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ON A QUASI EVERYWHERE EXISTENCE OF THE LOCAL TIME OF THE 1-DIMENSIONAL BROWNIAN MOTION

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1. Introduction

Recently quasi everywhere properties of the Brownian motion were discussed by many authors; Williams considered the quadratic variation (see [9]) and Fukushima [3] considered the nowhere differentiability, Lévy's Hölder continuity, the law of iterated logarithm etc. By the way, the *local time* plays an important role in stochastic analysis. The existence of the local time of the 1-dimensional Brownian motion was proved by Trotter [10]. He proved that the local time of the 1-dimensional Brownian motion exists almost everywhere (a.e.) with respect to the Wiener measure. In this paper we shall prove that it exists *quasi everywhere* (q.e.) with respect to the *Ornstein-Uhlenbeck process* on the Wiener space.

Fukushima's study is based on a concept of *capacity* related to the Ornstein-Uhlenbeck process. The term "quasi everywhere" means "except on a set of capacity 0". A set of capacity 0 is characterized by the Ornstein-Uhlenbeck process as follows (see [2], [6]). Let W_0^1 be a set of all continuous paths $w: [0, \infty) \rightarrow \mathbf{R}$ vanishing at 0 with the compact uniform topology and μ be the Wiener measure on W_0^1 . Let $(X_\tau)_{\tau \geq 0}$ be a W_0^1 -valued Ornstein-Uhlenbeck process with the initial distribution μ defined on an auxiliary probability space (Ω, \mathcal{F}, P) . Then for any $A \subset W_0^1$, A is of capacity 0 if and only if

$$(1.1) \quad P[X_\tau \notin A \quad \text{for all } \tau > 0] = 1.$$

On the other hand, by the Tanaka formula the local time $(\phi(\tau, t, a))$ of a Brownian motion $(X_\tau(t))_{t \geq 0}$ is given by

$$\phi(\tau, t, a) = (X_\tau(t) - a)^+ - (X_\tau(0) - a)^+ - \int_0^t 1_{(a, \infty)}(X_\tau(s)) X_\tau(ds)$$

(cf. [4], [8]). Then our main theorem is stated as follows.

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Theorem. *There exists a continuous version of $(\phi(\tau, t, a))$ in (τ, t, a) with respect to the measure P .*

By the above theorem, we can show the quasi everywhere existence of the local time. To see this, denote a continuous version also by $\phi(\tau, t, a)$. Then, for fixed $\tau > 0$,

$$(1.2) \quad \int_0^t f(X_\tau(s))ds = 2 \int_{\mathbf{R}} \phi(\tau, t, a)f(a)da, \quad \forall t \geq 0, \forall f \in C_0^\infty(\mathbf{R})$$

P-a.e. (the almost everywhere existence of the local time). By the continuity of $(\tau, t) \mapsto X_\tau(t)$ and $(\tau, t, a) \mapsto \phi(\tau, t, a)$, (1.2) holds for all $\tau > 0$ P-a.e. which asserts (1.1). Hence the local time exists quasi everywhere.

We will give a proof of the theorem in Section 2.

2. Proof of the theorem

First we give a realization of the Ornstein-Uhlenbeck process $(X_\tau)_{\tau \geq 0}$ on the probability space (Ω, \mathcal{F}, P) as follows. Let $(X_0(t))_{t \geq 0}$ be a 1-dimensional Brownian motion and $(W(\tau, t))_{\tau, t \geq 0}$ be a two parameter Brownian motion on (Ω, \mathcal{F}, P) , i.e., $(W(\tau, t))_{\tau, t \geq 0}$ is a Gaussian process with mean 0 and the covariance given by

$$E[W(\tau, t)W(\sigma, s)] = (\tau \wedge \sigma)(t \wedge s)$$

where E denotes the expectation relative to P . Assume moreover that $(X_0(t))_{t \geq 0}$ and $(W(\tau, t))_{\tau, t \geq 0}$ are independent. Then the Ornstein-Uhlenbeck process (X_τ) is given by

$$(1.1) \quad X_\tau(t) = e^{-\tau/2} X_0(t) + \int_0^\tau e^{-(\tau-\sigma)/2} W(d\sigma, t)$$

where the integral is the stochastic integral with respect to a martingale $\tau \mapsto W(\tau, t)$. Set $\mathcal{F}_t = \sigma\{X_0(s), W(\tau, s) | s \leq t, \tau \geq 0\}$. Then for fixed $\tau \geq 0$, $(X_\tau(t))_{t \geq 0}$ is an (\mathcal{F}_t) -Brownian motion and for fixed $\tau, \sigma \geq 0$, the quadratic variation of $(X_\tau(t))_{t \geq 0}$ and $(X_\sigma(t))_{t \geq 0}$ is given by

$$(1.2) \quad \langle X_\tau, X_\sigma \rangle_t = \exp\left(-\frac{1}{2}|\tau - \sigma|\right)t.$$

Hereafter we consider this specific Ornstein-Uhlenbeck process.

