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On the Geometry of Hopf Manifolds

By Mikio Ise

§1. Introduction

The purpose of the present note is to compute the cohomology groups $H^q(X, \Omega^p(F))$, $(0 \leqq q \leqq n)$ of an n-dimensional Hopf manifold X, where $\Omega^p(F)$ denotes the analytic sheaf of germs of holomorphic p-forms with values in a complex line bundle F over X. Throughout the arguments we make use of the fact that the Hopf manifold is a homogeneous compact complex manifold. Recently, R. Bott [3] and the author [7] have made some researches concerning the complex line bundles over a class of homogeneous compact complex manifolds ($=C$-manifolds in the sense of Wang). The essential difference between Hopf manifolds and Wang's C-manifolds lies in the non-triviality of the fundamental group of the former. But a Hopf manifold admits the so-called *Hopf fibering*, which plays the quite analogous role to the fundamental fiberings of non-kählerian C-manifolds (cf. [7]), and which allows us to have the similar results for them. Our principal tools are Leray's spectral sequences and the knowledge of the cohomology groups $H^q(P^n, \Omega^p(F))$ of the n-dimensional complex projective space P^n, which have been computed by Bott [3] and Matsumura [10].

The author wishes to express here his hearty thanks to Prof. S. Murakami, who has read through the manuscript and has given kind criticisms.

§2. Hopf manifolds

We recall here the definition of Hopf manifolds (cf. [6]). Let C^n denote a complex n-dimensional euclidean space and W^n the complement of the origin $o=(0, \cdots, 0) \in C^n$, and take a non-zero complex number d with the absolute value $|d| \neq 1$. Let Δ_d be the cyclic group generated by d in the multiplicative group C^* of all non-zero complex numbers. We denote also by Δ_d, for brevity, the subgroup of $GL(n, C)$ generated by the scaler matrix $d \cdot I$ (I is the unit matrix of $GL(n, C)$):

$$\Delta_d = \{d^m \cdot I | m \in Z\}$$

(Z is the ring of all integers)

The group Δ_d being a properly discontinuous group of W^n without fixed points, the quotient manifold $W^n/\Delta_d = X$ has a natural complex analytic structure. The complex manifold X thus obtained is, by definition, an *n-dimensional Hopf manifold* (corresponding to the number d)[1]. As is well known, X is diffeomorphic to the product manifold $S^1 \times S^{2n-1}$ of two odd dimensional spheres, since the group Δ_d is isomorphic to Z. The complex general linear group $GL(n, C)$ operates on W^n effectively and transitively; as Δ_d is contained in the centre of $GL(n, C)$, $GL(n, C)$ operates on X also transitively and holomorphically. Now we take the basic point $w_0 = (1, 0, \cdots, 0) \in W^n$ and the corresponding point $x_0 \in X$ modulo Δ_d. Then the isotropy subgroup U_d of $GL(n, C)$ at x_0 consists of the matrices of the form:

$$u = \left(\begin{array}{c|c} d^m & * \cdots\cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right), \quad d^m \in \Delta_d .$$

So we can identify X with the complex coset space:

$$X = GL(n, C)/U_d .$$

Note that the action of $GL(n, C)$ is not effective; if we take the quotient groups $\tilde{G} = GL(n, C)/\Delta_d$ and $\tilde{U}_d = U_d/\Delta_d$, then $\tilde{G}$ acts on X effectively and we can put

$$X = \tilde{G}/\tilde{U}_d .$$

The connected complex reductive Lie group $\tilde{G}$ is considered as a Lie subgroup of the connected analytic automorphism group $A(X)$ of X. As a matter of fact, we show the following

Theorem 1. $A(X) = \tilde{G}$.

The proof is quite similar to the case of non-kählerian C-manifolds ([7], Proposition 6), but we state it here for the completeness sake and for the convenience in the later discussions. First we need some definitions:

Definition 1. Let $G(1, n-1; C)$ denote the subgroup of $GL(n, C)$ consisting of the matrices:

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1) The 1-dimensional Hopf manifold in this definition is nothing but an elliptic curve T^1. We shall exclude this trivial case in the sequel; therefore assume $n \geqq 2$.

$$\begin{pmatrix} * & 0 \cdots\cdots 0 \\ \hline * & \\ \vdots & * \\ * & \end{pmatrix}.$$

Then we can identify $GL(n, C)/GL(1, n-1; C)$ with the $(n-1)$-dimensional complex projective space P^{n-1} and $GL(1, n-1; C)/U_d$ with an elliptic curve T^1 respectively. Therefore we have a natural (holomorphic) principal fibering of X with P^{n-1} as base, T^1 as group and the natural mapping ϕ of X onto P^{n-1} as projection; this fibering $X(P^{n-1}, T^1, \phi)$ is called the *Hopf fibering* of X.

DEFINITION 2. Let $X=G/U$ be a complex homogeneous space with a connected complex Lie group G and a (not nec. connected) closed complex Lie subgroup U, and let ρ be a holomorphic homomorphism of U into another complex Lie group B. Then the coset bundle $G(X, U, \pi)$ (π is the canonical projection of G onto X) and ρ induce a new holomorphic principal bundle $P(X, B, \varpi)$ over X, which is called a *homogeneous B-bundle over X with respect to the Klein form G/U.* In particular, when $B=GL(m, C)$, we have the associated m-dimensional vector bundle $E(X, C^m, GL(m, C), \varpi)$ which is simply written as $E_X(\rho, C^m)$ and which is called a *homogeneous vector bundle over X (with respect to the Klein form G/U).* This is the quotient space of $G\times C^m$ by the equivalence relation:

$$(g, \xi) \sim (gu, \rho(u^{-1})\xi)$$

for $g\in G$, $u\in U$ and $\xi\in C^m$.

