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Osaka University
Introduction

Let \( M \) be a compact Einstein Kahler manifold. Then the first Chern class \( c_1(M) \) of \( M \) is positive, negative or zero. We can ask whether the converse is true or not, that is, does a compact Kahler manifold \( M \) with the first Chern class \( c_1(M) > 0 \) (resp. \( c_1(M) < 0 \), \( c_1(M) = 0 \)) admit an Einstein Kahler metric? In the case when \( c_1(M) < 0 \), T. Aubin [2] has proved that a compact Kahler manifold \( M \) with \( c_1(M) < 0 \) admits a unique Einstein Kahler metric. As is well-known, in the case when \( c_1(M) = 0 \), our question is yes if the Calabi conjecture is true. The purpose of this note is to give some examples of a compact Kahler manifold with \( c_1(M) > 0 \) which does not admit any Einstein Kahler metric. Let \( X \) be a compact connected complex manifold. By a theorem of Bochner-Montogomery, the group \( \text{Aut}(X) \) of all holomorphic transformations of \( X \) is a complex Lie group and the map \( (f, x) \rightarrow f(x) \) is holomorphic. For a holomorphic vector bundle \( E \) over a compact complex manifold \( M \) let \( P(E) \) denote the associated complex projective bundle. Let \( \text{Aut}_o(X) \) denote the identity component of \( \text{Aut}(X) \). By a theorem of Blanchard, we can define a homomorphism \( \Pi: \text{Aut}_o(P(E)) \rightarrow \text{Aut}_o(M) \). In section 1 we shall show that the Lie algebra of the \( \text{Ker} \Pi \) is isomorphic with the Lie algebra \( H^0(M, \text{End}(E))/\mathbb{C} \cdot 1 \) where \( H^0(M, \text{End}(E)) \) denotes all holomorphic sections of the vector bundle \( \text{End}(E) \) over \( M \) and \( 1 \) denotes the element of \( H^0(M, \text{End}(E)) \) defined by the identity map of \( \text{End}(E)_x (x \in M) \). In section 2 we consider Kahler C-spaces with the second Betti number \( b_2 = 1 \) as \( M \). In this case we know that the group of all holomorphic line bundles \( H^1(M, \mathbb{C}^*) \) over \( M \) is generated by a homogeneous line bundle. From now on we shall exclusively consider holomorphic vector bundles \( E \) generated by holomorphic line bundles. Then the homomorphism \( \Pi: \text{Aut}_o(P(E)) \rightarrow \text{Aut}_o(M) \) is surjective and we can determine the structure of the Lie algebra of the \( \text{Ker} \Pi \). In particular, we can compute the dimension of \( \text{Aut}_o(P(E)) \) in these cases. In section 3 we shall compute the Chern class of \( P(E) \). The result in section 2 has been obtained by Brieskorn [6], Röhrl [13]

1) The authors would like to express their thanks to the referee for his kind suggestion.
for the case of the complex projective space $\mathbb{P}^1(C)$ of dimension 1 and by Ise [9] for the case of the complex projective space $\mathbb{P}^n(C)$. The result in section 3 has been obtained by Brieskorn [6] for the case of the complex projective space $\mathbb{P}^1(C)$. In section 4 we shall show that some of complex projective bundles over $M$ are examples of a compact Kähler manifold with $c_1(M) > 0$ which does not admit any Einstein Kähler metric. We remark that nothing is mentioned on Einstein Kähler metric in [6] [9] [13].

1. The automorphism group of a complex projective bundle

Let $M$ be a compact connected complex manifold and $E$ a holomorphic vector bundle over $M$. Let $P(E)$ denote the complex projective bundle over $M$ induced by $E$. Since $P(E)$ is a compact complex manifold, it is known that the group $\text{Aut}(P(E))$ of all holomorphic automorphisms of $P(E)$ is a complex Lie group and the map $\text{Aut}(P(E)) \times P(E) \to P(E)$ defined by $(f, x) \mapsto f(x)$ is holomorphic. Let $F(P(E))$ denote the subgroup of all fiber preserving automorphisms of $P(E)$.

Proposition 1.1 (Blanchard [3]). Let $\text{Aut}_0(P(E))$ (resp. $F_0(P(E))$) denote the identity component of $\text{Aut}(P(E))$ (resp. $F(P(E))$). Then $\text{Aut}_0(P(E)) = F_0(P(E))$.