Define $\psi(\tau, t, a)$ by

$$\psi(\tau, t, a) = \int_0^t 1_{(a, \infty)}(X_\tau(s))X_\tau(ds).$$

To prove the theorem it suffices to show that there exists a continuous version

of $\psi(\tau, t, a)$, i.e., there exists a continuous process $(\hat{\psi}(\tau, t, a))$ in (τ, t, a) P-a.e. such that for fixed $(\tau, t, a) \in [0, \infty) \times [0, \infty) \times \mathbf{R}$, $\psi(\tau, t, a) = \hat{\psi}(\tau, t, a)$ P-a.e. Since for fixed $\tau \geq 0, a \in \mathbf{R}$, $t \mapsto \psi(\tau, t, a)$ is a continuous process, we regard $\psi(\tau, \cdot, a)$ as a $C([0, \infty) \rightarrow \mathbf{R})$ -valued random variable. Hence if we show the following proposition, we can get a desired result by Kolmogorov's theorem ([1]).

Proposition. *For any $T > 0$, there exists a constant $K = K(T) > 0$ such that for any $a, b \in \mathbf{R}$ and $0 \leq \tau, \sigma \leq T$,*

$$E \left[\sup_{0 \leq t \leq T} |\psi(\tau, t, a) - \psi(\sigma, t, b)|^{10} \right] \leq K(|\tau - \sigma|^{9/4} + |a - b|^5).$$

For the proof, we need the following lemmas.

Lemma 1. *Let $(B(t))$ be a 1-dimensional Brownian motion. Then for any $p \geq 1$ and any $T > 0$, there exists a constant $C = C(p, T) > 0$ such that for any $a, b \in \mathbf{R}, a \leq b$,*

$$E \left[\left\{ \int_0^T 1_{(a,b]}(B(t)) dt \right\}^p \right] \leq C(b - a)^p.$$

The proof is easy and will be omitted.

Lemma 2.*) *Let $(B_1(t), B_2(t))$ be a 2-dimensional Brownian motion starting at 0, m be a positive constant and $C_a, a \in \mathbf{R}$, be a subset of \mathbf{R}^2 defined by*

$$C_a = \{(x, y) \in \mathbf{R}^2 \mid (y + mx + a)(y - mx + a) \leq 0\}.$$

Then it holds that for $T > 0, p \geq 1$,

$$E \left[\left\{ \int_0^T 1_{C_a}(B_1(t), B_2(t)) dt \right\}^p \right] \leq E \left[\left\{ \int_0^T 1_{C_0}(B_1(t), B_2(t)) dt \right\}^p \right].$$

Proof. Define a stopping time η by

$$\eta = \inf \left\{ t \geq 0 \mid B_2(t) = \frac{1}{2} a \right\}$$

and a continuous process $(\tilde{B}_2(t))$ by

$$\tilde{B}_2(t) = \begin{cases} B_2(t) & \text{if } t < \eta \\ a - B_2(t) & \text{if } t \geq \eta. \end{cases}$$

Then by the reflection principle of the Brownian motion, $(B_1(t), \tilde{B}_2(t))$ is also a 2-dimensional Brownian motion starting at 0. Moreover it holds that

*) The proof is due to H. Kaneko, who simplified the author's original one.

$$1_{c_a}(B_1(t), B_2(t)) \leq 1_{c_0}(B_1(t), \tilde{B}_2(t)).$$

Hence we can easily get a desired result. \square

Proof of the Proposition. Set

$$I = E \left[\sup_{0 \leq t \leq T} |\psi(\tau, t, a) - \psi(\tau, t, b)|^{10} \right] \quad \text{and}$$

$$J = E \left[\sup_{0 \leq t \leq T} |\psi(\tau, t, b) - \psi(\sigma, t, b)|^{10} \right].$$

Then we easily have

$$E \left[\sup_{0 \leq t \leq T} |\psi(\tau, t, a) - \psi(\sigma, t, b)|^{10} \right] \leq 2^9(I+J).$$

