When is $\Lambda_1 \otimes \Lambda_2$ hereditary?

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WHEN IS $\Lambda_1 \otimes \Lambda_2$ HEREDITARY?

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0. Introduction

Let $R$ be a complete discrete valuation ring with the quotient field $K$. Assuming that $R$ has a finite residue field, Janusz [4] gave a criterion for a tensor product of two $R$-orders $\Lambda_1 \otimes_R \Lambda_2$ to be hereditary or maximal. We shall extend his results by dropping the assumption that $R$ has a finite residue field. In [4], finiteness of a residue field was mainly used to calculate the discriminant. In this paper, we shall do fairly ring theoretical argument and reduces the question to the center $Z(\Lambda)$ of an order $\Lambda$ and $Z(\Lambda/J)$ of the residue ring modulo its radical $J$. These things enable us to handle the problem in a general setting. As for terminology, we mostly follow that of [1].

Notation 0.0. For a ring $A$, we shall consistently write as: $Z(A) :=$ center of $A$, $J(A) :=$ Jacobson radical of $A$, $\overline{A} := A/J(A)$, and $s(A)$ denote the number of isomorphism classes of indecomposable projective left $A$-modules.

Let $\pi$ denote a prime of $R$, $J(R) = \pi R$. For an $R$-order $\Lambda$, put $e(\Lambda|R) := \min\{\nu \in \mathbb{N} : J(\Lambda)^\nu \subseteq \pi \Lambda\}$. An $R$-order $\Lambda$ will be called unramified (over $R$) if and only if $e(\Lambda|R) = 1$ (i.e. $J(\Lambda) = \pi \Lambda$), $\Lambda$ will be called residually separable if and only if $\overline{\Lambda}$ is a separable $\overline{R}$-algebra. An unspecified tensor product $\otimes$ always means that over $R$. Note however, for $R$-orders $\Lambda_i$, $\overline{\Lambda}_1 \otimes \overline{\Lambda}_2 := \overline{\Lambda}_1 \otimes_R \overline{\Lambda}_2 \approx \overline{\Lambda}_1 \otimes_{\overline{R}} \overline{\Lambda}_2$, so that in this case $\otimes$ is in fact over the field $\overline{R}$.

Theorem 0.1. Let $\Lambda_i$ ($i = 1, 2$) be $R$-orders and assume that the following condition is satisfied

\[(*) \quad \overline{\Lambda}_1 \otimes \overline{\Lambda}_2 \text{ is a semisimple ring.}\]

Then:

(A) $\Lambda_1 \otimes \Lambda_2$ is hereditary if and only if both of $\Lambda_i$ are hereditary and one of $\Lambda_i$, say $\Lambda_1$, is unramified.

(B) $\Lambda_1 \otimes \Lambda_2$ is maximal if and only if $\Lambda_1 \otimes \Lambda_2$ is hereditary and moreover the following condition is satisfied

\[(**) \quad s(Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2)) = s(Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2)).\]
Proof of (A), (B) and the next (B1) will be given in §2, as direct consequences of our Main Lemma 2.7. While, if one of \( \Lambda_i \) is residually separable, the condition (*) is certainly satisfied, so that we don’t need to explicitly assume it in the following corollaries, where we can reduce the condition (**) into simpler forms.

**Corollary 0.2.** (B1) Let \( \Lambda_1 \) be an unramified \( R \)-order such that \( Z(\Lambda_1) = \overline{R} \) and \( \Lambda_2 \) be any \( R \)-order. Then:

\[ \Lambda_1 \otimes \Lambda_2 \text{ is maximal if and only if } \Lambda_2 \text{ is maximal.} \]

(B2) Let \( \Lambda_i \) (\( i = 1, 2 \)) be connected residually separable maximal orders. Assume that \( \Lambda_1 \) is unramified and moreover \( Z(\Lambda_1) \) is a Galois extension of \( \overline{R} \). Then:

\[ \Lambda_1 \otimes \Lambda_2 \text{ is maximal if and only if } Z(\Lambda_1) \cap Z(\Lambda_2) = Z(\Lambda_1) \cap Z(\Lambda_2), \]

where the intersection is taken in a fixed separable closure of \( \overline{R} \) (cf. §3 for detail).

