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<tr>
<td>Author(s)</td>
<td>Shigekawa, Ichirō</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 21(3) P.621–P.627</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1984</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/4535">https://doi.org/10.18910/4535</a></td>
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<td>DOI</td>
<td>10.18910/4535</td>
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Osaka University
ON A QUASI EVERYWHERE EXISTENCE OF THE LOCAL TIME OF THE 1-DIMENSIONAL BROWNIAN MOTION

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(Received November 5, 1983)

1. Introduction

Recently quasi everywhere properties of the Brownian motion were discussed by many authors; Williams considered the quadratic variation (see [9]) and Fukushima [3] considered the nowhere differentiability, Lévy's Hölder continuity, the law of iterated logarithm etc. By the way, the local time plays an important role in stochastic analysis. The existence of the local time of the 1-dimensional Brownian motion was proved by Trotter [10]. He proved that the local time of the 1-dimensional Brownian motion exists almost everywhere (a.e.) with respect to the Wiener measure. In this paper we shall prove that it exists quasi everywhere (q.e.) with respect to the Ornstein-Uhlenbeck process on the Wiener space.

Fukushima's study is based on a concept of capacity related to the Ornstein-Uhlenbeck process. The term "quasi everywhere" means "except on a set of capacity 0". A set of capacity 0 is characterized by the Ornstein-Uhlenbeck process as follows (see [2], [6]). Let \( W_1 \) be a set of all continuous paths \( w: [0, \infty) \to \mathbb{R} \) vanishing at 0 with the compact uniform topology and \( \mu \) be the Wiener measure on \( W_1 \). Let \( (X_t)_{t \geq 0} \) be a \( W_1 \)-valued Ornstein-Uhlenbeck process with the initial distribution \( \mu \) defined on an auxiliary probability space \( (\Omega, \mathcal{F}, P) \). Then for any \( A \subset W_1 \), \( A \) is of capacity 0 if and only if

\[
P[X_t \notin A \quad \text{for all } t > 0] = 1.
\]

On the other hand, by the Tanaka formula the local time \( (\phi(t, a)) \) of a Brownian motion \( (X_t(t))_{t \geq 0} \) is given by

\[
\phi(t, a) = (X_t(t) - a)^+ - (X_t(0) - a)^+ - \int_0^t 1_{(a, \infty)}(X_s(s)) X_s(ds)
\]

(cf. [4], [8]). Then our main theorem is stated as follows.

* This research was partially supported by Grant-in-Aid for Scientific Research.
**Theorem.** There exists a continuous version of $(\phi(\tau, t, a))$ in $(\tau, t, a)$ with respect to the measure $P$.

By the above theorem, we can show the quasi everywhere existence of the local time. To see this, denote a continuous version also by $\phi(\tau, t, a)$. Then, for fixed $\tau > 0$,

$$\int_0^t f(X_s(s))ds = 2 \int_\tau^T \phi(\tau, t, a)f(a)da, \quad \forall t \geq 0, \forall f \in C_c^\infty(\mathbb{R})$$

P-a.e. (the almost everywhere existence of the local time). By the continuity of $(\tau, t) \mapsto X_s(t)$ and $(\tau, t, a) \mapsto \phi(\tau, t, a)$, (1.2) holds for all $\tau > 0$ P-a.e. which asserts (1.1). Hence the local time exists quasi everywhere.

We will give a proof of the theorem in Section 2.

2. **Proof of the theorem**

First we give a realization of the Ornstein-Uhlenbeck process $(X_r(t))_{t \geq 0}$ on the probability space $(\Omega, \mathcal{F}, P)$ as follows. Let $(X_0(t))_{t \geq 0}$ be a 1-dimensional Brownian motion and $(W(\tau, t))_{\tau, t \geq 0}$ be a two parameter Brownian motion on $(\Omega, \mathcal{F}, P)$, i.e., $(W(\tau, t))_{\tau, t \geq 0}$ is a Gaussian process with mean 0 and the covariance given by

