq X_i \subseteq M_i$);
(2) *L is π-injective, if L has no simple summand;*
(3) *L is extending and any decomposition of L into indecomposable modules complements summands.*

Proof. Let L be a serial submodule of M.

(1) Suppose $J = \{i \in I \mid M_i$ is not simple$\}$. $A = \bigoplus_{i\in J} M_i$ and $B = \bigoplus_{i\in I\setminus J} M_i$. Then $L = (L\cap B)\oplus T$, where T is isomorphic to a submodule of A. T is also a serial module as any semisimple module has exchange property. Suppose $T = \bigoplus_{\alpha\in\Gamma} T_\alpha$. As A is π-injective (6.2 (1)) and is a direct sum of uniform modules, there exists a family $\{A_\alpha \mid \alpha \in \Gamma\}$ such that each $T_\alpha \trianglelefteq A_\alpha \subseteq A$ and $\bigoplus_{\alpha\in\Gamma} A_\alpha$ is a summand of A [12, Theorem 2.22]. Also any decomposition of A into indecomposables complements summands [12, Theorem 2.22] and so $T \simeq \bigoplus_{k\in J} T_k$, where each $T_k \subseteq M_k$. Thus $L \simeq \bigoplus_{i\in I} L_i$, where each $L_i \subseteq M_i$.

(2) Suppose $L = \oplus_{i\in I} L_i$, where each $L_i \subseteq M_i$. Define

$$\begin{aligned} I_1 &= \{i \in I \mid L_i \text{ is simple}\},\\ I_2 &= \{i \in I \mid L_i \text{ is non-simple and } L_i \neq M_i\},\\ I_3 &= \{i \in I \mid L_i \text{ is non-simple and } L_i = M_i\} \end{aligned}$$

and $U_k = \bigoplus_{i\in I_k} L_i$, for $k = 1, 2$, and 3. For every $i \in I_2$, $\ell(M_k) \geq 3$ and hence by 7.1 (2) U_2 is π-injective. By 6.2 (1) U_3 is π-injective. Suppose $j \in I_2$ and $k \in I_3$. Since $M_j \oplus M_k$ is π-injective, $L_k = M_k$ is L_j-injective. If $\ell(M_k) \geq 3$, then by 7.1 (2), L_j is L_k-injective. If $\ell(M_k) = 2$, then $M_j \oplus M_k$ is π-injective and 7.1 (1) imply that L_j is L_k-injective. Hence by 4.9 $U_2 \oplus U_3$ is π-injective. Thus if L has no simple summand, then L is π-injective.

(3) 7.1 (1) imply that for any $k \in I_1$, L_k is $\overline{U_2 \oplus U_3}$-injective. By 4.8 L is extending and any decomposition of L into indecomposable modules complements summands.

Suppose L is a subfactor of M and $L \simeq Y/X$, where $X \subseteq Y \subseteq M$. Then $M/Y \simeq \bigoplus_{i\in I} M_i/Y_i$ (6.2 (7)) and M/Y satisfies the hypothesis of the Proposition. Hence any subfactor of M also satisfies (1) through (3) of the Proposition. □

Suppose M satisfies the hypothesis of 7.2. We saw that every serial subfactor of M is extending. We prove that the converse is true if M is also a self-generator. We need the following Lemma which can be proved by just imitating the first part of the proof of Proposition 1.5 proved by Garcia and Dung [5].

Lemma 7.3. *Suppose every submodule of a module N is generated by $\{N_i\}_{i\in I}$ and each N_i has ACC on the submodules $\{Kef \mid f \in Hom(N_i, N)\}$. Then any local summand of N is closed in N.*

Proposition 7.4. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is local, be an FE module and a self-generator. Any subfactor T of M is serial if and only if it is extending.*

Proof. By 6.6 each $\overline{M_i}$ is homo-uniserial. It is enough to prove that an extending subfactor T of M is serial (7.2).

By 7.1 (3) we can assume without loss of generality that if $\ell(M_i) \geq 3$, then $Soc\, M_i$ is simple and $\not\simeq Top\, M_i$.

