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ON COMPLEX PROJECTIVE BUNDLES OVER A KÄHLER C-SPACE

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Introduction

Let M be a compact Einstein Kähler manifold. Then the first Chern class $c_1(M)$ of M is positive, negative or zero. We can ask whether the converse is true or not, that is, does a compact Kähler manifold M with the first Chern class $c_1(M)>0$ (resp. $c_1(M)<0$, $c_1(M)=0$) admit an Einstein Kähler metric? In the case when $c_1(M)<0$, T. Aubin [2] has proved that a compact Kähler manifold M with $c_1(M)<0$ admits a unique Einstein Kähler metric. As is well-known, in the case when $c_1(M)=0$, our question is yes if the Calabi conjecture is true. The purpose of this note is to give some examples of a compact Kähler manifold with $c_1(M)>0$ which does not admit any Einstein Kähler metric. Let X be a compact connected complex manifold. By a theorem of Bochner-Montogomery, the group $\mathrm{Aut}(X)$ of all holomorphic transformations of X is a complex Lie group and the map $\mathrm{Aut}(X)\times X\to X$ defined by $(f, x)\mapsto f(x)$ is holomorphic. For a holomorphic vector bundle E over a compact complex manifold M let $P(E)$ denote the associated complex projective bundle. Let $\mathrm{Aut}_0(X)$ denote the identity component of $\mathrm{Aut}(X)$. By a theorem of Blanchard, we can define a homomorphism $\Pi\colon \mathrm{Aut}_0(P(E))\to\mathrm{Aut}_0(M)$. In section 1 we shall show that the Lie algebra of the $\mathrm{Ker}\,\Pi$ is isomorphic with the Lie algebra $H^0(M, \mathrm{End}(E))/\boldsymbol{C}\cdot 1$ where $H^0(M, \mathrm{End}(E))$ denotes all holomorphic sections of the vector bundle $\mathrm{End}(E)$ over M and 1 denotes the element of $H^0(M, \mathrm{End}(E))$ defined by the identity map of $\mathrm{End}(E)_x (x\in M)$. In section 2 we consider Kähler C-spaces with the second Betti number $b_2=1$ as M. In this case we know that the group of all holomorphic line bundles $H^1(M, \boldsymbol{C}^*)$ over M is generated by a homogeneous line bundle. From now on we shall exclusively consider holomorphic vector bundles E generated by holomorphic line bundles. Then the homomorphism $\Pi\colon \mathrm{Aut}_0(P(E))\to\mathrm{Aut}_0(M)$ is surjective and we can determine the structure of the Lie algebra of the $\mathrm{Ker}\,\Pi$. In particular, we can compute the dimension of $\mathrm{Aut}_0(P(E))$ in these cases. In section 3 we shall compute the Chern class of $P(E)$. The result in section 2 has been obtained by Brieskorn [6], Röhrl [13]

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for the case of the complex projective space $P^1(\boldsymbol{C})$ of dimension 1 and by Ise [9] for the case of the complex projective space $P^n(\boldsymbol{C})$. The result in section 3 has been obtained by Brieskorn [6] for the case of the complex projective space $P^1(\boldsymbol{C})$. In section 4 we shall show that some of complex projective bundles over M are examples of a compact Kähler manifold with $c_1(M)>0$ which does not admit any Einstein Kähler metric. We remark that nothing is mentioned on Einstein Kähler metric in [6] [9] [13].

1. The automorphism group of a complex projective bundle

Let M be a compact connected complex manifold and E a holomorphic vector bundle over M. Let $P(E)$ denote the complex projective bundle over M induced by E. Since $P(E)$ is a compact complex manifold, it is known that the group $\mathrm{Aut}(P(E))$ of all holomorphic automorphisms of $P(E)$ is a complex Lie group and the map $\mathrm{Aut}(P(E))\times P(E)\to P(E)$ defined by $(f, x)\mapsto f(x)$ is holomorphic. Let $F(P(E))$ denote the subgroup of all fiber preserving automorphisms of $P(E)$.

Proposition 1.1 (Blanchard [3]). *Let* $\mathrm{Aut}_0(P(E))$ (*resp.* $F_0(P(E))$) *denote the identity component of* $\mathrm{Aut}(P(E))$ (*resp.* $F(P(E))$). *Then* $\mathrm{Aut}_0(P(E))=F_0(P(E))$.

Note that an element of $F_0(P(E))$ is a fiber preserving automorphism in the sense of Steenrod [14].

Let $P(M\ G, \pi)$ denote a principal holomorphic fiber bundle over M with the structure group G. Let $F(P(M, G, \pi))$ be the group of all fiber preserving holomorphic automorphisms of the principal bundle $P(M, G, \pi)$, that is, a biholomorphic map $\tilde{f}$ of $P(M, G, \pi)$ is an element of $F(P(M, G, \pi))$ if and only if $\tilde{f}(x\cdot g)=\tilde{f}(x)\cdot g$ for all $x\in M$ and $g\in G$.