Note that an element of $F_0(P(E))$ is a fiber preserving automorphism in the sense of Steenrod [14].

Let $P(M, G, \pi)$ denote a principal holomorphic fiber bundle over $M$ with the structure group $G$. Let $F(P(M, G, \pi))$ be the group of all fiber preserving holomorphic automorphisms of the principal bundle $P(M, G, \pi)$, that is, a biholomorphic map $\tilde{f}$ of $P(M, G, \pi)$ is an element of $F(P(M, G, \pi))$ if and only if $\tilde{f}(x \cdot g) = \tilde{f}(x) \cdot g$ for all $x \in M$ and $g \in G$.

Theorem 1.2 (Morimoto [11]). The group $F(P(M, G, \pi))$ equipped with the compact open topology can be given the structure of a complex Lie group which acts holomorphically on $P(M, G, \pi)$. Its Lie algebra is isomorphic to the Lie algebra of all holomorphic vector fields $X$ over $P(M, G, \pi)$ for which $R_g X = X$ for every $g \in G$, where $R_g \cdot$ denotes the differential mapping induced by the action $R_g$ of an element $g$ of $G$.

Let $\bar{P}$ (resp. $P$) denote the principal bundle associated to a complex projective bundle $P(E)$ (resp. a holomorphic vector bundle $E$) over $M$. Then $F(P)$ and $F(P(E))$ are naturally isomorphic. In fact, $P(E)$ is the quotient of $\bar{P} \times P^n(C)$ by the equivalence relation $(y, \xi) \sim (ya, a^{-1} \xi)$ ($y \in P$, $\xi \in P^n(C)$, $a \in PGL(m + 1, C)$). Let $\rho$ be the projection of $\bar{P} \times P^n(C)$ onto $P(E)$. For an element $f \in F(\bar{P})$, we can define a mapping $f' : P(E) \to P(E)$ by $f'(\rho(y, \xi)) = \rho(f(y), \xi)$ ($y \in \bar{P}, \xi \in P^n(C)$). Then $f' \in F(P(E))$ and $f, f'$ induce the same automorphism $\tilde{f}$ of $M$. Moreover the mapping $\theta : F(\bar{P}) \to F(P(E))$ defined by $\theta(f) = f'$ is an isomorphism of the
group $F(\hat{P})$ into the group $F(P(E))$. Conversely, let $f'$ be an element of $F(P(E))$. For every element $y \in \hat{P}$, there is an element $w \in \hat{P}$ such that $f'(\rho(y, \xi)) = \rho(w, \xi)$ for all $\xi \in P_n(C)$. Put $f(y) = w$. Then $f \in F(\hat{P})$ and $\theta(f) = f'$.

Let $PGL(m+1, C)$ denote the projective transformation group corresponding to $GL(m+1, C)$. Then we have an exact sequence

\[ 0 \to \mathbb{C}^* \to GL(m+1, C) \to PGL(m+1, C) \to 0. \]

Since $P$ (resp. $\hat{P}$) is the principal bundle associated to the vector bundle $E$ (resp. $P(E)$), we have an exact sequence of complex Lie groups

\[ 0 \to \mathbb{C}^* \to F_0(P) \to F_0(\hat{P}). \]

Since each element $g \in F(P)$ induces an element $g$ of $\text{Aut}(M)$, there is a canonical homomorphism $\Pi_P: F_0(P) \to \text{Aut}_0(M)$ for each principal fiber bundle $P$ over $M$.

**Proposition 1.3.** If $M$ is simply connected, we have an exact sequence

\[ 0 \to \mathbb{C}^* \to \text{Ker} \Pi_P \to \text{Ker} \Pi \to 0. \]

