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On the Geometry of Hopf Manifolds

By Mikio ISE

§1. Introduction

The purpose of the present note is to compute the cohomology groups $H^q(X, \Omega^p(F))$, ($0 \leq q \leq n$) of an n -dimensional Hopf manifold X , where $\Omega^p(F)$ denotes the analytic sheaf of germs of holomorphic p -forms with values in a complex line bundle F over X . Throughout the arguments we make use of the fact that the Hopf manifold is a homogeneous compact complex manifold. Recently, R. Bott [3] and the author [7] have made some researches concerning the complex line bundles over a class of homogeneous compact complex manifolds (= C -manifolds in the sense of Wang). The essential difference between Hopf manifolds and Wang's C -manifolds lies in the non-triviality of the fundamental group of the former. But a Hopf manifold admits the so-called *Hopf fibering*, which plays the quite analogous role to the fundamental fiberings of non-kählerian C -manifolds (cf. [7]), and which allows us to have the similar results for them. Our principal tools are Leray's spectral sequences and the knowledge of the cohomology groups $H^q(P^n, \Omega^p(F))$ of the n -dimensional complex projective space P^n , which have been computed by Bott [3] and Matsumura [10].

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§2. Hopf manifolds

We recall here the definition of Hopf manifolds (cf. [6]). Let C^n denote a complex n -dimensional euclidean space and W^n the complement of the origin $o=(0, \dots, 0) \in C^n$, and take a non-zero complex number d with the absolute value $|d| \neq 1$. Let Δ_d be the cyclic group generated by d in the multiplicative group C^* of all non-zero complex numbers. We denote also by Δ_d , for brevity, the subgroup of $GL(n, C)$ generated by the scalar matrix $d \cdot I$ (I is the unit matrix of $GL(n, C)$):

$$\Delta_d = \{d^m \cdot I \mid m \in Z\}$$

(Z is the ring of all integers)

The group Δ_d being a properly discontinuous group of W^n without fixed points, the quotient manifold $W^n/\Delta_d=X$ has a natural complex analytic structure. The complex manifold X thus obtained is, by definition, an *n-dimensional Hopf manifold* (corresponding to the number d)¹⁾. As is well known, X is diffeomorphic to the product manifold $S^1 \times S^{2n-1}$ of two odd dimensional spheres, since the group Δ_d is isomorphic to Z . The complex general linear group $GL(n, C)$ operates on W^n effectively and transitively; as Δ_d is contained in the centre of $GL(n, C)$, $GL(n, C)$ operates on X also transitively and holomorphically. Now we take the basic point $w_0=(1, 0, \dots, 0) \in W^n$ and the corresponding point $x_0 \in X$ modulo Δ_d . Then the isotropy subgroup U_d of $GL(n, C)$ at x_0 consists of the matrices of the form :

$$u = \left(\begin{array}{c|ccc} d^m & * & \dots & * \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & * \end{array} \right), \quad d^m \in \Delta_d.$$

So we can identify X with the complex coset space :

$$X = GL(n, C)/U_d.$$

Note that the action of $GL(n, C)$ is not effective; if we take the quotient groups $\tilde{G}=GL(n, C)/\Delta_d$ and $\tilde{U}_d=U_d/\Delta_d$, then \tilde{G} acts on X effectively and we can put

$$X = \tilde{G}/\tilde{U}_d.$$

The connected complex reductive Lie group \tilde{G} is considered as a Lie subgroup of the connected analytic automorphism group $A(X)$ of X . As a matter of fact, we show the following

Theorem 1. $A(X)=\tilde{G}$.

The proof is quite similar to the case of non-kählerian C -manifolds ([7], Proposition 6), but we state it here for the completeness sake and for the convenience in the later discussions. First we need some definitions :

DEFINITION 1. Let $G(1, n-1; C)$ denote the subgroup of $GL(n, C)$ consisting of the matrices :

1) The 1-dimensional Hopf manifold in this definition is nothing but an elliptic curve T^1 . We shall exclude this trivial case in the sequel; therefore assume $n \geq 2$.

$$\left(\begin{array}{c|cccc} * & 0 & \dots & 0 \\ \hline * & & & \\ \vdots & & & \\ * & & * & \end{array} \right).$$

Then we can identify $GL(n, C)/GL(1, n-1; C)$ with the $(n-1)$ -dimensional complex projective space P^{n-1} and $GL(1, n-1; C)/U_a$ with an elliptic curve T^1 respectively. Therefore we have a natural (holomorphic) principal fibering of X with P^{n-1} as base, T^1 as group and the natural mapping ϕ of X onto P^{n-1} as projection; this fibering $X(P^{n-1}, T^1, \phi)$ is called the *Hopf fibering* of X .

DEFINITION 2. Let $X=G/U$ be a complex homogeneous space with a connected complex Lie group G and a (not nec. connected) closed complex Lie subgroup U , and let ρ be a holomorphic homomorphism of U into another complex Lie group B . Then the coset bundle $G(X, U, \pi)$ (π is the canonical projection of G onto X) and ρ induce a new holomorphic principal bundle $P(X, B, \varpi)$ over X , which is called a *homogeneous B-bundle over X with respect to the Klein form G/U*. In particular, when $B=GL(m, C)$, we have the associated m -dimensional vector bundle $E(X, C^m, GL(m, C), \varpi)$ which is simply written as $E_X(\rho, C^m)$ and which is called a *homogeneous vector bundle over X (with respect to the Klein form G/U)*. This is the quotient space of $G \times C^m$ by the equivalence relation :

$$(g, \xi) \sim (gu, \rho(u^{-1})\xi)$$

for $g \in G, u \in U$ and $\xi \in C^m$.