Theorem 1.2 (Morimoto [11]). *The group $F(P(M, G, \pi))$ equipped with the compact open topology can be given the structure of a complex Lie group which acts holomorphically on $P(M, G, \pi)$. Its Lie algebra is isomorphic to the Lie algebra of all holomorphic vector fields X over $P(M, G, \pi)$ for which $R_g'X=X$ for every $g\in G$, where R_g' denotes the differential mapping induced by the action R_g of an element g of G.*

Let $\tilde{P}$ (resp. P) denote the principal bundle associated to a complex projective bundle $P(E)$ (resp. a holomorphic vector bundle E) over M. Then $F(P)$ and $F(P(E))$ are naturally isomorphic. In fact, $P(E)$ is the quotient of $\tilde{P}\times P^m(\boldsymbol{C})$ by the equivalence relation $(y, \xi)\sim(ya, a^{-1}\xi)$ $(y\in P, \xi\in P^m(\boldsymbol{C}), a\in PGL(m+1, \boldsymbol{C}))$. Let ρ be the projection of $\tilde{P}\times P^m(\boldsymbol{C})$ onto $P(E)$. For an element $f\in F(\tilde{P})$, we can define a mapping $f'\colon P(E)\to P(E)$ by $f'(\rho(y, \xi))=\rho(f(y), \xi)$ $(y\in\tilde{P}, \xi\in P^m(\boldsymbol{C}))$. Then $f'\in F(P(E))$ and f, f' induce the same automorphism $\bar{f}$ of M. Moreover the mapping $\theta\colon F(\tilde{P})\to F(P(E))$ defined by $\theta(f)=f'$ is an isomorphism of the group $F(\tilde{P})$ into the group $F(P(E))$. Conversely, let f' be an element of $F(P(E))$. For every element $y\in\tilde{P}$, there is an element $w\in\tilde{P}$ such that $f'(\rho(y,\xi))=\rho(w,\xi)$ for all $\xi\in P_m(\boldsymbol{C})$. Put $f(y)=w$. Then $f\in F(\tilde{P})$ and $\theta(f)=f'$.

Let $PGL(m+1,\boldsymbol{C})$ denote the projective transformation group corresponding to $GL(m+1,\boldsymbol{C})$. Then we have an exact sequence

$$(1)\quad 0\to\boldsymbol{C}^*\to GL(m+1,\boldsymbol{C})\to PGL(m+1,\boldsymbol{C})\to 0\,.$$

Since P (resp. $\tilde{P}$) is the principal bundle associated to the vector bundle E(resp. $P(E)$), we have an exact sequence of complex Lie groups

$$(2)\quad 0\to\boldsymbol{C}^*\to F_0(P)\to F_0(\tilde{P})\,.$$

Since each element $g\in F(P)$ induces an element $\bar{g}$ of $\mathrm{Aut}(M)$, there is a canonical homomorphism Π_P: $F_0(P)\to\mathrm{Aut}_0(M)$ for each principal fiber bundle P over M.

Proposition 1.3. *If M is simply connected, we have an exact sequence*

$$0\to\boldsymbol{C}^*\to\mathrm{Ker}\,\Pi_P\to\mathrm{Ker}\,\Pi_{\tilde{P}}\to 0\,.$$

Proof. Take a simple open covering $\{U_\alpha\}_\alpha$ of M such that, for each α, $\pi^{-1}{}_P(U_\alpha)\simeq U_\alpha\times GL(m+1,\boldsymbol{C})$ and $\pi^{-1}{}_{\tilde{P}}(U_\alpha)\simeq U_\alpha\times PGL(m+1,\boldsymbol{C})$. Moreover let $(g_{\alpha\beta})$ be the system of transition functions of the principal bundle P associated to the open covering $\{U_\alpha\}_\alpha$. Then $(g_{\alpha\beta})$ induces the system of transition functions $(\tilde{g}_{\alpha\beta})$ of the principal bundle $\tilde{P}$. Let $\tilde{\varphi}$ be an element of $\mathrm{Ker}\,\Pi_{\tilde{P}}$. Then there is a system of functions $\{\tilde{\varphi}_\alpha\}$ such that $\tilde{\varphi}_\alpha$: $U_\alpha\to PGL(m+1,\boldsymbol{C})$ and $\tilde{g}_{\alpha\beta}\cdot\tilde{\varphi}_\beta=\tilde{\varphi}_\alpha\cdot\tilde{g}_{\alpha\beta}$ on $U_\alpha\cap U_\beta$. Since U_α is simply connected, there is a holomorphic map φ_α: $U_\alpha\to SL(m+1,\boldsymbol{C})$ such that $\tilde{\varphi}_\alpha=p\cdot\tilde{\varphi}_\alpha$ where p: $SL(m+1,\boldsymbol{C})\to PGL(m+1,\boldsymbol{C})$ is the canonical map. Then