Let $T \simeq X/Y$, where $Y \subseteq X \subseteq M$. Now $M/Y \simeq \bigoplus_{i\in I} M_i/Y_i$ by 6.2 (7). Any indecomposable submodule of M/Y is uniform and hence uniserial. It is enough to show that T is a direct sum of indecomposable modules.

Let $J = \{i \in I \mid Y_i \neq 0 \text{ and } \ell(M_i/Y_i) \geq 3\}$. Suppose $K = I \setminus J$. If $k \in K$ and $\ell(M_k/Y_k) \geq 3$, then $Y_k = 0$. By 7.1 (3) applied to M/Y we have

$$\sigma[M/Y] = \bigoplus_{j\in J} \sigma[M_j/Y_j] \oplus \sigma\left[\bigoplus_{k\in K} M_k/Y_k\right].$$

We have $X/Y = \bigoplus_{j\in J} X_j/Y_j \oplus Z$, where Z is a submodule of $\bigoplus_{k\in K} M_k/Y_k$ and for all $j \in J$, $Y_j \subseteq X_j \subseteq M_j$.

We note that $\{M_i\}_{i\in I}$ generates every submodule of Z. By 7.3 and [12, 2.17] Z will be a direct sum of indecomposable modules, if for all $i \in I$, M_i has ACC on $\{Ke\, f \mid f \in Hom(M_i, Z)\}$.

Fix $i \in I$. Let $f : M_i \to Z$ be a map, for some $i \in I$. Since M_i is uniserial $f(M_i) \trianglelefteq U$, a uniform summand of $\bigoplus_{k\in K} M_k/Y_k$. As the decomposition $\bigoplus_{k\in K} M_k/Y_k$ complements summands (6.2 (3)), $U \simeq M_k/Y_k$ for some $k \in K$. Suppose $\ell(M_i) \geq 3$. By 7.1 (1) and (2) we get that, for $k \neq i$, $Hom(M_i, M_k/Y_k) = 0$ (7.1). If $k = i$ and $\ell(M_k/Y_k) \geq 3$, then any non-zero $f : M_i \to M_k/Y_k$ must be a monomorphism, for in this case $Y_k = 0$ and $Top\, M_i \not\simeq Soc\, M_i$. Thus M_i has ACC on $\{Ke\, f \mid f \in Hom(M_i, Z)\}$. Therefore Z and hence T is serial. □

Next we show that if M satisfies the hypothesis of 7.2 and is also an SE module, then any submodule of M is serial.

Proposition 7.5. *Let $M = \bigoplus_{i\in I} M_i$ be a module such that each M_i is a local module and $\overline{M_i}$ is homo-uniserial. If M is SE and FE, then any submodule N of M is isomorphic to $\bigoplus_{i\in I} N_i$, where each $N_i \subseteq M_i$.*

Proof. Any indecomposable submodule of M is uniform and hence uniserial [13, 7.5]. It is enough to prove that N is a direct sum of indecomposable modules (7.2). Suppose $J = \{i \in I \mid M_i \text{ is not simple}\}$. Let $A = \bigoplus_{i\in J} M_i$ and $B = \bigoplus_{i\in I\setminus J} M_i$. If $N \subseteq M$, then $N = (N \cap B) \oplus L$, where L is isomorphic to a submodule of A. Hence without loss of generality we can assume that $N \subseteq A$, and that if $j \in J$ and $\ell(M_j) \geq 3$, then $Soc\, M_j$ is simple and $\not\simeq Top\, M_j$ (7.1 (3)). Every submodule of A is extending and any cyclic submodule of A has finite dimension and hence is a direct sum of uniform modules. As any uniform submodule of A is isomorphic to a submodule of M_j, for some $j \in J$, the collection of all cyclic submodules of M_j, for all $j \in J$, generates every submodule of A. By 7.3 it is enough to show that every cyclic submodule N_j of M_j, $j \in J$, has ACC on the submodules $\{Ke\,f \mid f \in Hom(N_j, N)\}$. Since N_j is uniserial $f(N_j)$ is uniserial and hence isomorphic to a submodule of M_i, $i \in J$, i may be equal to j. If $\ell(N_j) \geq 3$, tnen $0 \neq Soc\, M_j \not\simeq Top\, M_j$ and 7.1 (1) gives us that f is either a zero map or a monomorphism. Thus N is a direct sum of indecomposables. □

We next prove the main theorem of this section.

