F. M.J. Van Oystaeyen
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ON ORDERS OVER GRADED KRULL DOMAINS

FREDDY M.J. VAN OYSTAEYEN

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0. Introduction

Recently various authors have investigated the concept of a non-commutative Krull ring, e.g. in [3], [9], [11] or [12], different types of non-commutative generalizations of Krull domains have been considered. We may observe that many of the examples have a graded structure or involve graded constructions like adding variables etc... All notions of non-commutative Krull rings introduced in loc. cit. agree in case one considers rings satisfying polynomial identities (P.I. rings), as a matter of fact a P.I. ring Λ is a Krull ring if and only if it is a maximal order over a Krull domain R in some central simple algebra (c.s.a.) Σ over the field of fractions K of R. If R is moreover a graded Krull domain, then the fact that the invariants i.e. Picard group $Pic(R)$, class group $Cl(R)$, are determined by graded data leads naturally to the question whether results like this hold in the non-commutative case. One of the specific problems of graded nature that arises is to decide whether a graded R-order maximal amongst graded R-orders is a maximal order of Σ. The positive answer to this problem, given in this note, also provides an easy argument for the existence of maximal R-orders in Σ which are also graded R-algebras. Along the way we prove that gr-tame R-orders are tame R-orders of Σ in the sense of R. Fossum [7]. The general theory of graded orders, in particular that of gr-hereditary orders has been strictly limited to the necessary. In the final paragraph of this note we pay particular attention to orders over generalized Rees rings and deduce some exact sequences for the class groups involved. The graded rings in this note are $\mathbf{Z}$-graded but it is an easy exercise, using a result of [1], to derive most results in case the rings are graded by a group G which is a torision-free abelian group satisfying the ascending chain condition on cyclic subgroups.

1. Old hat and new trivialities

For some basic theory on $\mathbf{Z}$-graded rings we recommend the reading of [13]. If a Krull domain R is graded by an abelian group G with neutral element 0 then R_0 is a Krull domain because in $K=Q(R)$ we have $R_0=k\cap R$, where k is the field of fractions of R_0. Let $X^1(R)$, resp. $X_g^1(R)$, denote the set of prime, resp. graded prime, ideals of height 1 of R. Let K^g be the graded field of fractions of R; $K^g=(K^g)_0[T, T^{-1}]$ where deg $T=e$ is the lowest positive degree occurring in R. Since R is a Krull domain: $R=\bigcap_{P\in X^1(R)} Q_P(R)$ and since R is moreover graded we have: $R=K^g\cap(\bigcap_{P\in X^1(R)} Q_P(R))=\bigcap_{P\in X_g^1(R)} Q_P^g(R)$, where Q_P^g is just graded localization at P, cf. [13] i.e. inverting homogeneous elements which are not in P. If $P\in\operatorname{Spec} R$ then P_g stands for the (prime !) ideal generated by the homeogeneous elements of P. If $P\in X^1(R)$ then either $P=P_g$ or $P_g=0$; in the latter case $Q_P^g(R)=K^g$ since $P_g=0$ means $h(R)\cap P=\{0\}$. Gathering some results from [13], [17] we obtain that for each $P\in X_g^1(R)$ the localized ring $Q_P^g(R)$ is a *gr*-disrete valuation ring i.e. for every homogeneous $y\in K^g$ we have either $y\in Q_P^g(R)$ or $y^{-1}\in Q_P^g(R)$.

Then $Q_P(R)$ is a discrete valuation ring of K and the associated valuation v is "graded" in the sense that for $x\in R$ with homogeneous decomposition $x=x_1+\cdots+x_n$ we have $v(x)=\min\{v(x_i)\}$.

Lemma 1.1 (Van Oystaeyen, [18]). *If A is a P.I. ring graded by an arbitrary group G such that the center $Z(A)$ is a gr-field then A is an Azumaya algebra.*

Proof. cf. [ ]; note that the proof given there in the case $G=\mathbf{Z}$ remains valid for arbitrary groups G.

