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# TRAPPING OBSTACLES WITH A SEQUENCE OF POLES OF THE SCATTERING MATRIX CONVERGING TO THE REAL AXIS

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1. Introduction. We consider the scattering of the acoustic equation by bounded obstacles. Let  $\mathcal{O}$  be a bounded open set in  $\mathbb{R}^3$  with sufficiently smooth boundary. We set  $\Omega = \mathbb{R}^3 - \overline{\mathcal{O}}$ . Suppose that  $\Omega$  is connected. Consider the following problem

$$\begin{cases} \Box u = \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^3 \frac{\partial^2 u}{\partial x_j^2} = 0 & \text{ in } (-\infty, \infty) \times \Omega \\ u(t, x) = 0 & \text{ on } (-\infty, \infty) \times \Gamma . \end{cases}$$

Denote by S(z) the scattering matrix for this problem. About the definition and the fundamental properties of the scattering matrix, see Lax and Phillips [8], especially Theorems 5.1 and 5.6 of Chapter V.

On relationships between geometric properties of  $\mathcal{O}$  and the location of poles of  $\mathcal{S}(z)$  Lax and Phillips gave a conjecture [8, page 158] (see also Ralston [16, 17]), that is, for a nontrapping obstacle the scattering matrix  $\mathcal{S}(z)$  is free for poles in  $\{z; \text{Im } z \leq \alpha\}$  for some constant  $\alpha > 0$ , and for a trapping obstacle  $\mathcal{S}(z)$  has a sequence of poles  $\{z_j\}_{j=1}^{\infty}$  such that  $\text{Im} z_j \rightarrow 0$  as  $j \rightarrow \infty$ . Concerning this conjecture Morawetz, Ralston and Strauss [14] and Melrose [11] proved that the part for nontrapping obstacles is correct. On the other hand, Bardos, Guillot and Ralston [1], Petkov [15] and Ikawa [4, 5, 6] made considerations on some simple cases of trapping obstacles. Among them the result of Ikawa [4, 5] shows that the part of the conjecture for trapping obstacles is not correct in general, namely for two strictly convex objects  $\mathcal{S}(z)$  is free for poles in  $\{z; \text{Im} z \leq \alpha\}$  ( $\alpha > 0$ ). Yet it seems very sure that the conjecture remains to be correct for a great part of trapping obstacles. In spite of the conjecture we have not known even an example of obstacle  $\mathcal{O}$  for which is proved the existence of a sequence of poles of the scattering matrix converging to the real axis.<sup>1)</sup>

The purpose of this paper is to show an example of  $\mathcal{O}$  whose scattering

<sup>&</sup>lt;sup>1)</sup> Ralston [16] gives examples of the scattering by the inhomogeneity of medium such that the scattering matrix has a sequence of poles converging to the real axis.

matrix has such a sequence of poles.

**Theorem 1.** Let  $\mathcal{O}_j$ , j=1, 2, be convex open sets in  $\mathbb{R}^3$  with sufficiently smooth boundary  $\Gamma_j$ , and let  $a_j \in \Gamma_j$ , j=1, 2, be the point such that  $|a_1-a_2| = \operatorname{dis}(\mathcal{O}_1, \mathcal{O}_2)$ . Suppose that the principal curvatures  $\kappa_{jl}(x)$ , l=1, 2 of  $\Gamma_j$  at  $x \in \Gamma_j$  satisfy

(1.1) 
$$C |x-a_j|^{\epsilon} \ge \kappa_{jl}(x) \ge C^{-1} |x-a_j|^{\epsilon} \quad \text{for all } x \in \Gamma_j$$

for some

$$(1.2) \qquad \qquad \infty > e \ge 2$$

and C>0. Then the scattering matrix for  $\mathcal{O}=\mathcal{O}_1\cup\mathcal{O}_2$  has a sequence of poles  $\{z_j\}_{j=1}^{\infty}$  such that

$$\operatorname{Im} z_j \to 0 \qquad as \quad j \to \infty$$

In the proof of this theorem we start from a trace formula proved by Bardos, Guillot and Ralston [1]:

$$\operatorname{tr}_{L^{2}(\mathbb{R}^{3})}\int \rho(t) \left(\cos t\sqrt{-\Delta} \oplus 0 - \cos t\sqrt{-\Delta_{0}}\right) dt$$
$$= \frac{1}{2} \sum_{\text{poles}} \hat{\rho}(\lambda_{j}) \qquad \text{for } \rho \in C_{0}^{\infty}(2\mathbb{R}, \infty)$$

(explanation of the notation will be given in §2). The main differences of the treatment of this formula in this article from in [1] are (i) we substitute in the place of  $\rho(t)$  a sequence of functions  $\rho_q(t)$ ,  $q=1, 2, \cdots$  such that min $\{t; t \in \text{supp}\rho_q\} \rightarrow \infty$  as  $q \rightarrow \infty$ , (ii) all the eigenvalues of the Poincaré mapping of the periodic ray are 1, which is a consequence of the assumption (1.1) subject to (1.2).

It should be remarked that the result in [4] can be extended to a case of two convex objects such that the Poincaré mapping of the periodic ray has not 1 as an eigenvalue. Namely, in this case all the poles of S(z) have the imaginary part $\geq \alpha$  for some  $\alpha > 0$ . Therefore in order to find an example of an obstacle composed of two convex objects with a sequence of poles converging to the real axis we have to consider obstacles whose Poincaré mapping has 1 as an eigenvalue. Of course these differences give rise to an essential difficulty in the proof, especially in the estimate of the left hand side of the trace formula for large q. To overcome this difficulty we represent the kernel of  $\cos t\sqrt{-\Delta}$ by a superposition of asymptotic solutions constructed following the process in [2, 4], and apply Varčenko's theorem [19, 7] to an estimation of integrals of asymptotic solutions.

#### 2. On the trace formula and a reduction of the problem

We denote by  $\Delta$  the selfadjoint realization in  $L^2(\Omega)$  of the Laplacian in  $\Omega$ 

with the Dirichlet boundary condition and by  $\Delta_0$  the selfadjoint realization in  $L^2(\mathbf{R}^3)$  of the Laplacian in  $\mathbf{R}^3$ . Bardos, Guillot and Ralston shows in [1] that the following trace formula

(2.1) 
$$\operatorname{tr}_{L^{2}(\mathbb{R}^{3})} \int_{\mathbb{R}} \rho(t) \left( \cos t \sqrt{-\Delta} \oplus 0 - \cos t \sqrt{-\Delta_{0}} \right) dt$$
$$= \frac{1}{2} \sum_{\text{poles}} \hat{\rho}(\lambda_{j})$$

holds for all  $\rho \in C_0^{\infty}(2R, \infty)^2$ , where R=diameter of  $\mathcal{O}$ ,

$$\hat{\rho}(\lambda) = \int e^{i\lambda t} \rho(t) dt$$

and  $\cos t\sqrt{-\Delta} \oplus 0$  is an operator in  $L^2(\mathbb{R}^3)$  defined for  $f=f_1+f_2$ ,  $f_1 \in L^2(\Omega)$ .  $f_2 \in L^2(\mathcal{O})$  by

$$((\cos t\sqrt{-\Delta}\oplus 0)f)(x) = \begin{cases} (\cos t\sqrt{-\Delta}f_1)(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathcal{O} \end{cases}$$

Remark that an estimate of the right hand side of (2.1)

(2.2) 
$$\sum_{\text{poles}} |\hat{\rho}(\lambda_j)| \leq C(T) ||\rho||_{H^4(\mathbf{R})}, \quad \forall \rho \in C_0^\infty(2\mathbf{R}, T)$$

is shown in §3 of [1], where C(T) is a constant depending on T.

Let  $\rho_0(t) \in C_0^{\infty}(-1, 1)$  and define  $\rho_q(t), q=1, 2, \cdots$  by

(2.3) 
$$\rho_q(t) = \rho_0((q+1)^l(t-2dq)),$$

where  $d = \operatorname{dis}(\mathcal{O}_1, \mathcal{O}_2)$  and l is a positive integer determined later.

**Lemma 2.1.** Suppose that all the poles  $\{\lambda_j\}_{j=1}^{\infty}$  of S(z) verify

$$(2.4) Im \lambda_j \ge \alpha$$

for some constant  $\alpha > 0$ . Then we have

(2.5) 
$$\sum_{j=1}^{\infty} |\hat{\rho}_q(\lambda_j)| \leq C(q+1)^{4l} e^{-2d\varpi q} \quad \text{for all } q$$

where C is a constant independent of q and l.

Proof. Set

$$\rho_{p,q}(t) = \rho_0((p+1)^l(t-2dq)).$$

Fix  $q_0$  in such a way  $2dq_0-1 \ge 2R$ . Then we have  $\rho_{p,q_0}(t) \in C_0^{\infty}(2R, T)$  ( $T = 2dq_0+1$ ) for all p. Applying (2.2) for  $\rho_{p,q_0}$  we have

<sup>1)</sup> Melrose [12] shows that (2.1) holds for all  $\rho \in C_0^{\infty}(\mathbb{R}^+)$ .

$$\sum_{j=1}^{\infty} |\hat{
ho}_{p,q_0}(\lambda_j)| \leqslant C(T) ||
ho_{p,q_0}||_{\operatorname{H}^4(\mathbf{R})} \ \leqslant C(T) C(p+1)^{4l} \,.$$

Since  $\hat{\rho}_{p,q}(\lambda) = e^{i2d(q-q_0)\lambda} \hat{\rho}_{p,q_0}(\lambda)$  we have, under the assumption (2.4), for all  $\lambda_j$ 

$$\begin{split} |\hat{\rho}_{p,q}(\lambda_j)| &\leqslant e^{-2d(q-q_0)\mathrm{Im}\lambda_j} |\hat{\rho}_{p,q_0}(\lambda_j)| \\ &\leqslant e^{-2d\omega(q-q_0)} |\hat{\rho}_{p,q_0}(\lambda_j)| \;. \end{split}$$

Then

$$\sum_{j=1}^{\infty} |\hat{
ho}_{p,q}(\lambda_j)| \leqslant e^{-2d lpha (q-q_0)} \sum_{j=1}^{\infty} |\hat{
ho}_{p,q_0}(\lambda_j)| \ \leqslant e^{-2d(q-q_0) lpha} C(T) C(p\!+\!1)^{4l} \ \leqslant C(T) C e^{2d q_0 lpha} (p\!+\!1)^{4l} e^{-2d q lpha} .$$

Note that  $\rho_q(t) = \rho_{q,q}(t)$ . Then we have (2.5) by setting p=q in the above estimate. Q.E.D.

Concerning the left hand side of (2.1) we have the following

**Proposition 2.2.** Suppose that  $\mathcal{O}$  satisfies the condition in Theorem 1. Choose  $\rho_0(t) \in C_0^{\infty}(-1, 1)$  so that

(2.6) 
$$\rho_0(t) \ge 0, \quad \int_{-\infty}^{\infty} \rho_0(t) dt = 1$$

and

(2.7) 
$$\hat{\rho}_0(-k) = \hat{\rho}_0(k) \ge 0 \quad \text{for all } k \in \mathbb{R} .$$

Then we have

(2.8) 
$$|\operatorname{tr}_{L^{2}(\mathbf{R}^{3})}\int_{-\infty}^{\infty}\rho_{q}(t)\left(\cos t\sqrt{-\Delta}\oplus 0-\cos t\sqrt{-\Delta_{0}}\right)dt| \\ \geqslant cq^{(1-2/\epsilon_{0})(l+1)-2}-C_{l}q^{(1-5/2\epsilon_{0})l}$$

for all  $q \ge q_0$  if  $l \ge l_0$ , where  $e_0 = e+2$  and  $l_0$  is a some fixed positive integer, c is a positive constant independent of l.

The remaining sections of this paper will be devoted to the proof of this proposition. Theorem 1 can be proved immediately by Lemma 2.1 and Proposition 2.2. Indeed, choose  $\rho_0$  so that (2.6) and (2.7) are verified. Suppose that there is no sequence of poles which converges to the real axis. Then there exists  $\alpha > 0$  such that

Im 
$$\lambda_j \ge \alpha$$
 for all  $j$ .

Then we have (2.5) for all large q. By using (2.5) and (2.8) we have from (2.1)

$$cq^{(1-2/e_0)(l+1)-2} - C_l q^{(1-5/2e_0)l} \leq C(q+1)^{4l} e^{-2daq}$$

for large q if  $l \ge l_0$ . Letting q tend to  $\infty$  the above inequality shows a contradiction. Thus Theorem 1 is proved.

We would like to remark that if we use the result of Melrose [12] Theorem 1 can be made better in the following form.

**Theorem 2.** Suppose that O satisfies the condition in Theorem 1. There exists a positive constant  $\gamma$  such that for any  $\varepsilon > 0$  a region

 $\{z; \operatorname{Im} z \leq \mathcal{E}(|\operatorname{Re} z|+1)^{-\gamma}\}$ 

contains an infinite number of poles of S(z).