Now we denote the Lie algebras of $GL(n, C), GL(1, n-1; C)$ and U_a by $\mathfrak{g}, \hat{\mathfrak{u}}$ and \mathfrak{u} respectively. Then \mathfrak{u} is an ideal of $\hat{\mathfrak{u}}$ and the exact sequence of modules :

$$(1) \quad 0 \rightarrow \hat{\mathfrak{u}}/\mathfrak{u} \rightarrow \mathfrak{g}/\mathfrak{u} \rightarrow \mathfrak{g}/\hat{\mathfrak{u}} \rightarrow 0$$

is considered as an exact sequence of U_a -modules under the adjoint actions. Therefore we can construct the exact sequence of homogeneous vector bundles over X with respect to the Klein form $GL(n, C)/U_a$:

$$0 \rightarrow E_X(Ad, \hat{\mathfrak{u}}/\mathfrak{u}) \rightarrow E_X(Ad, \mathfrak{g}/\mathfrak{u}) \rightarrow E_X(Ad, \mathfrak{g}/\hat{\mathfrak{u}}) \rightarrow 0.$$

If we denote here by Θ and $\hat{\Theta}$ the tangential vector bundles over X and $\hat{X}=P^{n-1}$ respectively, then, as is easily verified,

$$E_X(Ad, \mathfrak{g}/\mathfrak{u}) = \Theta, \quad E_X(Ad, \mathfrak{g}/\hat{\mathfrak{u}}) = \phi^*\hat{\Theta}$$

and $E_X(Ad, \hat{\mathfrak{u}}/\mathfrak{u})$ is the trivial line bundle (which is denoted by I).

Therefore we have the exact sequence :

$$(2) \quad 0 \rightarrow I \rightarrow \Theta \rightarrow \phi^*\hat{\Theta} \rightarrow 0.$$

This induces the corresponding exact sequence of the complex vector spaces of cross sections :

$$0 \rightarrow \Gamma_X(I) \rightarrow \Gamma_X(\Theta) \rightarrow \Gamma_X(\phi^*\hat{\Theta}).$$

Here, $\Gamma_X(\Theta)$ may be identified with the Lie algebra $\mathfrak{a}(X)$ of all holomorphic vector fields on X , and $\Gamma_X(I)$ is then identified with the 1-dimensional ideal C of $\mathfrak{a}(X)$. Moreover $\Gamma_X(\phi^*\hat{\Theta})$ is isomorphic with $\Gamma_{\hat{X}}(\hat{\Theta})$ or with the Lie algebra $\mathfrak{a}(P^{n-1})$ of all holomorphic vector fields on P^{n-1} , since the fibres of $\phi: X \rightarrow P^{n-1}$ are compact, connected. Hence, we obtain the exact sequence of Lie algebras :

$$0 \rightarrow C \rightarrow \mathfrak{a}(X) \xrightarrow{\dot{\phi}} \mathfrak{a}(P^{n-1}),$$

where the homomorphism $\dot{\phi}$ means that every holomorphic vector field on X is constant on each fibre and that it induces a vector field over P^{n-1} (This fact has been recognized by Blanchard [2] in general case; see [2], Proposition 1.1). Now we know that $\mathfrak{a}(P^{n-1})$ is isomorphic to the Lie algebra $\mathfrak{sl}(n, C)$ of $SL(n, C)$ and that $\mathfrak{a}(X)$ contains the Lie algebra \mathfrak{g} of \tilde{G} which is clearly isomorphic to that of $GL(n, C)$. Therefore $\dot{\phi}$ is surjective and $\mathfrak{a}(X) = \mathfrak{g}$. Hence $A(X) = \tilde{G}$. This proves Theorem 1.

Here we add the following theorem which can be proved in the same way as in [7], Proposition 7 (see, also Remark 2 in § 6).

Theorem 2. $\dim H^1(X, \Theta) = n^2$ and $H^q(X, \Theta) = \{0\}$ for $q \geq 2$.

The first result is known by Kodaira-Spencer [8] in the case $n=2$. The discussions in [8], § 15 about deformations of complex analytic structures of a 2-dimensional Hopf manifold might be immediately extended to the case $n > 2$.

§ 3. Complex line bundles

In this section we shall concern ourselves with the homogeneous line bundles over a Hopf manifold X . For this sake, it is more appropriate to take the Klein form $GL(n, C)/U_d$ rather than to take G/U_d . We call, from now on, a homogeneous bundle with respect to the Klein form $GL(n, C)/U_d$ simply a *homogeneous bundle*. Then we have

Theorem 3. *Every complex line bundle over a Hopf manifold X is*

homogeneous and has an integrable holomorphic connection. Moreover the group $H^1(X, \mathbf{C}^*)$ of all complex line bundles over X is isomorphic to \mathbf{C}^* .

Proof. First consider the sheaf exact sequence over X :

$$(1) \quad 0 \rightarrow Z \rightarrow \mathbf{C} \xrightarrow{\varepsilon} \mathbf{C}^* \rightarrow 0,$$

where \mathbf{C} (resp. \mathbf{C}^*)²⁾ denotes the sheaf of germs of holomorphic (resp. non-vanishing holomorphic) functions on X , Z the constant sheaf of integers and ε the homomorphism induced from the homomorphism ε of \mathbf{C} onto \mathbf{C}^* defined by $\varepsilon(\xi) = \exp 2\pi\sqrt{-1}\xi$ for any $\xi \in \mathbf{C}$. Because $H^2(X, Z) = \{0\}$, we have from (1) the following exact sequence:

$$0 \rightarrow H^1(X, Z) \rightarrow H^1(X, \mathbf{C}) \xrightarrow{\varepsilon} H^1(X, \mathbf{C}^*) \rightarrow 0.$$

On the other hand, we consider the exact sequence of the abelian groups:

$$(2) \quad 0 \rightarrow \text{Hom}(U_d, Z) \rightarrow \text{Hom}(U_d, \mathbf{C}) \xrightarrow{\varepsilon} \text{Hom}(U_d, \mathbf{C}^*),$$

where $\text{Hom}(U_d, B)$ means the abelian group of all holomorphic homomorphisms of U_d into the complex abelian Lie group B (If B is discrete, then holomorphic homomorphisms should be understood as abstract ones). We see readily that $\text{Hom}(U_d, B) \cong \text{Hom}(\Delta_d \times GL(n-1, \mathbf{C}), B) \cong \text{Hom}(\Delta_d \times \mathbf{C}^*, B) \cong \text{Hom}(\Delta_d, B) \times \text{Hom}(\mathbf{C}^*, B)$; hence we have

$$(3) \quad \begin{cases} \text{Hom}(U_d, Z) \cong \text{Hom}(\Delta_d, Z) \cong Z, \\ \text{Hom}(U_d, \mathbf{C}) \cong \text{Hom}(\Delta_d, \mathbf{C}) \cong \mathbf{C}, \\ \text{Hom}(U_d, \mathbf{C}^*) \cong \text{Hom}(\Delta_d, \mathbf{C}^*) \times \text{Hom}(\mathbf{C}^*, \mathbf{C}^*) \cong \mathbf{C}^* \times Z. \end{cases}$$