$$E[W(\tau, t)W(\sigma, s)] = (\tau \wedge \sigma)(t \wedge s)$$

where $E$ denotes the expectation relative to $P$. Assume moreover that $(X_0(t))_{t \geq 0}$ and $(W(\tau, t))_{\tau, t \geq 0}$ are independent. Then the Ornstein-Uhlenbeck process $(X_r(t))_{t \geq 0}$ is given by

$$X_r(t) = e^{-\gamma t}X_0(t) + \int_0^t e^{-\gamma(t-s)/2} W(ds, t)$$

where the integral is the stochastic integral with respect to a martingale $\gamma \mapsto W(\tau, t)$. Set $\mathcal{F}_\tau = \sigma\{X_0(s), W(\tau, s)|s \leq t, \tau \geq 0\}$. Then for fixed $\tau \geq 0$, $(X_r(t))_{t \geq 0}$ is an $(\mathcal{F}_\tau)$-Brownian motion and for fixed $\tau, \sigma \geq 0$, the quadratic variation of $(X_r(t))_{t \geq 0}$ and $(X_\sigma(t))_{t \geq 0}$ is given by

$$\langle X_r, X_\sigma \rangle_t = \exp \left(-\frac{1}{2}|\tau - \sigma|\right)t.$$  

Hereafter we consider this specific Ornstein-Uhlenbeck process.

Define $\varphi(\tau, t, a)$ by

$$\varphi(\tau, t, a) = \int_0^t \mathbb{1}_{[\tau, \infty)}(X_r(s))X_s(ds).$$

To prove the theorem it suffices to show that there exists a continuous version
of \( \psi(\tau, t, a) \), i.e., there exists a continuous process \( (\psi(\tau, t, a)) \) in \((\tau, t, a) \) \( P \)-a.e. such that for fixed \((\tau, t, a) \in [0, \infty) \times [0, \infty) \times R \), \( \psi(\tau, t, a) = \tilde{\psi}(\tau, t, a) \) \( P \)-a.e. Since for fixed \( \tau \geq 0, a \in R \), \( t \rightarrow \psi(\tau, t, a) \) is a continuous process, we regard \( \psi(\tau, \cdot, a) \) as a \( C([0, \infty) \rightarrow R) \)-valued random variable. Hence if we show the following proposition, we can get a desired result by Kolmogorov's theorem ([1]).

**Proposition.** For any \( T > 0 \), there exists a constant \( K = K(T) > 0 \) such that for any \( a, b \in R \) and \( 0 \leq \tau, \sigma \leq T \),

\[
E[\sup_{s \leq t \leq \tau} |\psi(\tau, t, a) - \psi(\sigma, t, b)|^p] \leq K(|\tau - \sigma|^{\frac{p}{2}} + |a - b|^p).
\]

For the proof, we need the following lemmas.

**Lemma 1.** Let \((B(t))\) be a 1-dimensional Brownian motion. Then for any \( p \geq 1 \) and any \( T > 0 \), there exists a constant \( C = C(p, T) > 0 \) such that for any \( a, b \in R \), \( a \leq b \),

\[
E[\left\{ \int_0^T 1_{(a,b]}(B(t))dt \right\}^p] \leq C(b-a)^p.
\]

The proof is easy and will be omitted.

**Lemma 2.** Let \((B_1(t), B_2(t))\) be a 2-dimensional Brownian motion starting at 0, \( m \) be a positive constant and \( C_a, a \in R \), be a subset of \( R^2 \) defined by

\[
C_a = \{(x, y) \in R^2 | (y + mx + a)(y - mx + a) \leq 0\}.
\]

Then it holds that for \( T > 0, \ p \geq 1 \),

\[
E[\left\{ \int_0^T 1_{C_a}(B_1(t), B_2(t))dt \right\}^p] \leq E[\left\{ \int_0^T 1_{C_a}(B_1(t), B_2(t))dt \right\}^p].
\]

**Proof.** Define a stopping time \( \eta \) by

\[
\eta = \inf\left\{ t \geq 0 | B_2(t) = \frac{1}{2} a \right\}
\]

and a continuous process \((\tilde{B}_2(t))\) by

\[
\tilde{B}_2(t) = \begin{cases} 
B_2(t) & \text{if } t < \eta \\
\frac{1}{2} a - B_2(t) & \text{if } t \geq \eta.
\end{cases}
\]