Recall that Melrose [12] shows that

$$(2.9) N(K) \leq C(1+K)^p$$

for some p>0 where N(K)=the number of  $\lambda_j$  such that  $|\lambda_j| \leq K$ . By using (2.9) we have the following lemma, and Theorem 2 is derived immediately from Proposition 2.2 and Lemma 2.3.

**Lemma 2.3.** Suppose that  $\{z; \text{ Im } z \leq \varepsilon_0 (|\text{Re } z|+1)^{-\gamma}\}$   $(\varepsilon_0 > 0)$  has no poles. Then it holds that

$$\sum_{j=1}^{\infty} |\hat{\rho}_q(\lambda_j)| \leqslant C_{\mathfrak{e}_0,l} \quad \text{for all } q$$

if  $0 < \gamma < l^{-1}$ .

Proof. Let  $0 < \gamma < l^{-1}$ . Choose  $\alpha > 0$  so that  $1 - \alpha \gamma > 0$ ,  $\alpha > l$ . We classify the poles into three groups:

Group  $I = \{\lambda_j; Im \lambda_j \ge \mathcal{E}\}$ , Group  $II = \{\lambda_j; \mathcal{E} > Im \lambda_j \ge \mathcal{E}_0(|\operatorname{Re} \lambda_j| + 1)^{-\gamma}, |\operatorname{Re} \lambda_j| \le q^{\mathfrak{a}}\}$ , Group  $III = \{\lambda_j: \mathcal{E} > Im \lambda_j \ge \mathcal{E}_0(|\operatorname{Re} \lambda_j| + 1)^{-\gamma}, |\operatorname{Re} \lambda_j| \ge q^{\mathfrak{a}}\}$ .

By the same argument as Lemma 2.1 we have

$$\sum_{j\in \text{Group I}} |\hat{\rho}_q(\lambda_j)| \leq C_l (q+1)^{4l} e^{-2dqt}$$

From (2.9) the number of the poles of Group II is less than  $C(1+q^{\alpha})^{p}$ . Then

$$\sum_{\lambda_j \in \text{Group II}} |\beta_q(\lambda_j)| \leq C_l e^{-2dq \mathfrak{e}_0(q^{\mathfrak{G}})^{-\gamma}} (1+q^{\mathfrak{G}})^p \leq C_l (1+q^{\mathfrak{G}})^p e^{-2d\mathfrak{e}_0 q^{1-\mathfrak{G}}\gamma}.$$

Since an estimate  $|\hat{\rho}_q(z)| \leq C_N \left(\frac{|z|}{q^l}\right)^{-N}$  holds for any N we have

$$\sum_{n<\operatorname{Re}\lambda_j\leq (n+1)} |\hat{\beta}_q(\lambda_j)| \leq C_N (n+1)^p (nq^{-l})^{-N},$$

and

$$\sum_{\substack{\lambda_j \in \text{Group III} \\ \ll C_N q^{lN}(q^{\mathfrak{G}})^{-N+p+2}}} |\hat{\rho}_q(\lambda_j)| \leq \sum_{\substack{n=\lfloor q^{\mathfrak{G}} \\ m \neq l \\ m \neq l}}^{\infty} C_N q^{lN} (n+1)^q n^{-N}$$

Then summing up these estimates, if we choose N so large that  $(-\alpha+l)N$  $+p+2\leq 0$ , it holds that

$$\sum_{\text{poles}} |\beta_q(\lambda_j)| \leqslant C(1+q^{\alpha})^p e^{-2d\epsilon_0 q^{(1-\alpha')}} + C_N \leqslant C'_N .$$
Q.E.D.

## 3. Program of the proof of Proposition 2.2

Denote the kernel distribution of  $\cos t\sqrt{-\Delta_0}$  and  $\cos t\sqrt{-\Delta}$  by  $E_0(t; x, y)$ and E(t, x, y) respectively. Then the kernel distribution e(t; x, y) of  $\cos t\sqrt{-\Delta}$  $\oplus 0 - \cos t\sqrt{-\Delta_0}$  is written as

$$e(t; x, y) = \tilde{E}(t; x, y) - E_0(t; x, y)$$

where

$$\widetilde{E}(t; x, y) = \begin{cases} E(t; x, y) & \text{for } x, y \in \Omega \\ 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 - \Omega \times \Omega \\ \end{cases}.$$

Set

$$c_q(x, y) = \int_{-\infty}^{\infty} \rho_q(t) e(t; x, y) dt .$$

In order to show Proposition 2.2 it suffices to prove the following facts:

$$(3.1) \qquad \qquad \operatorname{supp} c_q \subset \overline{\Omega} \times \overline{\Omega} ,$$

$$(3.2) c_q(x, y) \in C_0^{\infty}(\overline{\Omega} \times \overline{\Omega})$$

and

(3.3) 
$$|\int_{\mathbf{R}^3} c_q(x, x) dx - c_0 q^{(1-2/e_0)(l+1)-2}| \leq C_l q^{(1-5/2e_0)l}$$
 for all  $q$ 

where  $c_0$  is a positive constant determined by  $\mathcal{O}$  and  $\rho_0$ .

Since  $E_0(t; x, y)$  is well known the essential part of the proof is the consideration of E(t; x, y). To take out properties of E, first we construct an approximation of E as a superposition of asymptotic solutions, secondly we pick out the principal behavior of E as  $t \to \infty$ . The construction of asymptotic solutions is done by a method essentially same as in [2] and [4]. But the assumption that all the principal curvatures of the boundary vanish at  $a_1$  and  $a_2$ gives rise to another behavior of asymptotic solutions than those in [2, 4]. Then in order to pick up this behavior of asymptotic solutions we have to make other

considerations than in the previous papers.

Fix  $\delta_2$ ,  $\delta_3$  so that Corollary of Lemma 3.3 of [2] holds. Let  $S_j(\delta_l)$ , j=1, 2, l=2, 3 be the ones introduced in §3 of [2]. Denote by  $\omega(\delta_l)$  a domain surrounded by  $S_j(\delta_l)$ , j=1, 2 and  $\{y; \operatorname{dis}(y, L)=\delta_l\}$ . Let

(3.4) 
$$\psi(x) \in C_0^{\infty}(\Omega)$$
 such that  $\operatorname{supp} \psi \subset \omega(\delta_2)$ .

Then for  $f \in C^{\infty}(\Omega)$  we have by Fourier's inversion formula

(3.5) 
$$\psi(x)f(x) = w(x)\int_{S^2} d\omega \int_0^\infty k^2 dk \int_\Omega dy \ e^{ik\langle x-y,\omega\rangle} \psi(y)f(y) ,$$

where w(x) is a function in  $C_0^{\infty}(\omega(\delta_3))$  verifying

$$(3.6) w(x) = 1 on supp \psi$$

Let  $u(t, x; k, \omega)$  be the solution of an initial-boundary value problem

(3.7) 
$$\begin{cases} \Box u = 0 & \text{in } (0, \infty) \times \Omega \\ u(t, x) = 0 & \text{on } (0, \infty) \times \Gamma \\ u(0, x) = w(x)e^{ik\langle x, w \rangle} \\ \frac{\partial u}{\partial t}(0, x) = 0. \end{cases}$$

Then

$$a(t, x) = \int_{S^2} d\omega \int_0^\infty k^2 dk \int_\Omega dy \, u(t, x; k, \omega) e^{-ik\langle y, \omega \rangle} \psi(y) f(y)$$

satisfies

$$\begin{cases} \Box a = 0 & \text{in } (0, \infty) \times \Omega \\ a(t, x) = 0 & \text{on } (0, \infty) \times \Gamma \\ a(0, x) = \psi(x) f(x) \\ \frac{\partial a}{\partial t}(0, x) = 0 . \end{cases}$$

This means that  $a(t, \cdot) = (\cos t \sqrt{-\Delta} \psi) f$ . Therefore the kernel distribution  $E(t; x, y)\psi(y)$  of  $\cos t \sqrt{-\Delta} \psi$  is given by

(3.8) 
$$E(t; x, y)\psi(y) = \int_{S^2} d\omega \int_0^\infty k^2 dk \ u(t, x; k, \omega) e^{-ik\langle y, \omega \rangle} \psi(y) ,$$

here we interpret the integral as an oscillatory integral (cf. Kumano-go [8, §6 of Chapter 1]).

As an approximation of  $u(t, x; k, \omega)$  we construct an asymptotic solution of (3.7) in a way that we can make clear the reflexion of geometric properties of  $\mathcal{O}$  to the behavior of u. For the Cauchy problem with an oscillatory data

$$\begin{cases} \Box h = 0 & \text{in } (0, \infty) \times \mathbf{R}^3 \\ h(0, x) = w(x)e^{ik\langle x, w \rangle} & \text{in } \mathbf{R}^3 \\ \frac{\partial h}{\partial t}(0, x) = 0 & \text{in } \mathbf{R}^3 \end{cases}$$

admits an asymptotic solution

$$egin{aligned} h^{(N)}(t,\,x;\,k,\,\omega) &= e^{ik(\langle x,\omega
angle -t)}\sum\limits_{j=0}^N g_j(t,\,x;\,\omega)\;(ik)^{-j} \ &+ e^{ik\langle\langle x,\omega
angle +t)}\sum\limits_{j=0}^N ilde g_j(t,\,x;\,\omega)\;(ik)^{-j} \ &= h^{(N)}_+(t,\,x;\,k,\,\omega) + h^{(N)}_-(t,\,x;\,k,\,\omega)\,. \end{aligned}$$

Set

$$m^{(N)}(t, x; k, \omega) = h^{(N)}(t, x; k, \omega)|_{(0,\infty) \times \Gamma}$$
  
=  $h^{(N)}_+|_{(0,\infty) \times \Gamma} + h^{(N)}_-|_{(0,\infty) \times \Gamma} = m^{(N)}_+ + m^{(N)}_-$ 

Note that from the location of the support of  $h_{\pm}^{(N)}$ , the support of  $m_{\pm}^{(N)}$  is contained in one of  $(0, \infty) \times \Gamma_1$  and  $(0, \infty) \times \Gamma_2$ . For example when  $\omega_3 < 0$ 

supp 
$$m_{+}^{(N)} \subset (0, \infty) \times \Gamma_1$$
, supp  $m_{-}^{(N)} \subset (0, \infty) \times \Gamma_2$ .

Since all the rays starting from  $\sup \psi$  and hitting  $S(\delta_3)$  do not tangent to  $\Gamma$ in  $S(\delta_3)$  and the Gaussian curvature does not vanish in the outside of  $S(\delta_3)$ , the method of construction of asymptotic solution in [2] can be applied without any modification. We see from Corollary of Lemma 3.3 of [2] that it suffices to consider  $z^{(N)}$  constructed in §8 of [2] when we consider the behavior in  $\omega(\delta_3)$  of asymptotic solutions with oscillatory boundary data  $m_{\pm}^{(N)}$ . Let us denote by  $z_{\pm}^{(N)} = w_{\pm}^{(N)} + y_{\pm}^{(N)}$  the asymptotic solution  $z^{(N)}$  constructed by the process of Proposition 8.1 of [2] for boundary data  $m_{\pm}^{(N)}$ . Now consider  $z_{\pm}^{(N)}$ . For the simplicity of description we omit the suffix +. Recall that  $w^{(N)}$  is of the form

(3.9) 
$$\begin{cases} w^{(N)} = \sum_{q=0}^{\infty} u_q^{(N)}, \\ u_q^{(N)}(t, x; k, \omega) = e^{ik(\varphi_q(x, \omega) - t)} \sum_{j=0}^{N} v_{q,j}(t, x; \omega) (ik)^{-j} \end{cases}$$

and that  $y^{(N)}$  satisfies

(3.10) 
$$\operatorname{supp} y^{(N)} \cap ((0, \infty) \times \omega(\delta_2)) = \phi .$$

The fact that the principal curvatures of  $\Gamma_1$  and  $\Gamma_2$  vanish at  $a_1$  and  $a_2$  brings other behaviors of  $\varphi_q$  and  $v_{q,j}$  than those of [2, 4]. In this case  $\{\nabla \varphi_q\}_{q=0}^{\infty}$  is not bounded in  $C^{\infty}(\omega(\delta_3))$  and the sequences  $v_{q,j}$ , q=0. 1, 2 ... do not decrease exponentially. Concerning their estimate we have

**Lemma 3.1.** There exist positive integers l(j, m) depending on j and m such that

(3.11) 
$$\sum_{|\beta| \leq j} |\partial_{\omega}^{\beta} \nabla \varphi_{q}(\cdot; \omega)|_{m}(\omega(\delta_{1})) \leq C_{j,m} q^{l(j,m)},$$

(3.12) 
$$\sum_{|\beta| \leq h} |\partial_{\omega}^{\beta} \nu_{q,j}(\cdot; \omega)|_{m} (\boldsymbol{R} \times \omega(\delta_{1})) \leq C_{j+h,m} q^{l(j+h,m)}$$

hold.