Now there is a natural homomorphism η_B of the group $\text{Hom}(U_d, B)$ into the group $H^1(X, \mathbf{B})$ of all holomorphic B -bundles over X by assigning to every $\rho \in \text{Hom}(U_d, B)$ the corresponding homogeneous B -bundle over X defined by ρ . Then we have the commutative diagram:

$$(4) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(U_d, Z) & \rightarrow & \text{Hom}(U_d, \mathbf{C}) & \rightarrow & \text{Hom}(U_d, \mathbf{C}^*) \\ & & \downarrow \eta_Z & & \downarrow \eta_C & & \downarrow \eta_{\mathbf{C}^*} \\ 0 & \rightarrow & H^1(X, Z) & \rightarrow & H^1(X, \mathbf{C}) & \rightarrow & H^1(X, \mathbf{C}^*) \rightarrow 0. \end{array}$$

On the other hand, the universal covering manifold \tilde{X} of X is given by

2) Hereafter, for a given complex Lie group B , we denote by \mathbf{B} the sheaf (of group) of holomorphic mappings of a certain complex manifold X into B . Similarly, for a given complex analytic vector bundle E over X , we denote by \mathbf{E} the sheaf of germs of holomorphic sections of E .

$\tilde{X} = GL(n, C)/U_1$ where U_1 is the subgroup of $GL(n, C)$ consisting of matrices of the form :

$$\left(\begin{array}{c|cccc} 1 & * & \dots & * & * \\ \hline 0 & & & & \\ \vdots & & & & \\ 0 & & & * & \end{array} \right)$$

and $U_d/U_1 \cong \Delta_d$ is the covering transformation group of the covering $\psi : \tilde{X} \rightarrow X$. The bundle $\tilde{X}(X, \Delta_d, \psi)$ is obtained from the coset bundle $GL(n, C)(X, U_d, \pi)$ by the natural bundle homomorphism $\tau : GL(n, C) \rightarrow GL(n, C)/U_1 = \tilde{X}$, and is therefore the homogeneous Δ_d -bundle defined by the natural homomorphism $\tau : U_d \rightarrow U_d/U_1 = \Delta_d$. Now we say a *holomorphic B-bundle over X is defined by an (abstract) representation of the fundamental group* if it is induced from the bundle $\tilde{X}(X, \Delta_d, \psi)$ by a group homomorphism of Δ_d into B , and the homomorphism of $\text{Hom}(\Delta_d, B)$ into $H^1(X, B)$ which is obtained in this procedure is denoted by ζ_B .

We know that a holomorphic B -bundle has an integrable holomorphic connection if and only if it is defined by a representation of the fundamental group [1]. On the other hand, the homomorphism $U_d \rightarrow \Delta_d$ induces naturally a homomorphism τ_B of $\text{Hom}(\Delta_d, B)$ into $\text{Hom}(U_d, B)$ and the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}(U_d, B) & \xleftarrow{\tau_B} & \text{Hom}(\Delta_d, B) \\ \eta_B \searrow & & \swarrow \zeta_B \\ & H^1(X, B) & \end{array}$$

Returning to the diagram (4), if we show that η_C is bijective, the theorem will be proved. In fact, in this case, it is obvious that η_{C^*} is surjective and this means that every line bundle is homogeneous. Moreover, $H^1(X, C^*) \cong \text{Hom}(U_d, C)/\text{Hom}(U_d, Z) \cong C/Z \cong C^*$, and the first isomorphism implies that $\zeta_{C^*} : \text{Hom}(\Delta_d, C^*) \rightarrow H^1(X, C^*)$ is bijective. Therefore, every line bundle is defined by a representation of the fundamental group Δ_d of X . Now we show first that η_C is injective. In fact, if we take the element $\rho_0 \in \text{Hom}(\Delta_d, Z)$ which is defined by $\rho_0(d^m) = m$ for every $m \in Z$, then the bundle $\zeta_Z(\rho_0)$ is isomorphic with the bundle $\tilde{X}(X, \Delta_d, \psi)$ and so is not trivial, which implies that η_Z is injective. Then η_C being a linear homomorphism, η_C must be also injective. Next we shall prove that $H^1(X, C) \cong C$. For this sake we employ the spectral sequence associated to the Hopf fibering $X(P^{n-1}, T^1, \phi)$ and the sheaf C . That is to say, there exists a spectral sequence $\{E_k\}$ with $E_2^{r,s} = H^r(P^{n-1}, \phi^s(C))$ and with the final term E_∞^q associated to $H^q(X, C)$, where $\phi^s(C)$

is the sheaf defined by the presheaf $\phi^s(\mathbf{C})_N = H^s(\phi^{-1}(N), \mathbf{C})$ (for every open set $N \subset P^{n-1}$). In our discussion, it needs only the case $q=1$; so we are concerned only with $E_2^1 = E_2^{1,0} + E_2^{0,1}$ and with $\phi^s(\mathbf{C})$ ($s=0,1$). If we choose N as a Stein open set on which the bundle $\phi^{-1}(N)$ is trivial, then $\phi^{-1}(N) = N \times T^1$ and so by the Künneth relation we have:

$$\begin{aligned} H^0(\phi^{-1}(N), \mathbf{C}) &\cong H^0(N, \mathbf{C}) \\ H^1(\phi^{-1}(N), \mathbf{C}) &\cong H^1(N, \mathbf{C}) \otimes H^0(T^1, \mathbf{C}) + H^0(N, \mathbf{C}) \otimes H^1(T^1, \mathbf{C}). \end{aligned}$$

Because $H^1(N, \mathbf{C}) = \{0\}$, $H^s(T^1, \mathbf{C}) \cong \mathbf{C}$ ($s=0, 1$), it follows that $\phi^0(\mathbf{C}) = \mathbf{C}$, $\phi^1(\mathbf{C}) = \mathbf{C}$. Therefore $E_2^{1,0} = H^1(P^{n-1}, \mathbf{C}) = \{0\}$, $E_2^{0,1} = H^0(P^{n-1}, \mathbf{C}) \cong \mathbf{C}$. While, the d_2 -differential operator sends $E_2^{0,1}$ into $E_2^{2,0} = H^2(P^{n-1}, \mathbf{C}) = \{0\}$. This implies that $E_2^{0,1} = E_3^{0,1} = E_\infty^{0,1} = E_\infty^1$, and consequently that $H^1(X, \mathbf{C}) \cong \mathbf{C}$. The proof is now completed.