Firstly we give the estimate for I . Without loss of generality, we may assume $a \leq b$. Then by the Burkholder-Davis-Gundy inequality and Lemma 1, we have

$$\begin{aligned} I &= E \left[\sup_{0 \leq t \leq T} \left| \int_0^t 1_{(a,b]}(X_\tau(s)) X_\tau(ds) \right|^{10} \right] \\ &\leq c_1 E \left[\left\{ \int_0^T 1_{(a,b]}(X_\tau(t)) dt \right\}^5 \right]^{*}) \\ &\leq c_2 |a-b|^5. \end{aligned}$$

Secondly we give the estimate for J . By (2.2) and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} J &\leq c_3 E \left[\langle \psi(\tau, \cdot, b) - \psi(\sigma, \cdot, b) \rangle_T^5 \right] \\ &= c_3 E \left[\left\{ \langle \psi(\tau, \cdot, b) \rangle_T - 2 \langle \psi(\tau, \cdot, b), \psi(\sigma, \cdot, b) \rangle_T + \langle \psi(\sigma, \cdot, b) \rangle_T \right\}^5 \right] \\ &= c_3 E \left[\left\{ \int_0^T 1_{(b,\infty)}(X_\tau(t))^2 dt - 2e^{-|\tau-\sigma|/2} \right. \right. \\ &\quad \times \int_0^T 1_{(b,\infty)}(X_\tau(t)) 1_{(b,\infty)}(X_\sigma(t)) dt + \int_0^T 1_{(b,\infty)}(X_\sigma(t))^2 dt \left. \right\}^5 \right] \\ &= c_3 E \left[\left\{ \int_0^T (1_{(b,\infty)}(X_\tau(t)) - 1_{(b,\infty)}(X_\sigma(t)))^2 dt \right. \right. \\ &\quad \left. \left. + 2(1 - e^{-|\tau-\sigma|/2}) \int_0^T 1_{(b,\infty)}(X_\tau(t)) 1_{(b,\infty)}(X_\sigma(t)) dt \right\}^5 \right] \\ &\leq c_4 E \left[\left\{ \int_0^T (1_{(b,\infty)}(X_\tau(t)) - 1_{(b,\infty)}(X_\sigma(t)))^2 dt \right\}^5 \right] \\ &\quad + c_4 (1 - e^{-|\tau-\sigma|/2})^5 E \left[\left\{ \int_0^T 1_{(b,\infty)}(X_\tau(t)) 1_{(b,\infty)}(X_\sigma(t)) dt \right\}^5 \right] \\ &:= J_1 + J_2. \end{aligned}$$

As for J_2 , we easily obtain

* \rangle c_1, c_2, \dots are positive constants which depend only on T .

$$J_2 \leq c_5 |\tau - \sigma|^5.$$

As for J_1 , we define a subset $A_b \subseteq \mathbf{R}^2$ by

$$A_b = \{(x, y) \in \mathbf{R}^2 \mid (x-b)(y-b) \leq 0\}.$$

Then we get

$$J_1 = c_4 E[\{\int_0^T 1_{A_b}(X_\tau(t), X_\sigma(t)) dt\}^5].$$

On the other hand, we define a matrix $U(\tau, \sigma)$ by

$$\begin{aligned} & U(\tau, \sigma) \\ &= (U_{ij}(\tau, \sigma)) \\ &= \frac{1}{2\sqrt{1-e^{-|\tau-\sigma|}}} \begin{pmatrix} \sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} & -\sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} \\ -\sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} + \sqrt{1-e^{-|\tau-\sigma|/2}} \end{pmatrix} \end{aligned}$$

and 2-dimensional continuous process $(B_1(t), B_2(t))$ by

$$B_i(t) = U_{i1}(\tau, \sigma)X_\tau(t) + U_{i2}(\tau, \sigma)X_\sigma(t), \quad i = 1, 2.$$

Then it is easy to see that $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion starting at 0. Moreover set

$$\hat{B}_i(t) = \sum_{j=1}^2 O_{ij} B_j(t), \quad i = 1, 2,$$

where $O = (O_{ij})$ is an orthogonal matrix defined by

$$O = (O_{ij}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then $(\hat{B}_1(t), \hat{B}_2(t))$ is also a 2-dimensional Brownian motion starting at 0 by the rotation invariance. Thus we have

$$J_1 = c_4 E[\{\int_0^T 1_{OU(\tau, \sigma)A_b}(\hat{B}_1(t), \hat{B}_2(t)) dt\}^5].$$