Theorem 7.6. *Let $M = \bigoplus_{i \in I} M_i$ be an FE module such that each M_i is a local module and each $\overline{M_i}$ is homo-uniserial. Then the following are equivalent*:
(a) *M is SFE*;
(b) *M_i and M_j are relatively projective, for all $i \neq j \in I$ with $\ell(M_i) = \ell(M_j) = 2$*;
(c) *the direct sum of the non-simple M_i's is quasi-discrete*;
(d) *M is SE.*
In this case any subfactor T of M is serial and if every indecomposable summand of T is non-simple local, then T is quasi-discrete.

Proof. (a) ⇒ (b). Suppose $i \neq j \in I$ and $\ell(M_j) = \ell(M_i) = 2$. Let $N = M_i \oplus M_j$. Then N is SFE and hence is π-injective and SE. Suppose X is not small in N. Then X properly contains $Soc\, N$. Since X is extending X contains an indecomposable summand of length 2 and this is also a summand of N. Hence N is lifting. It is easy to see if A and B are proper summands of N such that $N = A + B$, then $A \cap B = 0$, and hence is trivially a summand of N. Thus N is quasi-discrete. By [12, 4.48], M_i and M_j are relatively projective.

(b) ⇒ (c). Let $J = \{i \in I \mid M_i \text{ not simple}\}$. Let $A = \bigoplus_{i \in J} M_i$ and $B = \bigoplus_{i \in I \setminus J} M_i$. By 7.1 (4) and (b), for $i \neq j \in J$, M_i and M_j are relatively projective. As the above decomposition of A complements summands, A is quasi-discrete [12, 4.53].

(c) ⇒ (d). It is enough to show that every submodule of M is serial (7.2). Let X be submodule of M. Define the summands A and B of M as in the proof of (b) ⇒ (c). Then $X = (X \cap B) \oplus Y$, where $Y \simeq Z \subseteq A$. Thus it is enough to show that every submodule of A is serial.

Define $J_1 = \{j \in J \mid \ell(M_j) \geq 3\}$ and $J_2 = J \setminus J_1$. For $k \in J_1$ and $j \in J_2$, $Rad\, M_j = Soc\, M_j \not\simeq Soc\, M_k$ or $Top\, M_k$ by 7.1 (1) and the fact that $M_j \oplus M_k$ is

π-injective. Hence

$$\sigma[Rad\,A] = \bigoplus_{j\in J_1} \sigma[Rad\,M_j] \oplus \sigma\left[\bigoplus_{j\in J_2} Rad\,M_j\right].$$

Therefore any small submodule of A is serial. As any decomposition of A ito indecomposables complements summands (8.6 (7)) any direct summand of A is serial. Since A is a lifting module any submodule of A is serial.

(d) $\Rightarrow$ (a). Let $X \subseteq M$. It is enough to show that $Z = M/X$ is SE. By 6.2 (7) $Z \simeq \bigoplus_{i\in I} M_i/X_i$. Define sets I_j and modules A_j, for $j = 1,2,3,4$, as in (f) $\Rightarrow$ (a) of Theorem 6.3. Then $Z = A_1 \oplus A_2 \oplus A_3 \oplus A_4$. Now 7.1 (1) and 7.1 (2) imply that $\sigma[A_1] \cap \sigma[A_2 \oplus A_3 \oplus A_4] = 0$ and that A_1 is SE. So it is enough to prove that $L = A_2 \oplus A_3 \oplus A_4$ is SE.

Let $Y \subset L$. $Y = (Y \cap (A_3 \oplus A_4)) \oplus C$ and C is isomorphic to a submodule of A_2. We note that A_2 is SE and FE and hence by 7.5 any submodule of A_2 is $\simeq \bigoplus_{i\in I_2} C_i$, where each $C_i \subseteq M_i$. Thus Y is serial and hence by 7.2, Y is extending.

By 7.5 applied to factor modules of M, we get that any subfactor T of M is serial. By (a) $\Longleftrightarrow$ (c) of Theorem applied to T, we get that T is quasi-discrete.

