Hijikata, H. and Nishida K.
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WHEN IS $\Lambda_1 \otimes \Lambda_2$ HEREDITARY ?

HIROAKI HIJIKATA and KENJI NISHIDA

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0. Introduction

Let R be a complete discrete valuation ring with the quotient field K. Assuming that R has a finite residue field, Janusz [4] gave a criterion for a tensor product of two R-orders $\Lambda_1 \otimes_R \Lambda_2$ to be hereditary or maximal. We shall extend his results by dropping the assumption that R has a finite residue field. In [4], finiteness of a residue field was mainly used to calculate the discriminant. In this paper, we shall do fairly ring theoretical argument and reduces the question to the center $Z(\Lambda)$ of an order Λ and $Z(\Lambda/J)$ of the residue ring modulo its radical J. These things enable us to handle the problem in a general setting. As for terminology, we mostly follow that of [1].

NOTATION 0.0. For a ring A, we shall consistently write as: $Z(A) :=$ center of A, $J(A) :=$ Jacobson radical of A, $\overline{A} := A/J(A)$, and $s(A)$ denote the number of isomorphism classes of indecomposable projective left A-modules.

Let π denote a prime of R, $J(R) = \pi R$. For an R-order Λ, put $e(\Lambda|R) := \min\{\nu \in \mathbb{N} : J(\Lambda)^{\nu} \subset \pi\Lambda\}$. An R-order Λ will be called *unramified* (over R) if and only if $e(\Lambda|R) = 1$ (i.e. $J(\Lambda) = \pi\Lambda$), Λ will be called *residually separable* if and only if $\overline{\Lambda}$ is a separable $\overline{R}$-algebra. An unspecified tensor product $\otimes$ always means that over R. Note however, for R-orders Λ_i, $\overline{\Lambda}_1 \otimes \overline{\Lambda}_2 := \overline{\Lambda}_1 \otimes_R \overline{\Lambda}_2 \simeq \overline{\Lambda}_1 \otimes_{\overline{R}} \overline{\Lambda}_2$, so that in this case $\otimes$ is in fact over the field $\overline{R}$.

Theorem 0.1. *Let Λ_i $(i = 1, 2)$ be R-orders and assume that the following condition is satisfied*

$$(*) \qquad \overline{\Lambda}_1 \otimes \overline{\Lambda}_2 \quad \text{is a semisimple ring.}$$

Then:

(A) $\Lambda_1 \otimes \Lambda_2$ *is hereditary if and only if both of Λ_i are hereditary and one of Λ_i, say Λ_1, is unramified.*

(B) $\Lambda_1 \otimes \Lambda_2$ *is maximal if and only if $\Lambda_1 \otimes \Lambda_2$ is hereditary and moreover the following condition is satisfied*

$$(**) \qquad s(Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2)) = s(\overline{Z(\Lambda_1)} \otimes \overline{Z(\Lambda_2)}).$$

Proof of (A), (B) and the next (B1) will be given in §2, as direct consequences of our Main Lemma 2.7. While, if one of Λ_i is residually separable, the condition $(*)$ is certainly satisfied, so that we don't need to explicitly assume it in the following corollaries, where we can reduce the condition $(**)$ into simpler forms.

Corollary 0.2. (B1) *Let* Λ_1 *be an unramified* R*-order such that* $Z(\overline{\Lambda}_1) = \overline{R}$ *and* Λ_2 *be any* R*-order. Then*:

$$\Lambda_1 \otimes \Lambda_2 \textit{ is maximal if and only if } \Lambda_2 \textit{ is maximal.}$$

(B2) *Let* Λ_i $(i = 1, 2)$ *be connected residually separable maximal orders. Assume that* Λ_1 *is unramified and moreover* $\overline{Z(\Lambda_1)}$ *is a Galois extension of* $\overline{R}$. *Then*:

$$\Lambda_1 \otimes \Lambda_2 \textit{ is maximal if and only if } \overline{Z(\Lambda_1)} \cap \overline{Z(\Lambda_2)} = \overline{Z(\Lambda_1)} \cap Z(\overline{\Lambda}_2),$$

where the intersection is taken in a fixed separable closure of $\overline{R}$ *(cf.* §3 *for detail).*

REMARK 0.3. (i) If R has a finite residue field, our (A) (respectively, (B2)) specializes to Theorem (a) (respectively, (b)) of [4].

