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# MOUFANG TREES AND GENERALIZED TRIANGLES

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## 1. Introduction

Let $\Gamma$ be an undirected graph, let $V(\Gamma)$ denote the vertex set of $\Gamma$ and let $G$ be a subgroup of $\mathrm{aut}(\Gamma)$. For $x\in V(\Gamma)$, we will denote by $\Gamma_x$ the set of vertices adjacent to $x$ in $\Gamma$ and by $G_x^{[1]}$ the pointwise stablizer of $\Gamma_x$ in the stabilizer $G_x$. An $n$-path of $\Gamma$ for any $n\geq 0$ is an $(n+1)$-tuple $(x_0,x_1,\cdots,x_n)$ of vertices such that $x_i\in\Gamma_{x_{i-1}}$ for $1\leq i\leq n$ and $x_i\neq x_{i-2}$ for $2\leq i\leq n$. Let

$$G_{x,y,\cdots,z}^{[1]}=G_x^{[1]}\cap G_y^{[1]}\cap\cdots\cap G_z^{[1]}$$

for any subset $\{x,y,\cdots,z\}$ of $V(\Gamma)$. The graph $\Gamma$ will be called thick if $|\Gamma_u|\geq 3$ for all $u\in V(\Gamma)$. An apartment of $\Gamma$ is a connected subgraph $\Delta$ such that $|\Delta_u|=2$ for every $u\in V(\Delta)$. When there is no danger of confusion, we will often use integers to denote vertices of $\Gamma$.

A generalized $n$-gon (for $n\geq 2$) is a bipartite graph of diameter $n$ and girth $2n$. A generalized $n$-gon $\Gamma$ for $n\geq 3$ is called Moufang if $G_{1,\cdots,n-1}^{[1]}$ acts transitively on $\Gamma_n\backslash\{n-1\}$ for every $(n-1)$-path $(1,\cdots,n)$ of $\Gamma$ for some $G\leq\mathrm{aut}(\Gamma)$. In [6], Tits showed that thick Moufang $n$-gons exist only for $n=3,4,6$ and 8. If $\Gamma$ is a thick generalized $n$-gon and $G\leq\mathrm{aut}(\Gamma)$, then $G_{0,1}^{[1]}\cap G_{0,\cdots,n}=1$ for every $n$-path $(0,\cdots,n)$ of $\Gamma$. (This is a special case of [5,(4.1.1)]; see Theorem 2 of [8].) Thus, the following (Theorem 1 of [8]) is a generalization of Tits' result:

**Theorem 1.1.** *Let $\Gamma$ be a thick connected graph, let $G\leq\mathrm{aut}(\Gamma)$ and let $n\geq 3$. Suppose that for each n-path $(0,1,\cdots,n)$ of $\Gamma$,*

(i) $G_{1,\cdots,n-1}^{[1]}$ *acts transitively on* $\Gamma_n\backslash\{n-1\}$ *and*

(ii) $G_{0,1}^{[1]}\cap G_{0,\cdots,n}=1$.

*Then* $n=3,4,6$ or 8.

We will say that a graph $\Gamma$ is $(G,n)$-Moufang if it is thick, connected and $\Gamma$, $G$ and $n$ fulfill conditions (i) and (ii) of (1.1). In this paper, we will be mainly concerned with the case that $\Gamma$ is a tree.

In [1,(3.6)], the following beautiful connection between trees and generalized polygons was established:

**Theorem 1.2.** *Let $n \geq 3$. Suppose $\Gamma$ is a tree and $\mathscr{A}$ a family of apartments of $\Gamma$ such that*

(i) *every $(n+1)$-path of $\Gamma$ lies on a unique element of $\mathscr{A}$ and*

(ii) *if $(x_0, \cdots, x_{2n})$ and $(x'_0, \cdots, x'_{2n})$ are two $2n$-paths with $x_i = x'_i$ for $0 \leq i \leq n$ but $x_{n+1} \neq x'_{n+1}$ each lying on an element of $\mathscr{A}$, then there is a third element of $\mathscr{A}$ containing $(x_{2n}, \cdots, x_n, x'_{n+1}, \cdots, x'_{2n})$.*

*For vertices $u$ and $v$ of $\Gamma$, let $u \sim v$ if there is an element of $\mathscr{A}$ containing them both and $\mathrm{dist}_\Gamma(u,v) = 2n$. Let $\approx$ be the transitive closure of $\sim$, let $\bar{u}$ be image of a vertex $u$ of $\Gamma$ in $V(\Gamma)/\approx$ and let $\bar{\Gamma}$ be the graph with vertex set $V(\Gamma)/\approx$, where two equivalency classes are adjacent in $\bar{\Gamma}$ whenever they contain elements adjacent in $\Gamma$. Then $\bar{\Gamma}$ is a generalized $n$-gon and the natural map from $V(\Gamma)$ to $V(\Gamma)/\approx$ induces a bijection from $\Gamma_u$ to $\bar{\Gamma}_{\bar{u}}$ for every $u \in V(\Gamma)$.*

For the sake of completeness (and because [1,(3.6)] is phrased differently), we include a proof of (1.2) in §6 below.

If $\Gamma$ is a $(G,n)$-Moufang graph, we will denote by $G^\circ$ the subgroup of $G$ generated by the groups $G^{[1]}_{1,\cdots,n-1}$ for all $(n-2)$-paths $(1,\cdots,n-1)$ of $\Gamma$. (Thus, of course, $\Gamma$ is also $(G^\circ,n)$-Moufang.)

Suppose now that $\Gamma$ is a $(G,n)$-Moufang tree with $G = G^\circ$ containing a $G$-invariant family of apartments fulfilling the conditions of (1.2). Let $\bar{\Gamma}$ be as in (1.2) and let $\bar{G}$ denote the subgroup of $\mathrm{aut}(\bar{\Gamma})$ induced by $G$. Then by (1.2) and the action of $\bar{G}$, the graph $\bar{\Gamma}$ is a Moufang $n$-gon. Thus, $\bar{\Gamma}$ and $\bar{G}$ are known, for $n=8$ by [7] and for $n=4$ and 6 by forthcoming work of Tits (see also [2] for partial results); the case $n=3$ is classical and can be found, for instance, in [3]. In particular, the structure of the amalgam $(G_x, G_y; G_{xy})$ for an edge $\{x,y\}$ is known since, for every $u \in V(\Gamma)$, the stabilizer $G_u$ acts faithfully on the set of vertices of $\Gamma$ at distance at most $n-1$ from $u$ and, by (1.2), the restriction of the natural map from $V(\Gamma)$ to $V(\Gamma)/\approx$ to this set is injective. Since $G \cong G_x *_{G_{x,y}} G_y$ and $\Gamma$ is isomorphic to the coset graph associated with this free amalgamated product (as defined in [4,(I.4.1)], it follows that the pair $(\Gamma, G)$ can be reconstructed from the pair $(\bar{\Gamma}, \bar{G})$.