Proof. Take a simple open covering $\{U_\alpha\}_\alpha$ of $M$ such that, for each $\alpha$, $\pi^{-1}(U_\alpha) = U_\alpha \times GL(m+1, C)$ and $\pi^{-1}(\hat{P}) = U_\alpha \times PGL(m+1, C)$. Moreover let $(g_{\alpha\beta})$ be the system of transition functions of the principal bundle $P$ associated to the open covering $\{U_\alpha\}_\alpha$. Then $(g_{\alpha\beta})$ induces the system of transition functions $(\tilde{g}_{\alpha\beta})$ of the principal bundle $\hat{P}$. Let $\tilde{\phi}$ be an element of $\text{Ker} \Pi_{\hat{P}}$. Then there is a system of functions $\{\phi_\alpha\}$ such that $\phi_\alpha: U_\alpha \to PGL(m+1, C)$ and $\tilde{g}_{\alpha\beta} \cdot \phi_\beta = \phi_\alpha \cdot \tilde{g}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. Since $U_\alpha$ is simply connected, there is a holomorphic map $\phi_\alpha: U_\alpha \to SL(m+1, C)$ such that $\phi_\alpha = \tilde{\phi} \cdot \phi_\alpha$ where $\tilde{\phi}: SL(m+1, C) \to PGL(m+1, C)$ is the canonical map. Then

\[ \tilde{g}_{\alpha\beta} \cdot \phi_\alpha = c_{\alpha\beta} \phi_\alpha \cdot g_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \]

and $c_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$ is holomorphic. By taking the determinant, we get $c_{\alpha\beta} = 1$ on $U_\alpha \cap U_\beta$. Since $U_\alpha \cap U_\beta$ is connected, $c_{\alpha\beta}$ is constant on $U_\alpha \cap U_\beta$ and $c_{\alpha\beta} \in \mathbb{Z}((m+1)\mathbb{Z})$. Moreover note that $c_{\alpha\beta} c_{\beta\gamma} c_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

**Lemma** (Principle of monodromy). Let $M$ be a simply connected manifold and $\mathcal{U} = \{U_\alpha\}$ be a simple open covering. Then $H^1(\mathcal{U}, \mathbb{Z}((m+1)\mathbb{Z})) = 0$.

Proof. See Weil [17].

Applying the lemma in our case, we get a system of constant functions $\{a_\alpha\}$ such that $c_{\alpha\beta} = a_\alpha \cdot a_{\beta}^{-1}$, $a_\alpha: U_\alpha \to \mathbb{Z}((m+1)\mathbb{Z})$. Hence, we have $g_{\alpha\beta} \delta_\beta \phi_\beta = a_\alpha \phi_\alpha g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ and we completes our proof.

**Corollary.** If $M$ is simply connected and $\Pi_P: F_0(P) \to \text{Aut}_0(M)$ is onto, then the following sequences is exact.
(3) \[ 0 \to \mathbb{C}^* \to F_0(P) \to F_0(\tilde{P}) \to 0 \]

Proof. Obvious from the following diagram.

\[
\begin{array}{ccc}
0 & 0 & \\
\downarrow & \downarrow & \\
0 \to \mathbb{C}^* & \to \text{Ker } \Pi_P & \to \text{Ker } \Pi_{\tilde{P}} & \to 0 \quad \text{(exact)} \\
\downarrow & \downarrow & \downarrow & \\
0 \to \mathbb{C}^* & \to F_0(P) & \to F_0(\tilde{P}) & \\
\Pi_P & \subseteq & \subseteq & \Pi_{\tilde{P}}
\end{array}
\]

Now we recall the exact sequence of holomorphic vector bundle over \( M \) associated to the holomorphic principal fiber bundle \( P \) on \( M \) with the structure group \( G \), due to Atiyah [1]. Let \( T(P) \) be the holomorphic tangent bundle of \( P \). Since \( G \) operates on \( P \), it also operates on \( T(P) \). We put \( Q = T(P)/G \), so that a point of \( Q \) is a field of tangent vectors to \( P \), defined along one of its fibers, and invariant under \( G \). Then we can show that \( Q \) has a natural vector bundle structure over \( M \). Let \( L(P) \) denote the vector bundle associated to \( P \) by the adjoint representation of \( G \). Note that \( L(P) \) is a bundle of Lie algebra, each fiber \( L(P)_x = L(P)_x \) being a Lie algebra isomorphic with \( L(G) \). Under these notations, there exists an exact sequence of holomorphic vector bundles over \( M \):

\[ (4) \quad 0 \rightarrow L(P) \rightarrow Q \rightarrow T(M) \rightarrow 0 \]

where \( T(M) \) is the holomorphic tangent bundle over \( M \).