There estimates are proved by induction of j, h, m by using Lemmas 5.2, 5.3 and their remarks of [2].

Taking account of the location of the support of  $z^{(N)}$  the estimates (3.11) and (3.12) give

supp 
$$z^{(N)} \subset (0, \infty) \times \Omega$$

(3.13) 
$$\sum_{|\beta| \leq \hbar} |\partial_{\omega}^{\beta}(z_{\pm}^{(N)}(\cdot, \cdot; k, \omega))|_{m}(t, \Omega) \leq C_{N,j,m} k^{m+\hbar} \left(\frac{t}{2d}\right)^{l(N+2+\hbar,m)},$$

(3.14) 
$$\sum_{|\beta| \leq h} |\partial_{\omega}^{\beta}(\Box z_{\pm}^{(N)}(\cdot, \cdot; k, \omega)|_{\mathfrak{m}}(t, \Omega) \leq C_{N, j, \mathfrak{m}} k^{-N+\mathfrak{m}} \left(\frac{t}{2d}\right)^{l(N+2+h, \mathfrak{m})},$$

(3.15) 
$$z_{\pm}^{(N)} = h_{\pm}^{(N)}$$
 on  $(0, \infty) \times \Gamma$ .

Set  $u^{(N)} = -(z_+^{(N)} + z_-^{(N)}) + h^{(N)}$ . Then

$$u^{(N)}(0, x; k, \omega) = w(x)e^{ik\langle x, \omega \rangle},$$
  
$$\frac{\partial u^{(N)}}{\partial t}(0, x; k, \omega) = 0$$

and  $\Box u^{(N)}$  has an estimate of the type (3.14). Concerning the difference between the actual solution u of (3.7) and  $u^{(N)}$  we have from the above remarks

(3.16) 
$$\sum_{|\beta| \leq h} \left| \partial_{\omega}^{\beta}(u - u^{(N)})(\cdot, \cdot; k, \omega) \right|_{m}(t, \Omega)$$
$$\leq C_{N,h,m} k^{-N+m+2} \left( \frac{t}{2d} \right)^{l(N+2+h,m)+1}.$$

We see immediately from Lemma 3.1 and (3.16) that

$$\int \rho(t) E(t; x, y) \psi(y) dt \in C^{\infty}(\overline{\Omega} \times \overline{\Omega}) \quad \text{for any } \rho \in C^{\infty}_{0}(\mathbf{R}).$$

Since supp  $E_0(t; \cdot, \cdot) \subset \{(x, y); |x-y| = |t|\}$ 

(3.17) 
$$\int_{\mathbf{R}^3} c_q(x, x) \psi(x) dx = \iint_{\mathbf{Q}} E(t, x, x) \psi(x) \rho_q(t) dt dx$$

for large q. From (3.8), (3.17)

.

$$\int_{\mathbf{R}^3} c_q(x, x) \psi(x) dx$$

$$= \int_{\Omega} dx \int dt \int_{S^2} d\omega \int_0^{\infty} k^2 dk \, \rho_q(t) u(x, t; k, \omega) e^{-ik\langle x, \omega \rangle} \psi(x)$$

$$= \int \cdots \int_{k>1} \rho_q(t) w_+^{(N)}(t, x; k, \omega) e^{-ik\langle x, \omega \rangle} \psi(x) dx dt d\omega k^2 dk$$

$$+ \int \cdots \int_{k>1} \rho_q(t) (w_-^{(N)}(t, x; k, \omega) e^{-ik\langle x, \omega \rangle} \psi(x) dx dt d\omega k^2 dk$$

$$+ \int \cdots \int_{k>1} \rho_q(t) (y_+^{(N)} + y_-^{(N)}) (t, x; k, \omega) e^{-ik\langle x, \omega \rangle} \psi(x) dx dt d\omega k^2 dk$$

$$+ \int \cdots \int_{k>1} \rho_q(t) (u(t, x; k, \omega) - u^{(N)}(t, x; k, \omega)) e^{-ik\langle x, \omega \rangle} \psi(x) dx dt d\omega k^2 dk$$

$$+ \int_{\Omega} dx \int dt \int_{S^2} d\omega \int_0^1 k^2 dk \, \rho_q(t) u(t, x; k, \omega) e^{-ik\langle x, \omega \rangle} \psi(x)$$

$$= I_+ + I_- + II + III + IV.$$

Since we have for  $0 \leq k \leq 1$ 

$$|u(t, x; k, \omega)| \leq C$$
 in  $[0, \infty) \times \Omega$ ,

it holds that

$$|IV| \leq C \int \psi(x) dx \int \rho_q(t) dt \leq C \int \psi(x) dx q^{-1}$$

From (3.4) and (3.10) the integrand of II vanishes identially. Thus II=0. Next consider III. Set

$$\begin{split} \int \cdots \int dx dt d\omega \int_{1}^{\infty} k^{2} dk \{ \} &= \int \cdots \int dx dt d\omega \int_{1}^{q} k^{2} dk \{ \} + \int \cdots \int dx dt d\omega \int_{q}^{\infty} k^{2} dk \{ \} \\ &= III_{1} + III_{2} . \\ &|III_{1}| \leqslant C \int \rho_{q}(t) dt \int_{\Omega} \psi(x) dx \int_{1}^{q} k^{2} dk \leqslant C q^{-1} q^{3} \leqslant C q^{-1+3} . \end{split}$$

Since supp  $\rho_q \subset [2dq - q^{-1}, 2dq + q^{-1}]$ , by using (3.16)

$$|III_2| \leq C \int \psi(x) dx \int \rho_q(t) dt \ q^{I(N+2,0)} \int_q^\infty k^{-N+2} dk \leq C \ q^{-I+I(N+2,0)-N+3} \ .$$

Thus we have

Lemma 3.2. If we choose l > l(N+2, 0) - N+3 it holds that (3.18)  $|\int_{\mathbf{R}^3} c_q(x, x) \psi(x) dx - (I_+ + I_-)| \leq C_{N,l}$ ,

for all q.

Now we set about the estimation of  $I_+$ . Set

(3.19) 
$$I_{r,j}(t, k) = \int_{S^2} d\omega \int_{\Omega} dx \, e^{ik \Phi_r(x,\omega)} v_{r,j}(t, x; \omega) \psi(x) \, ,$$

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(3.20) 
$$\Phi_r(x, \omega) = \varphi_r(x, \omega) - \langle x, \omega \rangle.$$

Then we have

(3.21) 
$$I_{+} = \int_{1}^{\infty} k^{2} dk \sum_{j=0}^{N} (ik)^{-l} \int e^{-ikt} \rho_{q}(t) I_{r,j}(t, k) dt$$

Note that except r=2q-1, 2q, 2q+1 supp  $\rho_q \cap \text{supp } v_{r,j}(t, x; \omega) = \phi$ . Since for  $r=2q\pm 1$ 

$$|\partial_{x_3}\Phi_r(x,\omega)| \ge 1$$
 for all  $(x,\omega) \in \omega(\delta_3) \times S^2$ ,

if  $v_{r,j}(t, x; \omega) \neq 0$ , we have

$$|I_{r,j}(t, k)| \leq C_M k^{-M} q^{l(M)}$$
 for  $r = 2q \pm 1$ 

where l(M) is an integer depending on M. Therefore we have

(3.22) 
$$|I_{+} - \int_{1}^{\infty} k^{2} dk \sum_{j=0}^{N} (ik)^{-j} \int e^{-ikt} I_{2q,j}(t,k) \rho_{q}(t) dt | \leq C \quad \text{for all } q$$

if *l* is large. Set

(3.23) 
$$\int_{q,j}(x_3,t;k) = \int_{S^2} d\omega \int_{\mathbf{R}^2} dx' e^{ik\Phi_{2q}(x,\omega)} v_{2q,j}(t,x;\omega) \psi(x) \, .$$

**Proposition 3.3.** 

(3.24) 
$$|J_{q,j}(x_3, t; k) - k^{-1-2/e_0 i 2dq} \{ c_{q,j}^0(x_3, t) + \sum_{k=1}^{\lfloor 3e_0/2 \rfloor} \sum_{m=1}^{m_k} c_{q,j}^{h,m}(x_3, t) k^{-h/e_0} (\log k)^{m-1} \} | \leq Cq^{l_1} k^{-4}$$

where  $l_1$  is a constant,  $c_{q,j}^{h,m}(x_3, t)$  are determined by  $\Phi_{2q}$  and  $v_{q,j}$  and they satisfy

$$\sum_{l=0}^{2} |\partial_{t}^{l} c_{q,j}^{h,m}(x_{3}, t)| \leq C q^{l_{1}} \quad \text{for all } x_{3} \in (0, d) \text{ and } t > 0,$$

especially

$$c_{q,j}^{0}(x_{3}, t) = c v_{q,j}(t, 0, x_{3}; \omega_{0}) q^{-1-2/e_{0}}$$

for some fixed non zero constant c determined by the shape of  $\Gamma_j$  near  $a_j$  and  $\omega_0 = (0, 0, 1)$ .

The above proposition will be proved in sections 4 and 5. Now admit this result. To evaluate  $v_{2q,0}$  we use (5.9) of [4]. For  $\omega_0$  we see from Lemma 4.1 of [2] that the principal curvatures at  $a_1$  and  $a_2$  of the wave front of  $\varphi$ , are zero for all r. Then we have  $\Lambda_{2q-j}(X_{-j}(x, \nabla \varphi_{2q}))=1$  for all j when x'=0. Therefore we have for  $\omega_0$ 

$$v_{2q,0}(t, 0, 0, x_3; \omega_0) = w(0, 0, (2qd+x_3)-t).$$

Note that from (3.6)  $w(0, 0, (2dq+x_3)-t)=1$  holds for  $(0, 0, x_3) \in \text{supp } \psi$  and

 $t \in \text{supp } \rho_q \subset [2dq - q^{-l}, 2dq + q^{-l}]. \text{ Therefore}$   $(3.25) \qquad \int dx_3 \int_0^\infty k^2 dk \int dt \ c_{q,0}^0(x_3, t) e^{-ikt} k^{-1-2/\epsilon_0} e^{ik2dq} \rho_q(t) \psi(0, x_3)$   $= c \int \psi(0, x_3) dx_3 \int_1^\infty k^{1-2/\epsilon_0} \hat{\rho}_0(k/q^l) q^{-l} \ dk \ q^{-1-2/\epsilon_0}$   $= c_0 \int_0^d \psi(0, x_3) dx_3 \ q^{(1-2/\epsilon_0)(l+1)-2} + O(q^{-l}) ,$ 

where  $c_0 = c \int_0^\infty k^{1-2/\epsilon_0} \hat{\rho}_0(k) dk \neq 0$  from (2.7). Next we shall show the following estimate for  $h \ge 1$  and for all j, m

(3.26) 
$$|\int_{0}^{d} dx_{3} \int_{1}^{\infty} k^{2} dk \int e^{-ikt} k^{-1-j-(2+h)/\epsilon_{0}} (\log k)^{m-1} c_{q,j}^{h,m}(x_{3}, t) \rho_{q}(t) dt | \\ \leq C_{l} q^{l_{1}} q^{(1-11/4\epsilon_{0})l} .$$

Set

$$I = \int_{1}^{\infty} k^{2} dk \int e^{-ikt} k^{-1-j-(2+k)/e_{0}} (\log k)^{m-1} c_{q,J}^{k,m}(x_{3}, t) \rho_{q}(t) dt$$
  
=  $\int_{1}^{(q+1)^{l}} k^{2} dk \int \cdots dt + \int_{(q+1)^{l}}^{\infty} k^{2} dk \int \cdots dt$   
=  $I_{1} + I_{2}$ .

Substituting an estimate  $|c_{q,j}^{h,m}| \leq C q^{l_1}$  we have

$$|I_1| \leq C q^{l_1} \int_1^{(q+1)^l} k^{1-j-(h+2)/e_0} (\log(q+1)^l)^{m-1} dk \cdot \int \rho_q(t) dt$$
  
$$\leq C q^{l_1} (l \log(1+q))^m q^{l(1-j-(2+h)/e_0)}.$$

About  $I_2$ , we make integration by parts in t variable two times, and we have

$$I_{2} = \int_{(q+1)^{j}}^{\infty} k^{2} dk \int (ik)^{-2} e^{-ikt} k^{-1-j-(2+k)/\epsilon_{0}} \cdot (\log k)^{m-1} \left(\frac{\partial}{\partial t}\right)^{2} (c_{q,j}^{h,m}(x_{3}, t) \rho_{q}(t)) dt$$

By using estimates of  $c_{q,j}^{h,m}$  and the definition of  $\rho_q(t)$  we have

$$\int \left| \left( \frac{\partial}{\partial t} \right)^2 (c_{q,j}^{h,m}(x_3,t) \rho_q(t)) \right| dt \leq C q^{l_1} (q+1)^l .$$

Thus it follows that

$$|I_2| \leq C q^{I_1} (q+1)^{I} \int_{(q+1)^{I}}^{\infty} k^{-1-j-(2+k)/\epsilon_0} (\log k)^{m-1} dk$$
  
$$\leq C_{\epsilon} q^{I_1} (q+1)^{I (1-j-(2+k)/\epsilon_0+\epsilon)} (\varepsilon > 0) .$$

Taking account of  $h \ge 1$  we have

$$|I| \leq C_l q^{l_1} q^{(1-11/4e_0)l}$$

Since  $C_1$  is independent of  $x_3$  the above estimate implies (3.26). Combining the above estimates we have

(3.27) 
$$|I_{+}-c_{0}\int_{0}^{d}\psi(0, x_{3})dx_{3} q^{(1-2/e_{0})(l+1)-2}| \leq C_{l}q^{(1-5/2e_{0})l}$$

if l is sufficiently large. For  $I_{-}$  we have the same estimate as  $I_{+}$ . Then form (3.27) and Lemma 3.2 it follows that

$$(3.28) \qquad |\int_{\mathbf{R}^3} c_q(x, x)\psi(x)dx - 2c_0 \int_0^d \psi(0, x_3)dx_3 q^{(1-2/e_0)(l+1)-2}| \leq C_l q^{(1-5/2e_0)l}$$

for any  $\psi \in C_0^{\infty}(\omega(\delta_3))$  when *l* is large.