DEFINITION 1.3. Let $\Gamma$ be a $(G,n)$-Moufang tree. We will say that the pair $(\Gamma, G)$ has property $(*)$ if there is a $G^\circ$-invariant family of apartments of $\Gamma$ fulfilling conditions (1.2.i) and (1.2.ii).

CONJECTURE 1.4. Suppose that $\Gamma$ is a $(G,n)$-Moufang tree. Then the pair $(\Gamma,G)$ has property $(*)$.

By the remarks of the previous paragraph, (1.4) would imply the classification of $(G,n)$-Moufang trees. In [12] and [13], (1.4) is proved for $n=6$ and $n=8$. In this paper, we will prove (1.4) for $n=3$:

**Theorem 1.5.** *If $\Gamma$ is a $(G,3)$-Moufang tree, then the pair $(\Gamma,G)$ has property $(*)$.*

Note added in proof: The case $n=4$ of (1.4) has been handled in [11]. By (1.1), this completes the proof of (1.4).

In the course of proving (1.5), we will require the following result which is perhaps of independent interest.

**Theorem 1.6.** *Let $\Gamma$ be a $(G,3)$-Moufang graph. Let $(1,2,3,4)$ be a 3-path of $\Gamma$, let $U_i=G^{[1]}_{i,i+1}$ for $i=1,2$ and 3 and let $U_+=\langle U_1,U_2,U_3\rangle$. Let $U_{12}=\langle U_1,U_2\rangle$ and $U_{23}=\langle U_2,U_3\rangle$. Let $\Delta$ be the graph with vertex set consisting of the sets of right-cosets in $U_+$ of $U_1$, $U_{12}$, $U_{23}$ and $U_3$ together with two other elements called L and R and the following adjacencies: L with R, L with every coset of $U_{12}$, a coset of $U_{12}$ with every coset of $U_1$ contained in it, R with every coset of $U_{23}$, a coset of $U_{23}$ with every coset of $U_3$ contained in it and a coset of $U_1$ with a coset of $U_3$ whenever their intersection is non-empty. Then $\Delta$ is a Moufang 3-gon.*

It will be clear that for given $\Gamma$, the graph $\Delta$ of (1.6) is isomorphic to the graph $\bar{\Gamma}$ arising from (1.2) and (1.5); see the remarks at the end of §5 below.

The proof of (1.5) is heavily dependent on Tits' work on Moufang polygons. In particular, the idea for (1.6) was suggested by some comments of J. Tits made in his recent lectures on the classification of Moufang polygons at the Collège de France.

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## 2. Preliminary observations

A version of the following can be found in both [1] and [9].

**Proposition 2.1.** *Let $n\geq 3$ and suppose that $\Gamma$ is a $(G,n)$-Moufang tree with $G=G^\circ$ such that for each $(n+1)$-path $(0,\cdots,n+1)$ of $\Gamma$,*

(i) *there is a unique vertex in $\Gamma_{n+1}\backslash\{n\}$ fixed by $G_{0,\cdots,n+1}$ and*

(ii) *the only fixed points of* $G_0^{[1]} \cap G_{0,\cdots,n}$ *in* $\Gamma_1 \cup \cdots \cup \Gamma_{n-1}$ *are* $0,1,\cdots,n-1$ *and* $n$.

*Then the pair* $(\Gamma,G)$ *has property* $(*)$.

Proof. Let $\mathscr{A}$ be the family of apartments $\Delta$ in $\Gamma$ such that $\Delta$ is fixed by $G_{0,\cdots,n+1}$ for every $(n+1)$-path $(0,\cdots,n+1)$ contained in $\Delta$. Condition (i) implies then that $\mathscr{A}$ fulfills (1.2.i). To show that condition (ii) implies that $\mathscr{A}$ fulfills (1.2.ii), we choose two elements $\Delta$ and $\Delta'$ of $\mathscr{A}$ containing $2n$-paths $(0,\cdots,2n)$ and $(0',\cdots,(2n)')$ such that $i=i'$ for $0 \leq i \leq n$. Let $\Delta''$ be the unique element of $\mathscr{A}$ containing $(2n,\cdots,n,(n+1)')$ and let $H=G_n^{[1]} \cap G_{0,\cdots,n}$. By (i), $H=G_n^{[1]} \cap G_{n,\cdots,2n}$. In particular, $G$ fixes $\Delta''$. Also $H=G_n^{[1]} \cap G_{n',\cdots,(2n)'}$, again by (i). Thus $n',\cdots,(2n)'$ are the only fixed points of $H$ in $\Gamma_{(n+1)'} \cup \cdots \cup \Gamma_{(2n-1)'}$ by (ii). This implies that $(n',\cdots,(2n)')$ lies on $\Delta''$. q.e.d.

Let $\Gamma$ be a $(G,3)$-Moufang tree. If $\Gamma$ is not trivalent, then (1.5) implies that condition (2.1.i) holds. This means that the family of $G^\circ$-invariant apartments fulfilling (1.2.i) is unique and, since $G^\circ$ is a normal subgroup of $G$, that this family is, in fact, $G$-invariant. If $\Gamma$ is trivalent, then there are exactly two families of $G^\circ$-invariant apartments fulfilling both (1.2.i) and (1.2.ii), one for each of the two $G^\circ$-orbits of "unordered 5-paths" in $\Gamma$: to each 5-path $(0,\cdots,5)$, we associate the family of apartments $\Delta$ such that every 5-path lying on $\Delta$ is in the same $G^\circ$-orbit as either $(0,\cdots,5)$ or $(5,\cdots,0)$. These two families of apartments are not necessarily $G$-invariant in the case that $G \neq G^\circ$; see, for instance, [10].

## 3. The proof of (1.5): First part

Suppose that $\Gamma$ is a $(G,3)$-Moufang graph. Let $(0,1,2,3,4)$ be an arbitrary 4-path in $\Gamma$ and let $U_i=G_{i,i+1}^{[1]}$ for $0 \leq i \leq 3$. If $H$ is a group, $H^*$ will denote the set of nontrivial elements of $H$.

**Proposition 3.1.** *The following hold:*

(i) $[U_i,U_{i+1}]=1$ *for* $0 \leq i \leq 2$,

(ii) $[U_i,U_{i+2}] \leq U_{i+1}$ *for* $0 \leq i \leq 1$ *and*

(iii) $[a_1,a_3] \neq 1$ *whenever* $a_1 \in U_1^*$ *and* $a_3 \in U_3^*$.