$$g_{\alpha\beta}\cdot\varphi_\alpha=c_{\alpha\beta}\varphi_\alpha\cdot g_{\alpha\beta}\quad\text{on}\quad U_\alpha\cap U_\beta\,\cdot$$

and $c_{\alpha\beta}$: $U_\alpha\cap U_\beta\to\boldsymbol{C}^*$ is holomorphic. By taking the determinant, we get $c_{\alpha\beta}^{n+1}=1$ on $U_\alpha\cap U_\beta$. Since $U_\alpha\cap U_\beta$ is connected, $c_{\alpha\beta}$ is constant on $U_\alpha\cap U_\beta$ and $c_{\alpha\beta}\in\boldsymbol{Z}/(m+1)\boldsymbol{Z}$. Moreover note that $c_{\alpha\beta}c_{\beta\gamma}c_{\gamma\alpha}=1$ on $U_\alpha\cap U_\beta\cap U_\gamma$.

Lemma (Principle of monodromy). *Let M be a simply connected manifold and $\mathfrak{U}=\{U_\alpha\}$ be a simple open covering. Then $H^1(\mathfrak{U},\boldsymbol{Z}/(m+1)\boldsymbol{Z})=(0)$.*

Proof. See Weil [17].

Applying the lemma in our case, we get a system of constant functions $\{a_\alpha\}$ such that $c_{\alpha\beta}=a_\alpha\cdot a_\beta^{-1}$, a_α: $U_\alpha\to\boldsymbol{Z}/(m+1)\boldsymbol{Z}$. Hence, we have $g_{\alpha\beta}a_\beta\varphi_\beta=a_\alpha\varphi_\alpha g_{\alpha\beta}$ on $U_\alpha\cap U_\beta$ and we completes our proof. q.e.d.

Corollary. *If M is simply connected and Π_P: $F_0(P)\to\mathrm{Aut}_0(M)$ is onto, then the following sequences is exact.*

(3) $0 \to \boldsymbol{C}^* \to F_0(P) \to F_0(\tilde{P}) \to 0$.

Proof. Obvious from the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \boldsymbol{C}^* \arrow[r] \arrow[d, equal] & \operatorname{Ker}\Pi_P \arrow[rr] \arrow[d, hook] & & \operatorname{Ker}\Pi_{\tilde{P}} \arrow[r] \arrow[d, hook] & 0 \quad \text{(exact)} \\
0 \arrow[r] & \boldsymbol{C}^* \arrow[r] & F_0(P) \arrow[rr] \arrow[dr, "\Pi_P"'] & \arrow[d, phantom, "\cap"] & F_0(\tilde{P}) \arrow[dl, "\Pi_{\tilde{P}}"] & \\
 & & & \operatorname{Aut}_0(M) \arrow[d] & & \\
 & & & 0 & &
\end{tikzcd}$$

Now we recall the exact sequence of holomorphic vector bundle over M associated to the holomorphic principal fiber bundle P on M with the structure group G, due to Atiyah [1]. Let $T(P)$ be the holomorphic tangent bundle of P. Since G operates on P, it also operates on $T(P)$. We put $Q=T(P)/G$, so that a point of Q is a field of tangent vectors to P, defined along one of its fibers, and invariant under G. Then we can show that Q has a natural vector bundle structure over M. Let $L(P)$ denote the vector bundle associated to P by the adjoint representation of G. Note that $L(P)$ is a bundle of Lie algebra, each fiber $L(P)_x=L(P)_x$ being a Lie algebra isomorphic with $L(G)$. Under these notations, there exists an exact sequence of holomorphic vector bundles over M:

(4) $0 \to L(P) \to Q \to T(M) \to 0$

where $T(M)$ is the holomorphic tangent bundle over M.

Then we have the exact sequence of cohomology

(5) $0 \to H^0(M, L(P)) \to H^0(M, Q) \to H^0(M, T(M)) \to H^1(M, L(P)) \to \cdots$

Now we can identify the Lie algebra of $F_0(P)$ (resp. $\operatorname{Ker}\Pi_P$, $\operatorname{Aut}_0(M)$) with $H^0(M, Q)$ (resp. $H^0(M, L(P))$, $H^0(M, T(M))$) (cf. Morimoto [11]). Note that the structure of the Lie algebra $H^0(M, L(P))$ is given by the following way. For $X, Y \in H^0(M, L(P))$, $X_x, Y_x \in L(P)_x$ $(x \in M)$. Since $L(P)_x$ has the Lie algebra structure, we have $[X_x, Y_x] \in L(P)_x$. On the other hand, $[X, Y] \in H^0(M, L(P))$ as holomorphic vector fields. Then it is easy to see that $[X, Y]_x=[X_x, Y_x]$ for every $x \in M$. That is, the Lie algebra structure of $H^0(M, L(P))$ as the subalgebra of $H^0(M, Q)$ coincides with the one induced by the Lie algebra $L(G)$ of G.