Then by the reflection principle of the Brownian motion, \((B_1(t), \tilde{B}_2(t))\) is also a 2-dimensional Brownian motion starting at 0. Moreover it holds that

\[\text{(*) The proof is due to H. Kaneko, who simplified the author's original one.}\]
Hence we can easily get a desired result. □

Proof of the Proposition. Set

\[ I = E\left[ \sup_{t \leq s \leq T} |\psi(\tau, t, \sigma) - \psi(\tau, t, b)|^2 \right] \quad \text{and} \]

\[ J = E\left[ \sup_{t \leq s \leq T} |\psi(\tau, t, b) - \psi(\sigma, t, b)|^2 \right]. \]

Then we easily have

\[ E\left[ \sup_{t \leq s \leq T} |\psi(\tau, t, a) - \psi(\sigma, t, b)|^2 \right] \leq 2^k (I + J). \]

Firstly we give the estimate for \( I \). Without loss of generality, we may assume \( a \leq b \). Then by the Burkholder-Davis-Gundy inequality and Lemma 1, we have

\[ I \leq c_1 E\left[ \int_0^T 1_{(a, b)}(X_s(t)) X_s(ds) \right]^2 \]

\[ \leq c_1 E\left[ \int_0^T 1_{(a, b)}(X_s(t)) dt \right]^2 \]

\[ \leq c_2 |a - b|^2. \]

Secondly we give the estimate for \( J \). By (2.2) and the Burkholder-Davis-Gundy inequality, we have

\[ J \leq c_3 E[\langle \psi(\tau, \cdot, b) - \psi(\sigma, \cdot, b) \rangle_T^2] \]

\[ = c_3 E\left[ \int_0^T 1_{(a, b)}(X_s(t)) dt - 2 e^{-|\tau - \sigma|/2} \int_0^T 1_{(a, b)}(X_s(t)) dt + \int_0^T 1_{(a, b)}(X_s(t)) dt \right]^2 \]

\[ = c_3 E\left[ \int_0^T 1_{(a, b)}(X_s(t)) - 1_{(a, b)}(X_s(t)) dt + 2(1 - e^{-|\tau - \sigma|/2}) \int_0^T 1_{(a, b)}(X_s(t)) dt \right]^2 \]

\[ \leq c_4 (1 - e^{-|\tau - \sigma|/2})^2 E\left[ \int_0^T 1_{(a, b)}(X_s(t)) dt \right]^2 \]

\[ := J_1 + J_2. \]

As for \( J_2 \), we easily obtain

\[ c_1, c_2, \ldots \text{ are positive constants which depend only on } T. \]
As for $J_1$, we define a subset $A_b \subseteq \mathbb{R}^2$ by

$$A_b = \{(x, y) \in \mathbb{R}^2 | (x-b)(y-b) \leq 0\}.$$ 

Then we get

$$J_1 = c_b E[\{ \int_0^T 1_{A_b}(X_\tau(t), X_\sigma(t)) dt \}]^5.$$ 

On the other hand, we define a matrix $U(\tau, \sigma)$ by

$$U(\tau, \sigma) = (U_{ij}(\tau, \sigma)) = \frac{1}{2\sqrt{1-e^{-|\tau-\sigma|/2}}} \begin{pmatrix} \sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} & -\sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} \\ -\sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} \end{pmatrix}$$

and $2$-dimensional continuous process $(B_1(t), B_2(t))$ by

$$B_i(t) = U_{ii}(\tau, \sigma) X_\tau(t) + U_{i2}(\tau, \sigma) X_\sigma(t), \quad i = 1, 2.$$ 

Then it is easy to see that $(B_1(t), B_2(t))$ is a $2$-dimensional Brownian motion starting at $0$. Moreover set

$$\hat{B}_i(t) = \sum_{j=1}^2 O_{ij} B_j(t), \quad i = 1, 2,$$

where $O = (O_{ij})$ is an orthogonal matrix defined by

$$O = (O_{ij}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$ 

Then $(\hat{B}_1(t), \hat{B}_2(t))$ is also a $2$-dimensional Brownian motion starting at $0$ by the rotation invariance. Thus we have

$$J_1 = c_b E[\{ \int_0^T 1_{OU(\tau, \sigma)A_\tau}(\hat{B}_1(t), \hat{B}_2(t)) dt \}]^5.$$ 