Proof. We have $[U_i,U_{i+1}] \leq U_i \cap U_{i+1}$, so $[U_i,U_{i+1}]=1$ by (1.1.ii) for $0 \leq i \leq 2$. This proves (i); (ii) is clear. Let $a_i \in U_i$ for $i=1$ and 3. If $[a_1,a_3]=1$, then $a_3 \in (G_4^{[1]})^{a_1}=G_x^{[1]}$ for $x=4^{a_1}$. If $a_1 \neq 1$, then $x \neq 4$ by (1.1.ii), so $a_3 \in G_{4,3,x}^{[1]}=1$, also by (1.1.ii). q.e.d.

**Proposition 3.2.** *The following hold:*

(i) *If* $a_1 \in U_1^*$ *and* $a_2 \in U_2$, *then there exists a unique element* $a_3$ *in* $U_3$ *such that* $[a_1,a_3]=a_2$.

(ii) $[U_i, U_{i+2}]=U_{i+1}$ *for* $0 \leq i \leq 1$.

Proof. Let $a_1 \in U_1^*$ and $a_2 \in U_2$. Choose $5 \in \Gamma_4 \backslash \{3\}$ and let $x=4^{a_1^{-1}}$. By (1.1.ii), $x=4^{a_2 a_1^{-1}} \neq 4$. Thus $5^{a_1^{-1}}$ and $5^{a_2 a_1^{-1}} \in \Gamma_x \backslash \{3\}$. By (1.1.i), therefore, there exists $a_3 \in U_3$ such that $5^{a_1^{-1} a_3^{-1}}=5^{a_2 a_1^{-1}}$. Thus $5^{[a_1,a_3]}=5^{a_2}$. By (1.1.ii) and (3.1.ii), it follows that $[a_1,a_3]a_2^{-1} \in U_2 \cap G_5=1$. Thus $a_2=[a_1,a_3]$. If also $a_2=[a_1,b_3]$ with $b_3 \in U_3$, then $[a_1,a_3 b_3^{-1}]=1$ by (3.1.i), so $b_3=a_3$ by (3.1.iii). This proves (i). By (i) and (3.1.ii), $[U_1,U_3]=U_2$. By a similar argument, $[U_0,U_2]=U_1$. q.e.d.

**Proposition 3.3.** $U_1 U_2$ *is abelian.*

Proof. It follows by (3.1.i) and (3.2.ii) that $U_1$ and $U_2$ are both abelian. The claim follows by another application of (3.1.i). q.e.d.

The next two steps are derived from (1.4.1) and Lemma 9 of [6].

**Proposition 3.4.** *The following hold:*

(i) *If* $a \in U_0^*$, *then there exist unique elements* $v_i(a) \in U_i$ *for* $0 \leq i \leq 3$ *such that* $(0,\cdots,4)^{\mu(a)}=(4,\cdots,0)$ *for* $\mu(a)=av_3(a)v_0(a)v_1(a)v_2(a)$.

(ii) If $a \in U_3^*$, *then there exist unique elements* $v_i(a) \in U_i$ *for* $0 \leq i \leq 3$ *such that* $(0,\cdots,4)^{\mu(a)}=(4,\cdots,0)$ *for* $\mu(a)=av_0(a)v_3(a)v_2(a)v_1(a)$.

(iii) $\mu(a)=\mu(v_0(a))$ *for all* $a \in U_3^*$.

(iv) $v_0$ *regarded as a function from* $U_3^*$ *to* $U_0^*$ *is a bijection.*

Proof. Let $a \in U_0^*$. By (1.1.i), there exist elements $v_0(a) \in U_0$ and $v_3(a) \in U_3$ such that $3^{av_3(a)}=1$ and $1^{v_3(a)v_0(a)}=3$. Thus $(1,2,3)^{av_3(a)v_0(a)}=(3,2,1)$. By (1.1.i) again, there exist elements $v_1(a) \in U_1$ and $v_2(a) \in U_2$ such that $0^{av_3(a)v_0(a)v_1(a)}=4$ and $4^{av_3(a)v_0(a)v_2(a)}=0$. Also, $[v_1(a),v_2(a)]=1$ by (3.1.i). Thus $\mu(a)$, as defined in (i), reflects the 4-path $(0,\cdots,4)$. Suppose $\mu'(a)=av_3'(a)v_0'(a)v_1'(a)v_2'(a)$ with $v_i'(a) \in U_i$ has the same property. Since $\mu'(a)$ reflects (1,2,3), it follows that $v_i'(a)=v_i(a)$ for $i=0$ and 3 by (1.1.ii). Therefore, $\mu(a)^{-1}\mu'(a)=v_1(a)^{-1}v_1'(a) \cdot v_2(a)^{-1}v_2'(a) \in U_1 U_2 \cap G_{0,\ldots,4}$. It follows by (1.1.ii) that $v_1(a)^{-1}v_1'(a) \in U_1 \cap G_4=1$ and $v_2(a)^{-1}v_2'(a) \in G_0 \cap U_2=1$. This proves (i); (ii) follows by applying (i) to the path $(4,\cdots,0)$.

If $a \in U_3^*$, then

$$\begin{aligned}\mu(a)&=av_0(a)v_3(a)v_2(a)v_1(a)\\&=v_0(a)v_3(a)v_2(a)v_1(a)a^{\mu(a)}\end{aligned}$$

$$= v_0(a)v_3(a)ev_2(a)^e v_1(a)^e,$$

where $e = a^{\mu(a)} \in U_0$; by (3.1), $v_2(a)^e v_1(a)^e \in U_1 U_2$. By the uniqueness of $\mu(f)$ for given $f \in U_0^*$, it follows that $\mu(v_0(a)) = \mu(a)$. This proves (iii).

We have $1^{a v_0(a)} = 3$ for all $a \in U_3^*$. By (1.1.i) and (1.1.ii), the maps from $U_0^*$ and $U_3^*$ to $\Gamma_2 \backslash \{1,3\}$ sending $a_0 \in U_0$ to $3^{a_0}$ and $a_3 \in U_3$ to $1^{a_3}$ are both bijections. Assertion (iv) follows. q.e.d.

**Proposition 3.5.** *The following hold:*

(i) *If* $[a_0, a_2] = a_1$ *with* $a_i \in U_i^*$ *for* $0 \leq i \leq 2$, *then* $a_2^{\mu(a_0)} = a_1^{-1}$ *and* $[a_1, v_3(a_0)] = a_2$.

(ii) *If* $[a_1, a_3] = a_2$ *with* $a_i \in U_i^*$ *for* $1 \leq i \leq 3$, *then* $a_1^{\mu(a_3)} = a_2$ *and* $[v_0(a_3), a_2] = a_1$.