In the case of vector bundles, we have the following proposition due to Atiyah.

Proposition 1.4. *Let E be a holomorphic vector bundle over M and P the* *associated principal bundle. Then* $L(P) \cong \mathrm{End}(E)$.

Proof. See Atiyah [1] Proposition 9.

Note that $H^0(M, \mathrm{End}(E))$ contains $\boldsymbol{C}$ in the center and the Lie algebra of $\mathrm{Ker}\,\Pi(\Pi: F_0(P(E)) \to \mathrm{Aut}_0(M))$ is isomorphic with $H^0(M, \mathrm{End}(E))/\boldsymbol{C}$. We now summarize our result as follows:

Theorem 1.5. *Let M be a simply connected compact complex manifold, E a holomorphic vector bundle over M and $P(E)$ the projective bundle induced by E. If* $\Pi: \mathrm{Aut}_0(P(E)) \to \mathrm{Aut}_0(M)$ *is surjective,*

$$\dim_C \mathrm{Aut}_0(P(E)) = \dim_C \mathrm{Aut}_0(M) + \dim_C H^0(M, \mathrm{End}(E)) - 1 .$$

Moreover the Lie algebra of $\mathrm{Ker}\,\Pi$ *is isomorphic with* $H^0(M, \mathrm{End}(E))/\boldsymbol{C}$.

REMARK 1. Let f, g be elements of $H^0(M, \mathrm{End}(E))$. Then the Lie algebra structure of $H^0(M, \mathrm{End}(E))$ is given by

$$[f, g](x) = [f(x), g(x)] = f(x) \circ g(x) - g(x) \circ f(x)$$

$(f(x), g(x) \in \mathrm{End}(E_x))$ for every $x \in M$.

2. Complex projective bundles over a Kähler C-space

We shall recall the following facts on Kähler C-spaces and holomorphic line bundles over these manifolds. A simply connected compact Kähler homogeneous manifold is called a Kähler C-space. Kähler C-spaces have been classified by H. C. Wang [16]. From now on we assume that the second Betti number $b_2(M)$ of a Kähler C-space M is 1. Note that such a class contains the class of irreducible hermitian symmetric spaces. We shall use the following known results on holomorphic line bundles over Kähler C-spaces with $b_2 = 1$ (cf. [4] [8]).

2.1. *The group of all holomorphic line bundles $H^1(M, \boldsymbol{C}^*)$ over a Kähler C-space M is isomorphic to $\boldsymbol{Z}$.*

2.2. *There is a homogeneous holomorphic line bundle L over M such that L is very ample. Moreover L is a generator of $H^1(M, \boldsymbol{C}^*)$. In particular, every holomorphic line bundle is homogeneous.*

2.3. *Let $f: M \to P^N(\boldsymbol{C})$ be the associated imbedding for L and H the holomorphic line bundle over $P^N(\boldsymbol{C})$ corresponding to a hyperplane of $P^N(\boldsymbol{C})$. Then L is the induced bundle f^*H over M and the homomorphism*

$$\gamma_k : H^0(P^N(\boldsymbol{C}), H^k) \to H^0(M, L^k) \qquad (k \geqq 0)$$

induced by the imbedding $f: M \to P^N(\boldsymbol{C})$ is surjective.

We shall consider a holomorphic vector bundle $E=L^{b_0}\oplus\cdots\oplus L^{b_m}$ $(b_0\leqq\cdots\leqq b_m)$ over a Kähler C-space M. We consider the structure of the automorphism group $\mathrm{Aut}_0(P(E))$ of the projective bundle $P(E)$ over M. Note that, for a holomorphic line bundle F and a holomorphic vector bundle E, the projective bundles $P(E)$ and $P(F\otimes E)$ are isomorphic. Thus we may assume that

$$E = 1\oplus L^{a_1}\oplus\cdots\oplus L^{a_m} \qquad \text{where}$$

a_k $(k=0, 1, \cdots, m)$ are integers such that $0=a_0\leqq a_1\leqq\cdots\leqq a_m$.

Lemma 2.1. *Let $E=1\oplus L^{a_1}\oplus\cdots\oplus L^{a_m}$ be a holomorphic vector bundle over $M=G/U$ and $P(E)$ the associated projective bundle. Then* Π: $\mathrm{Aut}_0(P(E))\to \mathrm{Aut}_0(M)$ *is surjective.*

Proof. Let $\tilde{G}$ denote $\mathrm{Aut}_0(M)$. Then we can write M as a homogeneous manifold $\tilde{G}/\tilde{U}$ for some closed connected complex Lie subgroup $\tilde{U}$ of $\tilde{G}$. Since the holomorphic line bundle L over M can be written as a homogeneous line bundle $\tilde{G}\times_{\tilde{\rho}} \boldsymbol{C}$ over $\tilde{G}/\tilde{U}$, where $\tilde{\rho}$: $\tilde{U}\to\boldsymbol{C}^*$ is a holomorphic representation, and $E=1\oplus L^{a_1}\oplus\cdots\oplus L^{a_m}$, it is easy to see that Π: $\mathrm{Aut}_0(P(E))\to\mathrm{Aut}_0(M)$ is surjective.
q.e.d.