It is easy to see that a graded Azumaya algebra over a *gr*-field is the same as a *gr*-simple *gr*-Artinian $Z(A)$-algebra (*gr*-c.s.a.). In case $G=\mathbf{Z}$ the latter rings are exactly matrix rings over *gr*-skewfield i.e. of the form $A=M_n(\Delta[X, X^{-1}, \varphi])(d)$, where Δ is a skewfield, φ is an automorphism of Δ such that some φ^n, $n\in N$, is inner in Δ, $\Delta[X, X^{-1}, \varphi]$ is the skew polynomial ring defined by $X^n\lambda=\varphi^n(\lambda)X^n$, $n\in\mathbf{Z}$ and $(d)\in\mathbf{Z}^n$ defines the gradation on A as follows: $(A_m)_{ij}=\Delta X^{m_{ij}}$ where $m_{ij}=m+d_i-d_j$; $i, j\in\{1, \cdots, n\}$.

Fix notations as follows: R is a graded Krull domain with *gr*-field of fractions $K^g\approx k[T, T^{-1}]$ and field of fractions K, Σ is a c.s.a. over K containing a graded R-algebra Λ which is an R-order in Σ, let $A=K^g\otimes_R\Lambda$ be the graded ring of fractions of Λ. By the lemma, A is an Azumaya algebra over $k[T, T^{-1}]$ hence a maximal K^g-order in Σ and thus also a prime Dedekind P.I. ring in the sence of [16]. A graded R-subalgebra of A which is a finite R-module spanning Σ over K is called a graded order of Σ and if no other graded order properly contains it then it is called a *gr-maximal R-order* of Σ. Starting from an arbitrary c.s.a. over the field of fractions K of R it is clear that graded R-orders need not exist in Σ, indeed, an obvious condition on Σ is that it should represent on element of $Br^g K^g$, the graded Brauer group of K^g in the sense of [19]. One easily verifies that this condition is also sufficient.

Lemma 1.2. *Let Λ be a gr-maximal R-order of Σ then Λ is a reflexive (divisorial) R-module, i.e. $\Lambda=\Lambda^{**}$ where $\Lambda^*=\mathrm{Hom}_R(\Lambda, R)$.*

Proof. Since Λ is a finitely generated R-module, $\mathrm{HOM}_R(\Lambda, R)=\mathrm{Hom}_R(\Lambda, R)$ (cf, [13]) and thus Λ^* and also Λ^{**} are graded R-modules. Following the lines of proof of Proposition 1.3 in [2], taking into account that the Noetherian hypothesis assumed there is not needed in this proof, it follows that $\Lambda^{**}\supset\Lambda$ is a graded order consequently $\Lambda=\Lambda^{**}$.

A graded R-order Λ is said to be *gr*-hereditary if graded left ideals of Λ are projective. A graded R-order Λ is *gr*-tame if $Q^g_P(\Lambda)$ is *gr*-hereditary for each $P\in X^1_g(R)$. We omit a general treatment of *gr*-hereditary orders here, (there is some interest in this when R is a *gr*-Dedekind or in particular a generalized Rees ring), but focus on *gr*-maximal orders over *gr*-discrete valuation rings in the following paragraph.

Let S be a *gr*-D.V.R. in K^g and let Γ be a *gr*-maximal S-order in Σ. Mimicking classical results of M. Deuring [5], p. 74 and p. 108 in particular, we obtain that Γ is a *gr*-local ring in the sense that it has a unique *gr*-maximal ideal, M say, and Γ/M is a *gr*-c.s.a..

Lemma 1.3. *Let S and Γ be as above. If E is a finitely generated graded Γ-module then: $hd^g_\Gamma E=hd^g_S E$ where hd^g_Γ, resp. hd^g_S, denotes the homological dimension in Γ-gr, resp. S-gr.*

Proof. In [13] the relations between certain dimensions (global dimension, weak (flat) dimension, Krull dimension...) and their analogous in the category of graded modules has been studied extensively. Combining these results with an easy graded version of Theorem 2.2 in [2] (note that S is Noetherian now!) the result follows.