When  $\psi \in C_0^{\infty}(\omega(\delta_3) \cup S(\delta_3))$  we have to modify the procedure of the construction of the kernel of  $\cos t \sqrt{-\Delta} \psi$ . Namely, when  $\operatorname{supp} \psi \cap S(\delta_3) \neq \phi$  we cannot choose w(x) in (3.6) as a function in  $C_0^{\infty}(\Omega)$ . Therefore the solution of (3.7) is not smooth function, and  $z_{\pm}^{(N)}$  has discontinuities, which make the argument more complicated. But as we shall show in §6 the same estimate also holds in this case. Thus

**Lemma 3.4.** For any  $\psi \in C_0^{\infty}(\omega(\delta_3) \cup S(\delta_3))$  the estimate (3.28) holds if we choose l sufficiently large.

Next consider the case  $\psi \in C_0^{\infty}(\overline{\Omega})$  and

(3.29) 
$$\operatorname{supp} \psi \cap \omega(\delta_3) = \phi .$$

Suppose that in addition to (3.29) any ray starting from supp  $\psi$  does not tangent to  $\Gamma$  at  $S(\delta_3)$ . Then the procedure of construction of an approximation of  $c_q(x, y)\psi(y)$  is same as before. In the representation of  $I_{r,j}(t, k)$  the amplitude function  $\Phi_r(x, \omega)$  has no critical point, that is,

$$|\partial_x \Phi_r(x, \omega)| + |\partial_\omega \Phi_r(x, \omega)| \neq 0$$
 for all  $(x, \omega) \in \text{supp } \psi \times S^2$ .

Thus we have for any M

$$|I_{+}(t, k)| \leq C_{M} q^{l(M)} k^{-M}$$

where l(M) is a constant depending on M. Therefore we have

$$\left|\int_{R^{3}}c_{q}(x, x)\psi(x)dx\right| \leq C$$
 for all  $q$ .

By employing the argument in §6 of [2] the additional condition may be removed easily. Then

**Lemma 3.5.** Let  $\psi \in C_0^{\infty}(\overline{\Omega})$  such that

supp 
$$\psi \cap \omega(\delta_3) = \phi$$
.

Then an estimate

$$(3.30) \qquad \qquad |\int_{\mathbf{R}^3} c_q(x, x)\psi(x)dx| \leq C$$

holds where the constant C depends of  $\mathcal{O}$  and  $\psi$  but independent of q.

Note that for  $\psi$  of the form  $\psi(x) = \psi_0(x-\zeta)$  for a fixed  $\psi_0 \in C_0^{\infty}(\mathbb{R}^3)$ and some  $\zeta \in \mathbb{R}^3$  the constant C in (3.30) is independent of  $\psi$ , namely C is depends on  $\mathcal{O}$  and  $\psi_0$  only. Since

$$supp(E(t; \cdot, \cdot) - E_0(t; \cdot, \cdot)) \subset \{(x, y); |x|, |y| \leq R + |t|\}$$

the estimate (2.8) is derived from Lemmas 3.3 and 3.4.

## 4. On the critical points of $\Phi_{2q}(x, \omega)$

Let  $\varphi_0(x, \omega) = \langle x, \omega \rangle$  and let  $\varphi_1, \varphi_2, \dots, \varphi_{2q}$  we be the sequence of phase functions in (3.9). For  $x \in \omega(\delta_3)$  set  $X_0(x, \omega) = x$  and, if  $\{x+l\omega; l \ge 0\} \cap \Gamma \neq \phi$ 

$$egin{aligned} &l_0(x,\,\omega) = \inf \left\{l;\,l \ge 0,\,x + l\omega \in \Gamma 
ight\}\,,\ &X_1(x,\,\omega) = x + l_0\omega\,,\ &\Xi_1(x,\,\omega) = \omega - 2 \langle n(X_1(x,\,\omega),\,\omega 
angle n(X_1(x,\,\omega))) \end{aligned}$$

Following the process of §3 of [2] define successively  $l_j(x, \omega)$ ,  $X_j(x, \omega)$ ,  $\Xi_j(x, \omega)$ ,  $L_j(x, \omega)$ ,  $\mathcal{L}_j(x, \omega)$  for  $j=1, 2, \cdots$ . For  $x \in \omega(\delta_3)$  set

$$egin{aligned} &l_{-1}(x,\,
ablaarphi_{2q}(x,\,\omega)) = \inf\left\{l;\,l\geqslant 0,\,x-l\,
ablaarphi_{2q}(x,\,\omega)\in\Gamma
ight\}\,,\ &X_{-1}(x,\,
ablaarphi_{2q}(x,\,\omega)) = x-l_{-1}(x,\,
ablaarphi_{2q}(x,\,\omega))
ablaarphi_{2q}(x,\,\omega)\,. \end{aligned}$$

Define successively  $X_{-j}(x, \nabla \varphi_{2q}(x, \omega))$  following §4 of [4]. For  $x \in \mathbb{R}^3$  and  $\omega \in S^2$  set

$$\mathscr{D}(x, \omega) = \{y; \langle y - x, \omega \rangle = 0\}$$
.

Let us denote by  $Y_{2q}(x, \omega)$  the point

$$\mathscr{L}(x, \omega) \cap \{X_{-2q}(x, \nabla \varphi_{2q}(x, \omega)) - l\omega; l \ge 0\}$$
.

Remark that, if we set  $y = Y_{2q}(x, \omega)$ , we have

$$X_{2q-j}(y, \omega) = X_{-1-j}(x, \nabla \varphi_{2q}(x, \omega)), \qquad j = 1, 2, ..., 2q-1.$$

 $Y_{2q}(x, \omega) \in \rho(x, \omega)$  means that

(4.1) 
$$\langle Y_{2q}(x, \omega), \omega \rangle = \langle x, \omega \rangle.$$

Now we have by using (4.1)

(4.2) 
$$\Phi_{2q}(x, \omega) = |X_1(y, \omega) - y| + |X_2(y, \omega) - X_1(y, \omega)|$$

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$$+\cdots+|X_{2q}(y,\omega)-X_{2q-1}(y,\omega)|+|x-X_{2q}(y,\omega)|$$

where we put  $y = Y_{2q}(x, \omega)$ . Recall that the broken ray  $\mathscr{X}(Y_{2q}(x, \omega), \omega)$  is a path starting from a point on a plane  $\mathscr{P}(x, \omega)$  and reach at x after 2q times reflexion on  $\Gamma$  according to the geometric optics. The path of the geometric optics can be characterized as a path that has a minimal length among the ones which start from on  $\mathscr{P}(x, \omega)$  and arrive at x after passing 2q times points on  $\Gamma$ . Namely,

(4.3) 
$$\Phi_{2q}(x, \omega) = \inf \{ |x^{(1)} - x^{(0)}| + |x^{(2)} - x^{(1)}| + \dots + |x^{(2q)} - x^{(2q-1)}| + |x - x^{(2q)}| \}$$

where the infimum is taken on  $x^{(0)}$ ,  $x^{(1)}$ ,  $\cdots$ ,  $x^{(2q)}$  running over

$$\begin{aligned} x^{(0)} &\in \mathcal{P}(x, \omega) , \\ x^{(1)}, x^{(3)}, \cdots, x^{(2q-1)} &\in \Gamma_2(\Gamma_1) , \\ x^{(2)}, x^{(4)}, \cdots, x^{(2p)} &\in \Gamma_1(\Gamma_2) , \end{aligned}$$

if  $\omega_3 > 0$  (if  $\omega_3 < 0$ ). Let us set

$$S_{\pm}^2 = \{\!(\omega_1\!,\,\omega_2\!,\,\pm\sqrt{1\!-\!\omega_1^2\!-\!\omega_2^2})\!;\,\omega_1^2\!+\!\omega_2^2\!\!<\!\!1\}\;.$$

**Lemma 4.1.** Let  $\omega \in S^2_+$  and  $x \in \omega(\delta_3)$ . Suppose that

$$(4.4) Y_{2q}(x, \omega) \text{ exists.}$$

Then for  $\omega = \omega(\omega_1, \omega_2) = (\omega_1, \omega_2, \sqrt{1 - \omega_1^2 - \omega_2^2})$ 

(4.5) 
$$\frac{\partial \Phi_{2q}(x,\omega)}{\partial \omega_j} = \langle y - x, \frac{\partial \omega}{\partial \omega_j} \rangle$$
$$= (y_j - x_j) - \omega_j (1 - \omega_1^2 - \omega_2^2)^{-1/2} (y_3 - x_3)$$

for j=1, 2, where  $y=(y_1, y_2, y_3)=Y_{2q}(x, \omega)$ .

Proof. Let  $\tilde{\omega} = \omega(\omega_1 + \Delta \omega_1, \omega_2)$  and  $\tilde{\mathcal{Y}} = Y_{2q}(x, \tilde{\omega})$ . Since  $X_{-j}(x, \nabla \varphi_{2q}(x, \omega))$  is continuous in x and  $\omega$  we have

(4.6)  $\tilde{y} \to y$  as  $\Delta \omega_1 \to 0$ .

Set

$$egin{aligned} &z = \mathscr{P}(x,\,\widetilde{\omega}) \cap \{y{+}l\omega;\,l{\in}m{R}\}\ &z = \mathscr{P}(x,\,\omega) \cap \{\mathfrak{I}{+}l\widetilde{\omega};\,l{\in}m{R}\}\ . \end{aligned}$$

Then from (4.3)

$$\begin{split} \Phi_{2q}(x,\,\tilde{\omega}) &= \inf \left\{ |\,x^{(1)} - x^{(0)} \,| + \cdots + |\,x - x^{(2q)} \,| \right\} \\ &\leq |\,X_1(y,\,\omega) - z \,| + |\,X_2(y,\,\omega) - X_1(y,\,\omega) \,| \end{split}$$

+ ... +  $|X_{2q}(y, \omega) - X_{2q-1}(x, \omega)| + |x - X_{2q}(y, \omega)|$ .

Since we have  $|X_1(y,\omega)-z| = |X_1(y,\omega)-y| + |y-z|$  if z is on the prolongation of a segment  $X_1(y,\omega)y$  it holds that

(4.7) 
$$\Phi_{2q}(x, \tilde{\omega}) \leq \Phi_{2q}(x, \omega) + |y-z|.$$

If z is on the prolongation of  $X_1(y, \omega)y \tilde{z}$  must be on a segment  $X_1(\tilde{y}, \tilde{\omega})\tilde{y}$ , and we have

$$|X_{\mathbf{1}}(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\omega}}) - \tilde{\boldsymbol{z}}| = |X_{\mathbf{1}}(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\omega}}) - \tilde{\boldsymbol{y}}| - |\tilde{\mathbf{y}} - \tilde{\boldsymbol{z}}|.$$

Then similarly we have

(4.8) 
$$\Phi_{2q}(x, \omega) \leq \Phi_{2q}(x, \tilde{\omega}) - |\tilde{y} - \tilde{z}| .$$

Taking account of  $\overline{X_1(y, \omega)y} \perp \mathscr{L}(x, \omega)$  and  $\overline{X_1(\tilde{y}, \tilde{\omega})}\tilde{y} \perp \mathscr{L}(x, \tilde{\omega})$  we have

$$|y-z| = \langle y-x, \,\tilde{\omega}-\omega \rangle + o(|\tilde{\omega}-\omega|)$$
$$|\tilde{y}-\tilde{z}| = \langle \tilde{y}-x, \,\tilde{\omega}-\omega \rangle + o(|\tilde{\omega}-\omega|).$$

Thus from (4.7), (4.8) and (4.6) it follows that

$$\lim_{\Delta \omega_1 \to 0} \frac{\Phi_{2q}(x, \tilde{\omega}) - \Phi_{2q}(x, \omega)}{\Delta \omega_1} = \langle y - x, \frac{\partial \omega}{\partial \omega_1} \rangle. \qquad Q.E.D.$$

In the rest of this section we shall use the notation as in §3 of [2].