Note that $H^0(P^N(\boldsymbol{C}), H^k)$ can be identified with the vector space S_k of all homogeneous polynomials of degree k on $\boldsymbol{C}^{N+1}$. We shall identify M with the image of f in $P^N(\boldsymbol{C})$. Let S be the vector space of all polynomials on $\boldsymbol{C}^{N+1}$, let $I(M)$ denote the ideal $\{p\in S\,|\,p_{|M}=0\}$ and put $I_k=I(M)\cap S_k$. By 2.3, we see S_k/I_k is isomorphic with $H^0(M, L^k)$. Note that, if $k=0$, $H^0(P^N(\boldsymbol{C}), H^k)\cong\boldsymbol{C}$.

Theorem 2.2. *Let $E=L^{a_0}\oplus L^{a_1}\oplus\cdots\oplus L^{a_m}$ be a holomorphic vector bundle over M where $0=a_0\leqq a_1\leqq\cdots\leqq a_m$ and $P(E)$ the projective bundle over M associated to the vector bundle E. We shall choose the integers $q_1, \cdots, q_s$ with $q_1+\cdots+q_s=m$ in such a way that $a_0=\cdots=a_{q_1}$ and $a_{q_1+\cdots+q_{\sigma-1}+1}=\cdots=a_{q_1+\cdots+q_\sigma}$ $(\sigma=2, \cdots, s)$. Let $M(q_i, q_j)$ be the set of $q_i\times q_j$ matrices given by*

$$\{B\,|\,B=(b_{kl}),\ b_{kl}\in S_{a_{q_1+\cdots+q_j}-a_{q_1+\cdots+q_i}}/I_{a_{q_1+\cdots+q_j}-a_{q_1+\cdots+q_i}}\}$$

In particular, $M(q_i, q_i)$ is the set of $q_i\times q_i$ matrices whose components are complex numbers. Then the Lie algebra of the kernel of Π: $\mathrm{Aut}_0(P(E))\to\mathrm{Aut}_0(M)$ *is given by*

$$\left\{\left.\begin{pmatrix} A_{11} & \cdots & A_{1s}\\ & \ddots & \vdots\\ 0 & & A_{ss}\end{pmatrix}\ \right|\ \begin{array}{l} A_{11}\in M(q_1+1, q_1+1)\\ A_{1j}\in M(q_1+1, q_j)\\ A_{ij}\in M(q_i, q_j)\\ 2\leqq i\leqq j\leqq s\end{array}\right\}\Big/\boldsymbol{C}\cdot 1$$

where 1 *denotes the $(m+1)\times(m+1)$-identity matrix.*

Proof. By Theorem 1.5, the Lie algebra of the kernel of Π: $\mathrm{Aut}_0(P(E))\to \mathrm{Aut}_0(M)$ is isomorphic to $H^0(M, \mathrm{End}(E))/\boldsymbol{C}\cdot 1$. Let $\{g_{\alpha\beta}\}$ be a system of transition functions of holomorphic line bundle L on M. Then

$$\{h_{\alpha\beta}\}\left[h_{\alpha\beta}=\begin{pmatrix}1 & & & 0\\ & g_{\alpha\beta}^{a_1} & & \\ & & \ddots & \\ 0 & & & g_{\alpha\beta}^{a_m}\end{pmatrix}\right]$$

is a system of transition functions of the holomorphic vector bundle $E=1\oplus L^{a_1}\oplus\cdots\oplus L^{a_m}$. Now $f=\{(f_{kl}^{\alpha})\}_{\alpha}\in H^0(M, \mathrm{End}(E))$ if and only if $(f_{kl}^{\alpha})\cdot h_{\alpha\beta}=h_{\alpha\beta}\cdot(f_{kl}^{\beta})$. Thus we get $f_{kl}^{\alpha}=g_{\alpha\beta}^{-(a_l-a_k)}f_{kl}^{\beta}$ for k, $l=1, \cdots, m+1$ and hence $f_{kl}=\{f_{kl}^{\alpha}\}_{\alpha}$ is an element of $H^0(M, L^{a_l-a_k})$. Conversely if f_{kl} is an element of $H^0(M, L^{a_l-a_k})$ for k, $l=1, \cdots, m+1$, $f=\{(f_{kl}^{\alpha})\}$ is an element of $H^0(M, \mathrm{End}(E))$. Since $H^0(M, L^k)$ is isomorphic with S_k/I_k, $H^0(M, \mathrm{End}\,(E))$ is isomorphic with

$$\left\{\begin{pmatrix}A_{11}\cdots\cdots A_{1s}\\ \ddots \quad \vdots\\ 0 \quad \ddots \vdots\\ A_{ss}\end{pmatrix}\;\middle|\;\begin{array}{l}A_{11}\in M(q_1+1,\, q_1+1)\\ A_{1j}\in M(q_1+1,\, q_j)\\ A_{ij}\in M(q_i,\, q_j)\\ 2\leqq i\leqq j\leqq s\end{array}\right\}$$

as vector spaces. Now, by the Remark 1 in section 1, we see that the isomorphism above is a Lie algebra isomorphism. q e d.