Since

$$(OU(\tau, \sigma))^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} \\ -\sqrt{1-e^{-|\tau-\sigma|/2}} & \sqrt{1+e^{-|\tau-\sigma|/2}} \end{pmatrix},$$

a subset $OU(\tau, \sigma)A_b$ is given by

$$OU(\tau, \sigma)A_b = \left\{ (x, y) \in \mathbf{R}^2 \mid \left(y + \frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}} x - \frac{\sqrt{2}}{\sqrt{1+e^{-|\tau-\sigma|/2}}} b \right) \right. \\ \left. \times \left(y - \frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}} x - \frac{\sqrt{2}}{\sqrt{1+e^{-|\tau-\sigma|/2}}} b \right) \leq 0 \right\}.$$

By Lemma 2, we obtain

$$J_1 \leq c_4 E \left[\left\{ \int_0^T 1_{OU(\tau, \sigma)A_0}(\dot{B}_1(t), \dot{B}_2(t)) dt \right\}^5 \right].$$

For any $R > 0$, we define $S_R, V_R \subseteq \mathbf{R}^2$ by

$$S_R = \{ (x, y) \in \mathbf{R}^2 \mid \sqrt{x^2 + y^2} \geq R \} \\ V_R = \left\{ (x, y) \in \mathbf{R}^2 \mid |y| < \frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}} R \right\},$$

then it is easy to see that $OU(\tau, \sigma)A_0 \subseteq S_R \cup V_R$. Let η_R be a stopping time given by

$$\eta_R = \inf \{ t \geq 0 \mid (\dot{B}_1(t), \dot{B}_2(t)) \in S_R \},$$

then we have

$$J_1 \leq c_5 E \left[\left\{ \int_0^T 1_{S_R}(\dot{B}_1(t), \dot{B}_2(t)) dt \right\}^5 \right] + c_5 E \left[\left\{ \int_0^T 1_{V_R}(\dot{B}_1(t), \dot{B}_2(t)) dt \right\}^5 \right] \\ \leq c_5 T^5 P[\eta_R \leq T] + c_6 \left(\frac{\sqrt{1-e^{-|\tau-\sigma|}}}{1+e^{-|\tau-\sigma|}} R \right)^5 \\ \leq c_7 (P[\eta_R \leq T] + |\tau - \sigma|^{5/2} R^5).$$

Since $(\sqrt{\dot{B}_1(t)^2 + \dot{B}_2(t)^2})$ is a Bessel process with index 2, the following fact is well-known (see e.g., Itô-McKean [5]). For any $\alpha > 0$,

$$E[e^{-\alpha \eta_R}] = \frac{I_0(0)}{I_0(\sqrt{2\alpha R})},$$

where I_0 is the modified Bessel function:

$$I_0(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+1)} \left(\frac{x}{2} \right)^{2m}.$$

Hence by the Chebyshev inequality,

$$P[\eta_R \leq T] \leq e^T E[e^{-\eta_R}] = e^T \frac{1}{I_0(\sqrt{2} R)} \leq c_8 R^{-45}.$$

Then, by setting $R = |\tau - \sigma|^{-1/20}$, we get

$$J_1 \leq c_9 (R^{-45} + |\tau - \sigma|^{5/2} R^5) \leq 2c_9 |\tau - \sigma|^{9/4}.$$

This completes the proof. \square

Corollary. For $w \in W_0^1$, set $Z_w = \{t > 0 \mid w(t) = 0\}$. Then Z_w has a continuum cardinal number and its Lebesgue measure is 0 for q.e.w.

Proof. Let Ω_0 be a set of all $\omega \in \Omega$ which satisfy (1.2) for all $\tau > 0$. Then on Ω_0 , we have

$$\phi(\tau, t, 0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon} \int_0^t 1_{(-\varepsilon, \varepsilon)}(X_\tau(s)) ds.$$

Hence it holds that

$$\phi(\tau, t, 0) = \int_0^t 1_{\{0\}}(X_\tau(s)) \phi(\tau, ds, 0)$$

and

$$\int_0^t 1_{\{0\}}(X_\tau(s)) ds \leq 4\varepsilon \frac{1}{4\varepsilon} \int_0^t 1_{(-\varepsilon, \varepsilon)}(X_\tau(s)) ds \rightarrow 0 \quad (\text{as } \varepsilon \rightarrow 0).$$

Then the rest is easy. \square

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