Lemma 4.2. Let  $y = Y_{2q}(x, \omega)$ . Suppose that  $|x'-y'| \leq |x'|/2$  and (4.9)  $y' \cdot \omega' \geq 0$ .

Then it holds that

$$|x'-y'| \ge cq |x'|^{\bullet}.$$

Proof. First note that from the assumption on the principal curvatures we have

$$|n(x)'| \ge c |x'|^{\mathfrak{s}+1}$$
 for  $x \in S(\delta_3)$ .

Since  $d |x'(s)|^2/ds \ge d |x'(s)|^2/ds|_{s=0} = y' \cdot \omega' \ge 0$  for all s > 0 we have

$$\frac{d}{ds}|x'(s)|^2 \ge c|y'|^{e+1} \quad \text{for } s \ge s_1.$$

Therefore

$$|x'|^2 - |y'|^2 \ge 2dqc |y'|^{e+1}$$
,

from which it follows that

$$|x'| - |y'| \ge 2dqc |y'|^{e}$$

By using  $|y'| \ge |x'|/2$ , which is a consequence of the assumption, the assertion of Lemma follows. Q.E.D.

Lemma 4.3. Suppose that

(4.10) 
$$|x'-y'| \leq |x'|/2 \text{ and } x' \cdot \Xi_{2q}(y, \omega) \leq 0.$$

Then it holds that  $y' \cdot \omega \leq -qc |x'|^{e+1}$  and

$$|x'-y'| \ge cq |x'|^{e}.$$

Proof. Since  $d |x'(s)|^2 / ds$  is an increasing function and

$$0 \ge d |x'(s)|^2 / ds|_{s=s_{2q}+0} \ge d |x'(s)|^2 / ds|_{s=s_{2q}-0} + 2c(1-\delta) |x'|^{e-1}$$

we have

$$\frac{d}{ds}|x'(s)|^2 \leq -2(1-\delta)c|x'|^{\epsilon+1}. \quad \text{for all } s < s_{2q},$$

which implies

$$|y'|^2 - |x'|^2 = |x'(0)|^2 - |x'(s_{2q})|^2 \ge 2dqc(1-\delta)|x'|^{\epsilon+1}.$$

Thus we have

$$|y'| - |x'| \ge 2dqc(1-\delta)|x'|^{\epsilon}.$$

Lemma 4.4. When

$$|\omega'| \geq Cq(|x'|^{s+1}+|y'|^{s+1})$$

holds for some constant C independent of q, we have

$$|x'-y'| \ge dq |\omega'|$$
 and  $x' \cdot \Xi_{2q} \ge 2dq |y'|^{s+1}$ .

Proof. Since  $|x'(s)|^2$  is a convex function we have  $|x'(s)| \leq \max(|x'|, |y'|)$  for all s. Denote the right hand side by M. From the law of reflexion

$$\Xi_j(y, \omega) - \Xi_{j-1}(y, \omega) = 2(X_j(y, \omega), n(X_j(y, \omega))n(X_j(y, \omega)))$$

we have for j=1

$$|\Xi_{\mathbf{I}}(y,\omega)'-\omega'| \leq 2 |n(X_{\mathbf{I}}(y,\omega))'| \leq 2cM^{e+1}.$$

Similarly we have for all  $j \leq 2q$ 

$$|(\Xi_j(y, \omega) - \Xi_{j-1}(y, \omega))'| \leq 2CM^{e+1}.$$

Then by using the assumption we have

$$\begin{aligned} |(\Xi_j - \omega)'| &\leq 2qCM^{\epsilon+1} \leq |\omega'|/2 \quad \text{for all } j \leq 2q. \\ |(x - y)'| &= |(\sum_{j=1}^{2^q} l_j \Xi_{j-1})'| \\ &\geq |(\sum_{j=1}^{2^q} l_j \omega)'| - |\sum_{j=1}^{2^q} l_j (\Xi_j - \omega)'| \\ &\geq 2dq |\omega'| - dq |\omega'| \geq dq |\omega'|. \end{aligned}$$

Q.E.D.

Lemma 4.5. Let  $x=(0, 0, x_3), 0 < x_3 < d$ . If  $q^2|\omega'| < 1$  it holds that

(4.11) 
$$|(x - Y_{2q}(x, \omega))' - 2dq \,\omega'| \leq Cq^2 |\omega'|$$

where C is a constant independent of q.

Proof. Let us set  $y = Y_{2q}(x, \omega), -\Xi_{2q}(y, \omega) = \tilde{\omega}$ . Then we have  $X_j(x, \tilde{\omega}) = X_{2q-j}(y, \omega), \, \Xi_j(x, \tilde{\omega}) = -\Xi_{2q-j}(y, \omega)$ .

First we show that

(4.12) 
$$|X_{j}(x, \tilde{\omega})'| \leq Cj |\omega'|, \quad |\Xi_{j}(x, \tilde{\omega})' - \tilde{\omega}'| \leq Cj |\tilde{\omega}'|^{2}$$

holds for all  $j \leq 2q$ . Suppose that  $q^2 |\tilde{\omega}'| < 1$  and (4.12) holds for  $j \leq h$ . Then

$$\begin{aligned} |X_{h+1}(x, \tilde{\omega})'| &\leq |X_{h}(x, \tilde{\omega})'| + l_{h} |\Xi_{h}(x, \tilde{\omega})'| \\ &\leq Ch |\omega'| + C(2d + \delta_{3}) |\omega'| \leq C(h+1) |\tilde{\omega}'| , \\ |\Xi_{h+1}(x, \tilde{\omega})' - \tilde{\omega}'| &\leq C |X_{h+1}(x, \tilde{\omega})'|^{\epsilon+1} \\ &\leq C(h+1)^{3} |\tilde{\omega}'|^{3} \leq C(h+1) |\tilde{\omega}'|^{2} . \end{aligned}$$

Thus (4.12) holds for j=h+1. By induction (4.12) holds for all  $j \leq 2q$ . Since

$$egin{aligned} X_{j+1}(x,\,\widetilde{\omega}) &= X_j(x,\,\widetilde{\omega}) \equiv l_j(x,\,\widetilde{\omega}) \Xi_j(x,\,\widetilde{\omega}) \,, \ (X_{2q}(x,\,\widetilde{\omega}) - x)' &= \sum\limits_{j=1}^{2^q} l_j(x,\,\widetilde{\omega}) \Xi_j(x,\,\widetilde{\omega})' \ &= \sum\limits_{j=1}^{2^q} l_j(x,\,\widetilde{\omega}) \widetilde{\omega}' + \sum\limits_{j=1}^{2^q} l_j(x,\,\widetilde{\omega}) \, (\Xi_j(x,\,\widetilde{\omega}) - \widetilde{\omega})' \end{aligned}$$

Note that  $|l_j(x, \tilde{\omega}) - d| \leq C |X_j(x, \tilde{\omega})'|^2 \leq Cq^2 |\tilde{\omega}'|^2$ . Then

$$(4.13) \qquad |(X_{2q}(x,\tilde{\omega})-x)'-2dq\tilde{\omega}'| \leq 2dq |\tilde{\omega}'|^2 + Cq^2 |\tilde{\omega}'|^2 \leq C'q^2 |\tilde{\omega}'|^2.$$

Now from (4.12) and  $\Xi_{2q}(x, \tilde{\omega}) = \omega$ 

$$|(\omega - \tilde{\omega})'| \leq C2q |\tilde{\omega}'|^2 \leq Cq^{-1} |\tilde{\omega}'|$$
,

which implies  $|(\omega - \tilde{\omega})'| \leq Cq^{-1}|\omega'|$  for large q. From (4.13) and the above

estimate (4.11) follows immediately.

Corollary. On the assumption of Lemma 4.5 we have

$$\left|\frac{\partial \Phi_{2q}}{\partial \omega_j}(0, x_3, \omega) - 2dq \,\omega_j\right| \leq Cq^2 |\omega'|^2.$$

Proof. Since x and y are on  $\mathcal{P}(x, \omega) |x_3-y_3| \leq |(x-y)'| |\omega'|$ . From (4.11)  $x_j-y_j=2dq\omega_j+0(q^2|\omega'|^2)$ , and from (4.5)

$$\frac{\partial \Phi_{2q}}{\partial \omega_j}(0, x_3, \omega) - (y_j - x_j) = O(q^2 |\omega'|^2).$$

Combining these relations we have the assertion.

**Lemma 4.6.** Suppose that  $q^2|x'| < 1$ ,  $|\omega'| < |x'|^3$ . Then

$$|(X_j(x, \omega)-x)'| \leq C |x'|^2,$$
  
$$|\Xi_j(x, \omega)'| \leq Cj |x'|^3$$

hold for all  $j \leq 2q$ , where C is a constant independent of q.

Proof. From (4.11) we have

$$\begin{split} |\Xi_{1}(x,\,\omega)'| &\leq |\omega'| + C \, |x'|^{3} \leq (C + C_{1}) \, |x'|^{3} \,, \\ |X_{1}(x,\,\omega)'| &\leq |x'| + 2(d + \delta_{3}) \, |\omega'| \leq |x'| (1 + Cq^{-4}) \leq |x'| (1 + q^{-2}) \,. \end{split}$$

Suppose that

(4.14) 
$$|X_{j}(x, \omega)'| \leq |x'|(1+jq^{-2}), |\Xi_{j}(x, \omega)'| \leq C_{2}j|x'|^{3}$$

holds for  $j \leq h$ . Then by the same reasoning as the above

$$|X_{h+1}(x, \omega)'| \leq |X_{h}(x, \omega)'| + 2(d+\delta_{3})C_{2}h|x'|^{3}$$
$$\leq |x'|(1+hq^{-2}+2(d+\delta_{3})C_{2}q^{-4})$$
$$\leq |x'|(1+(h+1)q^{-2})$$

if  $2(d+\delta_3)C_2q^{-2} < 1$ , and

$$\begin{aligned} |\Xi_{h+1}(x,\,\omega)'| &\leq |\Xi_{h}(x,\,\omega)'| + C |X_{h+1}(x,\,\omega)'|^{3} \\ &\leq C_{2}h |x'|^{3} + C |x'|^{3} (1 + (h+1)q^{-2})^{3} \\ &\leq C_{2}(h+1) |x'|^{3} \end{aligned}$$

if  $C2^3 < C_2$ . Thus (4.14) holds for all  $j \leq 2q$ . Therefore

$$|(X_{2q}(x, \omega) - x)'| \leq \sum_{j=1}^{2^{q}} l_{j} |\Xi_{j}(x, \omega)'|$$
  
 
$$\leq 2d |x'|^{3} C_{2} \sum_{j=1}^{2^{q}} j \leq C |x'|^{2}.$$

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**Lemma 4.7.** Let x and  $y = Y_{2q}(x, \omega) \in \omega(\delta_3)$ . Then we have

(4.15) 
$$|\operatorname{grad}_{x',\omega}\Phi_{2q}(x', x_3; \omega)| \ge c \min(|x'|^{e+1}, q^{-1}|\omega'|).$$

Proof. When  $|\omega'| \ge Cq(|x'|^{\ell+1}+|y'|^{\ell+1})$  Lemma 4.4 shows

$$|\partial_{\omega}\Phi_{2q}(x,\omega)| \geq (1-C|\omega'|)|(x-y)'| \geq (1-C|\omega'|)2dq|\omega'|$$

Thus (4.15) holds. Now let

(4.16) 
$$|\omega'| \leq Cq(|x'|^{\ell+1} + |y'|^{\ell+1}) \leq 1.$$

If  $|(x-y)'| \ge \frac{1}{2} |x'|$ , (4.15) follows immediately from (4.5). Then hereafter we suppose  $|x'-y'| \le 1/2 |x'|$ . Note that from the above inequality  $|y'| \le 3/2 |x'|$ . When  $|x(s)'|^2$  is monotonically increasing or decreasing Lemma 4.2 or 4.3 can be applied and we have  $|\partial_{\omega} \Phi_{2q}(x, \omega)| \ge (1-C |\omega'|) |x'|^2$ , which implies (4.15). If  $|x(s)'|^2$  is not monotone, set

$$|X_{j}(y, \omega)'|^{2} = \min |x(s)'|^{2}.$$

Suppose that  $|X'_j| \ge 1/2 |x'|$ . Under the condition (4.16) applying Lemma 4.3 to a broken ray  $y \rightarrow X_j$ , we have

$$y \cdot \omega' \leq -Cj |X_j|^{\epsilon+1} \leq -Cj |x'|^{\epsilon+1}.$$

Similarly applying Lemma 4.4 to a broken ray  $X_j \rightarrow x$  we have

$$\Xi_{2q}(y, \omega) \cdot x' \geq c(2q-j) |x'|^{\epsilon+1}$$

Therefore

$$egin{aligned} &(x',\,(\omega-
ablaarphi_{2q}(x,\,\omega))')=(x,\,\omega'-\Xi_{2q}(y,\,\omega)')\ &=(x-y,\,\omega')+(y,\,\omega')-(x',\,
ablaarphi_{2q}(y,\,\omega))\,, \end{aligned}$$

from which it follows that

$$|x'| |\omega' - \nabla \varphi_{2q}(x, \omega)'| \ge -\frac{|x'|}{2} cq(|x'|^{\epsilon+1} + |y'|^{\epsilon+1}) + 2q|x'|^{\epsilon+1}$$
$$\ge cq|x'|^{\epsilon+1}.$$