**Remark 0.3.** (i) If \( R \) has a finite residue field, our (A) (respectively, (B2)) specializes to Theorem (a) (respectively, (b)) of [4].

(ii) In [1] (26.26), (26.29), the results of [4] are quoted without proof, as valid over any complete discrete valuation ring \( R \), provided that \( K \otimes \Lambda_i \) are separable over \( K \). However, not only the proof but also the statements of results of [4] do not apply for general \( R \). For example, if \( \overline{R} \) has a non-trivial Brauer group, there always exists a central division \( K \)-algebra \( D \neq K \) with the maximal order \( \Lambda_1 \) such that \( Z(\Lambda_1) = \overline{R} \) and \( e(\Lambda_1|R) = 1 \) (by [5, Satz 1]). For such a \( \Lambda_1 \), by (B1):

\[ \Lambda_1 \otimes \Lambda_2 \text{ is maximal if and only if } \Lambda_2 \text{ is maximal.} \]

(iii) The above remark was already recognized and effectively used in [5] (proof of Satz 2), to derive the following remarkable result.

(c) If \( \Lambda \) is a connected residually separable maximal order, then \( Z(\Lambda) \) is always a cyclic Galois extension of degree \( e(\Lambda|Z(\Lambda)) \) over \( Z(\Lambda) \).

In §3, we shall use (c) to derive our final Proposition 3.2, which contains (B2) as a special case.

By the way, relatively recently, (c) is (reproved in [3] in another way and) extensively used in [6].

1. Hereditary orders

1. Recall from [1] §23: an \( R \)-lattice means a finitely generated free \( R \)-module; an \( R \)-order means an \( R \)-algebra which is also an \( R \)-lattice. Let \( \Lambda \) be an \( R \)-order, then the \( K \)-algebra \( \Lambda := K \otimes \Lambda \) has the same free rank over \( K \) as the free rank of \( \Lambda \) over \( R \), \([\Lambda : K] = [\Lambda : R]\). A left (respectively, right) \( \Lambda \)-lattice means a left (respectively,
right) \Lambda\text{-module which is also an } R\text{-lattice. An } R\text{-order } \Lambda \text{ is called a \textit{hereditary order} if and only if any left (or equivalently right) } \Lambda\text{-ideal is projective as a } \Lambda\text{-module.}

For a general facts on hereditary orders, we refer to [7] §39, or [1] §26, where the results are stated under the assumption that \Lambda is separable over \( K \). However, if \Lambda is hereditary, then \Lambda is necessarily semisimple ([(2) 1.7.1]), and at least for local theory, as is easily seen, semisimplicity is enough.

In particular, an \( R\)-order \( \Lambda \) is hereditary if and only if its Jacobson radical \( J(\Lambda) \) is projective as a left (or right) \( \Lambda\)-module. An \( R\)-order \( \Lambda \) will be called a \textit{principal order} if and only if \( J(\Lambda) \) is a principal ideal. Thus we have the implications:

\[ \text{maximal } \implies \text{principal } \implies \text{hereditary.} \]

1.1 Let \( \Lambda \) be a connected (i.e. having no non-trivial central idempotents) hereditary \( R\)-order, then \( \Lambda \) is also connected so that has the form \( \Lambda = M_n(D) \) by some division \( K\)-algebra \( D \). Let \( \Delta \) be the unique maximal order of \( D \).

By the structure theorem [1] (26.28), there is associated a decomposition \( (n_1, ..., n_s) \) of \( n \) \( (n = \sum n_i, \ 0 < n_i \in \mathbb{N}) \), such that \( \Lambda \) is \( \Lambda^x\text{-conjugate to the suborder of } M_n(\Delta) \) defined by the block decomposition as

\[ \Lambda \simeq \{ (\Lambda_{ij})_{1 \leq i, j \leq s} : \Lambda_{ij} = M_{n_i, n_j}(\Delta) \ (i \leq j); \Lambda_{ij} = M_{n_i, n_j}(J(\Delta)) \ (i > j) \} \subset M_n(D). \]

Hence, it is straightforward to derive the following relations, in the notation of 0.0.