Then we have the exact sequence of cohomology

\[ (5) \quad 0 \rightarrow H^0(M, L(P)) \rightarrow H^0(M, Q) \rightarrow H^0(M, T(M)) \rightarrow H^1(M, L(P)) \rightarrow \cdots \]

Now we can identify the Lie algebra of \( F_0(P) \) (resp. \( \text{Ker } \Pi_P, \text{Aut}_0(M) \)) with \( H^0(M, Q) \) (resp. \( H^0(M, L(P)), H^0(M, T(M)) \)) (cf. Morimoto [11]). Note that the structure of the Lie algebra \( H^0(M, L(P)) \) is given by the following way. For \( X, Y \in H^0(M, L(P)), X_x, Y_x \in L(P)_x \ (x \in M) \). Since \( L(P)_x \) has the Lie algebra structure, we have \( [X_x, Y_x] \in L(P)_x \). On the other hand, \( [X, Y] \in H^0(M, L(P)) \) as holomorphic vector fields. Then it is easy to see that \( [X, Y]_x = [X_x, Y_x] \) for every \( x \in M \). That is, the Lie algebra structure of \( H^0(M, L(P)) \) as the subalgebra of \( H^0(M, Q) \) coincides with the one induced by the Lie algebra \( L(G) \) of \( G \).

In the case of vector bundles, we have the following proposition due to Atiyah.

**Proposition 1.4.** Let \( E \) be a holomorphic vector bundle over \( M \) and \( P \) the
associated principal bundle. Then \( L(P) \cong \text{End}(E) \).


Note that \( H^0(M, \text{End}(E)) \) contains \( \mathbb{C} \) in the center and the Lie algebra of \( \text{Ker} \Pi(\Pi: F_0(P(E)) \to \text{Aut}_0(M)) \) is isomorphic with \( H^0(M, \text{End}(E))/\mathbb{C} \). We now summarize our result as follows:

**Theorem 1.5.** Let \( M \) be a simply connected compact complex manifold, \( E \) a holomorphic vector bundle over \( M \) and \( P(E) \) the projective bundle induced by \( E \). If \( \Pi: \text{Aut}_0(P(E)) \to \text{Aut}_0(M) \) is surjective,

\[
\dim_{\mathbb{C}} \text{Aut}_0(P(E)) = \dim_{\mathbb{C}} \text{Aut}_0(M) + \dim_{\mathbb{C}} H^0(M, \text{End}(E)) - 1.
\]

Moreover the Lie algebra of \( \text{Ker} \Pi \) is isomorphic with \( H^0(M, \text{End}(E))/\mathbb{C} \).

**Remark 1.** Let \( f, g \) be elements of \( H^0(M, \text{End}(E)) \). Then the Lie algebra structure of \( H^0(M, \text{End}(E)) \) is given by

\[
[f, g](x) = [f(x), g(x)] = f(x) \ast g(x) - g(x) \ast f(x)
\]

\((f(x), g(x) \in \text{End}(E_x)) \) for every \( x \in M \).

**2. Complex projective bundles over a Kähler C-space**

We shall recall the following facts on Kähler C-spaces and holomorphic line bundles over these manifolds. A simply connected compact Kähler homogeneous manifold is called a Kähler C-space. Kähler C-spaces have been classified by H. C. Wang [16]. From now on we assume that the second Betti number \( b_2(M) \) of a Kähler C-space \( M \) is 1. Note that such a class contains the class of irreducible hermitian symmetric spaces. We shall use the following known results on holomorphic line bundles over Kähler C-spaces with \( b_2 = 1 \) (cf. [4] [8]).

2.1. The group of all holomorphic line bundles \( H^1(M, \mathbb{C}^*) \) over a Kähler C-space \( M \) is isomorphic to \( \mathbb{Z} \).

2.2. There is a homogeneous holomorphic line bundle \( L \) over \( M \) such that \( L \) is very ample. Moreover \( L \) is a generator of \( H^1(M, \mathbb{C}^*) \). In particular, every holomorphic line bundle is homogeneous.