Proof. Suppose $[a_0, a_2] = a_1$ with $a_i \in U_i^*$ for $0 \leq i \leq 2$. Then $a_2^{\mu(a_0)} \in U_1$, so $a_2^{\mu(a_0)} = a_2^{\mu(a_0) v_0(a_0)^{-1}}$ by (3.1.i). Thus $a_2^{\mu(a_0)} = a_2^{a_0 v_3(a) b}$ for $b = (v_1(a) v_2(a))^{v_0(a)^{-1}}$; by (3.1), $b \in U_1 U_2$. Thus

$$\begin{aligned} a_2^{\mu(a_0)} = a_2^{a_0 v_3(a) b} &= (a_2 \cdot [a_2, a_0])^{v_3(a) b} \\ &= (a_2 \cdot [a_2, a_0] \cdot [[a_2, a_0], v_3(a)])^b \\ &= a_2 \cdot [a_2, a_0] \cdot [[a_2, a_0], v_3(a)] \\ &= a_2 a_1^{-1} [a_1^{-1}, v_3(a)] \end{aligned}$$

and hence $a_1 a_2^{\mu(a_0)} = a_2 [a_1^{-1}, v_3(a)]$ by (3.1) and (3.3). Since $a_1 a_2^{\mu(a_3)} \in U_1$, $a_2 [a_1^{-1}, v_3(a)] \in U_2$ and, by (1.1.ii), $U_1 \cap U_2 = 1$, it follows that $a_2^{\mu(a_0)} = a_1^{-1}$ and $[a_1, v_3(a)] = a_2$. This proves (i); (ii) follows by applying (i) to the path $(4, \cdots, 0)$. q.e.d.

Now choose $e_3 \in U_3^*$ and let $h = \mu(e_3)^2$. We have $h \in G_{0,\ldots,4}$.

**Proposition 3.6.** $a_1^h = a_1^{-1}$ *for every* $a_1 \in U_1$.

Proof. Choose $a_1 \in U_1^*$ and let $a_2 = [a_1, e_3]$. Then $a_2 \in U_2^*$ by (3.1.iii) and $[v_0(e_3), a_2] = a_1$ by (3.5.ii), so $a_2^{\mu(v_0(e_3))} = a_1^{-1}$ by (3.5.i). Thus $a_2^{\mu(e_3)} = a_1^{-1}$ by (3.4.iii). Also $a_1^{\mu(e_3)} = a_2$ by (3.5.ii). It follows that $a_1^h = a_1^{-1}$. q.e.d.

**Proposition 3.7.** $a_2^h = a_2^{-1}$ *for every* $a_2 \in U_2$.

Proof. Since $U_2 = U_1^{\mu(e_3)}$ and $[h, \mu(e_3)] = 1$, the claim follows from (3.6). q.e.d.

**Proposition 3.8.** $[h, U_3] = 1$.

Proof. Choose $a_1 \in U_1^*$ and $a_3 \in U_3$ and let $a_2 = [a_1, a_3]$. Then $[a_1^h, a_3^h] = a_2^h$.

By (3.1.ii), (3.6) and (3.7), we have $[a_1^{-1},a_3^h]=a_2^{-1}$, so $[a_1,a_3^h]=a_2$. Thus $a_3^{-1}a_3^h=1$ by (3.2.i). q.e.d.

**Proposition 3.9.** *$G_{0,\cdots,4}$ acts transitively on $U_2^*$ (by conjugation).*

Proof. Let $a_1\in U_1^*$ and $a_2,a_2'\in U_2^*$. By (3.2.i), there exist $a_3,a_3'\in U_3^*$ such that $[a_1,a_3]=a_2$ and $[a_1,a_3']=a_2'$. By (3.5.ii), $a_1^{\mu(a_3)}=a_2$ and $a_1^{\mu(a_3')}=a_2'$. Thus $a_2^{\mu(a_3)^{-1}\mu(a_3')}=a_2'$. The claim follows since $\mu(a_3)^{-1}\mu(a_3')\in G_{0,\cdots,4}$. q.e.d.

**Proposition 3.10.** *If* $\exp(U_2)\neq 2$, *then $h$ has a unique fixed point in $\Gamma_4\backslash\{3\}$.*

Proof. Suppose $\exp(U_2)\neq 2$ and let $x\in\Gamma_4\backslash\{3\}$. By (3.9), the map from $U_2$ to itself which sends each element to its square is therefore onto. This observation and (1.1.i) imply that there exists $d\in U_2$ such that $x^h=x^{d^2}$. By (3.7), $hd^{-1}=dh$. It follows that $h$ fixes $x^d$. Thus $h$ has at least one fixed point in $\Gamma_4\backslash\{3\}$. By (3.9) and the assumption that $\exp(U_2)\neq 2$, the group $U_2$ does not contain any involutions; thus $C_{U_2}(h)=1$ by (3.7). Since $U_2$ acts faithfully and regularly on $\Gamma_4\backslash\{3\}$ by (1.1.i) and (1.1.ii), it follows that $h$ does not have more than one fixed point in $\Gamma_4\backslash\{3\}$. q.e.d.

**Proposition 3.11.** *If* $\exp(U_2)\neq 2$, *then $G_{0,\cdots,4}$ has a unique fixed point in $\Gamma_4\backslash\{3\}$.*

Proof. If $a\in N_G(U_i)$, then $[a,h]\in C_G(U_i)$ for $i=2$ and 3 by (3.7) and (3.8). By (1.1.i), $G_{0,\cdots,4}\cap C_G(U_i)\leq G_{i-1}^{[1]}$ for $i=2$ and 3. This implies that $[G_{0,\cdots,4},h]\leq G_{1,2}^{[1]}\cap G_4$. By (1.1.ii), therefore, $h$ is central in $G_{0,\cdots,4}$. The claim follows by (3.10). q.e.d.

Let 5 be the unique fixed point of $G_{0,\cdots,4}$ in $\Gamma_4\backslash\{3\}$.

**Proposition 3.12.** *If* $\exp(U_2)\neq 2$, *then the only fixed points of $G_2^{[1]}\cap G_{2,3,4,5}$ in $\Gamma_3\cup\Gamma_4$ are* 2,3,4 *and* 5.

Proof. By (1.1.i), the group $U_3$ acts regularly $\Gamma_2\backslash\{3\}$; by (3.8), it follows that $h\in G_2^{[1]}$. By (1.1.i) and (1.1.ii), the group $U_1$ acts faithfully and regularly on $\Gamma_3\backslash\{2\}$; by (3.6), it follows that $h$ has no fixed points in $\Gamma_3\backslash\{2,4\}$. The claim follows now by (3.10). q.e.d.