□

Corollary 7.7. *Let $M = \bigoplus_{i\in I} M_i$ be an FE module such that each M_i is a local module and $\overline{M_i}$ is homo-uniserial. If for each $i \in I$, $\ell(M_i) \neq 2$, then M is an SFE module.*

Using 6.6 we get the following Corollary.

Corollary 7.8. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module, be an FE module and a self-generator. Then conditions* (a) *through* (d) *of Theorem* 7.6 *are equivalent.*

Corollary 7.9. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is local, be a self-generator and self-projective module. Then M is SFE if and only if M is FE.*

Proof. In this case obviously M_i and M_j are relatively projective for $i \neq j \in I$ and the proof follows from 7.6. □

Corollary 7.10. *Suppose M is a module of type $(\mathcal{A})$ and is a self generator. Then M is FE if and only if M is SFE.*

Corollary 7.11. *Let R be a ring of type $(\mathcal{A})$. Then R is a right FE ring if and only if R is a right SFE ring.*

8. Fπ and SFπ modules

Finitely generated self-projective Fπ modules were studied by Huynh and Wisbauer in [8] and semiperfect Fπ rings were studied by Goel and Jain in [6]. Suppose $M = \bigoplus_{i\in I} M_i$, where each M_i is local. We show that M is Fπ if and only if $\bigoplus_{i\in I} M_i/X_i$ is π-injective, for all $X_i \subseteq M_i$, and if also M is a self-generator, then M is SFπ. M^2 is Fπ if and only if M^n in Fπ, for all $n \in \mathbb{N}$. $M^{(\mathbb{N})}$ is Fπ if and only if $M^{(K)}$ is Fπ, for any set K, if and only if M is locally noetherian and Fπ. We also study modules M such that M is a projective Fπ module in $\sigma[M]$ and is a direct sum of indecomposable modules which are not necessarily local modules.

The following Lemma has been proved by Huynh and Wisbauer in [8].

Lemma 8.1. *Let $M = \bigoplus_{i\in I} M_i$, where each M_i is uniform module, be an Fπ module. Then every non-zero $f \in Hom(M_i, M_j)$, with $i \neq j$ is an epimorphism. If M_j is M_i-projective, then f is an isomorphism.*

Lemma 8.2. *Let $M = M_1 \oplus M_2$, where M_1, M_2 are local and $Top\, M_1 \not\simeq Top\, M_2$, be a F$\pi$ module. Then $\sigma[M_1] \cap \sigma[M_2] = 0$*

Proof. Suppose $f : X \to M_2/Y$ be a non-zero map, where $X \subseteq M_1$. Then f has an extension to M_1, which must be an onto map by 8.1. This contradicts the fact that $Top\, M_1 \not\simeq Top\, M_2$. □

Proposition 8.3. *Let $M = \bigoplus_{i\in I} N_i$, $N_i = \bigoplus_{j\in K_i} M_{i_j}$, where each M_{i_j} is local and $Top\, M_{i_j} \simeq Top\, M_{k_l}$ if and only if $i = k$, for all $i, k \in I$, $j \in K_i$, $l \in K_k$. Then thie following are equivalent:*

(a) *M is Fπ;*

(b) (i) *$\sigma[M] = \bigoplus_{i\in I} \sigma[N_i]$;*

(ii) *each N_i is Fπ;*

(c) *$\bigoplus_{i\in I} \bigoplus_{j\in K_i} M_{i_j}/X_{i_j}$ is π-injective for all $X_{i_j} \subseteq M_{i_j}$.*

Proof. (a) $\Rightarrow$ (b) follows from 8.2 and 3.5.
(b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (c) are trivial.
(c) $\Rightarrow$ (a) follows by 6.2 (7). □

Now we give equivalent condition for a module which is a direct sum of local modules to be an FI module. We recall that a module M is called an FI module if every factor module of M is injective in $\sigma[M]$, i.e. M-injective.

Proposition 8.4. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of local modules. Then the following are equivalent:*

(a) *M is FI;*

(b) M^n *is FI, for all* $n \in \mathbb{N}$;
(c) M^n *is F*π, *for all* $n \in \mathbb{N}$;
(d) M^2 *is F*π.