§ 4. The cohomology groups $H^q(X, \Omega^p(\mathbf{F}))$

By Theorem 3 we can write every complex line bundle over a Hopf manifold X as F_λ ; $\lambda \in C^* = \text{Hom}(\Delta_d, C^*)$. Our next step is to compute the cohomology groups $H^q(X, \Omega^p(F_\lambda))$, ($0 \leq q \leq n$) with coefficients in the analytic sheaf $\Omega^p(F_\lambda)$ ($0 \leq p \leq n$) of germs of holomorphic p -forms with values in F_λ .

To state our results of computations, we remark first the following situation. As to the Hopf fibering $X(P^{n-1}, T^1, \phi)$, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}(U_d, C^*) & \xrightarrow{\eta_{C^*}} & H^1(X, C^*) \rightarrow 0 \\ \uparrow \sigma & & \swarrow \phi^* \\ \text{Hom}(GL(1, n-1; C), C^*) & \xrightarrow{\eta} & H^1(P^{n-1}, C^*) \rightarrow 0, \end{array}$$

where σ is the restriction mapping of homomorphisms and η is the assignment of the defining homogeneous line bundle to each homomorphism belonging to $\text{Hom}(GL(1, n-1; C), C^*)$ (cf. [7], § 4). Now by the proof of Theorem 3 and Theorem 1 in [7], the above diagram yields the following one:

$$(1) \quad \begin{array}{ccc} 0 \rightarrow \text{Hom}(\Delta_d, C^*) & \xrightarrow{\eta_{C^*}} & H^1(X, C^*) \rightarrow 0 \\ \uparrow \sigma & & \uparrow \phi^* \\ 0 \rightarrow \text{Hom}(C^*, C^*) & \xrightarrow{\eta} & H^1(P^{n-1}, C^*) \rightarrow 0, \end{array}$$

where each row is an exact sequence. Furthermore we shall identify, in the sequel, $\text{Hom}(\Delta_d, C^*)$ with C^* and $\text{Hom}(C^*, C^*) = \{\mu \in \text{Hom}(C, C) \mid \mu(Z) \subset Z\}$ with Z respectively by means of the correspondences :

$$\begin{aligned} \text{Hom}(\Delta_d, C^*) \ni \lambda &\leftrightarrow \lambda(d) \in C^*, \\ \text{Hom}(C^*, C^*) \ni \mu &\leftrightarrow \mu(1) \in Z. \end{aligned}$$

Under these identifications, the mapping σ is given by $\sigma(m) = d^m$ for $m \in Z$, so that σ is an isomorphism of Z into C^* and its image is nothing but Δ_d . The mapping ϕ^* is, therefore, injective.

Thus we can state our main result.

Theorem 4. *Set $h^{p,q}(\lambda) = \dim H^q(X, \Omega^p(F_\lambda))$. Then $h^{p,q}(\lambda) = 0$ for any p and q if $\lambda \notin \Delta_d$, and in the case $\lambda \in \Delta_d$ we have :*

(A) *if $\lambda = d^m$, $m \neq 0$, $p > 0$,*

($n > 2$)

(i) *$h^{p,q}(\lambda) = 0$, if $2 \leq q \leq n-2$*

(ii) *$h^{p,0}(\lambda) = h^{p,1}(\lambda)$*

$$= \begin{cases} 0, & \text{if } m < p \\ \binom{n}{p}, & \text{if } m = p \\ \binom{n+m-p-1}{n-p-1} \binom{m-1}{p} + \binom{n+m-p}{n-p} \binom{m-1}{p-1}^3, & \text{if } m > p. \end{cases}$$

(iii) *$h^{p,n-1}(\lambda) = h^{p,n}(\lambda)$*

$$= \begin{cases} 0, & \text{if } m > p-n \\ \binom{n}{p}, & \text{if } m = p-n \\ \binom{-m+p}{p} \binom{-m-1}{n-p-1} + \binom{-m+p-1}{p-1} \binom{-m-1}{n-p}, & \text{if } m < p-n. \end{cases}$$

($n = 2$)

In this case $h^{p,0}(\lambda)$ and $h^{p,2}(\lambda)$ are given by the same formula as the case $n > 2$, setting $n = 2$; but $h^{p,1}(\lambda)$ is not, and $h^{p,1}(\lambda) = h^{p,0}(\lambda) + h^{p,2}(\lambda)$.

3) For any given two integers r and s , $\binom{r}{s}$ means the usual combination if $r, s > 0$ and, otherwise we shall understand it as follow; $\binom{r}{s} = 0$ if r or s is negative and $\binom{r}{s} = 1$ if $r, s \geq 0$, $rs = 0$.

- (B) if $\lambda = d^m, p = 0,$
- (i) $h^{0,q}(\lambda) = 0, \text{ if } 2 \leq q \leq n - 2$
 - (ii) $(n > 2), h^{0,0}(\lambda) = h^{0,1}(\lambda) = \binom{n+m-1}{m}$
 $h^{0,n-1}(\lambda) = h^{0,n}(\lambda) = \binom{-m-1}{-m-n}$
 - (iii) $(n = 2), h^{0,0}(\lambda) = \binom{m+1}{m}$
 $h^{0,1}(\lambda) = \binom{-m-1}{1} + \binom{m+1}{m}$
 $h^{0,2}(\lambda) = \binom{-m-1}{1}$
- (C) if $\lambda = 1,$
- (i) $h^{n,q}(1) = \begin{cases} 0, & \text{if } q \leq n-2 \\ 1, & \text{if } q = n-1, n \end{cases}$
 - (ii) $h^{p,q}(1) = 0, \text{ if } 1 \leq p \leq n-1,$
 - (iii) $h^{0,q}(1) = \begin{cases} 0, & \text{if } q \geq 2 \\ 1, & \text{if } q = 0, 1 \end{cases}$

§ 5. Summary of some known results.