2.3. Let \( f: M \to P^N(\mathbb{C}) \) be the associated imbedding for \( L \) and \( H \) the holomorphic line bundle over \( P^N(\mathbb{C}) \) corresponding to a hyperplane of \( P^N(\mathbb{C}) \). Then \( L \) is the induced bundle \( f^*H \) over \( M \) and the homomorphism

\[
\gamma_k: H^0(P^N(\mathbb{C}), H^k) \to H^0(M, L^k) \quad (k \geq 0)
\]

induced by the imbedding \( f: M \to P^N(\mathbb{C}) \) is surjective.
We shall consider a holomorphic vector bundle $E = L^b_0 \oplus \cdots \oplus L^b_m$ over a Kähler $C$-space $M$. We consider the structure of the automorphism group $\text{Aut}_0(P(E))$ of the projective bundle $P(E)$ over $M$. Note that, for a holomorphic line bundle $F$ and a holomorphic vector bundle $E$, the projective bundles $P(E)$ and $P(F \otimes E)$ are isomorphic. Thus we may assume that

$$E = 1 \oplus L^a_1 \oplus \cdots \oplus L^a_m$$

where $a_k (k=0, 1, \cdots, m)$ are integers such that $0 = a_0 \leq a_1 \leq \cdots \leq a_m$.

**Lemma 2.1.** Let $E = 1 \oplus L^a_1 \oplus \cdots \oplus L^a_m$ be a holomorphic vector bundle over $M = G/U$ and $P(E)$ the associated projective bundle. Then $\Pi: \text{Aut}_0(P(E)) \to \text{Aut}_0(M)$ is surjective.

**Proof.** Let $\mathcal{G}$ denote $\text{Aut}_0(M)$. Then we can write $M$ as a homogeneous manifold $G/U$ for some closed connected complex Lie subgroup $U$ of $G$. Since the holomorphic line bundle $L$ over $M$ can be written as a homogeneous line bundle $G \times_{\rho} C$ over $G/U$, where $\rho: U \to C^*$ is a holomorphic representation, and $E = 1 \oplus L^a_1 \oplus \cdots \oplus L^a_m$, it is easy to see that $\Pi: \text{Aut}_0(P(E)) \to \text{Aut}_0(M)$ is surjective. 

q.e.d.

Note that $H^0(P^N(C), H^k)$ can be identified with the vector space $S_k$ of all homogeneous polynomials of degree $k$ on $C^{N+1}$. We shall identify $M$ with the image of $f$ in $P^N(C)$. Let $S$ be the vector space of all polynomials on $C^{N+1}$, let $I(M)$ denote the ideal $\{p \in S | p|_M = 0\}$ and put $I_k = I(M) \cap S_k$. By 2.3, we see $S_k/I_k$ is isomorphic with $H^0(M, L^k)$. Note that, if $k=0$, $H^0(P^N(C), H^k) \cong C$.

**Theorem 2.2.** Let $E = L^a_0 \oplus L^a_1 \oplus \cdots \oplus L^a_m$ be a holomorphic vector bundle over $M$ where $0 = a_0 \leq a_1 \leq \cdots \leq a_m$ and $P(E)$ the projective bundle over $M$ associated to the vector bundle $E$. We shall choose the integers $q_1, \cdots, q_s$ with $q_1 + \cdots + q_s = m$ in such a way that $a_0 = \cdots = a_{q_1}$ and $a_{q_1} + \cdots + a_{q_{\sigma-1}+1} = \cdots = a_{q_1} + \cdots + a_{q_{\sigma-1}}$ $(\sigma = 2, \cdots, s)$. Let $M(q_i, q_j)$ be the set of $q_i \times q_j$ matrices given by

$$\{B | B = (b_{ki}), b_{ki} \in S a_{\epsilon_1 + \cdots + \epsilon_j} - a_{\epsilon_1 + \cdots + \epsilon_j} | I a_{\epsilon_1 + \cdots + \epsilon_j} - a_{\epsilon_1 + \cdots + \epsilon_j}\}$$

In particular, $M(q_i, q_j)$ is the set of $q_i \times q_j$ matrices whose components are complex numbers. Then the Lie algebra of the kernel of $\Pi: \text{Aut}_0(P(E)) \to \text{Aut}_0(M)$ is given by

$$\left[\begin{array}{c}
A_{11} & \cdots & A_{1s} \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{ss}
\end{array}\right]$$

where $1$ denotes the $(m+1) \times (m+1)$-identity matrix.
Proof. By Theorem 1.5, the Lie algebra of the kernel of \( \Pi: \text{Aut}_0(P(E)) \to \text{Aut}_0(M) \) is isomorphic to \( H^0(M, \text{End}(E))/\mathbb{C} \cdot 1 \). Let \( \{g_{ab}\} \) be a system of transition functions of holomorphic line bundle \( L \) on \( M \). Then