(ii) In [1] (26.26), (26.29), the results of [4] are quoted without proof, as valid over any complete discrete valuation ring R, provided that $K \otimes \Lambda_i$ are separable over K. However, not only the proof but also the statements of results of [4] do not apply for general R. For example, if $\overline{R}$ has a non-trivial Brauer group, there always exists a central division K-algebra D $(\neq K)$ with the maximal order Λ_1 such that $Z(\overline{\Lambda}_1) = \overline{R}$ and $e(\Lambda_1|R) = 1$ (by [5, Satz 1]). For such a Λ_1, by (B1):

$$\Lambda_1 \otimes \Lambda_2 \text{ is maximal if and only if } \Lambda_2 \text{ is maximal.}$$

(iii) The above remark was already recognized and effectively used in [5] (proof of Satz 2), to derive the following remarkable result.

(c) If Λ is a connected residually separable maximal order, then $Z(\overline{\Lambda})$ is always a cyclic Galois extension of degree $e(\Lambda|Z(\Lambda))$ over $\overline{Z(\Lambda)}$.

In §3, we shall use (c) to derive our final Proposition 3.2, which contains (B2) as a special case.

By the way, relatively recently, (c) is (reproved in [3] in another way and) extensively used in [6].

1. Hereditary orders

1. Recall from [1] §23: an *R-lattice* means a finitely generated free R-module; an *R-order* means an R-algebra which is also an R-lattice. Let Λ be an R-order, then the K-algebra $\widetilde{\Lambda} := K \otimes \Lambda$ has the same free rank over K as the free rank of Λ over R, $[\widetilde{\Lambda} : K] = [\Lambda : R]$. A left (respectively, right) Λ*-lattice* means a left (respectively,

right) Λ-module which is also an R-lattice. An R-order Λ is called a *hereditary order* if and only if any left (or equivalently right) Λ-ideal is projective as a Λ-module.

For a general facts on hereditary orders, we refer to [7] §39, or [1] §26, where the results are stated under the assumption that $\widetilde{\Lambda}$ is separable over K. However, if Λ is hereditary, then $\widetilde{\Lambda}$ is necessarily semisimple ([2] 1.7.1), and at least for local theory, as is easily seen, semisimplicity is enough.

In particular, an R-order Λ is hereditary if and only if its Jacobson radical $J(\Lambda)$ is projective as a left (or right) Λ-module. An R-order Λ will be called a *principal order* if and only if $J(\Lambda)$ is a principal ideal. Thus we have the implications:

$$\text{maximal} \Longrightarrow \text{principal} \Longrightarrow \text{hereditary}.$$

1.1 Let Λ be a connected (i.e. having no non-trivial central idempotents) hereditary R-order, then $\widetilde{\Lambda}$ is also connected so that has the form $\widetilde{\Lambda} = M_n(D)$ by some division K-algebra D. Let Δ be the unique maximal order of D.

By the structure theorem [1] (26.28), there is associated a decomposition $(n_1, ..., n_s)$ of n ($n = \sum n_i$, $0 < n_i \in \mathbb{N}$), such that Λ is $\widetilde{\Lambda}^{\times}$-conjugate to the suborder of $M_n(\Delta)$ defined by the block decomposition as

$$\Lambda \simeq \{(\Lambda_{ij})_{1\le i,j\le s} : \Lambda_{ij} = M_{n_i,n_j}(\Delta)\ (i \le j);\ \Lambda_{ij} = M_{n_i,n_j}(J(\Delta))\ (i > j)\} \\ \subset M_n(D).$$

Hence, it is straightforward to derive the following relations, in the notation of 0.0.

(0) $Z(\overline{\Lambda}) \simeq Z(\overline{\Delta})^{(s)} := Z(\overline{\Delta}) \oplus ... \oplus Z(\overline{\Delta})$ (s-times).
(1) $s = s(\Lambda) = s(\overline{\Lambda}) = s(Z(\overline{\Lambda}))$.
(2) $Z(\Lambda) \simeq Z(\Delta)$.
(3) $f(\Lambda|R) := [\overline{\Lambda} : \overline{R}] = f(\Delta|R) \sum_{i=1}^{s} n_i^2$.
(4) $e(\Lambda|R) = se(\Delta|R)$.
(5) Λ is maximal if and only if $s = 1$.
(6) Λ is principal if and only if ($s|n$ and) $n_i = n/s$.
(7) Λ is basic if and only if $s = n$, $n_i = 1$.