(0) \( Z(\Lambda) \simeq Z(\Delta)^{(s)} : = Z(\Delta) \oplus ... \oplus Z(\Delta) \) (\( s\)-times).
(1) \( s = s(\Lambda) = s(\Lambda) = s(Z(\Lambda)). \)
(2) \( Z(\Lambda) \simeq Z(\Delta). \)
(3) \( f(\Lambda|R) := [\Lambda : R] = f(\Delta|R) \sum_{i=1}^{s} n_i^2. \)
(4) \( e(\Lambda|R) = se(\Delta|R). \)
(5) \( \Lambda \) is maximal if and only if \( s = 1. \)
(6) \( \Lambda \) is principal if and only if \( (s|n \text{ and}) n_i = n/s. \)
(7) \( \Lambda \) is basic if and only if \( s = n, \ n_i = 1. \)

Concerning the statement of Theorem (A) (B), we shall remark:
(i) An unramified order is maximal (by (4)).
(ii) If \( \Lambda_1 \otimes \Lambda_2 \) is hereditary and \( (\ast\ast) \) is satisfied, then both of \( \Lambda_i \) are maximal (by (0)).

1.2 Let \( \Lambda \) be a connected hereditary \( R\)-order, then

\[ e(\Lambda|R)f(\Lambda|R) \geq [\Lambda : R] = [\Lambda : K]. \]

The equality holds if and only if \( \Lambda \) is principal.
Proof. By (3) and (4), \( e(\Delta|R)f(\Delta|R) = e(\Delta|R)f(\Delta|R) s \sum n_i^2 \). As is well-known (and as is easily seen), \( e(\Delta|R)f(\Delta|R) = [D : K] \). While \( \sum n_i^2 = \sum (n/s + (n_i - n/s)^2 = \sum (n/s)^2 + \sum (n_i - n/s)^2 \geq \sum (n/s)^2 = n^2/s \), so that \( e(\Delta|R)f(\Delta|R) \geq [D : K]n^2 = [\Lambda : K] \), as wanted. The equality holds if and only if \( n_i = n/s \) so that \( \Lambda \) is principal by (6).

1.3 Let \( \Lambda \) be a hereditary \( R \)-order. Then:

\[ \Lambda \text{ is maximal if and only if } s(Z(\Lambda)) = s(Z(\Lambda)). \]

Proof. It obviously suffices to prove for a connected \( \Lambda \). When connected, the claim is a consequence of (1) (2) and (5).

2. Proof of theorems

2. Let \( \Lambda_i \) \((i = 1, 2)\) be \( R \)-orders. Put \( J_i := J(\Lambda_i) \), \( e_i := e(\Lambda_i|R) \). Since \( \Lambda_i \) is free over \( R \), one may consider \( J_1 \otimes \Lambda_2 \) and \( \Lambda_1 \otimes J_2 \) as submodules of \( \Lambda_1 \otimes \Lambda_2 \), and \( J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 \) is a two-sided ideal of \( \Lambda_1 \otimes \Lambda_2 \). Let \( \varphi_i : \Lambda_i \to \Lambda_i := \Lambda_i/J_i \) be the natural \( R \)-algebra epimorphism.

2.1 The \( R \)-algebra epimorphism \( \varphi_1 \otimes \varphi_2 : \Lambda_1 \otimes \Lambda_2 \to \Lambda_1 \otimes \Lambda_2 \) induces the exact sequence

\[ 0 \to J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 \to \Lambda_1 \otimes \Lambda_2 \to \Lambda_1 \otimes \Lambda_2 \to 0. \]

Proof. Let \( \iota_i : J_i \to \Lambda_i \) be the natural monomorphism. Then straightforward computation yields

\[ \text{Ker}(\varphi_1 \otimes \varphi_2) = \text{Im}(\iota_1 \otimes id_{\Lambda_2}) + \text{Im}(id_{\Lambda_1} \otimes \iota_2). \]

2.2 \((J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2)^{e_1 + e_2 - 1} \subset \pi(\Lambda_1 \otimes \Lambda_2)\).