Corollary 1.4. *Γ is gr-hereditary.*

Proposition 1.5. *A graded S-order Γ is gr-maximal if and only if the graded Jacobson radical $J^g(\Gamma)$ of Γ is a gr-maximal ideal of Γ whilst Γ is gr-hereditary, if and only if $J^g(\Gamma)$ is a gr-maximal ideal which is projective as a left (or right) Γ-module.*

Proof. Exercise, following Theorem 2.3 of [2]. One can also produce graded versions of J.C. Robson's characterization of P.I. Dedekind rings here.

REMARK 1.6. Using the obvious properties of the trace map one can prove that the different of a *gr*-maximal order Λ is a graded ideal. So, if a prime ideal P divides the different (i.e. if P is of the first kind this means that P^2 divides Λ $(P\cap Z(\Lambda))$ then P_g divides the different.

2. *Gr*-tame and *gr*-maximal orders over Krull domains

The main results of this section establish that the notions of *gr*-tame and *gr*-maximal R-orders reduce to tame and maximal orders which are moreover graded R-algebras. A consequence of this is that *gr*-maximal orders over Krull domains are actually non-commutative Krull rings in the sense of M. Chamarie [3], E. Jespers, L. Le Bruyn, P. Wouters [9], H. Marubayashi [11]. Another consequence is that the notion of divisorial ideal in each of the cited papers coincides with R. Fossum's definition used in [7]. This leads to $Cl(\Delta) = Cl^g(\Lambda)$ and $\boldsymbol{Pic}^g(\Lambda) = \boldsymbol{Pic}(\Lambda)$.

One may verify that a graded order Λ over a graded Krull domain R is *gr*-maximal if and only if $\Lambda = \Lambda^{**} = \bigcap_{P \in X_g^1(R)} Q_P^g(\Lambda)$ and each $Q_P^g(\Lambda)$ is a *gr*-maximal $Q_P^g(R)$-order. Let R, K^g, K, Λ, A, Σ be as in section 1.

Proposition 2.1. *If Λ is a gr-maximal order then Λ is a maximal R-order in Σ.*

Proof. Since $\Lambda = \bigcap_{P \in X^1(R)} Q_P(\Lambda)$ it will be sufficient to prove that each $Q_P(\Lambda)$ is a maximal $Q_P(R)$-order. If $P_g = 0$ then $Q_P(\Lambda) = Q_{PK}(A)$ is an Azumaya algebra over $Q_P(R)$ hence certainly a maximal order. If $P = P_g$ i.e. $P \in X_g^1(R)$ then $Q_P(\Lambda) = Q_{P'}(Q_P^g(\Lambda))$, where $P' = PQ_P^g(R)$. Now $\Lambda' = Q_P^g(\Lambda)$ is a *gr*-maximal order over $R' = Q_P^g(R)$ which is a *gr*-D.V.R. If $\mathcal{P}$ is a prime ideal of Λ' then $\mathcal{P} \cap R' \neq 0$ is a prime ideal of R'. Since P' is a minimal prime ideal of R' we have $\mathcal{P} \cap R' \not\subset P'$ unless $\mathcal{P} \cap R' = P'$. The remark preceding, Lemma 1.3. states that Λ' has a unique *gr*-maximal ideal M' and clearly $M' \cap R' = P'$. Incomparability of prime ideals of Λ' lying over P, yields that M' is the unique prime ideal of Λ' lying over P'. Localizing Λ' centrally at P' we obtain that $Q_{P'}(\Lambda') = Q_P(\Lambda)$ has unique maximal ideal $M'Q_{P'}(\Lambda) \approx M' \otimes_{R'} Q_{P'}(R')$. Since Λ' is *gr*-maximal, M' is a projective Λ'-module and therefore $M'Q'_P(\Lambda')$ is a projective $Q_P(\Lambda)$-module, proving that $Q_P(\Lambda)$ is a maximal $Q_P(R)$-order.