Proof. (a) $\Rightarrow$ (b). Let $M^n = \bigoplus_{j\in J} N_j$, where each $N_j = M_i$, for some $i \in I$. We have $\bigoplus_{j\in J} N_j/X_j \simeq \bigoplus_{i=1}^n M/A_i$. As each M/A_i is M-injective, $\bigoplus_{j\in J} N_j/X_j$ is M-injective. By 6.2 (7) any factor module of M^n is of the form $\bigoplus_{j\in J} N_j/X_j$. Hence M^n is FI.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (a) is trivial since, for any $X \subseteq M$, $M/X \oplus M$ is π-injective. □

Proposition 8.5. *Let* $M = \bigoplus_{i\in I} M_i$, *where each* M_i *is a local module. Then the following are equivalent*:

(a) $M^{(\mathbb{N})}$ *if F*π *and hence FI*;
(b) M *is locally noetherian and is FI*;
(c) $M^{(K)}$ *if F*π *and hence FI, for any infinite set* K.

Proof. (a) $\Rightarrow$ (b). Each M_i is noetherian (6.9). Therefore M is locally noetherian.

(b) $\Rightarrow$ (c). Let J be any infinite set and let $L = \bigoplus_{j\in J} A_j$ be such that, for each $j \in J$, $A_j \simeq \bigoplus_{i\in I} M_i/X_{j_i}$. Then each A_j is M-injective as it is a factor module of M. Since M is locally noetherian, L is M-injective. By 6.2 (7) applied to $M^{(J)}$ we get that $M^{(J)}$ is Fπ.

(c) $\Rightarrow$ (a) is clear. □

Proposition 8.6. *Let* M *be a direct sum of local modules such that* M *is a self-generator.* M *is an F*π *module if and only if* M *is an SF*π *module.*

Proof. Suppose M is an Fπ module. By 8.3 it is enough to prove the case when $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module and $Top\, M_i \simeq Top\, M_j$, for all $i, j \in I$. As M generates any simple module in $\sigma[M]$, each M_i is homo-uniserial (6.6). If M_i is not simple, then $M_i \oplus Top\, M_j$ is not π-injective. Hence either M is homo-uniserial or semisimple. Therefore M is an SFπ-module. □

From the proof of Proposition 8.6 we get

Corollary 8.7. *Let* M *be sefl-generator and of type* $\mathcal{A}$. *Then the following are equivalent*:

(a) M *is F*π;
(b) M *SF*π;
(c) M *is a direct sum of fully invariant submodules which are either homo-*

uniserial or semisimple.

Taking $M = R$, we get the following result in which the equivalence of (a) and (c) has been proved in Goel and Jain [6, Theorem 2.4].

Corollary 8.8. *Let R be a ring of type $\mathcal{A}$. Then the following are equivalent:*
(a) *R_R is Fπ;*
(b) *R_R is SFπ;*
(c) *R is a direct sum of rings which are right uniserial or semisimple.*

In general an FI module need not be an SFE module. For example the $\mathbb{Z}$-module $\mathbb{Q}\oplus\mathbb{Q}$ is FI but not SFE. In the following we consider Fπ and SFπ modules M which are projective in $\sigma[M]$ and is a direct sum of indecomposable modules which are not necessarily local modules. By [4, 9.3] if M is a finitely generated FE module which is projective in $\sigma[M]$, then M is a direct sum of uniform modules.

The decomposition of an Fπ finitely generated self-projective module M is studied by Huynh-Wisbauer in [8]. They do this by grouping together the indecomposable summands whose endomorphism rings are division ring and the indecomposable summands whose endomorphism rings are not division ring. We prefer to group together the indecomposables whose endomorphism rings are local and those whose endomorphism rings are not local.

Proposition 8.9. *Let M be an R-module which is projective in $\sigma[M]$ and is a direct sum of indecomposables. The following are equivalent:*
(a) *M is Fπ;*
(b) *There exists a decomposition*

$$M = \bigoplus_{i\in I} N_i^{(K_i)} \oplus \bigoplus_{j\in J} U_j,$$

where each $N_i^{(K_i)}$ is Fπ with $End\, N_i$ a local ring and each U_j is uniform with $End\, U_j$ and a local ring, such that

$$\sigma[M] = \bigoplus_{i\in I} \sigma[N_i^{(K_i)}] \oplus \bigoplus_{j\in J} \sigma[U_j].$$

If further M is a self-generator, then $|K_i| = 1$, for all $i \in I$.