In particular, \( J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 \subset J(\Lambda_1 \otimes \Lambda_2) \).

Proof. From \( (J_1 \otimes \Lambda_2)^{e_1} \subset \pi\Lambda_1 \otimes \Lambda_2 = \pi(\Lambda_1 \otimes \Lambda_2), \) \( (\Lambda_1 \otimes J_2)^{e_2} \subset \pi(\Lambda_1 \otimes \Lambda_2) \), the claim is obvious.

2.3 The following six conditions for \((\Lambda_1, \Lambda_2)\) are equivalent.

\((*) \) \( \Lambda_1 \otimes \Lambda_2 \) is a semisimple ring.

\((*1) \) \( J(\Lambda_1 \otimes \Lambda_2) = J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 \).

\((*2) \) \( \Lambda_1 \otimes \Lambda_2 \cong \Lambda_1 \otimes \Lambda_2 \).
WHEN IS $\Lambda_1 \otimes \Lambda_2$ HEREDITARY

(*3) $Z(\Lambda_1 \otimes \Lambda_2) \cong Z(\Lambda_1) \otimes Z(\Lambda_2)$.

(*4) $Z(\Lambda_1) \otimes Z(\Lambda_2)$ is a semisimple ring.

(*5) $k_1 \otimes k_2$ is a semisimple ring for any $R$-subalgebra $k_i$ of $Z(\Lambda_i)$.

Proof. (*) $\Rightarrow$ (1) by 2.1 and 2.2; (1) $\Rightarrow$ (2) by 2.1; (2) $\Rightarrow$ (3) obvious; (3) $\Rightarrow$ (4) since $\Lambda_1 \otimes \Lambda_2$ is semisimple; (4) $\Rightarrow$ (5) since $k_1 \otimes k_2$ cannot have nilpotent elements; (5) $\Rightarrow$ (4) obvious; (4) $\Rightarrow$ (*) It obviously suffices to prove the claim when $\Lambda_i$ are simple so that $k_i := Z(\Lambda_i)$ are finite extension fields of $k := \overline{R}$. Assume (4), so that $k_1 \otimes k_2 \simeq \bigoplus_{j=1}^{t} T_j$ by finite extension fields $T_i$. We have $\Lambda_1 \otimes \Lambda_2 = (\Lambda_1 \otimes k_1) \otimes (k_2 \otimes \Lambda_2) \simeq \bigoplus_{j=1}^{t} (\Lambda_1 \otimes k_1, T_j \otimes k_2 \Lambda_2)$. Since $\Lambda_1$ is central simple over $k_1$, $\Lambda_1 \otimes k_1, T_j$ is simple, which implies that $(\Lambda_1 \otimes k_1, T_j) \otimes k_2 \Lambda_2$ is also simple. $\square$

2.4 If $(\Lambda_1, \Lambda_2)$ satisfies the condition (*), then

$$e(\Lambda_1 \otimes \Lambda_2|R) \leq e_1 + e_2 - 1.$$ 

Proof. By 2.3 (1) and 2.2. $\square$

2.5 Assume that $\Lambda_1$ is unramified, $\Lambda_2$ is hereditary and moreover the condition (*) is satisfied, then $\Lambda_1 \otimes \Lambda_2$ is hereditary.

Proof. By 2.3 (1), $J(\Lambda_1 \otimes \Lambda_2) = J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 = \pi \Lambda_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 = \Lambda_1 \otimes \pi \Lambda_2 + \Lambda_1 \otimes J_2 = \Lambda_1 \otimes J_2$. Since $\Lambda_2$ is hereditary, we have $J_2 \otimes X \simeq \Lambda_2^{(\nu)}$, so that $(\Lambda_1 \otimes J_2) \otimes (\Lambda_1 \otimes X) \simeq \Lambda_1 \otimes (J_2 \otimes X) \simeq \Lambda_1 \otimes \Lambda_2^{(\nu)} \simeq (\Lambda_1 \otimes \Lambda_2)^{(\nu)}$, hence $J(\Lambda_1 \otimes \Lambda_2) = \Lambda_1 \otimes J_2$ is $\Lambda_1 \otimes \Lambda_2$-projective. $\square$

2.6 ([4, Proposition 3]). If $\Lambda_1 \otimes \Lambda_2$ is hereditary, then both of $\Lambda_i$ are hereditary.