Let $X(P^{n-1}, T^1, \phi)$ be the Hopf fibering of X and let $E = E_X(\rho, C^m)$ be the homogeneous vector bundle over X defined by the representation (ρ, C^m) of U_d and let \mathbf{E} be the sheaf of germs of holomorphic sections of E . The restriction E/T^1 of E on $T^1 = GL(1, n-1, C)/U_d$ is also homogeneous with respect to the Klein form $GL(1, n-1; C)/U_d$. Therefore every element of $GL(1, n-1; C)$ induces a bundle automorphism of $E|T^1$, and so a linear isomorphism of the cohomology group $H^s(T^1, \mathbf{E}|T^1)$, ($s=0, 1$). The holomorphic representation of $GL(1, n-1; C)$ thus obtained will be denoted by ρ^s ($s=0, 1$)⁴⁾, and the corresponding homogeneous vector bundle $E_{P^{n-1}}(\rho^s, H^s(T^1, \mathbf{E}|T^1))$ over P^{n-1} will be denoted simply by $\phi^s(E)$.

Now we take a spectral sequence $\{E_k\}$ whose final term E_∞ is associated to $H^*(X, \mathbf{E})$ and the second term E_2 is given by $E_2^{r,s} = H^r(P^{n-1}, \phi^s(\mathbf{E}))$, where $\phi^s(\mathbf{E})$ is the so-called s -dimensional direct image sheaf of \mathbf{E} by ϕ . While, in our case, it is known the following result of Bott (cf. [3], Theorem VI).

4) This representation ρ^s is called, according to Bott, the s -dimensional induced representation of ρ .

Lemma 1. $\phi^s(\mathbf{E})$ coincides with the sheaf of germs of holomorphic sections of $\phi^s(E)$; therefore $\phi^s(\mathbf{E})$ are zero sheaves for $s \geq 2$.

In particular, let E be a (homogeneous) line bundle. Then as to the restriction $E|T^1$ we know the following lemma (cf. [9], Proposition 3.6).

Lemma 2. The 0-dimensional cohomology group $H^0(T^1, \mathbf{E}|T^1)$ does not vanish if and only if $E|T^1$ is the trivial line bundle; therefore $\phi^0(\mathbf{E})$ is the zero sheaf unless $E|T^1$ is trivial.

Now, for the computations in the next section, we need to know the cohomology groups $H^q(P^{n-1}, \Omega^p(\hat{F}))$, where $\Omega^p(\hat{F})$ is the sheaf of germs of holomorphic p -forms with values in the line bundle \hat{F} over P^{n-1} ; the dimensions of these cohomology groups have been computed by Bott [3] and by Matsumura [10] independently. That is,

Lemma 3. Let $\hat{F}_m (m \in Z)$ be the line bundle over P^{n-1} corresponding to $m \in \text{Hom}(C^*, C^*)$ (cf. §4, (1)), and set $\hat{h}^{p,q}(m) = \dim H^q(P^{n-1}, \Omega^p(\hat{F}_m))$. Then we have,

- (i) $\hat{h}^{p,p}(0) = 1 \quad (0 \leq p \leq n-1)$
- (ii) $\hat{h}^{p,0}(m) = \binom{n+m-p-1}{n-p-1} \binom{m-1}{p} \quad (m > p)$
- (iii) $\hat{h}^{p,n-1}(m) = \binom{-m+p}{p} \binom{-m-1}{n-p-1} \quad (p-n+1 > m)$
- (iv) $\hat{h}^{p,q}(m) = 0$ for other cases.

§ 6. The proof of the main theorem.

For the proof of our Theorem 4, we need the following extension of vector bundles over P^{n-1} :

$$(1) \quad 0 \rightarrow I \rightarrow Q(X) \rightarrow \hat{\Theta} \rightarrow 0,$$

which are the homogeneous vector bundles over P^{n-1} induced by the exact sequence (1) in §2 of $GL(1, n-1; C)$ -modules. For instance $Q(X) = E_{P^{n-1}}(Ad, \mathfrak{g}/\mathfrak{u})$; and we note that $\Theta = \phi^*Q(X)$. By Ω and $\hat{\Omega}$ are meant the analytic sheaves of germs of holomorphic sections of Θ^* and $\hat{\Theta}^*$ respectively (* means the dual vector bundle). Moreover we denote by Ξ the sheaf of germs of holomorphic sections of $Q(X)^*$. From (1) we have the exact sequence of analytic sheaves on P^{n-1} :

$$0 \rightarrow \hat{\Omega} \rightarrow \Xi \rightarrow C \rightarrow 0.$$

From this we can construct the following exact sequences:

$$(2) \quad 0 \rightarrow \hat{\Omega}^p \rightarrow \Xi^p \rightarrow \hat{\Omega}^{p-1} \rightarrow 0 \quad (1 \leq p \leq n),$$

where Ξ^p is the sheaf of germs of holomorphic sections of the vector bundle $Q(X)^{*p}$ which is the p -exterior product of $Q(X)^*$ (see, for detail, [5], Satz 4.1.3*) and $\hat{\Omega}^p$ denotes the sheaf of germs of holomorphic p -forms on P^{n-1} .

Now we consider the spectral sequence $\{E_s\}$ associated to the Hopf fibering and the sheaf $\Omega^p(F_\lambda)$ over X . Then, the sheaf $\phi^s(\Omega^p(F_\lambda))=0$ except for $s=0,1$ by Lemma 1 and $\phi^s(\Omega^p(F_\lambda))=\phi^s(F_\lambda) \otimes \Xi^p$, as is known by an easy argument on the induced representation, since $\Theta^{*p}=\phi^*(\Xi^p)$. On the other hand, the theorem of Riemann-Roch concerning the elliptic curve T^1 and the line bundle $F_\lambda|T^1$ (=the restriction of F_λ on T^1) implies that

$$\dim \phi^0(F_\lambda) - \dim \phi^1(F_\lambda) = 0,$$

because $F_\lambda|T^1$ has a holomorphic connection by a theorem of Matsushima [9] and so has the vanishing Chern class (cf. Atiyah [1]). Moreover, by Lemma 2, $\dim \phi^0(F_\lambda) > 0$ if and only if $F_\lambda|T^1$ is trivial. The latter condition means that F_λ is induced from a line bundle \hat{F}_m over P^{n-1} by ϕ ; therefore in this case $\lambda=d^m$ (cf. (1) in §4). Hence, if $\lambda \notin \Delta_d$ then $\phi^s(\Omega^p(F_\lambda))=0$ for all s (and p), which implies $E_2=E_\infty=H^*(X, \Omega^p(F_\lambda))=\{0\}$.