Then  $|\partial_{x'}\Phi_{2q}(x,\omega)| = |\omega' - (\nabla \varphi_{2q}(x,\omega))'| \ge cq |x'|^e$ , which implies (4.15). Consider the case  $|X'_j| \le \frac{1}{2} |x'|$ . Since

$$d |x(s)'|^2 / ds|_{s=s_j+0} \ge 0, \ d |x(s)'|^2 / ds|_{s=s_j-0} \le 0$$

we have  $y \cdot \omega' < 0$ . Suppose that  $j \leq q$ .

$$2\Xi_{2q-1}(y, \omega)' \cdot x' = d |x(s)'|^2 / ds \Big|_{s=s_{2q}}$$

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$$\ge \frac{1}{q} (|x'|^2 - |X'_j|^2) \ge \frac{1}{2q} |x'|^2,$$

$$\frac{1}{2q} |x'|^2 \le \Xi_{2q-1}(y, \omega)' \cdot X_{2q}(y, \omega)' - y' \cdot \omega'$$

$$= (\Xi_{2q}(y, \omega)' - \omega') \cdot x' + (x - y)' \cdot \omega'$$

$$\le |\Xi_{2q}(y, \omega)' - \omega'| |x'| - \frac{|x'|}{2} Cq |x'|^{s+1}.$$

Then we have

$$|\Xi_{2q}(y,\omega)'-\omega'| \ge \frac{Cq}{2}|x'|^{s+1}$$
, or  $|\Xi_{2q}(y,\omega)'-\omega'| \ge \frac{|x'|}{q}$ .

This shows (4.15).

**Corollary.** For any fixed  $0 < x_3 < d$ ,  $\Phi_{2q}(x', x_3; \omega)$  as a function of x' and  $\omega$ , the critical points of  $\Phi_{2q}$  are  $(x', \omega)$  such that x'=0,  $\omega=(0, 0, \pm 1)$ .

**Lemma 4.8.** For  $\omega = (0, 0, \pm 1)$  it holds that for  $q^2 |x'| < 1$ 

$$(4.17) Cq|x'|^{\epsilon+2} \ge \Phi_{2q}(x, \omega) - 2dq \ge cq|x'|^{\epsilon+2}$$

(4.18) 
$$\left|\frac{\partial \Phi_{2q}}{\partial \omega}(x, \omega)\right| \leq Cq |x'|^2.$$

Proof. Let  $\omega'=0$  and  $q^2|x'|<1$ . For a broken ray  $\mathscr{X}(y,\omega), y=Y_{2q}(x,\omega)$ , since  $y'\cdot\omega'=0 |x(s)'|^2$  is increasing. Therefore  $|x'| \ge |y'|$ , which implies  $q^2|y'|<1$ . Apply Lemma 4.6 to  $\omega$  and y and we have

$$|(X_{j}(y, \omega)-y)'| \leq C |y'|^{2} \leq C |x'|^{2}$$
.

Setting j=2q we have  $|x'-y'| \leq C |x'|^2$  which shows (4.18). By using the above estimate we have

$$|X_j(y, \omega)' - x'| \leq C |x'|^2.$$

Therefore we have

$$C|x'|^{\mathfrak{e}+2} \geq |X_{j+1}(y, \omega) - X_j(y, \omega)| - d \geq c |x'|^{\mathfrak{e}+2}.$$

Summing up this inequality from j=0 to 2q-1 and we have (4.17).

#### 5. Proof of Proposition 3.3

From Corollary of Lemma 4.7 it suffices to consider the integration (3.23) near x'=0,  $\omega=(0, 0, \pm 1)$ . Since  $x_3$  and t are fixed we shall omit in the rest of this section to write them in the expression of calculus. First we apply the stationary phase method to the integration in  $\omega$  variables. Let us set

$$\omega(\omega') = (\omega_1, \omega_2, \sqrt{1-\omega_1^2-\omega_2^2}), \ \omega' = (\omega_1, \omega_2),$$

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$$\frac{\partial \Phi_{2q}}{\partial \omega_j}(x', x_3, \omega(\omega')) = f_{q,j}(x', \omega'), \ j = 1, 2.$$

From Corollary of Lemma 4.5 we have

(5.1) 
$$f_{q,j}(0, 0) = 0, \ j = 1, 2,$$

(5.2) 
$$\frac{\partial f_{q,j}}{\partial \omega_h}(0,0) = 2qd\,\delta_{jh}, \ j,h=1,2.$$

Concerning Lemma 3.1 we can easily verify from Lemmas 5.2 and 5.3 of [2] that l(2, 0)=2, i.e.

(5.3) 
$$|\frac{\partial f_{q,j}}{\partial \omega_h}(x', \omega')|_1 \leqslant Cq^2.$$

Then the implicit function theorem assures the existence of solution of the equations

(5.4) 
$$f_{q,j}(x',\omega') = 0, \quad j = 1, 2 \text{ for } |x'| \leq q^{-2}.$$

Let us denote this solution by  $\omega'_q(x')$ . Then from (3.11) we have

(5.5) 
$$|\partial_{x'\omega_q}^{\omega}(x')| \leq C_{\omega}q^{l(\omega)} \quad \text{for } |x'| \leq q^{-2}$$

where  $l(\alpha)$  denotes an integer depending on  $\alpha$ . In the rest of this section we shall use notation  $l(\alpha)$  for various integer depending on  $\alpha$ . For the phase function we have

$$\Phi_{2q}(x, \omega) = \Phi_{2q}(x, \omega(\omega_q'(x'))) + \frac{1}{2} \sum_{|\omega|=2} \frac{1}{\alpha!} (\omega' - \omega_q'(x'))^{\omega} F_{q,\omega}(x', \omega'),$$

where

$$F_{q,(j,k)}(x',\omega') = \int_0^1 \frac{\partial f_{q,j}}{\partial \omega_k} (x',\theta \omega'_q(x') + (1-\theta)\omega') d\theta$$

Then from (5.2) and (5.3) it holds that

$$\mathcal{F}_q(x',\,\omega')=[F_{q,(j,h)}(x',\,\omega')]_{j,h=1,2} \geq dqI.$$

By making a change of variables

$$\zeta = \mathscr{F}_q(x', \omega')^{1/2}(\omega' - \omega'_q(x'))$$

we have

(5.6) 
$$\Phi_{2q}(x, \omega(\omega')) = \Phi_{2q}(x, \omega(\omega'_q(x'))) + \frac{1}{2} \zeta^* \zeta$$

and an estimate

$$(5.7) |\partial_{x'}^{\omega}\zeta| \leqslant C_{\omega}q^{l(\omega)}.$$

Let  $\mathfrak{X}$  be a  $C^{\infty}$  function verifying

$$\mathfrak{X}(\omega') = egin{cases} 1 & |\omega'| \leqslant 1 \ 0 & |\omega'| \geqslant 2 \,. \end{cases}$$

**Lemma 5.1.** Let  $|x'| \leq q^{-2}$  and  $g(x', \omega') \in C^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$ . An oscillatory integral

$$H_q(k, x') = \int_{\mathbf{R}^2} e^{ik\Phi_{2q}(x,\omega(\omega'))} g(x', \omega') \chi(\omega'/\delta) d\omega' \quad (\delta > 0)$$

has an expansion

$$H_{q}(k, x') = e^{ik\Psi_{q}(x')} \{ \sum_{j=0}^{6} k^{-1-j/2} h_{q,j}(x') + k^{-4} h_{q}(x', k) \}$$

where

(5.8) 
$$\Psi_q(x') = \Phi_{2q}(x, \omega(\omega_q'(x'))),$$

(5.9) 
$$|\partial_{x'}^{\omega}h_{q,j}(x')| \leq C_{\omega}q^{l(\omega)}|g|_{|\omega|+2j},$$

(5.10) 
$$|\partial_{x'}^{\omega}h_q(x';k)| \leq C_{\omega}q^{l(\omega)}|g|_{|\omega|+12} \quad \text{for all } k.$$

Especially for j=0

$$h_{q,0}(x') = \frac{1}{2\pi} \left( \det \mathcal{F}_q(x', \omega_q'(x')) \right)^{-1/2} g(x', \omega_q'(x')) \,.$$

Proof. By (5.6) we can write

$$H_q(k, x') = e^{ik\Psi_q(x')} \int_{\mathbf{R}^2} e^{ik\zeta^*\zeta} g(x', \omega') \frac{D\omega'}{D\zeta} d\zeta .$$

By using (5.7) we have the assertion by a standard argument.

Then the proof of (3.24) is reduced to obtain an expansion of an oscillatory integral

(5.11) 
$$H_{q,j}(k) = \int e^{ik\Psi_q(x')} h_{q,j}(x') dx' \, .$$

To this end we apply Varčenko's theorem [18, 7]. First consider properties of  $\Psi_q(x')$ .

Let  $x_3 = -\gamma(x')$  be a representation of  $\Gamma_1$  near  $a_1$  and  $x_3 = d + \tilde{\gamma}(x')$  be a representation of  $\Gamma_2$  near  $a_2$ .

Lemma 5.2. It holds that

$$(5.12) \qquad |\Psi_q(x')-2q(d+\gamma(x')+\tilde{\gamma}(x'))| \leq C_q(\gamma(x')+\tilde{\gamma}(x'))|x'|^2,$$

where  $C_q$  has an estimate  $C_q \leq Cq^a$  for some a > 0.

Proof. Let x(s) be a representation of  $\mathscr{X}(x, \omega(\omega'_q(x')))$ . Setting  $|X'_j| = \min |x(s)'|$  we have  $\Xi_j X'_j \ge 0$ ,  $\Xi_{j-1} \cdot X'_j \le 0$ . Note that we have  $x = X_{2q}(x, \omega(\omega'_q(x')))$  from the definition of  $\omega'_q(x')$ . Since  $\Xi_j - 2(\Xi_j, n(X_j))n(X_j) = 0$  it holds that

$$|\Xi_j| \leq C |y'|^{\epsilon+1} = C |x'|^{\epsilon+1} \leq C |x'|^3$$
.

Applying Lemma 4.6 to broken rays  $X_j$  to  $X_{2q}$  and y=x to  $X_j$  we have, if  $q^2|x'| \leq 1$ ,

(5.13) 
$$|X_{k}(x, \omega)' - x'| \leq C |x'|^{2},$$
$$|\Xi_{k}(x, \omega)'| \geq Cq |x'|^{3}$$

for all h. Evidently we have

$$(X_h)_3 = \begin{cases} -\gamma(X'_h) & \text{if } X_h \in \Gamma_1 \\ d + \tilde{\gamma}(X'_h) & \text{if } X_h \in \Gamma_2 . \end{cases}$$

Thus we have

$$\begin{aligned} ((X_{h+1})_3 - (X_h)_3)^2 &= \{ (d + \gamma(x') + \gamma(x')) + ((X_{h+1})_3 - (-\gamma(x')) \\ &- ((X_h)_3 - (d + \tilde{\gamma}(x')))^2 \\ &= (d + \gamma(x') + \gamma(x'))^2 (1 + (O(\operatorname{grad}(\gamma + \tilde{\tau}) |x'|^2)^2) \,. \end{aligned}$$

On the other hand

$$|X'_{h+1}-X'_{h}| \leq Cq |x'|^{e+1}.$$

Then taking account of (1.1) we have

$$|\sum_{h=0}^{2^{q-1}} |X_{h+1} - X_h| - 2q(d + \gamma(x') + \gamma(x'))| \leq C_q |x'|^{2(e+1+2)}$$

For x' such that  $q^2|x'| > 1$  (5.12) holds for  $C_q = q^a$  if we choose a sufficiently large. Q.E.D.

Let  $\chi_1$  and  $\chi_2$  be functions in  $C^{\infty}(\mathbf{R}^2)$  such that

$$\chi_1 + \chi_2 = 1$$
 on  $\mathbb{R}^2$ ,  
supp  $\chi_1 \subset \{x'; |x'| \leq 2\}, \chi_1 = 1$  for  $|x'| \leq 1$ .