Corollary 2.2. *A gr-maximal order Λ is a non-commutative Krull ring.*

Proof. A P.I. ring Λ is a central Ω Krull ring (cf. [9]) if and only if it is a (symmetric) maximal order over a Krull domain, if and only if it is a Krull ring in the sense of [3] or [11]. The proof of these equivalences is an easy reformulation of a result of M. Chamarie in [3].

Theorem 2.3. *If Λ is a gr-tame R-order then Λ is a tame R-order in Σ.*

Proof. If $P \in X^1(R)$ such that $P_g = 0$ then $Q_P(\Lambda)$ is an Azumaya algebra over the D.V.R. $Q_P(R)$ (as in 2.1.) and therefore a fortiori hereditary (even maximal). If $P \in X_g^1(R)$ then $\Lambda' = Q_P^g(\Lambda)$ is a *gr*-hereditary $R' = Q_P^g(R)$-

order over the *gr*. D.V.R. $Q_P^g(R)$. Putting $P' = PQ_P^g(R)$, we only have to check that $Q_{P'}(\Lambda')$ is a hereditary $Q_{P'}(R')$-order. Argumentation similar to the proof of 1.2. yields that the graded Jacobsen radical $J^g(\Lambda')$ localizes to the Jacobson radical $J(Q_P(\Lambda))$ (note: in the P.I. case the Jacobson radical is the intersection of the maximal ideals; a *gr*-maximal ideal is the intersection of the maximal ideals containing it, see [15]). Since $J^g(\Lambda')$ is projective it follows that $J(Q_P(\Lambda))$ is projective too and thus $Q_P(\Lambda)$ is left and right hereditary.

Recall that a *fractional ideal* of Λ in Σ is a two-sided Λ-submodule I of Σ such that $cI \subset \Lambda$ for some $c \in R$. A *divisorial* ideal will be a fractional ideal which is a reflexive Λ-module. By the foregoing results these divisorial ideals form a group $\boldsymbol{D}(\Lambda)$ in case Λ is a *gr*-maximal order over R. Let $\boldsymbol{P}_R(\Lambda)$ be the subgroup of $\boldsymbol{D}(\Lambda)$ consisting of the ideals Λc with $c \in \boldsymbol{R}$. The central *class group* $Cl(\Lambda)$ is defined to be $\boldsymbol{D}(\Lambda)/\boldsymbol{P}_R(\Lambda)$. Let $\boldsymbol{Pic}(\Lambda)$ be the subgroup of $Cl(\Lambda)$ generated by the images of the invertible ideals i.e. fractional ideals I such that there is a fractional ideal J such that $IJ = JI = \Lambda$. Note that that $\boldsymbol{Pic}(\Lambda)$ is different from $Pic(\Lambda)$ defined by using the bimodule isomorphism classes of invertible Λ-bimodules (for *Pic* of orders see also A. Fröhlich [8]). There is an epimorphism $\boldsymbol{Pic}(\Lambda) \to Pic(\Lambda)$. In the graded case one notes that, if I is a graded fractional ideal of Λ then I^{**} is graded too and $I \subset I^{**} \subset A \subset \Sigma$. It is clear how the graded analogous of the definitions introduced above have to be defined. We obtain a subgroup $\boldsymbol{D}^g(\Lambda)$ of $\boldsymbol{D}(\Lambda)$. It is not hard to see that an ideal Λc, with $c \in K$, is graded if and only if c is homogeneous. It follows that $Cl^g(\Lambda)$ is a subgroup of $Cl(\Lambda)$ and $\boldsymbol{Pic}^g(\Lambda)$ is a subgroup of $\boldsymbol{Pic}(\Lambda)$ (Also $Pic^g(\Lambda) \subset Pic\Lambda$ if one takes isomorphism classes for graded bimodule morphisms and not necessarily only these of degree zero!). Since in the *P.I.* case the theory developed in [3], [9], [11] is valid, it follows that $\boldsymbol{D}(\Lambda)$ is the free abelian group generated by $X^1(\Lambda)$. In formally the same way one may derive that $\boldsymbol{D}^g(\Lambda)$ is the free abelian group generated by $X_g^1(\Lambda)$ but this also follows from the definition of $D^g(\Lambda)$ and the property of $\boldsymbol{D}(\Lambda)$.