Proof. Let $M$ be a (left) ideal of $\Lambda_2$. Since $\Lambda_1$ is free over $R$, $M$ is a direct summand of $\Lambda_1 \otimes M$. Since $\Lambda_1 \otimes \Lambda_2$ is hereditary, $\Lambda_1 \otimes M$ is $\Lambda_1 \otimes \Lambda_2$-projective, which implies, since $\Lambda_1$ is free over $R$, $\Lambda_1 \otimes M$ is $\Lambda_2$-projective so that $M$ is $\Lambda_2$-projective. $\square$

Main Lemma 2.7. Let $\Lambda_i$ ($i = 1, 2$) be connected hereditary orders satisfying the condition (*). If $\Lambda_1 \otimes \Lambda_2$ is hereditary, then one of $\Lambda_i$ is unramified.

Proof. (I) First we assume that both of $\Lambda_i$ are principal. Decompose $\Lambda_1 \otimes \Lambda_2$ into the connected components $\Gamma_j$ ($1 \leq j \leq t$), $\Lambda_1 \otimes \Lambda_2 = \bigoplus \Gamma_j$. Putting $f_i := [\Lambda_i : \overline{R}]$, $e_j := e(\Gamma_j|R)$ and $f_j := [\overline{\Gamma}_j : \overline{R}]$, we have

1) $\sum f_j = \sum [\overline{\Gamma}_j : \overline{R}] = [\Lambda_1 : \overline{R}] [\Lambda_2 : \overline{R}] = f_1 f_2$. 

Since $\Gamma_j$'s are hereditary and $A_i$'s are principal, by 1.2, we have

$$\sum e_j f_j \geq \sum [\Gamma_j : R] = [A_1 \otimes A_2 : R] = [A_1 : R][A_2 : R] = f_1 e_1 f_2 e_2.$$ 

Combining 1) and 2), we get

$$\sum (e_j' - e e_2) f_j' \geq 0.$$ 

From $e(\Theta \Gamma_j | \Lambda) = \max e(\Gamma_j | R) \geq e'$, using 2.4, we get $e_1 + e_2 - 1 \geq e(\Lambda_i \otimes \Lambda_2 | R) \geq e_j'$, so that

$$-\left( e_1 - 1 \right) (e_2 - 1) \sum f_j' = \sum (e_1 + e_2 - 1 - e_1 e_2) f_j' \geq \sum (e_j' - e_1 e_2) f_j' \geq 0,$$

where the last inequality is by 3). Since $e_i \geq 1$, one of $e_i = 1$.

(II) Let $A_i^j$ be a basic (hence principal) hereditary order which is Morita equivalent with $A_i$. We shall show that $A_i^j \otimes A_i^j$ is Morita equivalent with $A_i \otimes A_2$ (hence is also hereditary). Indeed, $A_2$ has the form $A_2 = \text{Hom}_{A_2}(P, P)$ by some progenerator $P$, $A_2$ is free (hence flat) over $R$, and $P$ is finitely presented as $A_2$-module, so that we have

$$A_i^j \otimes A_i^j \simeq A_i^j \otimes \text{Hom}_{A_2}(P, P) \simeq \text{Hom}_{A_i^j \otimes A_i^j}(A_i^j \otimes P, A_i^j \otimes P).$$

Since $A_i \otimes P$ is a progenerator for $A_i \otimes A_2$, $A_i^j \otimes A_i^j$ is Morita equivalent with $A_i \otimes A_2$. By the same reason, $A_i^j \otimes A_2$ is Morita equivalent with $A_i \otimes A_2$. 