Set

$$H_{q,j}^{(p)}(k) = \int e^{ik\Psi_q(x')} \chi_p(q^3x') h_{q,j}(x') dx', \ p = 1, 2.$$

From (5.12) it follows

$$|\nabla_{\mathbf{x}'}\Psi_q(\mathbf{x}')| \geq \frac{q}{2} |\operatorname{grad}(\gamma(\mathbf{x}')+\gamma(\mathbf{x}'))| \geq \frac{cq}{2} |\mathbf{x}'|^{\ell+1}.$$

Therefore on the support of  $\chi_2$  we have  $|\nabla_{x'}\Psi_q(x')| \ge cq^{-a}$  for some  $a \ge 0$ . Then using (5.9) we have

(5.13) 
$$|H_{q,j}^{(2)}(k)| \leq C_N q^{l(N)} k^{-N}$$
.

When we apply Varčenko's theorem to  $H_{q,j}^{(1)}$  we have to pay attention to parameter q, in other words, we have to obtain an expansion in k of  $H_{q,j}^{(1)}$  which is uniform in  $q \rightarrow \infty$ . To this end first we consider the Newtonian polyhedra of  $\Psi_q$ . Here we use freely the notation in [7]. (5.12) implies

$$\Psi_{q\Gamma} = q(\gamma + \tilde{\gamma})_{\Gamma}$$
 for large  $q$ .

Let Y and  $\pi$  be an analytic manifold and a projection constructed following Chapter II of [7] for  $(\gamma + \tilde{\gamma})$ , that is,

$$\begin{aligned} \pi \colon Y \to U \\ (\gamma + \tilde{\gamma}) \circ \pi(y) &= + y_1^{l_1} y_2^{l_2} \\ J(y) &= y_1^{m_1} y_2^{m_2} J_{\pi}(y), \ J_{\pi}(0) \neq 0 \,. \end{aligned}$$

Then it follows that

$$\Psi_{q} \circ \pi(y) = + q y_1^{l_1} y_2^{l_2} (1 \! + \! \psi_q(y))$$
 ,

where  $\psi_q(0)=0$  and  $|\partial_y \psi_q(y)| \leq C_{\alpha} q^{l(\alpha)}$ . Then we can find a change of variables  $\pi_q: y=y_q(z)$  for  $|z| \leq Cq^a$  such that

$$\begin{split} \Psi_q \circ \pi \circ \pi_q(z) &= +q z_1^{l_1} z_2^{l_2} ,\\ y_q(0) &= 0, \, \frac{\partial y_q}{\partial z} \left( 0 \right) = I ,\\ |\partial_z y_q| \leqslant C_a q^{l(\alpha)} . \end{split}$$

Then  $H_{q,j}^{(1)}$  may be represented in finite sum of integrals of the form

$$\int e^{ikqz_1^{I_1}z_2^{I_2}}(h_{q,j}\circ\pi\circ\pi_q) \ (z)J_q(z)dz \ .$$

Then Proposition 3.3 follows from Theorem 3.23 of [7] beside the representation of  $c_{q,j}^0$ . Note that  $\Psi_q$  verifies the condition 3) of Theorem 3.23 of [7] because  $\partial_{x'}^{\alpha'}\Psi_q(0)=0$  for  $|\alpha| \leq 4$ , and  $\Psi_q(x') > \Psi_q(0)$  for  $x' \neq 0$ . In Lemma 5.1 when we set  $g=v_{q,j}$  we have

$$h_{q,0} = \frac{1}{2\pi} q^{-1} v_{q,j}(x, \omega(\omega'_q(x')))$$

Then we have from Theorem 3.23 of [7] the desired relation.

# 6. Representation of the kernel of $\cos t \sqrt{-\Delta}$ near $a_1$ and $a_2$

Let  $\psi(x)$  be a  $C^{\infty}$  function with support contained in a small neighbor-

hood of  $a_1$ . We consider the behavior of

$$\int_{\Omega} (\int \rho_q(t) E(t, x, x) \psi(x) dt) dx \quad \text{as} \quad q \to \infty \; .$$

In this section we denote by s a point of  $\Gamma_1$  and by n(s) the unit outer normal of  $\Gamma_1$  at s. Correspond (r, s) to x near  $a_1$  by x=s+rn(s). First we state a result on the propagation of the solutions for oscillatory boundary data.

**Lemma 6.1.** Let *m* be an oscillatory boundary data on  $\mathbf{R} \times \Gamma_1$  of the form

 $m(t, s; p, p') = e^{i(p\zeta(s)-p't)}h(t, s; p)$ 

satisfying supp  $h \subset (0, 1) \times S_1(\delta_3)$  and

$$(6.1) \qquad |\partial_{t,s}^{\alpha}h| \leq C_{\alpha} p^{(1/2-\varepsilon_0)|\alpha|} \qquad (\varepsilon_0 > 0) .$$

If  $|p\nabla_s \zeta|/|p'| \ge 4 \delta_3/d$ , the solution of

(6.2) 
$$\begin{cases} \Box u = 0 & \text{in } \mathbf{R} \times \Omega \\ u = m & \text{on } \mathbf{R} \times \Gamma_1 \\ u = 0 & \text{on } \mathbf{R} \times \Gamma_2 \\ \text{supp } u \subset \{t \ge 0\} \end{cases}$$

verifies an estimate for any N

(6.3) 
$$|\partial_{t,x}^{\omega}u(t,x;p)| \leq C_{\omega,N}q^{l(\omega)}p^{-N} \quad on \ [2d, 2dq] \times \omega(\delta_3).$$

Except the case that  $|p\nabla_s \zeta|/|p'|$  is near 1 an asymptotic solution of (6.2) can be constructed by a usual manner and checked the propagation of solutions. For exceptional case we make use of the result of Melrose-Sjöstrand [13] on the propagation of singularities. We omit the proof.

As in §3 denoting by  $u(t, x; k, \omega)$  the solution of

(6.4) 
$$\begin{cases} \Box u = 0 & \text{in } (0, \infty) \times \Omega \\ u = 0 & \text{on } (0, \infty) \times \Gamma \\ u(0, x) = e^{ik\langle x, \omega \rangle} w(x) & \text{in } \Omega \\ \frac{\partial u}{\partial t}(0, x) = 0 & \text{in } \Omega \end{cases}$$

where w(x) = 1 on supp  $\psi$ , we have

$$E(t; x, y)\psi(y) = \int_0^\infty k^2 dk \int_{|\omega|=1} d\omega \, u(t, x; k, \omega) e^{-ik\langle y, \omega \rangle} \psi(y) \, .$$

In consideration of the behavior of  $u(t, x; k, \omega)$  the difference of the case (6.4) from u of §3 is that the initial data  $w(x)e^{ik\langle x,\omega\rangle}$  does not verify the compatibility condition of an initial-boundary value problem at  $\{t=0\} \times \Gamma$ . There-

fore the solution of (6.4) is not regular, and this fact gives rise to difficulties. Let  $\chi_1, \chi_2 \in C^{\infty}(\mathbf{R})$  such that

$$\chi_1 = \begin{cases} 1 & |r| \leq 1 \\ 0 & |r| \geq 2 \end{cases}$$

and  $\chi_1(r)^2 + \chi_2(r)^2 = 1$  on **R**. For  $\varepsilon > 0$  we have in  $\Omega$ 

$$w(x)e^{ik\langle x,\omega\rangle}$$

$$= Y(r)e^{ik\langle x,\omega\rangle}\chi_1(k^{1/2-\mathfrak{e}}r)^2w(x) + Y(r)e^{ik\langle x,\omega\rangle}\chi_2(k^{1/2-\mathfrak{e}}r)^2w(x)$$

$$= f_1 + f_2$$

where Y(r)=1 for  $r\geq 0$  and =0 for r<0. For  $u_2(t, x; k, \omega)=\cos t\sqrt{-\Delta}f_2$  we can use the method in §3~5 without large modification and we have

Lemma 6.2. It holds that

$$\begin{aligned} |\int_{\Omega} dx \int_{0}^{\infty} k^{2} dk \int_{|\omega|=1} d\omega \,\chi_{2}(k^{1/2-\epsilon})r) e^{-ik\langle x,\omega\rangle} (\int \rho_{q}(t) u_{2}(t,\,x;\,k,\,\omega) dt) \\ -c_{0} q^{(1-2/\epsilon_{0})(l+1)-2} \int_{0}^{d} \psi(0,\,x_{3}) dx_{3}| \leq C_{l} q^{(1-5/2\epsilon_{0})l} \end{aligned}$$

Hereafter we consider the behavior of  $u_1(x, t; k, \omega) = \cos t \sqrt{-\Delta} f_1$ . The asymptotic solution  $u_0$  for Cauchy problem

$$\left\{egin{array}{ll} \square u = 0 & ext{in } {oldsymbol{R}} imes {oldsymbol{R}}^3 \ u(0, \, x) = e^{ik\langle x, \, \omega 
angle} \chi_1(k^{1/2-e}r) \psi(x) & ext{in } {oldsymbol{R}}^3 \ rac{\partial u}{\partial t}(0, \, x) = 0 & ext{in } {oldsymbol{R}}^3 \end{array}
ight.$$

is obtained in a form

$$u_{0}(t, x; k, \omega) = e^{ik(\langle x, \omega \rangle - t)} \sum_{j=1}^{N} v_{j}^{+}(t, x; k) (ik)^{-j} + e^{ik(\langle x, \omega \rangle + t)} \sum_{j=1}^{N} v_{j}^{-}(t, x, k) (ik)^{-j} = u_{0}^{+} + u_{0}^{-}.$$

Then  $m^{\pm} = u_0^{\pm}|_{(0,\infty) \times r}$  is of the form

(6.5) 
$$m^{\pm}(t, s; k) = e^{ik(\langle s, \omega \rangle \pm t)} h^{\pm}(t, s; k)$$

(6.6)  $\left|\partial_{t,s}^{\omega}h^{\pm}(t,s;k)\right| \leqslant C_{\omega}k^{(1/2-\varepsilon)|\omega|}.$ 

Extend  $m^{\pm}$  to a function on  $\mathbf{R} \times \Gamma$  by setting  $m^{\pm}=0$  for t<0. Denote by  $u^{\pm}$  the solution of

(6.7) 
$$\begin{cases} \Box u = 0 & \text{in } \mathbf{R} \times \Omega \\ u = m^{\pm} & \text{on } \mathbf{R} \times \Gamma \\ \text{supp } u \subset \{t \ge 0\} \times \Omega . \end{cases}$$

Then we have  $u_1 = -u^+ - u^-$  on  $\omega(\delta_3)$  for  $t \ge 2R$ . Then it suffices to consider  $u^{\pm}$ .

Since  $m^{\pm}|_{(0,\infty) \times \Gamma_2} \in C_0^{\infty}$  we can apply the method in §3~5 for  $m^{\pm}$  on  $\Gamma_2$ . Therefore we consider only the solution for  $m^{\pm}$  on  $\Gamma_1$ . First consider the case  $|\omega'| \ge 1/2$ . Since there is no difference for  $m^+$  and  $m^-$  we consider the solution for  $m^+$  and omit + for brevity. By Fourier's inversion formula

(6.8) 
$$m(t, s; k, \omega) = w(t) \iint e^{ik'(t-t')} m^+(t', s; k, \omega) dt' dk'$$
$$= \int w(t) e^{ik't} e^{ik\langle s, \omega \rangle} \hat{h}(k'-k, s; k, \omega) dk'$$

for  $w(t) \in C_0^{\infty}(\mathbf{R})$  such that w(t) = 1 on supp  $m^+$ . Let us denote by  $b(t, x; k, \omega, k')$  the solution of (6.7) whose  $m^+$  is replaced by  $w(t)e^{ik't}e^{ik\langle s,\omega\rangle}\hat{h}(k'-k, s; k, \omega)$ . For  $|k'| \leq 2|k|$  we have

$$|k \nabla_s \langle s, \omega \rangle| / |k'| \ge 4 \delta_3 / d$$
,

from which

(6.9) 
$$|b(t, x; k, \omega, k')| \leq C_N q^{l(\mathfrak{A})} k^{-N} \quad \text{in } [2d, 2dq] \times \omega(\delta_3)$$

follows by an application of Lemma 6.1. For  $|k'| \ge 2|k|$ 

$$|\hat{h}(k'-k, s; k, \omega)| \leq C |k'-k|^{-1} \leq C' |k'|^{-1}$$
.