\[
\{h_{ab}\} \quad h_{ab} = \begin{pmatrix} 1 & 0 \\ g_{ab} & 0 \\ \vdots & \vdots \\ 0 & g_{ab} \end{pmatrix}
\]

is a system of transition functions of the holomorphic vector bundle \( E = 1 \oplus L^1 \oplus \cdots \oplus L^s \). Now \( f = \{f_{kl}^a\} \in H^0(M, \text{End}(E)) \) if and only if \((f_{kl}^a) \cdot h_{ab} = h_{ab} \cdot (f_{kl}^a)\). Thus we get \( f_{kl}^a = g_{ab}^{-1} g_{kl}^b f_{kl}^a \) for \( k, l = 1, \ldots, m+1 \) and hence \( f_{kl} = \{f_{kl}^a\}_a \) is an element of \( H^0(M, L^{i-a}) \). Conversely if \( f_{kl} \) is an element of \( H^0(M, L^{i-a}) \) for \( k, l = 1, \ldots, m+1 \), \( f = \{f_{kl}^a\} \) is an element of \( H^0(M, \text{End}(E)) \).

Since \( H^0(M, L^s) \) is isomorphic with \( S_{s,1} \), \( H^0(M, \text{End}(E)) \) is isomorphic with \( S_{s,1} \), \( \forall a^i \) as vector spaces. Now, by the Remark 1 in section 1, we see that the isomorphism above is a Lie algebra isomorphism. 

**Corollary 2.3.** Let \( E \) be as in Theorem 2.2. Then

\[
\dim \text{C} \text{Aut}_0(P(E)) = \dim \text{C} \text{Aut}_0(M) - 1 + \sum_{a^{i-a^j}} \dim H^0(M, L^{i-a^j})
\]

Proof. By Theorem 1.5 and Lemma 2.1,

\[
\dim \text{C} \text{Aut}_0(P(E)) = \dim \text{C} \text{Aut}_0(M) - 1 + \dim \text{C} H^0(M, \text{End}(E)).
\]

Now \( \dim \text{C} H^0(M, \text{End}(E)) = \sum_{a^{i-a^j}} \dim \text{C} H^0(M, L^{i-a^j}) \) by Theorem 2.2.

**Remark 2.** It is known that \( \dim H^0(M, L^{i-a^j}) \) can be computed by the dimension formula of Weyl. (cf. [5])

**Remark 3.** In the case when \( M \) is a complex projective space \( P^1(\mathbb{C}) \) of dimension 1, Theorem 2.2 and Corollary 2.3 are known (See [13] §2 and [6] §1).

**Corollary 2.4.** Let \( E \) be as in Theorem 2.2. If \( 0 = a_0 < a_1 < \cdots < a_m \), then the Lie algebra of the kernel \( \Pi: \text{Aut}_0(P(E)) \to \text{Aut}_0(M) \) is solvable, but is not abelian.

Proof. In this case the Lie algebra of the kernel \( \Pi \) is given by
3. Chern classes of certain complex projective bundles

Let $\pi$ denote the canonical projection $C^{n+1} - (0)$ onto the complex projective space $P^n(C)$. The triple $(C^{n+1} - (0), \pi, P^s(C))$ is a principal $C^*$-bundle over $P^n(C)$. Let $\xi$ be the standard line bundle over $P^n(C)$ associated to the above principal bundle. Note that the dual line bundle $\xi^*$ is the holomorphic line bundle $H$ corresponding to a hyperplane of $P^n(C)$. For an $m$-tuple $a=(a_1, \cdots, a_m)$ of non-negative integers, we denote by $H_a$ the holomorphic vector bundle $1 \oplus \xi^* \oplus \cdots \oplus \xi^* \oplus \Pi P^n(C)$. Let $P(\xi^*)$ denote the associated complex projective bundle over $P^n(C)$.