Lemma 2.4. *If Λ is a gr-maximal order over R then*: $Cl(R) \hookrightarrow Cl(\Lambda)$ *and* $Cl^g(R) \hookrightarrow Cl^g(\Lambda)$.

Proof. The map $Cl(R) \to Cl(\Lambda)$ derives from ϕ: $\boldsymbol{D}(R) \to \boldsymbol{D}(\Lambda)$ which is given by $p \in X^1(R) \to P^{e_P}$ where $P \in X^1(\Lambda)$ is lying over p (unique as such) and e_P is the ramification index defined by: $Q_P(\Lambda)p = \prod_P (Q_P(\Lambda)P)^{e_P}$.

Noting that $\Lambda c \cap K^g = Rc$ if $c \in K^g$. one easily deduces the statement. The graded statement may be proven in a similar way.

Proposition 2.5. *If Λ is a gr-maximal order over R then* $Cl^g(\Lambda) = Cl(\Lambda)$

and $\boldsymbol{Pic}^g(\Lambda)=\boldsymbol{Pic}(\Lambda)$.

Proof. The kernel of $Cl(\Lambda)\to Cl(A)$ is generated by the $P\in X^1(\Lambda)$ containing a homogeneous element but since $P_g\neq 0$ for a $P\in X^1(\Lambda)$ entails $P=P_g\in X_g^1(\Lambda)$ it follows that $Cl^g(\Lambda)$ is equal to this kernel.
Since A is an Azumaya algebra over a P.I.D. it follows that $Cl^g(\Lambda)=Cl(\Lambda)$. The fact that $Pic^g(\Lambda)=Pic(\Lambda)$ follows easily.

REMARK 2.6.
1. If Δ is another maximal order over R then $\boldsymbol{D}(\Delta)\cong\boldsymbol{D}(\Lambda)$ and this isomorphism is given by the conductor $(\Delta:\Lambda)$. So if Σ represents over K an element of Br^gK^g then the divisor group of any maximal order over R is given by graded data.
2. The structure of $A=M_n(\Delta[X, X^{-1}, \varphi])$ (d), $\Sigma=M_n(\Delta(X, X^{-1}, \varphi)$ for some skewfield Δ allows to reduce the study a gr-maximal orders in Σ to the study of gr-maximal orders in a skewfield $\Delta(X, X^{-1}, \varphi)$.
3. For tame orders R. Fossum introduced a class group $W(\Lambda)$ in [7], and he conjectures that the cannonical map $W(\Lambda)\to W(\Lambda[X])$ is an epimorphism. For Cl it is not hard to prove that $Cl(\Lambda)=Cl\,\Lambda[X]$, this actually follows from a nice general result of [10]. Now we conjecture that there is an epimorphism $Cl(\Lambda)\to W(\Lambda)$ (much like $\boldsymbol{Pic}(\Lambda)\to Pic(\Lambda)$) and if this is so then R. Fossum's conjecture also holds.

3. Graded orders over generalized Rees rings

Recall that a ring R graded by a group G is said to be strongly graded if $R_\sigma R_\tau=R_{\sigma\tau}$ holds for every $\sigma, \tau\in G$. The generalized crossed product theorem, cf. [14], states that a strongly graded ring R is necessarily of the form $R_0\langle G, \phi, \mathcal{F}_\phi\rangle$ where $\phi: G\to Pic(R_0)$ is a group homomorphism, $\mathcal{F}_\phi$ is a factor set associated to ϕ and the gradation on R is defined by: $R_\sigma\cong P_\sigma$, where P_σ represents $\phi(\sigma)$. In case $G=\boldsymbol{Z}$ this yields that $R\cong\sum_{n\in Z} I^nX^n$ for some $I\in PicR_0$. Extending the terminology of [17] to the Krull case we will say that a strongly graded Krull domain is a generalized Rees ring. Actually it is possible to consider divisorially strongly graded rings satisfying the condition $(RR_\sigma)^{**}(RR_\tau)^{**}=(RR_{\sigma\tau})^{**}$ for all $\sigma, \tau\in G$. It turns out that these have a crossed product structure with respect to some $\phi: G\to Cl(R_0)$, however we will not go into these details here. Recall from [13].