Now we denote the Lie algebras of $GL(n, C)$, $GL(1, n-1; C)$ and U_d by $\mathfrak{g}$, $\hat{\mathfrak{u}}$ and $\mathfrak{u}$ respectively. Then $\mathfrak{u}$ is an ideal of $\hat{\mathfrak{u}}$ and the exact sequence of modules:

$$(1) \qquad 0 \to \hat{\mathfrak{u}}/\mathfrak{u} \to \mathfrak{g}/\mathfrak{u} \to \mathfrak{g}/\hat{\mathfrak{u}} \to 0$$

is considered as an exact sequence of U_d-modules under the adjoint actions. Therefore we can construct the exact sequence of homogeneous vector bundles over X with respect to the Klein form $GL(n, C)/U_d$:

$$0 \to E_X(Ad, \hat{\mathfrak{u}}/\mathfrak{u}) \to E_X(Ad, \mathfrak{g}/\mathfrak{u}) \to E_X(Ad, \mathfrak{g}/\hat{\mathfrak{u}}) \to 0 .$$

If we denote here by Θ and $\hat{\Theta}$ the tangential vector bundles over X and $\hat{X}=P^{n-1}$ respectively, then, as is easily verified,

$$E_X(Ad, \mathfrak{g}/\mathfrak{u}) = \Theta, \quad E_X(Ad, \mathfrak{g}/\hat{\mathfrak{u}}) = \phi^*\hat{\Theta}$$

and $E_X(Ad, \hat{\mathfrak{u}}/\mathfrak{u})$ is the trivial line bundle (which is denoted by I).

Therefore we have the exact sequence:

$$0 \to I \to \Theta \to \phi^* \hat{\Theta} \to 0 . \tag{2}$$

This induces the corresponding exact sequence of the complex vector spaces of cross sections:

$$0 \to \Gamma_X(I) \to \Gamma_X(\Theta) \to \Gamma_X(\phi^* \hat{\Theta}) .$$

Here, $\Gamma_X(\Theta)$ may be identified with the Lie algebra $\mathfrak{a}(X)$ of all holomorphic vector fields on X, and $\Gamma_X(I)$ is then identified with the 1-dimensional ideal C of $\mathfrak{a}(X)$. Moreover $\Gamma_X(\phi^* \hat{\Theta})$ is isomorphic with $\Gamma_{\hat{X}}(\hat{\Theta})$ or with the Lie algebra $\mathfrak{a}(P^{n-1})$ of all holomorphic vector fields on P^{n-1}, since the fibres of $\phi : X \to P^{n-1}$ are compact, connected. Hence, we obtain the exact sequence of Lie algebras:

$$0 \to C \to \mathfrak{a}(X) \overset{\dot{\phi}}{\to} \mathfrak{a}(P^{n-1}) ,$$

where the homomorphism $\dot{\phi}$ means that every holomorphic vector field on X is constant on each fibre and that it induces a vector field over P^{n-1} (This fact has been recognized by Blanchard [2] in general case; see [2], Proposition 1. 1). Now we know that $\mathfrak{a}(P^{n-1})$ is isomorphic to the Lie algebra $\mathfrak{sl}(n, C)$ of $SL(n, C)$ and that $\mathfrak{a}(X)$ contains the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ which is clearly isomorphic to that of $GL(n, C)$. Therefore $\dot{\phi}$ is surjective and $\mathfrak{a}(X) = \tilde{\mathfrak{g}}$. Hence $A(X) = \tilde{G}$. This proves Theorem 1.

Here we add the following theorem which can be proved in the same way as in [7], Proposition 7 (see, also Remark 2 in § 6).

Theorem 2. $\dim H^1(X, \Theta)) = n^2$ *and* $H^q(X, \Theta) = \{0\}$ *for* $q \geqq 2$.

The first result is known by Kodaira-Spencer [8] in the case $n=2$. The discussions in [8], § 15 about deformations of complex analytic structures of a 2-dimensional Hopf manifold might be immediately extended to the case $n > 2$.

§ 3. Complex line bundles

In this section we shall concern ourselves with the homogeneous line bundles over a Hopf manifold X. For this sake, it is more appropriate to take the Klein form $GL(n, C)/U_d$ rather than to take G/U_d. We call, from now on, a homogeneous bundle with respect to the Klein form $GL(n, C)/U_d$ simply a *homogeneous bundle*. Then we have

Theorem 3. *Every complex line bundle over a Hopf manifold X is* *homogeneous and has an integrable holomorphic connection. Moreover the group* $H^1(X, \boldsymbol{C}^*)$ *of all complex line bundles over* X *is isomorphic to* C^*.

Proof. First consider the sheaf exact sequence over X:

$$(1) \qquad 0 \to Z \to \boldsymbol{C} \xrightarrow{\varepsilon} \boldsymbol{C}^* \to 0,$$

where $\boldsymbol{C}$ (resp. $\boldsymbol{C}^*$)[2] denotes the sheaf of germs of holomorphic (resp. non-vanishing holomorphic) functions on X, Z the constant sheaf of integers and ε the homomorphism induced from the homomorphism ε of C onto C^* defined by $\varepsilon(\xi) = \exp 2\pi\sqrt{-1}\,\xi$ for any $\xi \in C$. Because $H^2(X, Z) = \{0\}$, we have from (1) the following exact sequence:

$$0 \to H^1(X, Z) \to H^1(X, \boldsymbol{C}) \xrightarrow{\varepsilon} H^1(X, \boldsymbol{C}^*) \to 0.$$

On the other hand, we consider the exact sequence of the abelian groups:

$$(2) \qquad 0 \to \mathrm{Hom}\,(U_d, Z) \to \mathrm{Hom}\,(U_d, C) \xrightarrow{\varepsilon} \mathrm{Hom}\,(U_d, C^*),$$

where $\mathrm{Hom}\,(U_d, B)$ means the abelian group of all holomorphic homomorphisms of U_d into the complex abelian Lie group B (If B is discrete, then holomorphic homomorphims should be understood as abstract ones). We see readily that $\mathrm{Hom}\,(U_d, B) \cong \mathrm{Hom}\,(\Delta_d \times GL(n-1, C), B) \cong \mathrm{Hom}\,(\Delta_d \times C^*, B) \cong \mathrm{Hom}\,(\Delta_d, B) \times \mathrm{Hom}\,(C^*, B)$; hence we have

$$(3) \qquad \begin{cases} \mathrm{Hom}\,(U_d, Z) \cong \mathrm{Hom}\,(\Delta_d, Z) \cong Z, \\ \mathrm{Hom}\,(U_d, C) \cong \mathrm{Hom}\,(\Delta_d, C) \cong C, \\ \mathrm{Hom}\,(U_d, C^*) \cong \mathrm{Hom}\,(\Delta_d, C^*) \times \mathrm{Hom}\,(C^*, C^*) \cong C^* \times Z. \end{cases}$$