2.8 Proof of Theorem (A): 'If part' is by 2.5. 'Only if part' is easily derived from 2.7.

(B): By 1.3, $A_i \otimes A_2$ is maximal if and only if $A_i \otimes A_2$ is hereditary and

$s(Z(A_i \otimes A_2)) = s(Z(A_1 \otimes A_2)).$ 

By 2.3 (*3), we have $Z(A_i \otimes A_2) = Z(A_i) \otimes Z(A_2)$ since $Z(A_i)$ is an $R$-subalgebra of $Z(A_i)$, by 2.3 (*5), $Z(A_i) \otimes Z(A_2)$ is semisimple, hence by 2.3, $Z(A_i) \otimes Z(A_2) = Z(A_1) \otimes Z(A_2).$ Thus $A_i \otimes A_2$ is maximal if and only if $s(Z(A_i) \otimes Z(A_2)) = s(Z(A_1) \otimes Z(A_2))$.

(B1): Assume that $Z(A_1) = R$. Then $Z(A_1) \otimes Z(A_2) \simeq Z(A_2)$ and $Z(A_1) \otimes Z(A_2) \simeq Z(A_2)$. $A_1 \otimes A_2$: maximal $\iff s(Z(A_2)) = s(Z(A_2)) \iff A_2$: maximal (by 1.3).

3. Proof of Corollary (B2)

3. Let $A_i$ ($i = 1, 2$) be connected maximal $R$-orders satisfying (*). Put $k := \overline{R}$, $k_i := \overline{Z(A_i)}$ and $k'_i := Z(A_i)$. Then $k'_i$ is an extension field of $k$ containing $k_i$, and $k_1 \otimes k_2 = \oplus_{j=1}^T T_j$ is a direct sum of extension fields $T_j$ of $k$. Obviously the following two conditions are equivalent:

$$s(k'_1 \otimes k'_2) = s(k_1 \otimes k_2),$$

(II) Let $A_i^j$ be a basic (hence principal) hereditary order which is Morita equivalent with $A_i$. We shall show that $A_i^j \otimes A_i^j$ is Morita equivalent with $A_i \otimes A_2$ (hence is also hereditary). Indeed, $A_2$ has the form $A_2 = \text{Hom}_{A_2}(P, P)$ by some progenerator $P$, $A_2$ is free (hence flat) over $R$, and $P$ is finitely presented as $A_2$-module, so that we have

$$A_i^j \otimes A_i^j \simeq A_i^j \otimes \text{Hom}_{A_2}(P, P) \simeq \text{Hom}_{A_i^j \otimes A_i^j}(A_i^j \otimes P, A_i^j \otimes P).$$

Since $A_i \otimes P$ is a progenerator for $A_i \otimes A_2$, $A_i^j \otimes A_i^j$ is Morita equivalent with $A_i \otimes A_2$. By the same reason, $A_i^j \otimes A_2$ is Morita equivalent with $A_i \otimes A_2$. 

2.8 Proof of Theorem (A): 'If part' is by 2.5. 'Only if part' is easily derived from 2.7.

(B): By 1.3, $A_i \otimes A_2$ is maximal if and only if $A_i \otimes A_2$ is hereditary and

$s(Z(A_i \otimes A_2)) = s(Z(A_1 \otimes A_2)).$ 

By 2.3 (*3), we have $Z(A_i \otimes A_2) = Z(A_i) \otimes Z(A_2)$ since $Z(A_i)$ is an $R$-subalgebra of $Z(A_i)$, by 2.3 (*5), $Z(A_i) \otimes Z(A_2)$ is semisimple, hence by 2.3, $Z(A_i) \otimes Z(A_2) = Z(A_1) \otimes Z(A_2).$ Thus $A_i \otimes A_2$ is maximal if and only if $s(Z(A_i) \otimes Z(A_2)) = s(Z(A_1) \otimes Z(A_2))$.

(B1): Assume that $Z(A_1) = R$. Then $Z(A_1) \otimes Z(A_2) \simeq Z(A_2)$ and $Z(A_1) \otimes Z(A_2) \simeq Z(A_2)$. $A_1 \otimes A_2$: maximal $\iff s(Z(A_2)) = s(Z(A_2)) \iff A_2$: maximal (by 1.3).