As an approximation of b we have an asymptotic solution of the form

$$b' = \sum_{q=0}^{\infty} \sum_{j=0}^{N} e^{ik'(\varphi_q(x;\omega,k/k')-t)} v_{q,j}(t, x; k, \omega, k') (ik')^{-j}$$

where

(6.10) 
$$|v_{q,j}| \leq C (k^{1/2-\epsilon})^{2j} |k'-k|^{-1}.$$

Since  $|\partial \varphi_q / \partial x_3| \ge 1 - C \delta_3$  on  $\omega(\delta_3)$  and  $|\omega_3| \le \sqrt{3}/2$ , we have

$$\begin{aligned} &|\int_{\Omega} (\int \rho_q(t) b(t, v; k, \omega, k') dt) \chi_1(k^{1/2 - \varepsilon} r) e^{ik\langle x, \omega \rangle} dx| \\ &\leq C(\hat{\rho}(k'/q^{-1}) + q^{-1}C |\zeta(k'/q^{-1})|) |k' - k|^{-3} k^{1/2 - \varepsilon} \end{aligned}$$

by using (6.10) and the fact b'=0 on  $\Gamma_1$ , where  $\zeta$  is a rapidly decreasing function. Thus we have

$$\begin{split} &|\int_{\Omega} dx \int_{|\omega'| > 1/2} d\omega \int_{0}^{\infty} k^{2} dk \int_{|k'| > 2k} dk' (\int \rho_{q}(t) b dt) \chi_{1}(k^{1/2-\epsilon} r) e^{ik\langle x, \omega \rangle} | \\ &\leq C \int \int_{|k'| > 2k} k'^{-3} k^{2+1/2+\epsilon} (\hat{\rho}(k'/q^{-1}) + q^{-l} \zeta(k'/q^{-1})) dk' dk \\ &\leq C q^{l(1/2-\epsilon)} . \end{split}$$

Combining this estimate and (6.9) we have

Lemma 6.3. It holds that

$$\begin{aligned} &|\int_{\Omega} dx \int k^2 dk \int_{|\omega'| > 1/2} d\omega \int^{\natural} dk' (\int \rho_q(t) b dt) \mathcal{X}_1(k^{1/2-\mathfrak{e}} r) e^{ik\langle x, \omega \rangle} | \\ &\leqslant C_l(q^l)^{1/2-\mathfrak{e}} \,. \end{aligned}$$

Next we consider the case of  $|\omega'| \leq 1/2$ . In this case in addition to (6.6) another estimate

$$(6.11) \qquad |\partial_{s,t}^{\omega}h^{+}(t,s;k)| \leq C_{\omega} \qquad \text{for } (t,s) \in [0, t_{0}k^{-(1/2-\mathfrak{e})}] \times S_{1}(\delta_{3})$$

holds if we choose  $t_0 > 0$  small. Let us set

$$m^{\pm} = Y(t)\chi_{1}(Tk^{1/2-\mathfrak{e}}t)^{2}m^{\pm} + Y(t)\chi_{2}(Tk^{1/2-\mathfrak{e}}t)^{2}m^{\pm}$$
  
=  $m_{1}^{\pm} + m_{2}^{\pm}$ .

Denote by  $b_p^{\pm}$ , p=1, 2, the solution of (6.7) replaced  $m^{\pm}$  by  $m_p^{\pm}$ . Concerning  $b_2^{\pm}$  we can apply the method in §3~5 for construction of asymptotic solution and acheive the parallel argument.

Lemma 6.4. We have an estimate

$$|\int_{\Omega} dx \int_{|\omega'| < 1/2} d\omega \int k^2 dk (\int \rho_q(t) b_2^+ dt) \chi_1(k^{1/2-\epsilon} r) e^{ik\langle x, \omega \rangle}|$$
  
$$\ll C_1 q^{(1/2-\epsilon)l}.$$

Note that  $m_1^{\pm}$  is of the form

(6.12) 
$$m_1^{\pm} = e^{ik\langle\langle s,\omega\rangle \mp t\rangle} h_1^{\pm}(t, s; k, \omega),$$
$$|\partial_t^{\alpha} \partial_s^{\beta} h_1(t, s; k, \omega)| \leq C_{\alpha,\beta} k^{(1/2-\mathfrak{e})\alpha} \quad for \ t > 0$$

We consider only for  $m_1^+$ , and hereafter we omit the suffix + and 1 for brevity. In a same way as (6.8) we have

$$m(t, s; k, \omega) = \int w(t) e^{-ik't} e^{ik\langle s, \omega \rangle} \hat{h}(k'-k, s; k, \omega) dk'$$

where

(6.13) 
$$\hat{h}(k'-k, s; k, \omega) = \int e^{-i(k't'-kt')}h(t', s; k, \omega)dt'$$

Denote by  $b(t, x; k, \omega, k')$  the solution of (6.7) replaced  $m^{\pm}$  by  $w(t)e^{ik't}e^{ik\langle s,\omega\rangle}$  $\hat{h}(k'-k, s; k, \omega)$ . Then

(6.14) 
$$b_1^+(t, x; k, \omega) = \int b(t, x; k, \omega, k') dk'$$

Taking account of (6.12) we have for all k'

$$(6.15) \qquad \qquad |\partial_s^{\omega} \hat{h}(k'-k,s;k,\omega)| \leq C_{\omega} k^{-(1/2-\mathfrak{e})},$$

and for  $k' \neq k$  we have by integration by parts in (6.13)

(6.16) 
$$|\partial_s^{\alpha} \hat{h}(k'-k,s;k,\omega)| \leq C_{\omega} |k'-k|^{-1}.$$

For small  $\gamma$  let  $\varphi_1(x; \omega, \gamma)$  be a function verifying

$$\begin{cases} \varphi_1 = (1+\gamma) \langle s, \omega \rangle & \text{ on } \Gamma_1 \\ \frac{\partial \varphi_1}{\partial n} > 0 & \text{ on } \Gamma_1 \\ |\nabla \varphi_1| = 1 \,. \end{cases}$$

Then for  $\varphi_1$  we can define a sequence of phase functions  $\varphi_j(x; \omega, \gamma), j=2, 3, \cdots$  following the process in §3. Set

$$\Phi_{2q}(x; \omega, \gamma) = \varphi_{2q}(x; \omega, \gamma) - \langle x, \omega \rangle.$$

As a modification of considerations in §4 we have

**Lemma 6.5.** Let  $\gamma_0$  and  $r_0$  be small positive constants. Then there exists  $\omega(x, \gamma)$  satisfying

$$abla_{\omega'} \Phi_{2q}(x; \omega(x, \gamma), \gamma) = 0$$
 for  $|x-a_1| \leq r_0$ 

and this critical point is non-degenerate. If we set

$$\psi_q(x, \gamma) = \Phi_{2q}(x; \omega(x, \gamma), \gamma),$$

the critical point with respect to x' is only x'=0 and concerning the Newtonian polyhedra of  $\psi_q$  we have the same assertions as in §4 for all  $|\gamma| \leq \gamma_0$ .

For  $k' \in [(1-\gamma_0)k, (1+\gamma_0)k]$ , with the aid of the above lemma we estimate an oscillatory integral following the process of §5. Applying Varčenko's theorem we have

(6.17) 
$$J(t, r; k, k') = \int ds \int_{|\omega'| < 1/2} d\omega b(t, x; k, \omega, k') e^{ik\langle x, \omega \rangle} \chi_1(k^{1/2 - \mathfrak{e}} r)$$
$$= \chi_1(k^{1/2 - \mathfrak{e}} r) e^{ik'(t - (2dq + \gamma r))} \{ c_0(r; k, k') k'^{-1 - 2/\mathfrak{e}_0} + O(k'^{-1 - 5/3\mathfrak{e}_0}) \}$$

where

$$|c_0(r; k, k')| \leq Ck^{-1/2+\epsilon}$$

holds because of (6.15). Then

(6.18) 
$$|\int dr \int k^2 dk \int \rho_q(t) dt \int_{|k'-k| < k^{1/2+\varepsilon}} J(t, r; k, k') dk' |$$

$$\leq C \int \int_{|k'-k| < k^{1/2+\varepsilon}} (k'/q^{-1}) k'^{-1-2/\varepsilon} k^2 k^{-(1/2-\varepsilon)} k^{-(1/2-\varepsilon)} dk dk$$

$$\leq C_1 q^{(1/2-2/\varepsilon_0+\varepsilon)/}.$$

For  $k' \in [k+k^{1/2+\epsilon}, (1+\gamma_0)k]$  use (6.16) and make an integration by parts with respect to r in the left hand side of (6.18). Then since  $c_0(0; k, k')=0$  we have

(6.19) 
$$|\int k^{2} dk \int \rho_{q}(t) dt \int_{0}^{r_{0}} dr \int_{k+k^{1/2+\epsilon}}^{(1+\gamma_{0})k} J(t, r; k, k') dk' |$$

$$\leq C \int dk \int_{k+k^{1/2+\epsilon}}^{(1+\gamma_{0})k} \beta\left(\frac{k'}{q^{l}}\right) \frac{1}{|k'-k|^{3}} k'^{-1-2/\epsilon_{0}} k^{1/2-\epsilon} k^{2} dk'$$

$$\leq C_{l} q^{(1/2+\epsilon-2/\epsilon_{0})l} .$$

We have the same estimate for  $k' \in [(1-\gamma_0)k, k-k^{1/2+\epsilon}]$ . Thus it remains us to consider for  $|\omega'| < 1/2$  and  $|k'-k| \ge \gamma_0 k$ . For  $k' \ge (1+\gamma_0)k$  set

$$\tilde{J}(k, k') = \int dt \int dr \, \rho_q(t) J(t, r; k, k')$$

and we have from (6.16)

$$\widetilde{J}(k, k') \leq \zeta(k'/q') |k'-k|^{-3} k^{1/2-\epsilon}$$

where  $\zeta \in \mathcal{S}(\mathbf{R})$ . Thus

(6.20) 
$$|\int k^2 dk \int_{(1+\gamma_0)k}^{\infty} \widetilde{J}(k, k') dk'| \leq C_l q^{(1/2-\epsilon)l}$$

Suppose  $|k'| \leq (1-\gamma_0)k$ . When  $|k\omega'| \leq k^{\mathfrak{e}}$ ,  $|k'| \leq k^{\mathfrak{e}}$  we have immediately

$$|\partial_{t,x}^{\alpha}b(x, t; k, \omega, k')| \leq C_{\alpha}k^{(|\alpha|+2)^{2}}$$

from the energy estimate of solution of (6.7). Thus we have

$$\left|\int J(t, r; k, k')dr\right| \leq Ck^{-3+3\varepsilon},$$

from which it follows that

$$\left|\int k^{2}dk\int_{|\omega'|< k^{-1+\varrho}}d\omega\int_{|k'|< k^{\varrho}}dk'\int dt\,\rho_{q}(t)J(t,\,r;\,k,\,k')\right| \leq C$$

for all q. Let us suppose  $|k\omega'| \ge k^{\epsilon}$ ,  $|k'| \ge k^{\epsilon}$ . If  $|k\omega'|/|k'| \ge 4\delta_3/d$  an application of Lemma 6.1 gives

$$|J(t, r; k, k')| \leq C_N k^{-\varepsilon_N}$$
.

Thus

(6.21) 
$$|\int k^2 dk \int_{(1-\gamma_0)k>|k'|>k^2} dk' \int_{|k\omega'|/|k'|>d_0} d\omega \int dr \int \rho_q(t) J(t, r; k, k') dt | \leq C.$$

Let  $|k\omega'|/|k'| \leq d_0 = 4\delta_3/d$ ,  $(1-\gamma_0)k \geq |k'| \geq k^e$ . Then we have

$$|J(t, r; k, k')| \leq Ck^{-3}k^{1/2-\varepsilon}$$
.

Therefore

(6.22) 
$$|\int k^{2} dk \int_{k^{2}}^{(1-\gamma_{0})^{k}} dk' \int_{|k\omega'|/|k'| \leq d_{0}} d\omega \int dt \, dr \, \rho_{q}(t) J(t, \, r; \, k, \, k')|$$
$$\leq C \int k^{2} dk \int_{k^{2}}^{(1-\gamma_{0})^{k}} dk' \zeta \left(\frac{k'}{q^{l}}\right) k^{-3} k^{1/2-\epsilon} \left(\frac{k'}{k}\right)^{2}$$
$$\leq C \int_{-\infty}^{\infty} \zeta (k'q^{-l}) k'^{2} (\int_{(1-\gamma_{0})^{k'}}^{\infty} k^{-5/2-\epsilon} dk) dk'$$
$$\leq C \int_{-\infty}^{\infty} \zeta (k'q^{-l}) k'^{2-3/2-\epsilon} dk' \leq C q^{(1/2-\epsilon)l} .$$

Then the estimates  $(6.18) \sim (6.22)$  imply the following

Lemma 6.6. We have

$$\left|\int_{\Omega} dx \int_{|\omega| < 1/2} d\omega \int k^2 dk (\int \rho_q(t) b_1^+ dt) \chi_1(k^{1/2-\mathfrak{e}} r) e^{ik\langle x, \omega \rangle} \right| \leq C_1 q^{(1/2-\mathfrak{e})I}$$

From Lemmas  $6.2 \sim 6.6$  we have

**Proposition 6.7.** Let  $\psi(x)$  be a  $C^{\infty}$  function with support in a small neighborhood of  $a_1$ . Then an estimate

$$|\int_{\Omega} (\int \rho_q(t) E(t, x, x) dt) \psi(x) dx - c_0 q^{(1-2/\epsilon_0)(1-1)} \int_0^d \psi(0, x_3) dx_3| \leq C_1 q^{(1-5/2\epsilon_0)/2}$$

holds.

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