Now there is a natural homomorphism η_B of the group $\mathrm{Hom}\,(U_d, B)$ into the group $H^1(X, \boldsymbol{B})$ of all holomorphic B-bundles over X by assigning to every $\rho \in \mathrm{Hom}\,(U_d, B)$ the corresponding homogeneous B-bundle over X defined by ρ. Then we have the commutative diagram:

$$(4) \qquad \begin{array}{ccccccc} 0 \to \mathrm{Hom}\,(U_d, Z) & \to & \mathrm{Hom}\,(U_d, C) & \to & \mathrm{Hom}\,(U_d, C^*) & & \\ \downarrow \eta_Z & & \downarrow \eta_C & & \downarrow \eta_{C^*} & & \\ 0 \to H^1(X, Z) & \to & H^1(X, \boldsymbol{C}) & \to & H^1(X, \boldsymbol{C}^*) & \to & 0. \end{array}$$

On the other hand, the universal covering manifold $\tilde{X}$ of X is given by

2) Hereafter, for a given complex Lie group B, we denote by $\boldsymbol{B}$ the sheaf (of group) of holomorphic mappings of a certain complex manifold X into B. Similary, for a given complex analytic vector bundle E over X, we denote by $\boldsymbol{E}$ the sheaf of germs of holomorphic sections of E.

$\tilde{X}=GL(n, C)/U_1$ where U_1 is the subgroup of $GL(n, C)$ consisting of matrices of the form:

$$\left(\begin{array}{c|c} 1 & * \cdots\cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right).$$

and $U_d/U_1 \cong \Delta_d$ is the covering transformation group of the covering $\psi: \tilde{X} \to X$. The bundle $\tilde{X}(X, \Delta_d, \psi)$ is obtained from the coset bundle $GL(n, C)(X, U_d, \pi)$ by the natural bundle homomorphism $\tau: GL(n, C) \to GL(n, C)/U_1 = \tilde{X}$, and is therefore the homogeneous Δ_d-bundle defined by the natural homomorphism $\tau: U_d \to U_d/U_1 = \Delta_d$. Now we say a *holomorphic B-bundle over X is defined by an (abstract) representation of the fundamental group* if it is induced from the bundle $\tilde{X}(X, \Delta_d, \psi)$ by a group homomorphism of Δ_d into B, and the homomorphism of $\mathrm{Hom}(\Delta_d, B)$ into $H^1(X, \boldsymbol{B})$ which is obtained in this procedure is denoted by ζ_B.

We know that a holomorphic B-bundle has an integrable holomorphic connection if and only if it is defined by a representation of the fundamental group [1]. On the other hand, the homomorphism $U_d \to \Delta_d$ induces naturally a homomorphism τ_B of $\mathrm{Hom}(\Delta_d, B)$ into $\mathrm{Hom}(U_d, B)$ and the following diagram is commuative.

$$\begin{array}{ccc} \mathrm{Hom}(U_d, B) & \overset{\tau_B}{\leftarrow} & \mathrm{Hom}(\Delta_d, B) \\ & \eta_B \searrow \quad \swarrow \zeta_B & \\ & H^1(X, \boldsymbol{B}) & \end{array}$$

Returning to the diagram (4), if we show that η_C is bijective, the theorem will be proved. In fact, in this case, it is obvious that η_{C^*} is surjective and this means that every line bundle is homogeneous. Moreover, $H^1(X, \boldsymbol{C}^*) \cong \mathrm{Hom}(U_d, C)/\mathrm{Hom}(U_d, Z) \cong C/Z \cong C^*$, and the first isomorphism implies that $\zeta_{C^*}: \mathrm{Hom}(\Delta_d, C^*) \to H^1(X, \boldsymbol{C}^*)$ is bijective. Therefore, every line bundle is defined by a representation of the fundamental group Δ_d of X. Now we show first that η_C is injective. In fact, if we take the element $\rho_0 \in \mathrm{Hom}(\Delta_d, Z)$ which is defined by $\rho_0(d^m)=m$ for every $m \in Z$, then the bundle $\zeta_Z(\rho_0)$ is isomorphic with the bundle $\tilde{X}(X, \Delta_d, \psi)$ and so is not trivial, which implies that η_Z is injective. Then η_C being a linear homomorphism, η_C must be also injective. Next we shall prove that $H^1(X, \boldsymbol{C}) \cong C$. For this sake we employ the spectral sequence associated to the Hopf fibering $X(P^{n-1}, T^1, \phi)$ and the sheaf C. That is to say, there exists a spectral sequence $\{E_k\}$ with $E_2^{r,s}=H^r(P^{n-1}, \phi^s(\boldsymbol{C}))$ and with the final term E_∞^q associated to $H^q(X, \boldsymbol{C})$, where $\phi^s(\boldsymbol{C})$

is the sheaf defined by the presheaf $\phi^s(\boldsymbol{C})_N = H^s(\phi^{-1}(N), \boldsymbol{C})$ (for every open set $N \subset P^{n-1}$). In our discussion, it needs only the case $q=1$; so we are concerned only with $E_2^1 = E_2^{1,0} + E_2^{0,1}$ and with $\phi^s(\boldsymbol{C})$ ($s=0.1$). If we choose N as a Stein open set on which the bundle $\phi^{-1}(N)$ is trivial, then $\phi^{-1}(N) = N \times T^1$ and so by the Künneth relation we have:

$$H^0(\phi^{-1}(N), \boldsymbol{C}) \cong H^0(N, \boldsymbol{C})$$
$$H^1(\phi^{-1}(N), \boldsymbol{C}) \cong H^1(N, \boldsymbol{C}) \otimes H^0(T^1, \boldsymbol{C}) + H^0(N, \boldsymbol{C}) \otimes H^1(T^1, \boldsymbol{C}).$$