Lemma 3.1. *If R is strongly ($\boldsymbol{Z}$-) graded then*:
1. *The categories R-gr and R_0-mod are equivalent, the equivalences being given bij the functors $(-)_0$ and $(-)\underset{R_0}{\otimes}R$.*

2. *Every graded left ideal is generated as a left ideal by its homogenous part of degree zero.*
3. *Every graded localization in R-gr is induced by a kernel functor in R_0-mod.*

Proof. cf [13] except for 3. which is an easy consequence of 1. Now let R, K^g, K, Λ, A, Σ be as in Section 1 but we suppose moreover that R is strongly graded i.e. $RR_1=R$.

Lemma 3.2. *If R is a strongly graded Krull domain, then:*
1. *Λ is a central extension of Λ_0.*
2. *Λ_0 is a maximal order over the Krull domain R_0 in the c.s.a. A_0.*
3. *Λ is a left (and right) flat Λ_0-module.*
4. *The (central) extension of Krull rings $\Lambda_0 \hookrightarrow \Lambda$ satisfies condition P.D.E. (no blowing up).*
5. *The map φ: $\boldsymbol{D}(\Lambda_0) \to \boldsymbol{D}(\Lambda)$, $I \mapsto (\Lambda I)^{**}$ defines group homomorphisms $Cl(\Lambda_0) \to Cl(\Lambda)$ and $\boldsymbol{Pic}(\Lambda_0) \to \boldsymbol{Pic}(\Lambda)$. Actually $(\Lambda I)^{**}=\Lambda I$ for every $I \in \boldsymbol{D}(\Lambda_0)$.*

Proof.
1. If $x \in \Lambda_h$ then $x \in R_h R_{-h} \Lambda_h \in R\Lambda_0$.
2. If I is an ideal of Λ_0 then ΛI is a proper graded ideal of Λ and thus $[\Lambda I\colon \Lambda I]_A=\Lambda$ follows from the fact that Λ is a *gr*-maximal order. Consequently we have $[I\colon I]_{A_0} \subset \Lambda_0$. That A_0 is a c.s.a. is well-known, cf [19]. It is satrightforward to deduce that Λ_0 is a maximal R_0-order in A_0. In fact $\Lambda \cong \Lambda_0 \otimes_{R_0} R$.
3. Since Λ is a generalized crossed product we have that Λ_n is isomorphic in R_0-mod to some finitely generated projective R_0-module. From $\Lambda=\oplus_{n\in Z}\Lambda_n$ it then follows that Λ is a left (and right) flat Λ_0-module.
4. Follows from 3. in a very general context; here it also follows easily from Lemma 3.1., 2.
5. If $\kappa^g = \inf_{P \in X_g^1(\Lambda)} \kappa_P^g$ is the graded kernel functor with graded filter $L(\kappa^g)$ consisting of graded left ideals containing a graded ideal which is not contained in a prime ideal of height 1, then localization at κ coincides with localization at $\kappa_0 = \inf_{P \in X^1(\Lambda_0)} \kappa_P$ because of Lemma 3.1.3.

Now the graded divisorial ideals for Λ are exactly those fractional ideals I such that $Q_\kappa(I)=I$. Since $I=\Lambda I_0$ and $Q_\kappa(I)=Q_{\kappa_0}(I)=Q_{\kappa_0}(I_0) \underset{\Lambda_0}{\otimes} \Lambda = \Lambda\, Q_{\kappa_0}(I_0)$ it follows that the functor $(-)_0$ (as well as its adjoint $(-)\underset{\Lambda_0}{\otimes}\Lambda)$) takes κ-closed ideals to κ_0-closed ideals. Therefore if $I \in \boldsymbol{D}(\Lambda_0)$ then $\Lambda I \in \boldsymbol{D}(\Lambda)$. All other claims are easily verified.