We assume hereafter that $\lambda=d^m \in \Delta_d$, and that $F_\lambda=\phi^*\hat{F}_m$. Then $F_\lambda|T^1$ is trivial and $\phi^s(\Theta^{*p} \otimes F_\lambda) \cong Q(X)^{*p} \otimes \hat{F}_m$ for $s=0,1$ by an easy argument on the induced representations; hence we have $E_2^{r,s}=H^r(P^{n-1}, \Xi^p \otimes \hat{F}_m)$ ($s=0,1$) and $E_2^{r,s}=\{0\}$ ($s \geq 2$), which implies that

$$(3) \quad E_2^q = E_2^{q,0} + E_2^{q-1,1} = H^q(P^{n-1}, \Xi^p \otimes \hat{F}_m) + H^{q-1}(P^{n-1}, \Xi^p \otimes \hat{F}_m)$$

for $0 \leq q \leq n$. Now we shall divide the subsequent discussions into three cases.

(A) The case $m \neq 0, p > 0$.

The sequence (2) implies the following sheaf exact sequences:

$$(4) \quad 0 \rightarrow \hat{\Omega}^p(\hat{F}_m) \rightarrow \Xi^p \otimes \hat{F}_m \rightarrow \hat{\Omega}^{p-1}(\hat{F}_m) \rightarrow 0 \quad (1 \leq p \leq n).$$

The corresponding cohomology exact sequence is

$$\begin{aligned} &\rightarrow H^{q-1}(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \rightarrow H^{q-1}(P^{n-1}, \Xi^p \otimes \hat{F}_m) \rightarrow H^{q-1}(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \\ &\rightarrow H^q(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \rightarrow H^q(P^{n-1}, \Xi^p \otimes \hat{F}_m) \rightarrow H^q(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \rightarrow \end{aligned}$$

Therefore, if $1 \leq q \leq n-2$, then $H^q(P^{n-1}, \hat{\Omega}^p(\hat{F}_m))=\{0\}$ for any p by Lemma 3, so that $E_2^{q,0}=\{0\}$, and $E_2^{q-1,1}=\{0\}$ also for the case $q > 1$. Let $n > 2$. If $q=0$, $E_2^0=E_\infty^0$ and $E_2^0=E_2^{0,1}=H^0(P^{n-1}, \Xi^p \otimes \hat{F}_m)$ is given by

$$0 \rightarrow H^0(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \rightarrow E_2^0 \rightarrow H^0(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \rightarrow 0;$$

hence $\dim E_2^0 = \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m)$. If $q = n-1$, $E_2^{n-2,1} = E_2^{n-2,0} = \{0\}$ and $E_2^{n-1,0} = E_2^{n-1}$ is given by

$$0 \rightarrow H^{n-1}(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \rightarrow E_2^{n-1} \rightarrow H^{n-1}(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \rightarrow 0;$$

hence $\dim E_2^{n-1} = \hat{h}^{p,n-1}(m) + \hat{h}^{p-1,n-1}(m)$. If $q = n$, $E_2^{n,0} = H^n(P^{n-1}, \Xi^p \otimes \hat{F}_m) = \{0\}$ and $E_2^{n-1,1} = E_2^{n-1,0}$. Thus the spectral sequence is trivial and we obtain

- (i) $h^{p,q}(\lambda) = 0$ for $2 \leq q \leq n-2$
- (ii) $h^{p,0}(\lambda) = h^{p,1}(\lambda) = \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m)$
- (iii) $h^{p,n}(\lambda) = h^{p,n}(\lambda) = \hat{h}^{p,n-1}(m) + \hat{h}^{p-1,n-1}(m)$.

In case $n=2$, from (3), (4) and Lemma 3, we can deduce readily the following results :

- (i) $h^{p,0}(\lambda) = \begin{cases} \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m), & \text{if } m \geq p-1 \\ \hat{h}^{p,0}(m), & \text{if } m \leq p-1. \end{cases}$
- (ii) $h^{p,2}(\lambda) = \begin{cases} \hat{h}^{p-1,1}(m), & \text{if } m \geq p-1 \\ \hat{h}^{p,1}(m) + \hat{h}^{p-1,1}(m), & \text{if } m \leq p-1. \end{cases}$
- (iii) $h^{p,1}(\lambda) = h^{p,0}(\lambda) + h^{p,2}(\lambda)$.

(B) The case $p=0$.

By Lemma 3 and (3), we have $E_2^q = \{0\}$ for $2 \leq q \leq n-2$. Furthermore, if $n > 2$, we have $E_2^0 = E_2^1 = H^0(P^{n-1}, \hat{F}_m)$, $E_2^{n-1} = E_2^n = H^{n-1}(P^{n-1}, \hat{F}_m)$, and if $n=2$, we have $E_2^0 = H^0(P^1, \hat{F}_m)$, $E_2^1 = H^0(P^1, \hat{F}_m) + H^1(P^1, \hat{F}_m)$ and $E_2^2 = H^1(P^1, \hat{F}_m)$. Moreover the spectral sequence is trivial, and we obtain :

- (i) $h^{0,q}(\lambda) = 0$, for $2 \leq q \leq n-2$
- (ii) $(n > 2) \begin{cases} h^{0,0}(\lambda) = h^{0,1}(\lambda) = \hat{h}^{0,0}(m) \\ h^{0,n-1}(\lambda) = h^{0,n}(\lambda) = \hat{h}^{0,n-1}(m) \end{cases}$
- (iii) $(n = 2) \begin{cases} h^{0,0}(\lambda) = \hat{h}^{0,0}(m), \\ h^{0,1}(\lambda) = \hat{h}^{0,0}(m) + \hat{h}^{0,1}(m) \\ h^{0,2}(\lambda) = \hat{h}^{0,1}(m). \end{cases}$

(C) The case $m=0$, $p > 0$.