Because $H^1(N, \boldsymbol{C}) = \{0\}$, $H^s(T^1, \boldsymbol{C}) \cong C (s=0, 1)$, it follows that $\phi^0(\boldsymbol{C}) = \boldsymbol{C}$, $\phi^1(\boldsymbol{C}) = \boldsymbol{C}$. Therefore $E_2^{1,0} = H^1(P^{n-1}, \boldsymbol{C}) = \{0\}$, $E_2^{0,1} = H^0(P^{n-1}, \boldsymbol{C}) \cong C$. While, the d_2-differential operator sends $E_2^{0,1}$ into $E_2^{2,0} = H^2(P^{n-1}, \boldsymbol{C}) = \{0\}$. This implies that $E_2^{0,1} = E_3^{0,1} = E_\infty^{0,1} = E_\infty^1$, and consequently that $H^1(X, \boldsymbol{C}) \cong C$. The proof is now completed.

§4. The cohomology groups $H^q(X, \Omega^p(F))$

By Theorem 3 we can write every complex line bundle over a Hopf manifold X as F_λ; $\lambda \in C^* = \mathrm{Hom}\,(\Delta_d, C^*)$. Our next step is to compute the cohomology groups $H^q(X, \Omega^p(F_\lambda))$, $(0 \leqq q \leqq n)$ with coefficients in the analytic sheaf $\Omega^p(F_\lambda)$ $(0 \leqq p \leqq n)$ of germs of holomorphic p-forms with values in F_λ.

To state our results of computations, we remark first the following situation. As to the Hopf fibering $X(P^{n-1}, T^1, \phi)$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}\,(U_d, C^*) & \xrightarrow{\eta_{C^*}} & H^1(X, \boldsymbol{C}^*) \to 0 \\ \uparrow \sigma & & \nwarrow \phi^* \\ \mathrm{Hom}\,(GL(1, n-1; C), C^*) & \xrightarrow{\eta} & H^1(P^{n-1}, \boldsymbol{C}^*) \to 0, \end{array}$$

where σ is the restriction mapping of homomorphisms and η is the assignment of the defining homogeneous line bundle to each homomorphism belonging to $\mathrm{Hom}\,(GL(1, n-1; C), C^*)$ (cf. [7], §4). Now by the proof of Theorem 3 and Theorem 1 in [7], the above diagram yields the following one:

$$\begin{array}{ccc} 0 \to \mathrm{Hom}\,(\Delta_d, C^*) & \xrightarrow{\eta_{C^*}} & H^1(X, \boldsymbol{C}^*) \to 0 \\ \uparrow \sigma & & \uparrow \phi^* \\ 0 \to \mathrm{Hom}\,(C^*, C^*) & \xrightarrow{\eta} & H^1(P^{n-1}, \boldsymbol{C}^*) \to 0, \end{array} \tag{1}$$

where each row is an exact sequence. Furthermore we shall identify, in the sequel, $\mathrm{Hom}(\Delta_d, C^*)$ with C^* and $\mathrm{Hom}(C^*, C^*)=\{\mu\in\mathrm{Hom}(C, C)\mid \mu(Z)\subset Z\}$ with Z respectively by means of the correspondences:

$$\mathrm{Hom}(\Delta_d, C^*)\ni\lambda\leftrightarrow\lambda(d)\in C^*,$$

$$\mathrm{Hom}(C^*, C^*)\ni\mu\leftrightarrow\mu(1)\in Z.$$

Under these identifications, the mapping σ is given by $\sigma(m)=d^m$ for $m\in Z$, so that σ is an isomorphism of Z into C^* and its image is nothing but Δ_d. The mapping ϕ^* is, therefore, injective.

Thus we can state our main result.

Theorem 4. *Set* $h^{p,q}(\lambda)=\dim H^q(X, \Omega^p(F_\lambda))$. *Then* $h^{p,q}(\lambda)=0$ *for any* p *and* q *if* $\lambda\notin\Delta_d$, *and in the case* $\lambda\in\Delta_d$ *we have*:

(A) *if* $\lambda=d^m$, $m\neq 0$, $p>0$,

($n>2$)

(i) $h^{p,q}(\lambda)=0$, *if* $2\leqq q\leqq n-2$

(ii) $h^{p,0}(\lambda)=h^{p,1}(\lambda)$

$$=\begin{cases}0, & \text{if } m<p\\ \binom{n}{p}, & \text{if } m=p\\ \binom{n+m-p-1}{n-p-1}\binom{m-1}{p}+\binom{n+m-p}{n-p}\binom{m-1}{p-1}^{3)}, & \text{if } m>p.\end{cases}$$

(iii) $h^{p,n-1}(\lambda)=h^{p,n}(\lambda)$

$$=\begin{cases}0, & \text{if } m>p-n\\ \binom{n}{p}, & \text{if } m=p-n\\ \binom{-m+p}{p}\binom{-m-1}{n-p-1}+\binom{-m+p-1}{p-1}\binom{-m-1}{n-p}, & \text{if } m<p-n.\end{cases}$$

($n=2$)

In this case $h^{p,0}(\lambda)$ *and* $h^{p,2}(\lambda)$ *are given by the same formula as the case* $n>2$, *setting* $n=2$; *but* $h^{p,1}(\lambda)$ *is not, and* $h^{p,1}(\lambda)=h^{p,0}(\lambda)+h^{p,2}(\lambda)$.

3) For any given two integers r and s, $\binom{r}{s}$ means the usual combination if $r, s>0$ and, otherwise we shall understand it as follow; $\binom{r}{s}=0$ if r or s is negative and $\binom{r}{s}=1$ if $r, s\geqq 0$, $rs=0$.