**Proposition 3.13.** *Suppose that* $\exp(U_2)\neq 2$ *and that $\Gamma$ is a tree. Then the pair $(\Gamma,G)$ has property $(*)$.*

Proof. Since $(0,\cdots,4)$ is an arbitrary 4-path of $\Gamma$, it follows by (3.11) that for each 4-path $(x_0,\cdots,x_4)$ of $\Gamma$, $G_{x_0,\cdots,x_4}$ has a unique fixed point in $\Gamma_{x_4}\backslash\{x_3\}$. By

(1.1.i), for each $x\in V(\Gamma)$, the stabilizer $G_x$ acts transitively on the set of 4-paths $(x_0,\cdots,x_4)$ with $x_0=x$. Since 2 is an arbitrary vertex of $\Gamma$, it follows by (3.12) that for each 3-path $(x_0,\cdots,x_3)$ of $\Gamma$, the only fixed points of $G_{x_0}^{[1]}\cap G_{x_0,\cdots,x_3}$ in $\Gamma_{x_1}\cup\Gamma_{x_2}$ are $x_0$, $x_1$, $x_2$ and $x_3$. The claim follows, therefore, by (2.1). q.e.d.

## 4. The proof of (1.6)

Now let $\Delta$ be as in (1.6) and let $D=\mathrm{aut}(\Delta)$. Observe that $\Delta$ is bipartite, that the shortest circuit through the edge $\{L,R\}$ is of length six and that every vertex of $\Delta$ is a distance at most three from both $L$ and $R$. Thus, to prove that $\Delta$ is a generalized 3-gon, it will suffice to show that $D$ acts transitively on the edge set of $\Delta$. From the action of $U_+$ on $\Delta$ by right multiplication, we see that $D_{L,R}$ acts transitively on both $\Delta_L\backslash\{R\}$ and $\Delta_R\backslash\{L\}$. If we can show that neither $D_L$ nor $D_R$ lies in $D_{L,R}$, it will follow that $D_u$ acts transitively on $\Delta_u$ for both $u=L$ and $R$ and hence that, in fact, $D$ acts transitively on the edge set of $\Delta$.

By (3.1), we have $U_+=U_1U_2U_3$. Let $\kappa=\mu(e_3)$, where $\mu$ is as defined in (3.4) and $e_3\in U_3^*$. Since $(0,\cdots,4)^\kappa=(4,\cdots,0)$, we have $U_i^\kappa=U_{3-i}$ for $0\le i\le 3$. Let $\sigma$ be the function from $V(\Delta)$ to itself which fixes $L$, exchanges $R$ and $U_{12}$ and sends

$$\begin{aligned}
&U_1a_2 \text{ to } U_{23}a_2^\kappa,\\
&U_{23}a_1 \text{ to } U_1a_1^\kappa,\\
&U_3a_1a_2 \text{ to } U_3a_1^\kappa a_2^\kappa,\\
&U_{12}a_3 \text{ to } U_{12}v_0(a_3)^{-\kappa} \text{ and}\\
&U_1a_2a_3 \text{ to } U_1[v_0(a_3),a_2]^{-\kappa}v_0(a_3)^{-\kappa}
\end{aligned}$$

for all $a_1\in U_1$, $a_2\in U_2$ and $a_3\in U_3^*$. By (3.4.iv), $\sigma$ restricted to the set of cosets of $U_{12}$ different from $U_{12}$ itself is a permutation. For given $a_0\in U_0^*$, the map from $U_2$ to $U_1$ which sends $a_2$ to $[a_0,a_2]$ is a bijection by (3.1). It follows that $\sigma$ is a permutation of $V(\Delta)$.

We show now that $\sigma\in D$. Let $a_1\in U_1$, $a_2,a_2'\in U_2$ and $a_3\in U_3^*$. By (3.1), $U_1a_2a_3$ and $U_3a_1a_2'$ are adjacent vertices of $\Delta$ if and only if $[a_1,a_3]=a_2^{-1}a_2'$. The images of these two vertices under $\sigma$ are adjacent if and only if $[(a_2')^\kappa,v_0(a_3)^{-\kappa}]=[v_0(a_3),a_2]^\kappa a_1^\kappa$, or equivalently, $[v_0(a_3),a_2^{-1}a_2']=a_1$. By (3.5), $[v_0(a_3),a_2^{-1}a_2']=(a_2^{-1}a_2')^{-\mu(v_0(a_3))}$ and $[a_1,a_3]=a_1^{\mu(a_3)}$. By (3.4.iii) and (3.7), it follows that $U_1a_2a_3$ and $U_3a_1a_2'$ are adjacent vertices if and only if their images under $\sigma$ are. It is easy to check that the same assertion holds for any other pair of vertices of $\Delta$. Thus, $\sigma\in D$ and hence $D_L\nleq D_{L,R}$.

To show that $D_R\nleq D_{L,R}$, we argue similarly. First choose $5\in\Gamma_4\backslash\{3\}$. We observe that the function $\mu$ introduced in (3.4) depends on the 4-path $(0,\cdots,4)$; we rename it $\mu_{(0,\cdots,4)}$ to emphasize this dependency and then set $\nu=\mu_{(5,\cdots,1)}$. Thus, $\nu(a)=aw_4(a)w_1(a)w_2(a)w_3(a)$ for $a\in U_1^*$ and $\nu(a)=aw_1(a)w_4(a)w_3(a)w_2(a)$ for $a\in U_4^*$

with $w_i(a) \in U_i$ for $1 \leq i \leq 4$ and $(1,\cdots,5)^{v(a)} = (5,\cdots,1)$ for each $a \in U_1^* \cup U_4^*$; moreover, by (3.4.iii), (3.4.iv), (3.5) and (3.6), the following hold:

**Proposition 4.1.** *$v(a) = v(w_4(a))$ for all $a \in U_1^*$.*

**Proposition 4.2.** *$w_4$ regarded as a function from $U_1^*$ to $U_4^*$ is a bijection.*

**Proposition 4.3.i.** *$[a_1,a_3] = a_3^{-v(a_1)}$ for $a_1 \in U_1^*$ and $a_3 \in U_3$.*

**Proposition 4.3.ii.** *$[a_2,a_4] = a_2^{v(a_4)}$ for $a_2 \in U_2$ and $a_4 \in U_4^*$.*

**Proposition 4.4.** *$a_3^{v(a)^2} = a_3^{-1}$ for $a_3 \in U_3$ and $a \in U_1^*$.*

Now choose $e_1 \in U_1^*$ and let $\lambda = v(e_1)$. Then $U_i^\lambda = U_{5-i}$ for $1 \leq i \leq 4$. We define $\tau$ to be the function from $V(\Delta)$ to itself which fixes $R$, exchanges $L$ and $U_{23}$ and sends

$$\begin{aligned}
&U_3a_2 \text{ to } U_{12}a_2^\lambda,\\
&U_{12}a_3 \text{ to } U_3a_3^\lambda,\\
&U_1a_2a_3 \text{ to } U_1a_2^\lambda a_3^\lambda,\\
&U_{23}a_1 \text{ to } U_{23}w_4(a_1)^{-\lambda} \text{ and}\\
&U_3a_1a_2 \text{ to } U_3w_4(a_1)^{-\lambda}[w_4(a_1),a_2]^{-\lambda}
\end{aligned}$$

for all $a_1 \in U_1^*$, $a_2 \in U_2$ and $a_3 \in U_3$. By (4.2), $\tau$ restricted to the set of cosets of $U_{23}$ different from $U_{23}$ itself is a permutation. For given $a_4 \in U_4^*$, the map from $U_2$ to $U_3$ which sends $a_2$ to $[a_2,a_4]$ is a bijection by (3.1). It follows that $\tau$ is a permutation of $V(\Delta)$. Let $a_1 \in U_1^*$, $a_2, a_2' \in U_2$ and $a_3 \in U_3$. Then $U_1a_2a_3$ and $U_3a_1a_2'$ are adjacent vertices of $\Delta$ if and only if $[a_1,a_3] = a_2^{-1}a_2'$; by (4.3.i), this holds if and only if $a_3^{-v(a_1)} = a_2^{-1}a_2'$. The images of these two vertices under $\tau$ are adjacent if and only if $[w_4(a_1), a_2^{-1}a_2'] = a_3^{-1}$. By (4.3.ii), this holds if and only if $a_3 = (a_2^{-1}a_2')^{v(w_4(a_1))}$. By (4.1) and (4.4), it follows that $U_1a_2a_3$ and $U_3a_1a_2'$ are adjacent vertices if and only if their images under $\tau$ are. It is easy to check that the same assertion holds for any other pair of vertices of $\Delta$. Thus $\tau \in D$ and hence $D_R \not\leq D_{L,R}$.

We conclude that $D$ acts transitively on the edge set of $\Delta$. It follows that $\Delta$ is a generalized 3-gon and, from the action of $U_2$ on $\Delta$ by right multiplication, that $\Delta$ is Moufang. This completes the proof of (1.6).

## 5. The proof of (1.5) : Conclusion

We continue to assume that $\Gamma$ is a $(G,3)$-Moufang graph. Let $\Delta$ be as in (1.6) and let $(0,\cdots,5)$ and $U_i$ for $0 \leq i \leq 4$ be as in the previous sections. By (1.6) and the classification of Moufang 3-gons (see [3]), there exists an alternative division ring $F$ such that each $U_i$ is isomorphic to the additive group of $F$; moreover, we can choose isomorphisms from $F$ to $U_i$ for $1 \leq i \leq 3$ so that;

**Proposition 5.1.** $[x_1(s), x_3(t)] = x_2(st)$ *for all* $s, t \in F$, *where* $x_i(v)$ *denotes the image of* $v \in F$ *in* $U_i$ *for* $1 \leq i \leq 3$.

Recall that the elements $e_1 \in U_1^*$ and $e_3 \in U_3^*$ used to define $\kappa = \mu(e_3)$ and $\lambda = \nu(e_1)$ in the previous section were chosen arbitrarily; thus, we can assume now that $e_1 = x_1(1)$ and $e_3 = x_3(1)$. (We do this just to avoid introducing new letters.) From $[x_1(s), x_3(1)] = x_2(s)$ for $s \in F$, we obtain $x_1(s)^\kappa = x_2(s)$ by (3.5.ii). We now label the elements of $U_0$ by setting $x_0(t) = x_3(t)^\kappa$ for each $t \in F$. Conjugating equation (5.1) by $\kappa$, we find that $[x_2(s), x_0(t)] = x_2(st)^\kappa = x_1(st)^{\kappa^2}$. By (3.6), $x_1(st)^{\kappa^2} = x_1(-st)$. Hence;

**Proposition 5.2.** $[x_0(s), x_2(t)] = x_1(ts)$ *for all* $s, t \in F$.

From $[x_1(1), x_3(t)] = x_2(t)$ for $t \in F$, we obtain $x_3(t)^\lambda = x_2(-t)$ by (4.3.i). We now label the elements of $U_4$ by setting $x_4(s) = x_1(s)^\lambda$ for each $s \in F$. Conjugating equation (5.1) by $\lambda$, we find that $[x_4(s), x_2(-t)] = x_2(st)^\lambda = x_3(-st)^{\lambda^2}$. By (4.4), $x_3(-st)^{\lambda^2} = x_3(st)$. Hence;

**Proposition 5.3.** $[x_2(s), x_4(t)] = x_3(ts)$ *for all* $s, t \in F$.

**Proposition 5.4.** $U_{i-1}U_i = \bigcup_{u \in \Gamma_i} G_{i,u}^{[1]}$ *for* $i = 2$ *and* 3.

Proof. Let $i = 2$. For $s, t \in F$ with $s \neq 0$, we have $x_1(s)x_2(t) = x_1(s)^{x_3(s^{-1}t)} \in U_1^{x_3(s^{-1}t)}$ by (5.1) and the fact that $s(s^{-1}t) = t$ in an alternative division ring. Thus $x_1(s)x_2(t) \in G_{2,u}^{[1]}$ for $u = 1^{x_3(s^{-1}t)}$. If $u \in \Gamma_2 \backslash \{3\}$, then there exists $a \in U_3$ such that $G_{2,u}^{[1]} = U_1^a$ by (1.1.i) and $U_1^a \leq U_1 U_2$ by (3.1.i). The claim follows. The case $i = 3$ follows by a similar argument. q.e.d.

By (5.4), we have $U_1 U_2 \trianglelefteq G_2$ and $U_2 U_3 \trianglelefteq G_3$.