Proof. We first note that if M is finitely generated, then the assumption that M is a direct sum of indecomposables is superfluous [4, 9.3].

(a) $\Rightarrow$ (b). Let $M = \bigoplus_{k\in K} M_k$, where each M_k is indecomposable. Let N be the direct sum of all M_k's whose endomorphism rings are local (and hence are local modules) and L be the direct sum of the remaining summands. As M is projective, we can write $N = \bigoplus_{i\in I} N_i^{(K_i)}$ such that each N_i is local and $Top\,N_i \not\simeq Top\,N_j$, for all $i \neq j \in I$. By 8.3 $\sigma[N] = \bigoplus_{i\in I} \sigma[N_i^{(K_i)}]$. Let $L = \bigoplus_{j\in J} U_j$, where each U_j is indecomposable and $End\,U_j$ is not a local ring. It is enough to show that $\sigma[N_i] \cap \sigma[U_j] = 0$ and $\sigma[U_k] \cap \sigma[U_j] = 0$, for all $i \in I$ and $j \neq k \in J$.

Let $i \in I$ and $j \in J$. Suppose $Y \subseteq N_i$ and $f : Y \to U_j/X$ is a non-zero map. Then f can be extended to N_i. As N_i is projective in $\sigma[M]$ we get a non-zero map from $N_i \to U_j$. By 8.1 the above map must be an isomorphism, a contradiction. Hence $\sigma[N_i] \cap \sigma[U_j] = 0$.

Let $k \neq j$ and $k, j \in J$. As $End\,U_j$ is not local, U_j is not continuous and hence contains a proper submodule X isomorphic to it self. Suppose $f : U_j \to U_k$ is a non-zero homomorphism. By 8.1 f is an isomorphism. But then $f|_X : X \to U_k$ is not an isomorphism. Hence $Hom(U_j, U_k) = 0$. It is easy to verify that $\sigma[U_j] \cap \sigma[U_k] = 0$, for all $j \neq k \in J$.

(b) $\Rightarrow$ (a) is easy to prove.

If M is a self-generator, then each N_i must be homo-uniserial and hence $|K_i| = 1$, for all $i \in I$. □

Corollary 8.10. *Let M be as in* 8.9. *Then*

(1) *M^2 is $F\pi$ if and only if M^n is FI, for all $n \in \mathbb{N}$.*

(2) *$M^{(\mathbb{N})}$ is $F\pi$ if and only if M is locally noetherian and FI, if and only if $M^{(K)}$ is FI, for any set K.*

Proof. From 8.9 every indecomposable summand of M is local. The Corollary follows from 8.4 and 8.5. □

Corollary 8.11. *Suppose M is finitely generated and projective in $\sigma[M]$. If M^2 is an $F\pi$ module, then M is semiperfect in $\sigma[M]$.*

Proposition 8.12. *Let M be a projective module in $\sigma[M]$ such that $M = \bigoplus_{j\in J} M_j \oplus K$, where each M_i is indecomposable and non-simple, and K is semisimple. The following are equivalent*:

(a) *M is $SF\pi$;*

(b) *$\sigma[M] = \bigoplus_{j\in J} \sigma[M_j] \oplus \sigma[K]$ and M_j is $SF\pi$, for all $j \in J$.*

If M is finitely generated, then the assumption that M is a direct sum indecomposables is superfluous.

Proof. (b) $\Rightarrow$ (a) is obvious and we prove (a) $\Rightarrow$ (b). Using 8.9 we see that (b) follows if we prove that if A is a local non-simple module, then $A \oplus A$ is not SFπ. Let B be a cyclic proper submodule of A. As A is FE, A is uniserial. $A \oplus Top\,B$ is π-injective. But the map $f : B \to Top\,B$ cannot be extended to A, a contradiction. □

Taking $M = R$ we get

Corollary 8.13. *Let R be a ring. Then the following are equivalent*:
(a) *R is a right SFπ ring*;
(b) *R is a ring direct sum of rings R_i's, where each R_i as a right R-module is either a uniform SFπ module or a semisimple module.*

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