(B) *if* $\lambda = d^m$, $p=0$,

(i) $h^{0,q}(\lambda) = 0$, *if* $2 \leqq q \leqq n-2$

(ii) $(n>2)$, $h^{0,0}(\lambda) = h^{0,1}(\lambda) = \binom{n+m-1}{m}$

$$h^{0,n-1}(\lambda) = h^{0,n}(\lambda) = \binom{-m-1}{-m-n}$$

(iii) $(n=2)$, $h^{0,0}(\lambda) = \binom{m+1}{m}$

$$h^{0,1}(\lambda) = \binom{-m-1}{1} + \binom{m+1}{m}$$

$$h^{0,2}(\lambda) = \binom{-m-1}{1}$$

(C) *if* $\lambda = 1$,

(i) $h^{n,q}(1) = \begin{cases} 0, & if \quad q \leqq n-2 \\ 1, & if \quad q = n-1, n \end{cases}$

(ii) $h^{p,q}(1) = 0$, *if* $1 \leqq p \leqq n-1$,

(iii) $h^{0,q}(1) = \begin{cases} 0, & if \quad q \geqq 2 \\ 1, & if \quad q = 0, 1 \end{cases}$

§ 5. Summary of some known results.

Let $X(P^{n-1}, T^1, \phi)$ be the Hopf fibering of X and let $E = E_X(\rho, C^m)$ be the homogeneous vector bundle over X defined by the representation (ρ, C^m) of U_d and let $\boldsymbol{E}$ be the sheaf of germs of holomorphic sections of E. The restriction E/T^1 of E on $T^1 = GL(1, n-1, C)/U_d$ is also homogeneous with respect to the Klein form $GL(1, n-1; C)/U_d$. Therefore every element of $GL(1, n-1; C)$ induces a bundle automorphism of $E|T^1$, and so a linear isomorphism of the cohomology group $H^s(T^1, \boldsymbol{E}|T^1)$, $(s=0, 1)$. The holomorphic representation of $GL(1, n-1; C)$ thus obtained will be denoted by ρ^s $(s=0, 1)$[4], and the corresponding homogeneous vector bundle $E_{P^{n-1}}(\rho^s, H^s(T^1, \boldsymbol{E}|T^1))$ over P^{n-1} will be denoted simply by $\phi^s(E)$.

Now we take a spectral sequence $\{E_k\}$ whose final term E_∞ is associated to $H^*(X, \boldsymbol{E})$ and the second term E_2 is given by $E_2^{r,s} = H^r(P^{n-1}, \phi^s(\boldsymbol{E}))$, where $\phi^s(\boldsymbol{E})$ is the so-called *s-dimensional direct image sheaf of* $\boldsymbol{E}$ *by* ϕ. While, in our case, it is known the following result of Bott (cf. [3], Theorem VI).

4) This representation ρ^s is called, according to Bott, the s-dimensional induced representation of ρ.

Lemma 1. *$\phi^s(\boldsymbol{E})$ coincides with the sheaf of germs of holomorphic sections of $\phi^s(E)$; therefore $\phi^s(\boldsymbol{E})$ are zero sheaves for $s\geqq 2$.*

In particular, let E be a (homogeneous) line bundle. Then as to the restriction $E|T^1$ we know the following lemma (cf. [9], Proposition 3. 6).

Lemma 2. *The 0-dimensional cohomology group $H^0(T^1, \boldsymbol{E}|T^1)$ does not vanish if and only if $E|T^1$ is the trivial line bundle; therefore $\phi^0(E)$ is the zero sheaf unless $E|T^1$ is trivial.*

Now, for the computations in the next section, we need to know the cohomology groups $H^q(P^{n-1}, \Omega^p(\hat{F}))$, where $\Omega^p(\hat{F})$ is the sheaf of germs of holomorphic p-forms with values in the line bundle $\hat{F}$ over P^{n-1}; the dimensions of these cohomology groups have been computed by Bott [3] and by Matsumura [10] independently. That is,

Lemma 3. *Let $\hat{F}_m$ $(m\in Z)$ be the line bundle over P^{n-1} corresponding to $m\in\mathrm{Hom}\,(C^*, C^*)$ (cf. §4, (1)), and set $\hat{h}^{p,q}(m)=\dim H^q(P^{n-1}, \Omega^p(\hat{F}_m))$. Then we have,*

(i) $\hat{h}^{p,p}(0) = 1 \quad (o\leqq p\leqq n-1)$

(ii) $\hat{h}^{p,0}(m)=\binom{n+m-p-1}{n-p-1}\binom{m-1}{p} \quad (m>p)$

(iii) $\hat{h}^{p,n-1}(m)=\binom{-m+p}{p}\binom{-m-1}{n-p-1} \quad (p-n+1>m)$

(iv) $\hat{h}^{p,q}(m)=0$ *for other cases.*

§6. The proof of the main theorem.

For the proof of our Theorem 4, we need the following extension of vector bundles over P^{n-1}:

$$0\to I\to Q(X)\to\hat{\Theta}\to 0\,, \tag{1}$$

which are the homogeneous vector bundles over P^{n-1} induced by the exact sequence (1) in §2 of $GL(1, n-1; C)$-modules. For instance $Q(X)=E_{P^{n-1}}(Ad, \mathfrak{g}/\mathfrak{u})$; and we note that $\Theta=\phi^*Q(X)$. By Ω and $\hat{\Omega}$ are meant the analytic sheaves of germs of holomorphic sections of Θ^* and $\hat{\Theta}^*$ respectively ($*$ means the dual vector bundle). Moreover we denote by Ξ the sheaf of germs of holomorphic sections of $Q(X)^*$. From (1) we have the exact sequence of analytic sheaves on P^{n-1}:

$$0\to\hat{\Omega}\to\Xi\to\boldsymbol{C}\to 0\,.$$

From this we can construct the following exact sequences:

$$(2) \qquad 0 \to \hat{\Omega}^p \to \Xi^p \to \hat{\Omega}^{p-1} \to 0 \quad (1 \leq p \leq n),$$

where Ξ^p is the sheaf of germs of holomorphic sections of the vector bundle $Q(X)^{*p}$ which is the p-exterior product of $Q(X)^*$ (see, for detail, [5], Satz 4.1.3*) and $\hat{\Omega}^p$ denotes the sheaf of germs of holomorphic p-forms on P^{n-1}.