**Proposition 5.5.** $C_{G_2}(U_1 U_2) = U_1 U_2$ *and* $C_{G_3}(U_2 U_3) = U_2 U_3$.

Proof. Let $d \in C_{G_2}(U_1 U_2)$. Then $d \in N_{G_2}(U_i) = G_{i,i+1}$ for $i = 1$ and 2. By (1.1.i), there then exists an element $e \in U_1 U_2$ such that $de \in G_{0,\dots,4}$. By (3.3), $de \in C_{G_2}(U_1 U_2)$. Choose $a_1 \in U_1^*$ and $a_3 \in U_3$ arbitrarily and let $a_2 = [a_1, a_3]$. By (3.1.ii), $a_2 \in U_2$. Conjugating by $de$, we find that $a_2 = [a_1, a_3^{de}]$. The element $a_3^{de}$ lies in $U_3$ since $de \in G_{34}$; thus $a_3^{de} = a_3$ by (3.2.i). Hence $[de, U_3] = 1$. By (1.1.i), it follows that $de \in G_2^{[1]}$; similarly, $de \in G_1^{[1]}$ since $[de, U_2] = 1$. By (1.1.ii), we conclude that $de \in G_{12}^{[1]} \cap G_4 = 1$ and therefore $d \in U_1 U_2$. Thus $C_{G_2}(U_1 U_2) = U_1 U_2$; $C_{G_3}(U_2 U_3) = U_2 U_3$ follows by a similar argument. q.e.d.

Let $M_2=\langle U_0, U_3\rangle$ and $M_3=\langle U_1, U_4\rangle$. For $i=2$ and 3, let $X_i$ be the $M_i$-orbit containing the vertex $i-2$.

**Proposition 5.6.** *Suppose* $\exp(U_2)=2$. *Then for* $i=2$ *and* 3, *the vertex* $i+2$ *lies in* $X_i$ *and* $|X_i\cap\Gamma_u|=1$ *for each* $u\in\Gamma_i$.

Proof. Let $i=2$. By (1.1.ii), both $0^{U_3}$ and $4^{U_0}$ contain a unique element in $\Gamma_x$ for each $x\in\Gamma_2\backslash\{1,3\}$. It will thus suffice to show that $\{0\}\cup 4^{U_0}=\{4\}\cup 0^{U_3}$. Choose $u\in\Gamma_2\backslash\{1,3\}$. By (1.1.i) and (1.1.ii), there exist unique elements $a\in U_0$ and $b\in U_3$ such that $u^a=3$ and $u^b=1$. Choose $z\in\Gamma_u\backslash\{2\}$. By (1.1.i), there exists $d\in G_{u,z}^{[1]}$ mapping 1 to 3 and then $e\in G_{2,u}^{[1]}$ mapping $0^d$ to 4. Thus $(2,1,0)^c=(2,3,4)$ for $c=de$. Since $(2,u,z)$ and $(1,2,3)$ are in the same $G$-orbit, we can apply (5.4) to conclude that there is a vertex $v\in\Gamma_u\backslash\{2\}$ such that $c\in G_{u,v}^{[1]}$. Since $U_2$ and $G_{uv}^{[1]}$ are conjugate in $G, c^2=1$. Thus $(0,\cdots,,4)^c=(4,\cdots,0)$; from this, $a^c b\in G_1\cap U_3=1$ and hence $a^c=b$ follows. By a similar argument, there exists $w\in\Gamma_3\backslash\{2\}$ and $f\in G_{3,w}^{[1]}$ such that $(0,1,2,u,v)^f=(v,u,2,1,0)$. Then $a^f=c$, so $b=a^c=a^{faf}=f(af)^3$ and hence $(af)^3=fb$. Since $a$ and $f$ are involutions, we have $(af)^a=(af)^{-1}$ and therefore $(fb)^a=(fb)^{-1}$. By (3.3), $[bf, U_3]=1$. Thus, $\langle fb\rangle$ is normalized by $\langle a, U_3\rangle$. Since $bf\in G_3^{[1]}$ and $\langle a, U_3\rangle$ acts transitively on $\Gamma_2$, it follows that $bf\in G_x^{[1]}$ for all $x\in\Gamma_2$. By (1.1.ii), it follows that $b=f$. Thus $0^b=0^f=v$ and therefore $4^a=4^{cbc}=0^{bc}=v$. We conclude that $\{0\}\cup 4^{U_0}=\{4\}\cup 0^{U_3}$ as claimed. The case $i=3$ follows by a similar argument. q.e.d.

**Proposition 5.7.** *Suppose* $\exp(U_2)=2$. *Then* $\langle M_i, G_{i-2,\dots,i+2}\rangle\cap U_{i-1}U_i=1$ *for* $i=2$ *and* 3.

Proof. Let $i=2$. Since $G_{0,\dots,4}$ normalizes $M_2$ and fixes 0, the group $\langle M_2, G_{0,\dots,4}\rangle$ stabilizes $X_2$. By (5.6), $U_1U_2\cap G_{X_2}\le G_{0,\dots,4}$. By (1.1.ii), $U_1U_2\cap G_{0,\dots,4}=1$. The case $i=3$ follows by a similar argument. q.e.d.

We are now in a position to conclude the proof of (1.5). By (3.13), we can assume that $\exp(U_2)=2$. Let $H_2=\langle U_0, U_1, U_2, U_3\rangle$, $H_3=\langle U_1, U_2, U_3, U_4\rangle$, $K_2=\langle H_2, H_3\cap G_2\rangle$ and $K_3=\langle H_3, H_2\cap G_3\rangle$. Since $\Gamma$ is connected, $\langle K_2, K_3\rangle$ acts transitively on the edge set of $\Gamma$ and hence $G^\circ=\langle K_2, K_3\rangle$. The action of $M_i$ on $U_{i-1}U_i$ is determined by (5.1), (5.2) and (5.3) for $i=2$ and 3. By (5.5) and (5.7), this determines $H_i$ as a split extension of $M_i$ by $U_{i-1}U_i$ for $i=2$ and 3.

In particular, the action of $H_3\cap G_2$ on $U_1U_2$ is determined. For each $a\in H_3\cap G_2$, there exists by (1.1.i) an element $d\in U_1U_2U_3$ such that $ad\in H_3\cap G_{0,\dots,4}$. Thus $\langle M_2, H_3\cap G_2\rangle$ and $\langle M_2, H_3\cap G_{0,\dots,4}\rangle$ have the same action on $U_1U_2$. By (5.5) and (5.7), this determines the structure of $K_2$. By a similar argument for $K_3$, it follows that the structure of the amalgam $A_\Gamma=(K_2, K_3; K_2\cap K_3)$ is uniquely determined. In particular, there is an isomorphism from $A_\Gamma$ to an amalgam

$A_\Delta=(M_L,M_R;M_L\cap M_R)$ sitting inside of $(D_L,D_R;D_{L,R})$. If we now assume (for the first time!) that $\Gamma$ is a tree, then by [4,(I.4.1)], this isomorphism extends to an isomorphism $\phi$ from $G$ to the free amalgamated product $\tilde{M}=M_L*_{M_L\cap M_R}M_R$ and there is an isomorphism $\psi$ from $\Gamma$ to the coset graph $\Omega$ associated with $A_\Delta$ compatible with $\phi$ and the action of $\tilde{M}$ on $\Omega$ by right multiplication. ($\Omega$ is the graph with vertex set the union of the set of right cosets of $M_L$ and of $M_R$ in $\tilde{M}$, where two of these cosets are adjacent in $\Omega$ whenever their intersection contains a right coset of $M_L\cap M_R$.) The natural map from $\tilde{M}$ onto $D^\circ$ induces a map $\pi$ from $\Omega$ onto $\Delta$ which sends $\Omega_x$ bijectively to $\Delta_u$ for $u=x^\pi$ and for all $x\in V(\Omega)$. Let $\mathscr{A}$ be the family of apartments of $\Gamma$ which are mapped by $\psi\pi$ to 6-circuits of $\Delta$. Then $\mathscr{A}$ is $G$-invariant and fulfills conditions (1.2.i) and (1.2.ii); the graph $\bar{\Gamma}$ described in (1.2) is (up to isomorphism) precisely $\Delta$. The proof of (1.5) is now complete.