Corollary 2.3. *Let E be as in Theorem* 2.2. *Then*

$$\dim_C \mathrm{Aut}_0(P(E))=\dim_C \mathrm{Aut}_0(M)-1+\sum_{a_k\geqq a_l}\dim H^0(M, L^{a_k-a_l})$$

Proof. By Theorem 1.5 and Lemma 2.1,
$\dim_C \mathrm{Aut}_0(P(E))=\dim_C \mathrm{Aut}_0(M)-1+\dim_C H^0(M, \mathrm{End}\,(E))$. Now $\dim_C H^0(M, \mathrm{End}(E))=\sum_{a_k\geqq a_l}\dim_C H^0(M, L^{a_k-a_l})$ by Theorem 2.2. q.e.d.

REMARK 2. It is known that $\dim H^0(M, L^{a_k-a_l})$ can be computed by the dimension formula of Weyl. (cf. [5])

REMARK 3. In the case when M is a complex projective space $P^1(\boldsymbol{C})$ of dimension 1, Theorem 2.2 and Corollary 2.3 are known (See [13] §2 and [6] §1). In the case when M is a complex projective space $P^n(\boldsymbol{C})$, $\mathrm{Aut}_0(P(E))$ has been studied by Ise [9].

Corollary 2.4. *Let E be as in Theorem* 2.2. *If* $0=a_0<a_1<\cdots<a_m$, *then the Lie algebra of the kernel* Π: $\mathrm{Aut}_0(P(E))\to\mathrm{Aut}_0(M)$ *is solvable, but is not abelian.*

Proof. In this case the Lie algebra of the kernel Π is given by

$$\left\{ \begin{pmatrix} b_{00} \cdots b_{0m} \\ \ddots \vdots \\ 0 \quad b_{mm} \end{pmatrix} \middle| \begin{matrix} b_{ii} \in \boldsymbol{C} \ (i=0, \cdots, m) \\ b_{ij} \in S_{a_j - a_i} / I_{a_j - a_i} \end{matrix} \right\} \Big/ \boldsymbol{C} \cdot 1 .$$

Now it is easy to see our claim. q.e.d.

3. Chern classes of certain complex projective bundles

Let π denote the canonical projection $\boldsymbol{C}^{n+1}-(0)$ onto the complex projective space $P^n(\boldsymbol{C})$. The triple $(\boldsymbol{C}^{n+1}-(0), \pi, P^n(\boldsymbol{C}))$ is a principal $\boldsymbol{C}^*$-bundle over $P^n(\boldsymbol{C})$. Let ζ be the standard line bundle over $P^n(\boldsymbol{C})$ associated to the above principal bundle. Note that the dual line bundle ζ^* is the holomorphic line bundle H corresponding to a hyperplane of $P^n(\boldsymbol{C})$. For an m-tuple $a=(a_1, \cdots, a_m)$ of non-negative integers a_j $(j=1, \cdots, m)$ such that $a_1 \leqq \cdots \leqq a_m$, we denote by ζ^a the holomorphic vector bundle $1 \oplus \zeta^{a_1} \oplus \cdots \oplus \zeta^{a_m}$ over $P^n(\boldsymbol{C})$. Let $P(\zeta^a)$ denote the associated complex projective bundle over $P^n(\boldsymbol{C})$.

Now we shall recall that $P(\zeta^a)$ can be imbedded in $P^n(\boldsymbol{C}) \times P^{(n+1)m}(\boldsymbol{C})$ in a natural way (cf. [6] [8]). Let $y=(y_0, \cdots, y_n)$ be the homogeneous coordinates of $P^n(\boldsymbol{C})$ and $x=(x_{00}, \cdots, x_{ik}, \cdots)$ $(0 \leqq i \leqq n;\ 1 \leqq k \leqq m)$ the homogeneous coordinates of $P^{(n+1)m}(\boldsymbol{C})$. We define a projective algebraic manifold Σ_a by

$$\Sigma_a = \left\{ (\pi(y), \pi(x)) \in P^n(\boldsymbol{C}) \times P^{(n+1)m}(\boldsymbol{C}) \middle| \begin{matrix} y_j^{a_k} x_{ik} = y_i^{a_k} x_{jk} \\ (1 \leqq k \leqq m;\ 0 \leqq i, j \leqq n) \end{matrix} \right\} .$$