Now we consider the spectral sequence $\{E_k\}$ associated to the Hopf fibering and the sheaf $\Omega^p(F_\lambda)$ over X. Then, the sheaf $\phi^s(\Omega^p(F_\lambda))=0$ except for $s=0,1$ by Lemma 1 and $\phi^s(\Omega^p(F_\lambda))=\phi^s(\boldsymbol{F}_\lambda)\otimes\Xi^p$, as is known by an easy argument on the induced representation, since $\Theta^{*p}=\phi^*(\Xi^p)$. On the other hand, the theorem of Riemann-Roch concerning the elliptic curve T^1 and the line bundle $F_\lambda|T^1$ (=the restriction of F_λ on T^1) implies that

$$\dim \phi^0(F_\lambda) - \dim \phi^1(F_\lambda) = 0,$$

because $F_\lambda|T^1$ has a holomorphic connection by a theorem of Matsushima [9] and so has the vanishing Chern class (cf. Atiyah [1]). Moreover, by Lemma 2, $\dim\phi^0(F_\lambda)>0$ if and only if $F_\lambda|T^1$ is trivial. The latter condition means that F_λ is induced from a line bundle $\hat{F}_m$ over P^{n-1} by ϕ; therefore in this case $\lambda=d^m$ (cf. (1) in § 4). Hence, if $\lambda\notin\Delta_d$ then $\phi^s(\Omega^p(F_\lambda))=0$ for all s (and p), which implies $E_2=E_\infty=H^*(X, \Omega^p(F_\lambda))=\{0\}$.

We assume hereafter that $\lambda=d^m\in\Delta_d$, and that $F_\lambda=\phi^*\hat{F}_m$. Then $F_\lambda|T^1$ is trivial and $\phi^s(\Theta^{*p}\otimes F_\lambda)\cong Q(X)^{*p}\otimes\hat{F}_m$ for $s=0,1$ by an easy argument on the induced representations; hence we have $E_2^{r,s}=H^r(P^{n-1}, \Xi^p\otimes\hat{\boldsymbol{F}}_m)$ $(s=0,1)$ and $E_2^{r,s}=\{0\}$ $(s\geq 2)$, which implies that

$$(3) \qquad E_2^q = E_2^{q,0}+E_2^{q-1,1} = H^q(P^{n-1}, \Xi^p\otimes\hat{\boldsymbol{F}}_m)+H^{q-1}(P^{n-1}, \Xi^p\otimes\hat{\boldsymbol{F}}_m)$$

for $0\leq q\leq n$. Now we shall devide the subsequent discussions into three cases.

(A) The case $m\neq 0$, $p>0$.

The sequence (2) implies the following sheaf exact sequences:

$$(4) \qquad 0\to\hat{\Omega}^p(\hat{F}_m)\to\Xi^p\otimes\hat{\boldsymbol{F}}_m\to\hat{\Omega}^{p-1}(\hat{F}_m)\to 0 \qquad (1\leq p\leq n).$$

The corresponding cohomology exact sequence is

$$\to H^{q-1}(P^{n-1}, \hat{\Omega}^p(\hat{F}_m))\to H^{q-1}(P^{n-1}, \Xi^p\otimes\hat{\boldsymbol{F}}_m)\to H^{q-1}(P^{n-1}\hat{\Omega}^{p-1}(\hat{F}_m))$$
$$\to H^q(P^{n-1}, \hat{\Omega}^p(\hat{F}_m))\to H^q(P^{n-1}, \Xi^p\otimes\hat{\boldsymbol{F}}_m)\to H^q(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m))\to$$

Therefore, if $1\leq q\leq n-2$, then $H^q(P^{n-1}, \hat{\Omega}^p(\hat{F}_m))=\{0\}$ for any p by Lemma 3, so that $E_2^{q,0}=\{0\}$, and $E_2^{q-1,1}=\{0\}$ also for the case $q>1$. Let $n>2$. If $q=0$, $E_2^0=E_\infty^0$ and $E_2^0=E_2^{0,1}=H^0(P^{n-1}, \Xi^p\otimes\hat{\boldsymbol{F}}_m)$ is given by

$$0 \to H^0(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \to E_2^0 \to H^0(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \to 0\,;$$

hence $\dim E_2^0 = \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m)$. If $q=n-1$, $E_2^{n-2,1} = E_2^{n-2,0} = \{0\}$ and $E_2^{n-1,0} = E_2^{n-1}$ is given by

$$0 \to H^{n-1}(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \to E_2^{n-1} \to H^{n-1}(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \to 0\,;$$

hence $\dim E_2^{n-1} = \hat{h}^{p,n-1}(m) + \hat{h}^{p-1,n-1}(m)$. If $q=n$, $E_2^{n,0} = H^n(P^{n-1}, \Xi^p \otimes \hat{\boldsymbol{F}}_m) = \{0\}$ and $E_2^{n-1,1} = E_2^{n-1,0}$. Thus the spectral sequence is trivial and we obtain

(i) $h^{p,q}(\lambda) = 0$ for $2 \leqq q \leqq n-2$

(ii) $h^{p,0}(\lambda) = h^{p,1}(\lambda) = \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m)$

(iii) $h^{p,n-1}(\lambda) = h^{p,n}(\lambda) = \hat{h}^{p,n-1}(m) + \hat{h}^{p-1,n-1}(m)$.

In case $n=2$, from (3), (4) and Lemma 3, we can deduce readily the following results:

(i) $h^{p,0}(\lambda) = \begin{cases} \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m), & \text{if } m \geqq p-1 \\ \hat{h}^{p,0}(m), & \text{if } m \leqq p-1\,. \end{cases}$

(ii) $h^{p,2}(\lambda) = \begin{cases} \hat{h}^{p-1,1}(m), & \text{if } m \geqq p-1 \\ \hat{h}^{p,1}(m) + \hat{h}^{p-1,1}(m), & \text{if } m \leqq p-1\,. \end{cases}$

(iii) $h^{p,1}(\lambda) = h^{p,0}(\lambda) + h^{p,2}(\lambda)$.

(B) The case $p=0$.