Now we shall recall that $P(\xi^*)$ can be imbedded in $P^n(C) \times P^{(n+1)m}(C)$ in a natural way (cf. [6] [8]). Let $y=(y_0, \cdots, y_n)$ be the homogeneous coordinates of $P^n(C)$ and $x=(x_0, \cdots, x_m, \cdots) (0 \leq i \leq n; 1 \leq k \leq m)$ the homogeneous coordinates of $P^{(n+1)m}(C)$. We define a projective algebraic manifold $\Sigma_a$ by

$$
\Sigma_a = \left\{ (\pi(y), \pi(x)) \in P^n(C) \times P^{(n+1)m}(C) \mid y^*_k x_i = y^*_i x_{jk} \right\}.
$$

Let $\bar{\pi}: \Sigma_a \to P^n(C)$ be the projection defined by $\bar{\pi}(\pi(y), \pi(x)) = \pi(y)$. Then we can see that the complex projective bundle $(\Sigma_a, \bar{\pi}, P^n(C))$ is equivalent to $(P(\xi^*), \pi, P^n(C))$ (cf. Ise [8] p. 511). We shall identify $P(\xi^*)$ with $\Sigma_a$. Thus we get an imbedding $j: P(\xi^*) \to P^n(C) \times P^{(n+1)m}(C)$.

Now let $M$ be a Kahler $C$-space with the second Betti number $b_2(M)=1$ and let $f: M \to P^n(C)$ be the imbedding as in 2.3. For an $m$-tuple $a=(a_1, \cdots, a_m)$ of non-negative integers, let $L^a$ denote the holomorphic vector bundle $1 \oplus L^{-a_1} \oplus \cdots \oplus L^{-a_m}$ over $M$. Since the holomorphic line bundle $L^{-1}$ over $M$ is the induced bundle $f^*\xi$ of the standard line bundle $\xi$ over $P^n(C)$, we see that $L^{-a} = f^*\xi^a$ and $P(L^{-a})$ is the induced bundle $f^*P(\xi^a)$ of $P(\xi^*)$ by the imbedding $f: M \to P^n(C)$. Thus we have an imbedding $f: P(L^{-a}) \to f(\xi^*)$ such that the diagram is commutative:

$$
\begin{array}{ccc}
P(L^{-a}) & \xrightarrow{f} & P(\xi^*) \\
\downarrow \pi & \cap & \downarrow \pi \\
M & \xrightarrow{f} & P^n(C)
\end{array}
$$

Now we have an imbedding of $P(L^{-a})$ into $P^n(C) \times P^{(n+1)m}(C)$ such that the diagram is commutative.
Let $\xi$ be a holomorphic vector bundle with the fiber $\mathbb{C}^{*+1}$ over $M$, $P(\xi)$ the complex projective bundle over $M$ associated to $\xi$ and $\pi: P(\xi) \to M$ the bundle projection. Then in a natural way $\pi^*\xi$ has a holomorphic line bundle $\eta$ as sub-bundle such that $\eta$ induces the standard line bundle over each fiber $P^n(\mathbb{C})$ of $M$. Let $T_f$ denote the bundle along the fibers $P^n(\mathbb{C})$ of $P(\xi)$.

Now we have the following Lemma.

**Lemma 3.1.** Let $T(M)$ (resp. $T(P(\xi))$) denote the holomorphic tangent bundle over $M$ (resp. $P(\xi)$). Then the following sequences are exact.

$$0 \to T_f \to T(P(\xi)) \to \pi^* T(M) \to 0$$


Let $g \in H^2(P^{N+1}M(\mathbb{C}), \mathbb{Z})$ (resp. $h \in H^2(P^{N}(\mathbb{C}), \mathbb{Z})$) denote the Chern class $c(H_2)$ (resp. $c(H_1)$) of the holomorphic line bundle $H_2$ (resp. $H_1$) corresponding to a hyperplane of $P^{N+1}M(\mathbb{C})$ (resp. $P^{N}(\mathbb{C})$). We put $\xi = \pi^*(1 \times g)$ and $\nu = (\delta f)^*(1 \times 1)$. Then $H^2(P(L^{-a}), \mathbb{Z}) \cong \mathbb{Z}\xi + \mathbb{Z}\nu$.

**Corollary 3.2.** Let $c(M)$ denote the total Chern class of $M$. Then the total Chern class of $P(L^{-a})$ is given by

$$c(P(L^{-a})) = \pi^*c(M) \prod_{i=0}^n (1 + \xi - a_i \nu)$$

where $a_0 = 0$.

**Proof.** Let $1 \boxtimes H_2$ denote the holomorphic line bundle over $P^{N}(\mathbb{C}) \times P^{(N+1)^*}M(\mathbb{C})$ defined by the line bundle $H_2$ over $P^{(N+1)}(\mathbb{C})$. Then $\eta = (\delta f)^*(1 \boxtimes H_2)$.