Let $\tilde{\pi}$: $\Sigma_a \to P^n(\boldsymbol{C})$ be the projection defined by $\tilde{\pi}(\pi(y), \pi(x)) = \pi(y)$. Then we can see that the complex projective bundle $(\Sigma_a, \tilde{\pi}, P^n(\boldsymbol{C}))$ is equivalent to $(P(\zeta^a), \pi, P^n(\boldsymbol{C}))$ (cf. Ise [8] p. 511). We shall identify $P(\zeta^a)$ with Σ_a. Thus we get an imbedding j: $P(\zeta^a) \to P^n(\boldsymbol{C}) \times P^{(n+1)m}(\boldsymbol{C})$.

Now let M be a Kähler C-space with the second Betti number $b_2(M)=1$ and let f: $M \to P^N(\boldsymbol{C})$ be the imbedding as in 2.3. For an m-tuple $a=(a_1, \cdots, a_m)$ of non-negative integers a_j $(j=1, \cdots, m)$ such that $a_1 \leqq \cdots \leqq a_m$, let L^{-a} denote the holomorphic vector bundle $1 \oplus L^{-a_1} \oplus \cdots \oplus L^{-a_m}$ over M. Since the holomorphic line bundle L^{-1} over M is the induced bundle $f^*\zeta$ of the standard line bundle ζ over $P^N(\boldsymbol{C})$, we see that $L^{-a} = f^*\zeta^a$ and $P(L^{-a})$ is the induced bundle $f^*P(\zeta^a)$ of $P(\zeta^a)$ by the imbedding f: $M \to P^N(\boldsymbol{C})$. Thus we have an imbedding f: $P(L^{-a}) \to P(\zeta^a)$ such that the diagram is commutative:

$$\begin{array}{ccc} P(L^{-a}) & \xrightarrow{\tilde{f}} & P(\zeta^a) \\ \downarrow \pi & \cap & \downarrow \pi \\ M & \xrightarrow{f} & P^N(\boldsymbol{C}) . \end{array}$$

Now we have an imbedding of $P(L^{-a})$ into $P^N(\boldsymbol{C}) \times P^{(N+1)m}(\boldsymbol{C})$ such that the diagram is commutative:

$$\begin{array}{ccccc} P(L^{-a}) & \xrightarrow{\tilde{f}} & P(\zeta^a) & \xrightarrow{j} & P^N(\boldsymbol{C})\times P^{(N+1)m}(\boldsymbol{C}) \\ \downarrow \pi & \cap\!\!\!\downarrow & \downarrow \pi & \cap\!\!\!\downarrow & \downarrow p_1 \\ M & \xrightarrow{f} & P^N(\boldsymbol{C}) & \xrightarrow{id} & P^N(\boldsymbol{C}) . \end{array}$$

Let ξ be a holomorphic vector bundle with the fiber $\boldsymbol{C}^{n+1}$ over M, $P(\xi)$ the complex projective bundle over M associated to ξ and $\pi: P(\xi)\to M$ the bundle projection. Then in a natural way $\pi^*\xi$ has a holomorphic line bundle η as subbundle such that η induces the standard line bundle over each fiber $P^m(\boldsymbol{C})$ of M. Let T_f denote the bundle along the fibers $P^m(\boldsymbol{C})$ of $P(\xi)$.

Now we have the following Lemma.

Lemma 3.1. *Let $T(M)$ (resp. $T(P(\xi))$) denote the holomorphic tangent bundle over M (resp. $P(\xi)$). Then the following sequences are exact.*

1) $0\to T_f\to T(P(\xi))\to\pi^*T(M)\to 0$
2) $0\to\eta\to\pi^*\xi\to\eta\otimes T_f\to 0$

Proof. See [7] §13 (cf. [6] §2).

Let $g\in H^2(P^{(N+1(m}(\boldsymbol{C}), \boldsymbol{Z})$ (resp. $h\in H^2(P^N(\boldsymbol{C}), \boldsymbol{Z})$) denote the Chern class $c(H_2)$ (resp. $c_1(H_1)$) of the holomorphic line bundle H_2(resp. H_1) corresponding to a hyperplane of $P^{(N+1)m}(\boldsymbol{C})$ (resp. $P^N(\boldsymbol{C})$). We put $\varepsilon=(j\circ\tilde{f})^*(1\times g)$ and $\nu=(j\circ\tilde{f})^*(h\times 1)$. Then $H^2(P(L^{-a}), \boldsymbol{Z})\cong\boldsymbol{Z}\varepsilon+\boldsymbol{Z}\nu$.