By Lemma 3 and (3), we have $E_2^q = \{0\}$ for $2 \leqq q \leqq n-2$. Furthermore, if $n>2$, we have $E_2^0 = E_2^1 = H^0(P^{n-1}, \hat{\boldsymbol{F}}_m)$, $E_2^{n-1} = E_2^n = H^{n-1}(P^{n-1}, \hat{\boldsymbol{F}}_m)$, and if $n=2$, we have $E_2^0 = H^0(P^1, \hat{\boldsymbol{F}}_m)$, $E_2^1 = H^0(P^1, \hat{\boldsymbol{F}}_m) + H^1(P^1, \hat{\boldsymbol{F}}_m)$ and $E_2^2 = H^1(P^1, \hat{\boldsymbol{F}}_m)$. Moreover the spectral sequence is trivial, and we obtain:

(i) $h^{0,q}(\lambda) = 0$, for $2 \leqq q \leqq n-2$

(ii) $(n>2)$ $\begin{cases} h^{0,0}(\lambda) = h^{0,1}(\lambda) = \hat{h}^{0,0}(m) \\ h^{0,n-1}(\lambda) = h^{0,n}(\lambda) = \hat{h}^{0,n-1}(m) \end{cases}$

(iii) $(n=2)$ $\begin{cases} h^{0,0}(\lambda) = \hat{h}^{0,0}(m), \\ h^{0,1}(\lambda) = \hat{h}^{0,0}(m) + \hat{h}^{0,1}(m) \\ h^{0,2}(\lambda) = \hat{h}^{0,1}(m)\,. \end{cases}$

(C) The case $m=0$, $p>0$.

From (2) and (3) we have

$$\to H^{q-1}(P^{n-1}, \hat{\Omega}^p) \to E_2^{q-1,0} = E_2^{q-1,1} \to H^{q-1}(P^{n-1}, \hat{\Omega}^{p-1})$$
$$\to H^q(P^{n-1}, \hat{\Omega}^p) \to E_2^{q,0} = E_2^{q,1} \to H^q(P^{n-1}, \hat{\Omega}^{p-1})$$

If $p=n$, then $H^q(P^{n-1}, \hat{\Omega}^n)=\{0\}$ and so $E_2^q=E_2^{q,0}+E_2^{q-1,1}=H^q(P^{n-1}, \hat{\Omega}^{n-1})+H^{q-1}(P^{n-1}, \hat{\Omega}^{n-1})$; hence by Lemma 3 we have

$$h^{n,q}(1)=\begin{cases}0, & \text{if } q\leqq n-2\\ 1, & \text{if } q=n-1, n.\end{cases}$$

We assume hereafter that $1\leqq p\leqq n-1$. If $q\neq p\pm 1$, p, then we have $E_2^{q,0}=\{0\}$, $E_2^{q-1,1}=\{0\}$; hence $h^{p,q}(1)=0$. If $q=p-1$, then $E_2^{p-2,1}=\{0\}$, $E_2^{p-1,0}=E_\infty^{p-1}$. If $q=p+1$, then $E_2^{p+1,0}=\{0\}$, $E_2^{p,1}=E_\infty^{p+1}$. If $q=p$, then we have

$$0\to E_2^{p-1,0}\to H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1})\overset{\delta^*}{\to} H^p(P^{n-1}, \hat{\Omega}^p)\to E_2^{p,0}\to 0\,. \tag{5}$$

We remark here that $\dim H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1})=\dim H^p(P^{n-1}, \hat{\Omega}^p)=1$ and that $E_2^{p-1,0}=E_2^{p,0}=\{0\}$ if and only if δ^* is not the zero homomorphism (i.e. bijective). While by the following Lemma 4, we have in reality $E_2^{p-1,0}=E_2^{p-1,1}=E_2^{p,0}=E_2^{p,1}=\{0\}$ for $1\leqq p\leqq n-2$; hence we have then $h^{p,q}(1)=0$ also for $q=p\pm 1$, p. It remains only the case $p=n-1$; in this case we have $h^{n-1,q}(1)=h^{1,n-q}(1)=0$ by Serre's duality for $n>2$ (The case $n=2$ is contained in the proof of Lemma 4).

Lemma 4. *In the above exact sequence* (5), *if* $1\leqq p\leqq n-2$, δ^* *is bijective*; *hence* $E_2^{p-1,0}=E_2^{p,0}=\{0\}$.

Proof. First we shall consider the case $p=1$ (in (2) we set $\hat{\Omega}^0=\boldsymbol{C}$). In this case, $\delta^*: H^0(P^{n-1}, \boldsymbol{C})\to H^1(P^{n-1}, \hat{\Omega})$ is not the zero homomorphism; in fact, if otherwise, the extension: $0\to\hat{\Omega}\to\Xi\to\boldsymbol{C}\to 0$ is splittable by a lemma of Atiyah (Proc. London Math. Soc., 7 (1957), p. 429, Lemma 13), and then, by the same argument as in our previous paper [7] (cf. the foot-note 7)), the Hopf fibering must be trivial; however $S^1\times S^{2n-1}$ and $T^1\times P^{n-1}$ are clearly not homeomorphic. This proves the lemma in our case.

In general case we prove the lemma by induction on n. In case $n=2$, it must be $p=1$; therefore the lemma has been proved by the above discussions. We assume $n>2$ and consider the exact sequence (2) over the base space P^{n-2}, which will be written as:

$$0\to\hat{\Omega}_*^p\to\Xi_*^p\to\hat{\Omega}_*^{p-1}\to 0 \quad \text{over} \quad P^{n-2}\,. \tag{2_*}$$

On the other hand, the imbedding of $GL(n-1, C)$ into $GL(n, C)$, defined by $g\to\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}$ for $g\in GL(n-1, C)$, induces an imbedding of $P^{n-?}$ into P^{n-1} as a hyperplane, which we shall fix once for all. The sheaves in (2_*) are naturally extendable to the sheaves over P^{n-1} by assuming that the fibres on the complement of P^{n-1} vanish, and they shall be denoted with the same letters as in (2_*). Now we shall show that there are natural sheaf homomorphisms $\alpha_p: \Xi^p \to \Xi^p_*$ and $\beta_p: \hat{\Omega}^p \to \hat{\Omega}^p_*$ which yield the following commutative diagram:

$$
(6)\qquad
\begin{array}{ccccccccc}
0 \to & \hat{\Omega}^p & \to & \Xi^p & \to & \hat{\Omega}^{p-1} & \to 0 \\
 & \downarrow \beta_p & & \downarrow \alpha_p & & \downarrow \beta_{p-1} & \\
0 \to & \hat{\Omega}^p_* & \to & \Xi^p_* & \to & \hat{\Omega}^{p-1}_* & \to 0 .
\end{array}
$$