3. Proof of Corollary (B2)

3. Let $A_i$ ($i = 1, 2$) be connected maximal $R$-orders satisfying (*). Put $k := \overline{R}$, $k_i := \overline{Z(A_i)}$ and $k'_i := Z(A_i)$. Then $k'_i$ is an extension field of $k$ containing $k_i$, and $k_1 \otimes k_2 = \oplus_{j=1}^T T_j$ is a direct sum of extension fields $T_j$ of $k$. Obviously the following two conditions are equivalent:

$$s(k'_1 \otimes k'_2) = s(k_1 \otimes k_2),$$
WHEN IS $\Lambda_1 \otimes \Lambda_2$ HEREDITARY

$\Lambda_1 \otimes \Lambda_2$ is a field for any $j$ (1 $\leq j \leq t)$.

3.0 Assume that $\Lambda_1$ is unramified and residually separable over $R$. Then, by (c) 0.3 (or more elementary Hilfssatz 3 of [5]), we have

$$k_1' = Z(\Lambda_1) = \mathbb{Z}(\Lambda_1) = k_1.$$  

Being separable over $k$, $k_1$ has the form $k_1 = k[x]/f_k[x]$ by a separable polynomial $f_k$ in $k[x]$. The decomposition $k_1 \otimes k_2 = \oplus T_j$ corresponds to the decomposition of $f = \Pi f_j$ as irreducible factors in $k_2[x]$. Thus (**1) is equivalent with

(**2) $f_j$ is irreducible in $k_2'[x]$ for any $j$ (1 $\leq j \leq t$).

3.1 Further assume that $\Lambda_2$ is also residually separable over $R$, so that $k_2'$ is separable over $k$, and moreover $k_2'/k_2$ is a (cyclic) Galois extension by (c) 0.3. Since the condition (**2) depends only on the $k$-algebra structure of $k_2'$, we consider that $k_2'$ and $k_1$ are contained in a fixed separable closure $k_{sep}$ of $k$, and apply Galois theory.

Let $G := \text{Gal}(k_{sep}/k)$ and $G(L) := \{\sigma \in G : \sigma|_L = id_L\}$ for $L \subset k_{sep}$. The decomposition of $f$ in $k_2[x]$ (respectively $k_2'[x]$) corresponds to the double cosets decomposition $G(k_2) \backslash G/G(k_1)$ (respectively $G(k_2') \backslash G/G(k_1)$), so that (**2) is equivalent with

(**3) $G(k_2) \subset G(k_2') \sigma G(k_1) \sigma^{-1} = G(k_2') G(\sigma(k_1))$ for any $\sigma \in G$.

Proposition 3.2. Let $\Lambda_i$ (i = 1, 2) be connected residually separable maximal orders and $\Lambda_1$ be unramified over $R$. Then:

(i) $\Lambda_1 \otimes \Lambda_2$ is maximal if and only if

$$k_2 = k_2' \cap \sigma(k_1)k_2 \text{ for any } \sigma \in \text{Gal}(k_{sep}/k).$$

(ii) If further, one of $k_1$ or $k_2'$ is Galois over $k$, then:

$\Lambda_1 \otimes \Lambda_2$ is maximal if and only if

$$k_2 \cap \sigma(k_1) = k_2' \cap \sigma(k_1) \text{ for any } \sigma \in \text{Gal}(k_{sep}/k).$$

Proof. (i) Since $G(k_2')$ is a normal subgroup of $G(k_2)$:

$$G(k_2) \subset G(k_2') G(\sigma(k_1)) \Leftrightarrow G(k_2) = G(k_2') G(\sigma(k_1)) \cap G(k_2) = G(k_2') G(\sigma(k_1)k_2)$$

$$\Leftrightarrow k_2 = k_2' \cap \sigma(k_1)k_2.$$
(ii) \( G(k_2) \subset G(k'_2)G(\sigma(k_1)) \Rightarrow G(k_2) \subset G(k'_2), G(\sigma(k_1)) \Leftrightarrow k_2 \supset k'_2 \cap \sigma(k_1) \Leftrightarrow k_2 \cap \sigma(k_1) = k'_2 \cap \sigma(k_1) \).

At the first implication, the converse holds if one of \( k'_2 \) or \( k_1 \) is Galois over \( k \).

References


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