From (2) and (3) we have

$$\begin{aligned} &\rightarrow H^{q-1}(P^{n-1}, \hat{\Omega}^p) \rightarrow E_2^{q-1,0} = E_2^{q-1,1} \rightarrow H^{q-1}(P^{n-1}, \hat{\Omega}^{p-1}) \\ &\rightarrow H^q(P^{n-1}, \hat{\Omega}^p) \rightarrow E_2^{q,0} = E_2^{q,1} \rightarrow H^q(P^{n-1}, \hat{\Omega}^{p-1}) \end{aligned}$$

If $p=n$, then $H^q(P^{n-1}, \hat{\Omega}^n) = \{0\}$ and so $E_2^q = E_2^{q,0} + E_2^{q-1,1} = H^q(P^{n-1}, \hat{\Omega}^{n-1}) + H^{q-1}(P^{n-1}, \hat{\Omega}^{n-1})$; hence by Lemma 3 we have

$$h^{n,q}(1) = \begin{cases} 0, & \text{if } q \leq n-2 \\ 1, & \text{if } q = n-1, n. \end{cases}$$

We assume hereafter that $1 \leq p \leq n-1$. If $q \neq p \pm 1, p$, then we have $E_2^{q,0} = \{0\}, E_2^{q-1,1} = \{0\}$; hence $h^{p,q}(1) = 0$. If $q = p-1$, then $E_2^{p-2,1} = \{0\}, E_2^{p-1,0} = E_\infty^{p-1}$. If $q = p+1$, then $E_2^{p+1,0} = \{0\}, E_2^{p,1} = E_\infty^{p+1}$. If $q = p$, then we have

$$(5) \quad 0 \rightarrow E_2^{p-1,0} \rightarrow H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) \xrightarrow{\delta^*} H^p(P^{n-1}, \hat{\Omega}^p) \rightarrow E_2^{p,0} \rightarrow 0.$$

We remark here that $\dim H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) = \dim H^p(P^{n-1}, \hat{\Omega}^p) = 1$ and that $E_2^{p-1,0} = E_2^{p,0} = \{0\}$ if and only if δ^* is not the zero homomorphism (i.e. bijective). While by the following Lemma 4, we have in reality $E_2^{p-1,0} = E_2^{p-1,1} = E_2^{p,0} = E_2^{p,1} = \{0\}$ for $1 \leq p \leq n-2$; hence we have then $h^{p,q}(1) = 0$ also for $q = p \pm 1, p$. It remains only the case $p = n-1$; in this case we have $h^{n-1,q}(1) = h^{1,n-q}(1) = 0$ by Serre's duality for $n > 2$ (The case $n = 2$ is contained in the proof of Lemma 4).

Lemma 4. *In the above exact sequence (5), if $1 \leq p \leq n-2$, δ^* is bijective; hence $E_2^{p-1,0} = E_2^{p,0} = \{0\}$.*

Proof. First we shall consider the case $p=1$ (in (2) we set $\hat{\Omega}^0 = \mathbf{C}$). In this case, $\delta^*: H^0(P^{n-1}, \mathbf{C}) \rightarrow H^1(P^{n-1}, \hat{\Omega})$ is not the zero homomorphism; in fact, if otherwise, the extension: $0 \rightarrow \hat{\Omega} \rightarrow \Xi \rightarrow \mathbf{C} \rightarrow 0$ is splittable by a lemma of Atiyah (Proc. London Math. Soc., 7 (1957), p. 429, Lemma 13), and then, by the same argument as in our previous paper [7] (cf. the foot-note 7)), the Hopf fibering must be trivial; however $S^1 \times S^{2n-1}$ and $T^1 \times P^{n-1}$ are clearly not homeomorphic. This proves the lemma in our case.

In general case we prove the lemma by induction on n . In case $n=2$, it must be $p=1$; therefore the lemma has been proved by the above discussions. We assume $n > 2$ and consider the exact sequence (2) over the base space P^{n-2} , which will be written as:

$$(2_*) \quad 0 \rightarrow \hat{\Omega}_*^p \rightarrow \Xi_*^p \rightarrow \hat{\Omega}_*^{p-1} \rightarrow 0 \quad \text{over } P^{n-2}.$$

On the other hand, the imbedding of $GL(n-1, \mathbf{C})$ into $GL(n, \mathbf{C})$, defined by $g \rightarrow \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ for $g \in GL(n-1, \mathbf{C})$, induces an imbedding of P^{n-2} into P^{n-1} as a hyperplane, which we shall fix once for all. The sheaves in (2_{*}) are naturally extendable to the sheaves over P^{n-1} by assuming that the fibres on the complement of P^{n-2} vanish, and they shall be denoted with the

same letters as in (2_{*}). Now we shall show that there are natural sheaf homomorphisms $\alpha_p: \Xi^p \rightarrow \Xi^p_*$ and $\beta_p: \hat{\Omega}^p \rightarrow \hat{\Omega}^p_*$ which yield the following commutative diagram:

$$(6) \quad \begin{array}{ccccccc} 0 & \rightarrow & \hat{\Omega}^p & \rightarrow & \Xi^p & \rightarrow & \hat{\Omega}^{p-1} \rightarrow 0 \\ & & \downarrow \beta_p & & \downarrow \alpha_p & & \downarrow \beta_{p-1} \\ 0 & \rightarrow & \hat{\Omega}^p_* & \rightarrow & \Xi^p_* & \rightarrow & \hat{\Omega}^{p-1}_* \rightarrow 0. \end{array}$$

For this sake, we identify the exact sequence of \hat{U} -modules (1) in §2 with the one:

$$0 \rightarrow C^1 \rightarrow C^n \rightarrow C^{n-1} \rightarrow 0,$$

where \hat{U} acts on each module as the identity representation on C^1 , as

$$\frac{1}{b} \hat{u} = \begin{pmatrix} 1 & \frac{1}{b} * \\ 0 & \frac{1}{b} B \end{pmatrix} \text{ on } C^n \text{ and as } \frac{1}{b} B \text{ on } C^{n-1} \text{ respectively, for every element}$$