For this sake, we identity the exact sequence of $\hat{U}$-modules (1) in §2 with the one:

$$0 \to C^1 \to C^n \to C^{n-1} \to 0 ,$$

where $\hat{U}$ acts on each module as the identity representation on C^1, as $\frac{1}{b}\hat{u} = \begin{pmatrix} 1 & \frac{1}{b}* \\ 0 & \frac{1}{b}B \end{pmatrix}$ on C^n and as $\frac{1}{b}B$ on C^{n-1} respectively, for every element $\hat{u} = \begin{pmatrix} b & * \\ 0 & B \end{pmatrix} \in \hat{U}$. Then the restrictions of $\hat{\Theta}^{*p}$ and $Q(X)^{*p}$ on P^{n-2} are given by $GL(n-1, C) \times_{\hat{U}_*} (C^{n-1})^{*p}$ [5] and $GL(n-1, C) \times_{\hat{U}_*} (C^n)^{*p}$ respectively where $\hat{U}_* = GL(1, n-2; C)$ acts on $(C^n)^{*p}$ and $(C^{n-1})^{*p}$ as defined above. Then we have the commutative diagram of modules:

$$
\begin{array}{ccccccccc}
0 \to & (C^{n-1})^{*p} & \to & (C^n)^{*p} & \to & (C^{n-1})^{*p-1} & \to 0 \\
 & \downarrow \tilde{\beta}_p & & \downarrow \tilde{\alpha}_p & & \downarrow \tilde{\beta}_{p-1} & \\
0 \to & (C^{n-2})^{*p} & \to & (C^{n-1})^{*p} & \to & (C^{n-2})^{*p-1} & \to 0 ,
\end{array}
$$

where $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ denote the restriction mappings of alternating p-forms. This diagram, considered as the one of $\hat{U}_*$-modules, is commutative as is easily seen. Therefore it induces the commutative diagram of homogeneous vector bundles over P^{n-2}; this implies that there are corresponding sheaf homomorphisms α_p, β_p and β_{p-1} as in (6). Thus we have the following commutative diagram:

$$
\begin{array}{ccc}
H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) & \xrightarrow{\delta^*} & H^p(P^{n-1}, \hat{\Omega}^p) \\
\downarrow \beta_{p-1} & & \downarrow \beta_p \\
H^{p-1}(P^{n-2}, \hat{\Omega}^{p-1}_*) & \xrightarrow{\delta^*} & H^p(P^{n-2}, \hat{\Omega}^p_*) ,
\end{array}
$$

where the mappings β_{p-1} and β_p are bijective for $1 \leqq p \leqq n-2$ since they coincide with the restriction mappings of harmonic forms *via* the

5) $(C^{n-1})^{*p}$ denotes the vector space of all alternating p-forms on C^{n-1}.

Dolbealt isomorphisms. While, δ^* in the under column is bijective by induction assumption, so our δ^* must be bijective.

REMARK 1. Theorem 4 tells us that both Riemann-Roch's theorem with respect to any line bundle and Hodge's index theorem are valid for Hopf manifolds (cf. [8]). In fact, for any line bundle F_λ over a Hopf manifold X, we can readily check that

$$\chi(X, \boldsymbol{F}_\lambda) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, \boldsymbol{F}_\lambda) = \sum_{q=0}^{n} (-1)^q h^{0,q}(\lambda) = 0\,;$$

while the Todd genus $T(X, \boldsymbol{F}_\lambda)=0$ since $H^2(X, Z)=\{0\}$. Furthermore, the index $\tau(X)$ of X is clearly 0, since X is homeomorphic to $S^1\times S^{2n-1}$; while we see immediately, from (C) in Theorem 4, that

$$\sum_{p,q} (-1)^q h^{p,q}(1) = 0\,.$$

REMARK 2. Theorem 1 and Theorem 2 can be readily derived from Theorem 4. In fact, by Serre's duality theorem, we have $H^q(X, \Theta)\simeq H^{n-q}(X, \Omega^1(K))$ where K denotes the canonical line bundle of X. While, from the exact sequence (2) in §2, we get immediately $K=\phi^*\hat{K}$, where $\hat{K}$ is the canonical bundle of P^{n-1} and coincides with $\hat{F}_{-n}$. Therefore, Theorem 4, (A) yields that $\dim H^0(X, \Theta)=\dim H^1(X, \Theta)=n^2$, $H^q(X, \Theta)=\{0\}$ for $q\geqq 2$.

REMARK 3. The proof of Theorem 4 suggests us the possibilities of computing the cohomology groups $H^p(X, \Omega^p(F))$ for other class of C-manifolds with the *fundamental fibering* $X(\hat{X}, T^1, \phi)$ (cf. [7]) provided that the cohomology groups $H^q(\hat{X}, \Omega(\hat{F}))$ are known. For instance, Calabi-Eckmann's example (cf. [4], [7]) or $SU(3)$ with a left invariant complex structure is such a manifold. However, for them $\hat{X}=P^p\times P^q$ or $\hat{X}=F(3)$ (=the 3-dimensional flag manifold) respectively and the corresponding cohomology groups $H^q(\hat{X}, \Omega^p(\hat{F}))$ are rather complicated; consequently the computations of $H^q(X, \Omega^p(F))$ might be more difficult than for Hopf manifolds.

But we shall exhibit here the number $h^{p,q}=h^{p,q}(1)$ for $SU(3)$, since Bott's computations in [3] for then are incorrect.

$$h^{0,0} = h^{0,1} = h^{1,1} = h^{1,2} = 1\,,$$
$$h^{4,4} = h^{4,3} = h^{3,3} = h^{3,2} = 1\,,$$
$$h^{p,q} = 0 \quad \text{otherwise.}$$

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