It should be clear now that $\bar{\Gamma}\cong\Delta$ also when $\exp(U_2)\neq 2$. Here is a circuitous way to see this. By (1.2.i) and (1.2.ii), the element $\mu(e_3)$ as defined in (3.4) lies in $M_2=\langle U_0,U_3\rangle$. By (3.8), $h\in Z(M_2)$. By (3.6), 0 and 2 are the only fixed points of $h$ in $\Gamma_1$. This implies that the conclusions of (5.6) hold. Thus $\bar{\Gamma}\cong\Delta$ holds exactly as in the case that $\exp(U_2)=2$.

## 6. The proof of (1.2)

Let $\Gamma$, $\mathscr{A}$ and $n$ fulfill the hypotheses of (1.2). Let $\pi$ denote the natural map from $V(\Gamma)$ to $V(\bar{\Gamma})$.

**Proposition 6.1.** *Suppose $u\approx v$. Let $u_0,u_1,\cdots,u_m$ be a sequence of vertices of $\Gamma$ of minimal length $m$ such that $u_0=u$, $u_m=v$ and $u_i\sim u_{i-1}$ for $1\leq i\leq m$. Let $d_i=\mathrm{dist}_\Gamma(u_0,u_i)$ for $1\leq i\leq m$. Then $d_1<d_2<\cdots<d_m$.*

Proof. Suppose the conclusion is false. Then we can choose $e\geq 1$ minimal such that $d_{e+1}\leq d_e$. Let $(0,\cdots,2n)$ be the $2n$-path from $0=u_e$ to $2n=u_{e-1}$ and let $(0',\cdots,(2n)')$ be the $2n$-path from $0'=u_e$ to $(2n)'=u_{e+1}$. Then $i=i'$ for $0\leq i\leq n$ since $d_{e-1}<d_e$ and $d_{e+1}\leq d_e$. If $n+1=(n+1)'$ as well, then $u_{e-1}=u_{e+1}$ by (1.2.i). If $n+1\neq(n+1)'$, then $u_{e-1}\sim u_{e+1}$ by (1.2.ii). Both conclusions contradict the minimality of $m$. q.e.d.

**Proposition 6.2.** *Let $u$ and $v$ be distinct and let $m=\mathrm{dist}_\Gamma(u,v)$. If $u\approx v$, then $m\geq 2n$ and $m$ is even.*

Proof. If $u\sim v$, then $m=2n$. The first claim follows by (6.1) and the second by induction. q.e.d.

**Proposition 6.3.** *$\pi$ induces a bijection from $\Gamma_u$ to $\bar{\Gamma}_{\bar{u}}$ for each $u \in V(\Gamma)$.*

Proof. By (6.2), no two neighbors of a given vertex of $\Gamma$ are equivalent. It thus suffices to show that if $u \sim v$, then to each neighbor of $u$ there exists an equivalent neighbor of $v$. Let $u \sim v$ and choose $w \in \Gamma_u$. Let $\Delta$ be the element of $\mathscr{A}$ which contains $u$ and $v$ and let $(-1,0,\cdots,2n,2n+1)$ be the $(2n+2)$-path on $\Delta$ with $0=u$ to $2n=v$. Then $-1 \sim 2n-1$ and $1 \sim 2n+1$, so we can assume that $w \notin V(\Delta)$. By (1.2.i), there exists a $2n$-path $(0',\cdots,(2n)')$ lying on an element of $\mathscr{A}$ such that $0'=w$, $(n+1)'=n$ and $(n+2)' \neq n+1$. Again by (1.2.i), there exists a $2n$-path $(0'',\cdots,(2n)'')$ lying on an element of $\mathscr{A}$ such that $1''=2n$ and $(n+2)''=(n+2)'$. By (1.2.ii), $(2n)'=(2n)''$. Let $z=0''$. Then $z \in \Gamma_v$ and $w \sim (2n)'=(2n)'' \sim z$, so $w \approx z$. q.e.d.

**Proposition 6.4.** *The girth of $\bar{\Gamma}$ is $2n$ and $\bar{\Gamma}$ is bipartite.*

Proof. The image in $\bar{\Gamma}$ of a $2n$-path of $\Gamma$ which lies on an element of $\mathscr{A}$ is a circuit of $\bar{\Gamma}$, so the girth of $\bar{\Gamma}$ is less than or equal to $2n$. Let $(x_0,x_1,\cdots,x_m)$ be an $m$-path of $\bar{\Gamma}$ such that $x_0=x_m$ and $m>0$. By (6.3), there exists an $m$-path $(0,\cdots,m)$ of $\Gamma$ such that $\bar{e}=x_e$ for $0 \leq e \leq m$. Then $0 \approx m$ since $x_0=x_m$. Thus $m \geq 2n$ and $m$ is even by (6.2). q.e.d.

**Proposition 6.5.** *The diameter of $\bar{\Gamma}$ is $n$.*

Proof. Let $p$ and $q$ be two vertices of $\bar{\Gamma}$. Choose $u$ and $v \in V(\Gamma)$ such that $\bar{u}=p$, $\bar{v}=q$ and $m=\mathrm{dist}_\Gamma(u,v)$ is minimal. Let $(0,\cdots,m)$ be the $m$-path in $\Gamma$ with $0=u$ and $m=v$. If $m>n$, then by (1.2.i), there exists a $2n$-path $(0',\cdots,(2n)')$ lying on an element of $\mathscr{A}$ such that $0'=m$ and $(n+1)'=m-n-1$. Then $\mathrm{dist}_\Gamma(u,(2n)') < \mathrm{dist}_\Gamma(u,v)$ and $v=0' \sim (2n)'$. This contradicts the choice of $u$ and $v$. q.e.d.

With (6.3), (6.4) and (6.5), the proof of (1.2) is complete.

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