Corollary 3.2. *Let $c(M)$ denote the total Chern class of M. Then the total Chern class of $P(L^{-a})$ is given by*

$$c(P(L^{-a})) = \pi^*c(M)\prod_{i=0}^{m}(1+\varepsilon-a_i\nu)$$

where $a_0=0$.

Proof. Let $1\boxtimes H_2$ denote the holomorphic line bundle over $P^N(\boldsymbol{C})\times P^{(N+1)m}(\boldsymbol{C})$ defined by the line bundle H_2 over $P^{(N+1)}(\boldsymbol{C})$. Then $\eta=(j\circ\tilde{f})^*(1\boxtimes H_2^*)$. Thus $c(\eta)=-\varepsilon$. Since $L^{-1}=f^*(H_1^*)$, $c(\pi^*L^{-1})=-\nu$. Applying Lemma 3.1 for $\xi=L^{-a}$, we see that the total Chern class of T_f is given by

$$c(T_f) = c(\eta^{-1}\otimes\pi^*L^{-a}) = \prod_{i=0}^{m}c(\eta^{-1}\otimes\pi^*L^{-a_i}) = \prod_{i=0}^{m}(1+\varepsilon-a_i\nu)$$

and hence the total Chern class of $P(\xi)$ is given by

$$c(P(\xi)) = \pi^*c(M)\prod_{i=0}^{m}(1+\varepsilon-a_i\nu) .$$

q.e.d.

Since $H^2(M, \boldsymbol{Z})$ is generated by the first Chern class $c_1(L)$, we can write $c_1(M)=k(M)c_1(L)$.

Corollary 3.3. *The first Chern class $c_1(P(L^{-a}))$ of $P(L^{-a})$ is given by*

$$c_1(P(L^{-a})) = \{k(M)-\sum_{i=1}^{m} a_i\}\nu+(m+1)\varepsilon .$$

It is known that the integer $k(M)$ is positive (cf. [4]). In the case of compact irreducible hermitian symmetric spaces, the integer $k(M)$ is given as follows:

I $k(U(m+n)/U(m)\times U(n))=m+n$
II $k(SO(2n)/U(n))=2n-2$
III $k(Sp(n)/U(n))=n+1$
IV $k(SO(n+2)/SO(2)\times SO(n))=n \quad (n>2)$
V $k(E_6/\mathrm{Spin}\,(10)\times T^1)=12$
VI $k(E_7/E_6\times T^1)=18$.

4. A compact Kähler manifold which does not admit any Einstein Kähler metric

In this section we shall give example of a compact Kähler manifold with the positive first Chern class which does not admit any Einstein Kähler metric.

Theorem 4.1. *Let $P(L^{-a})$ denote a complex projective bundle over M defined in section 3. Then the first Chern class $c_1(M)$ is positive if $k(M)-\sum_{i=1}^{m} a_i>0$. But the compact Kähler manifold $P(L^{-a})$ does not admit any Einstein Kähler metric if $0<a_1<\cdots<a_m$.*

Proof. By Corollary 3.3, the first Chern class $c_1(P(L^{-a}))$ is given by

$$c_1(P(L^{-a}) = (k(M)-\sum_{i=1}^{m} a_i)\nu+(m+1)\varepsilon .$$

Note that if $a, b\in \boldsymbol{Z}$ are positive the element $a\nu+b\varepsilon\in H^2(P(L^{-a}), \boldsymbol{Z})$ is projectively induced (cf. [15] §2). Thus $c_1(P(L^{-a}))$ is positive if $k(M)-\sum_{i=1}^{m} a_i>0$.

Now we have a following Theorem due to Matsushima on a compact Einstein Kahler manifold.

Theorem (Matsushima [10]). *Let X be a compact Einstein Kähler manifold with nonzero Ricci tensor. Then the Lie algebra $\mathfrak{k}(X)$ of Killing vector fields on X is a real form of the Lie algebra $\mathfrak{a}(X)$ of holomorphic vector fields on X, that is,*

$$\mathfrak{a}(X) = \mathfrak{k}(X) + \sqrt{-1}\,\mathfrak{k}(X) .$$

Note that the Lie algebra $\mathfrak{k}(X)$ is compact and hence $\mathfrak{k}(X)$ is reductive. By Corollary 2.4, the holomorphic vector fields $\mathfrak{a}(P(L^{-a}))$ has a solvable ideal which is not abelian if $0 < a_1 < \cdots < a_m$. In particular, the Lie algebra $(P(L^{-a}))$ is not reductive. Hence $P(L^{-a})$ does not admit any Einstein Kähler metric. q.e.d.

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Added in proof.

After finishing this work, the authors learned that S. T. Yau proved that the complex projective bundle $P(1\oplus\zeta)$ over a complex projective space $P^1(\boldsymbol{C})$ of dimension 1 admits a Kähler metric with positive Ricci curvature but does not admit a Kähler metric with constant scalar curvature in his paper "*On the curvature of compact Hermitian manifolds*" Invent. math. **25** (1974), 213–239.