$\hat{u} = \begin{pmatrix} b & * \\ 0 & B \end{pmatrix} \in \hat{U}$. Then the restrictions of $\hat{\Theta}^{*p}$ and $Q(X)^{*p}$ on P^{n-2} are given by $GL(n-1, C) \times \hat{U}_*(C^{n-1})^{*p}$ ⁵⁾ and $GL(n-1, C) \times \hat{U}_*(C^n)^{*p}$ respectively where $\hat{U}_* = GL(1, n-2; C)$ acts on $(C^n)^{*p}$ and $(C^{n-1})^{*p}$ as defined above. Then we have the commutative diagram of modules:

$$\begin{array}{ccccccc} 0 & \rightarrow & (C^{n-1})^{*p} & \rightarrow & (C^n)^{*p} & \rightarrow & (C^{n-1})^{*p-1} \rightarrow 0 \\ & & \downarrow \tilde{\beta}_p & & \downarrow \tilde{\alpha}_p & & \downarrow \tilde{\beta}_{p-1} \\ 0 & \rightarrow & (C^{n-2})^{*p} & \rightarrow & (C^{n-1})^{*p} & \rightarrow & (C^{n-2})^{*p-1} \rightarrow 0, \end{array}$$

where $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ denote the restriction mappings of alternating p -forms. This diagram, considered as the one of \hat{U}_* -modules, is commutative as is easily seen. Therefore it induces the commutative diagram of homogeneous vector bundles over P^{n-2} ; this implies that there are corresponding sheaf homomorphisms α_p, β_p and β_{p-1} as in (6). Thus we have the following commutative diagram:

$$\begin{array}{ccc} H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) & \xrightarrow{\delta^*} & H^p(P^{n-1}, \hat{\Omega}^p) \\ \downarrow \beta_{p-1} & & \downarrow \beta_p \\ H^{p-1}(P^{n-2}, \hat{\Omega}^{p-1}_*) & \xrightarrow{\delta^*} & H^p(P^{n-2}, \hat{\Omega}^p_*), \end{array}$$

where the mappings β_{p-1} and β_p are bijective for $1 \leq p \leq n-2$ since they coincide with the restriction mappings of harmonic forms *via* the

5) $(C^{n-1})^{*p}$ denotes the vector space of all alternating p -forms on C^{n-1} .

Dolbeault isomorphisms. While, δ^* in the under column is bijective by induction assumption, so our δ^* must be bijective.

REMARK 1. Theorem 4 tells us that both Riemann-Roch's theorem with respect to any line bundle and Hodge's index theorem are valid for Hopf manifolds (cf. [8]). In fact, for any line bundle F_λ over a Hopf manifold X , we can readily check that

$$\chi(X, F_\lambda) = \sum_{q=0}^n (-1)^q \dim H^q(X, F_\lambda) = \sum_{q=0}^n (-1)^q h^{0,q}(\lambda) = 0;$$

while the Todd genus $T(X, F_\lambda) = 0$ since $H^2(X, Z) = \{0\}$. Furthermore, the index $\tau(X)$ of X is clearly 0, since X is homeomorphic to $S^1 \times S^{2n-1}$; while we see immediately, from (C) in Theorem 4, that

$$\sum_{p,q} (-1)^q h^{p,q}(1) = 0.$$

REMARK 2. Theorem 1 and Theorem 2 can be readily derived from Theorem 4. In fact, by Serre's duality theorem, we have $H^q(X, \Theta) \cong H^{n-q}(X, \Omega^1(K))$ where K denotes the canonical line bundle of X . While, from the exact sequence (2) in § 2, we get immediately $K = \phi^* \hat{K}$, where \hat{K} is the canonical bundle of P^{n-1} and coincides with \hat{F}_{-n} . Therefore, Theorem 4, (A) yields that $\dim H^0(X, \Theta) = \dim H^1(X, \Theta) = n^2$, $H^q(X, \Theta) = \{0\}$ for $q \geq 2$.

REMARK 3. The proof of Theorem 4 suggests us the possibilities of computing the cohomology groups $H^p(X, \Omega^p(F))$ for other class of C -manifolds with the *fundamental fibering* $X(\hat{X}, T^1, \phi)$ (cf. [7]) provided that the cohomology groups $H^q(\hat{X}, \Omega^q(\hat{F}))$ are known. For instance, Calabi-Eckmann's example (cf. [4], [7]) or $SU(3)$ with a left invariant complex structure is such a manifold. However, for them $\hat{X} = P^2 \times P^q$ or $\hat{X} = F(3)$ (=the 3-dimensional flag manifold) respectively and the corresponding cohomology groups $H^q(\hat{X}, \Omega^q(\hat{F}))$ are rather complicated; consequently the computations of $H^q(X, \Omega^p(F))$ might be more difficult than for Hopf manifolds.

But we shall exhibit here the number $h^{p,q} = h^{p,q}(1)$ for $SU(3)$, since Bott's computations in [3] for them are incorrect.

$$\begin{aligned} h^{0,0} &= h^{0,1} = h^{1,1} = h^{1,2} = 1, \\ h^{4,4} &= h^{4,3} = h^{3,3} = h^{3,2} = 1, \\ h^{p,q} &= 0 \quad \text{otherwise.} \end{aligned}$$

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References

- [1] M. F. Atiyah: Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181-207.
- [2] A. Blanchard: Sur les variétés analytiques complexes, *Ann. Ecol. Norm. Sup.* **73** (1956), 157-202.
- [3] R. Bott: Homogeneous vector bundles, *Ann. of Math.* **66** (1957), 203-248.
- [4] E. Calabi and B. Eckmann: A class of compact complex manifolds which are not algebraic, *Ann. of Math.* **98** (1953), 494-500.
- [5] F. Hirzebruch: Neue topologische Methoden in der algebraischen Geometrie, *Erg. de Math.* **9**, 1956.
- [6] H. Hopf: Zur Topologie der komplexen Mannigfaltigkeiten, in *Studies and Essays presented to R. Courant*, New York, 1948.
- [7] M. Ise: Some properties of complex analytic vector bundles over compact complex homogeneous spaces, *Osaka Math. J.* **12** (1960), 217-252.
- [8] K. Kodaira and D. C. Spencer: On deformations of complex analytic structures, I-II, *Ann. of Math.* **67** (1958), 328-466.
- [9] Y. Matsushima: Fibrés holomorphes sur un tore complexe, *Nagoya Math. J.* **14** (1958), 1-24.
- [10] H. Matsumura: Geometric structure of the cohomology rings in abstract algebraic geometry, *Mem. Coll. Sci. Univ. Kyoto, Ser. A Math.* **32** (1959), 33-84.