Thus $c(\eta) = -\xi$. Since $L^{-a} = f^*(H_2)$, $c(\pi^*L^{-a}) = -\nu$. Applying Lemma 3.1 for $\xi = L^{-a}$, we see that the total Chern class of $T_f$ is given by

$$c(T_f) = c(\eta^{-1} \boxtimes \pi^*L^{-a}) = \prod_{i=0}^n c(\eta^{-1} \boxtimes \pi^*L^{-a}) = \prod_{i=0}^n (1 + \xi - a_i \nu)$$

and hence the total Chern class of $P(\xi)$ is given by

$$c(P(\xi)) = \pi^*c(M) \prod_{i=0}^n (1 + \xi - a_i \nu)$$

q.e.d.
Since $H^2(M, \mathbb{Z})$ is generated by the first Chern class $c_1(L)$, we can write $c_1(M) = k(M)c_1(L)$.

**Corollary 3.3.** The first Chern class $c_1(P(L^{-a}))$ of $P(L^{-a})$ is given by

$$c_1(P(L^{-a})) = \{k(M) - \sum_{i=1}^n a_i\} \nu + (m+1)\xi.$$  

It is known that the integer $k(M)$ is positive (cf. [4]). In the case of compact irreducible hermitian symmetric spaces, the integer $k(M)$ is given as follows:

I $k(U(m+n)/U(m) \times U(n)) = m+n$

II $k(SO(2n)/U(n)) = 2n-2$

III $k(Sp(n)/U(n)) = n+1$

IV $k(SO(n+2)/SO(2) \times SO(n)) = n$ (n > 2)

V $k(E_6/Spin(10) \times T') = 12$

VI $k(E_7/E_6 \times T') = 18$.

4. A compact Kähler manifold which does not admit any Einstein Kähler metric

In this section we shall give example of a compact Kähler manifold which does not admit any Einstein Kähler metric.

**Theorem 4.1.** Let $P(L^{-a})$ denote a complex projective bundle over $M$ defined in section 3. Then the first Chern class $c_1(M)$ is positive if $k(M) - \sum_{i=1}^n a_i > 0$. But the compact Kähler manifold $P(L^{-a})$ does not admit any Einstein Kähler metric if $0 < a_1 < \cdots < a_n$.

**Proof.** By Corollary 3.3, the first Chern class $c_1(P(L^{-a}))$ is given by

$$c_1(P(L^{-a})) = (k(M) - \sum_{i=1}^n a_i) \nu + (m+1)\xi.$$  

Note that if $a, b \in \mathbb{Z}$ are positive the element $av + b\xi \in H^2(P(L^{-a}), \mathbb{Z})$ is projectively induced (cf. [15] §2). Thus $c_1(P(L^{-a}))$ is positive if $k(M) - \sum_{i=1}^n a_i > 0$.

Now we have a following Theorem due to Matsushima on a compact Einstein Kähler manifold.

**Theorem** (Matsushima [10]). Let $X$ be a compact Einstein Kähler manifold with nonzero Ricci tensor. Then the Lie algebra $\mathfrak{k}(X)$ of Killing vector fields on $X$ is a real form of the Lie algebra $\mathfrak{a}(X)$ of holomorphic vector fields on $X$, that is,
\[ \alpha(X) = \mathfrak{f}(X) + \sqrt{-1} \mathfrak{f}(X). \]

Note that the Lie algebra \( \mathfrak{f}(X) \) is compact and hence \( \mathfrak{f}(X) \) is reductive. By Corollary 2.4, the holomorphic vector fields \( \alpha(P(L^{-a})) \) has a solvable ideal which is not abelian if \( 0 < a_1 < \cdots < a_m \). In particular, the Lie algebra \( (P(L^{-a})) \) is not reductive. Hence \( P(L^{-a}) \) does not admit any Einstein Kähler metric. q.e.d.

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References

Added in proof.

After finishing this work, the authors learned that S. T. Yau proved that
the complex projective bundle $P(1 \oplus \xi)$ over a complex projective space $P^1(C)$ of
dimension 1 admits a Kähler metric with positive Ricci curvature but does not
admit a Kähler metric with constant scalar curvature in his paper "On the cur-