Since

$$(OU(\tau, \sigma))^{-1} = \frac{1}{2} \begin{pmatrix} \sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} \\ -\sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} \end{pmatrix},$$

a subset $OU(\tau, \sigma)A_\tau$ is given by
Let \( \sigma, \tau \) be any\( \in \mathbb{R}^2 \) by
\[
S_R = \{(x, y) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} \leq R\}
\]
\[
V_R = \{(x, y) \in \mathbb{R}^2 | |y| < \sqrt{\frac{1 - e^{-|\tau - \sigma|}}{1 + e^{-|\tau - \sigma|} R}}\},
\]
then it is easy to see that \( \text{OU}(\tau, \sigma) \mathcal{A}_0 \subseteq S_R \cup V_R \). Let \( \eta_R \) be a stopping time given by
\[
\eta_R = \inf \{t \geq 0 | (\hat{B}_1(t), \hat{B}_2(t)) \in S_R\},
\]
then we have
\[
J_1 \leq c_5 E[\{ \int_0^T \chi_{S_R}(\hat{B}_1(t), \hat{B}_2(t))dt\}^\frac{5}{2}] + c_6 E[\{ \int_0^T \chi_{V_R}(\hat{B}_1(t), \hat{B}_2(t))dt\}^5]
\]
\[
\leq c_5 T^5 \mathbb{P}[\eta_R \leq T] + c_6 \left( \sqrt{\frac{1 - e^{-|\tau - \sigma|}}{1 + e^{-|\tau - \sigma|} R}} \right)^5
\]
\[
\leq c_7 (T^{\frac{5}{2}} \mathbb{P}[\eta_R \leq T] + |\tau - \sigma|^{5/2} R^5).
\]
Since \( (\sqrt{\hat{B}_1(t)^2 + \hat{B}_2(t)^2}) \) is a Bessel process with index 2, the following fact is well-known (see e.g., Itô-McKean [5]). For any \( \alpha > 0 \),
\[
E[e^{-\alpha \eta_R}] = \frac{I_0(0)}{I_0(\sqrt{2\alpha R})},
\]
where \( I_0 \) is the modified Bessel function:
\[
I_0(x) = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha x)}{\Gamma(m+1)} \left( \frac{2}{x} \right)^{2m}.
\]
Hence by the Chebyshev inequality,
\[
\mathbb{P}[\eta_R \leq T] \leq e^T E[e^{-\alpha \eta_R}] = e^T \frac{1}{I_0(\sqrt{2 R})} \leq c_8 R^{-\frac{5}{2}}.
\]
Then, by setting \( R = |\tau - \sigma|^{-1/20} \), we get
\[
J_1 \leq c_9 (R^{-\frac{5}{2}} + |\tau - \sigma|^{5/2} R^5) \leq 2c_9 |\tau - \sigma|^{\frac{5}{4}}.
\]
This completes the proof. □

**Corollary.** For \( w \in W_0^6 \), set \( Z_w = \{ t > 0 \mid w(t) = 0 \} \). Then \( Z_w \) has a continuum cardinal number and its Lebesgue measure is 0 for q.e. \( w \).

Proof. Let \( \Omega_0 \) be a set of all \( \omega \in \Omega \) which satisfy (1.2) for all \( \tau > 0 \). Then on \( \Omega_0 \), we have

\[
\phi(\tau, t, 0) = \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^t 1_{(\varepsilon, \varepsilon)}(X_\tau(s))ds.
\]

Hence it holds that

\[
\phi(\tau, t, 0) = \int_0^t 1_{\Omega}(X_\tau(s))\phi(\tau, ds, 0)
\]
and

\[
\int_0^t 1_{\Omega}(X_\tau(s))ds \leq 4\varepsilon \frac{1}{4\varepsilon} \int_0^t 1_{(\varepsilon, \varepsilon)}(X_\tau(s))ds \to 0 \quad (\text{as } \varepsilon \to 0).
\]

Then the rest is easy. □

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**References**


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