Theorem 3.3. *The following is an exact and commutative diagram of abelian groups:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & Cl(R) & \xrightarrow{i} & Cl(\Lambda) & \xrightarrow{\gamma} & \bigoplus_{P\in X_g^1(\Lambda)} Z/e_P Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & Cl(R_0) & \xrightarrow{i_0} & Cl(\Lambda_0) & \xrightarrow{\gamma_0} & \bigoplus_{P_1\in X^1(\Lambda_0)} Z/e_{P_0} Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \langle I_0\rangle & & \langle \Lambda_0 I_0\rangle & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $e_P=e_{P_0}$ is as in Lemma 2.4.

Proof. The map γ is defined by: $I=P_1^{\nu_1}, \cdots, P_r^{\nu_r} \mapsto (\nu_1 \bmod e_{P_1}, \cdots, \nu_r \bmod e_{P_r})$. This map is well defined on classes in $Cl(\Lambda)$ because I and Ic, with $c\in R$ have the same image under γ.
Note that in defining γ we have used the fact that $Cl(\Lambda)=Cl^g(\Lambda)$. If $\gamma(\bar{I})=0$ for some $\bar{I}\in Cl^g(\Lambda)$ then e_i divides ν_i for $i=1, \cdots, r$, where $e_i=e_{P_i}$. Obviously we then have $\bar{I}=i(p_1^{\nu_1/e_1}\cdot \cdots \cdot p_r^{\nu/e_r})$ where $p_i=p_i\cap R$.
Similar argumentation proves the exactness of the second row in the diagram. Exactness of the first column is a direct consequence of the fact that R is a generalized Rees ring, cf. [17] for a detailed proof.
The diagram of extensions:

$$(*)\qquad \begin{array}{ccc}
R & \longrightarrow & \Lambda \\
\uparrow & & \uparrow \\
R_0 & \longrightarrow & \Lambda_0
\end{array}$$

consists of extensions satisfying the P.D.E. condition. Also it is clear that the ramification numbers e_P are equal to the e_{P_0} because graded ideals of R, Λ are generated by their parts in R_0, Λ_0 resp. The diagram $Cl(*)=Cl^g(*)$ is obviously commutative. Now the class of $P_{0,1}^{t_1}\cdot \cdots \cdot P_{0,q}^{t_q}$ in $Cl(\Lambda_0)$ maps to the class of $P_1^{t_1}\cdot \cdots \cdot P_q^{t_q}$ in $Cl(\Lambda)$ where $P_i=\Lambda P_{0,i}$ $i=1, \cdots, q$. Now the latter will be trivial in $Cl(\Lambda)$ if $e_i|t_i$, $i=1, \cdots, q$ and the class of $P_1^{t_1/e_1}\cdot \cdots \cdot P_q^{t_q/e_q}$ maps to 0 in $Cl(\Lambda)$, where $p_i=P_i\cap R$, $i=1, \cdots, q$. Putting $P_{0,i}$ equal to $p_i\cap R_0$, we may write the foregoing as $P_{0,1}^{t_1/e_1}\cdot \cdots P_{0,r}^{t_q/e_q}=I_0^m c$ for some $m\in N$, $c\in K^g$. Since the ramification numbers of P_0 over the $p_{0,i}$ equal e_i, for each $i=1, \cdots, q$, it follows that $P_{0,1}^{t_1}\cdot \cdots \cdot P_{0,q}^{t_q}=(\Lambda_0 I_0)^m c$ and exactness of the second column follows from the fact that $\Lambda_0 I_0$ is divisorial (here it is even invertible because $I_0\in Pic\, R_0$ and $R_0=Z(\Lambda_0)$).

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Department of Mathematics
University of Antwerp, U.I.A.
Antwerp, Universiteitsplein 1
B-2610 Wilrijk
Belgium