Concerning the statement of Theorem (A) (B), we shall remark:

(i) An unramified order is maximal (by (4)).
(ii) If $\Lambda_1 \otimes \Lambda_2$ is hereditary and (**) is satisfied, then both of Λ_i are maximal (by (0)).

1.2 Let Λ be a connected hereditary R-order, then

$$e(\Lambda|R)f(\Lambda|R) \ge [\Lambda : R] = [\widetilde{\Lambda} : K].$$

The equality holds if and only if Λ is principal.

Proof. By (3) and (4), $e(\Lambda|R)f(\Lambda|R) = e(\Delta|R)f(\Delta|R)s\sum n_i^2$. As is well-known (and as is easily seen), $e(\Delta|R)f(\Delta|R) = [D : K]$. While $\sum n_i^2 = \sum(n/s + (n_i - n/s))^2 = \sum(n/s)^2 + \sum(n_i - n/s)^2 \geq \sum(n/s)^2 = n^2/s$, so that $e(\Lambda|R)f(\Lambda|R) \geq [D : K]n^2 = [\widetilde{\Lambda} : K]$, as wanted. The equality holds if and only if $n_i = n/s$ so that Λ is principal by (6). $\square$

1.3 Let Λ be a hereditary R-order. Then:

$$\Lambda \text{ is maximal if and only if } s(\overline{Z(\Lambda)}) = s(Z(\overline{\Lambda})).$$

Proof. It obviously suffices to prove for a connected Λ. When connected, the claim is a consequence of (1) (2) and (5). $\square$

2. Proof of theorems

2. Let Λ_i $(i = 1, 2)$ be R-orders. Put $J_i := J(\Lambda_i)$, $e_i := e(\Lambda_i|R)$. Since Λ_i is free over R, one may consider $J_1 \otimes \Lambda_2$ and $\Lambda_1 \otimes J_2$ as submodules of $\Lambda_1 \otimes \Lambda_2$, and $J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2$ is a two-sided ideal of $\Lambda_1 \otimes \Lambda_2$. Let $\varphi_i : \Lambda_i \to \overline{\Lambda}_i := \Lambda_i/J_i$ be the natural R-algebra epimorphism.

2.1 The R-algebra epimorphism $\varphi_1 \otimes \varphi_2 : \Lambda_1 \otimes \Lambda_2 \to \overline{\Lambda}_1 \otimes \overline{\Lambda}_2$ induces the exact sequence

$$0 \longrightarrow J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 \longrightarrow \Lambda_1 \otimes \Lambda_2 \longrightarrow \overline{\Lambda}_1 \otimes \overline{\Lambda}_2 \longrightarrow 0.$$

Proof. Let $\iota_i : J_i \to \Lambda_i$ be the natural monomorphism. Then straightforward computation yields

$$\mathrm{Ker}(\varphi_1 \otimes \varphi_2) = \mathrm{Im}(\iota_1 \otimes id_{\Lambda_2}) + \mathrm{Im}(id_{\Lambda_1} \otimes \iota_2). \quad \square$$

2.2 $(J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2)^{e_1+e_2-1} \subset \pi(\Lambda_1 \otimes \Lambda_2)$.
In particular, $J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 \subset J(\Lambda_1 \otimes \Lambda_2)$.

Proof. From $(J_1 \otimes \Lambda_2)^{e_1} \subset \pi\Lambda_1 \otimes \Lambda_2 = \pi(\Lambda_1 \otimes \Lambda_2)$, $(\Lambda_1 \otimes J_2)^{e_2} \subset \pi(\Lambda_1 \otimes \Lambda_2)$, the claim is obvious. $\square$

2.3 The following six conditions for (Λ_1, Λ_2) are equivalent.
(*) $\overline{\Lambda}_1 \otimes \overline{\Lambda}_2$ is a semisimple ring.
(*1) $J(\Lambda_1 \otimes \Lambda_2) = J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2$.
(*2) $\overline{\Lambda_1 \otimes \Lambda_2} \simeq \overline{\Lambda}_1 \otimes \overline{\Lambda}_2$.

(*3) $Z(\overline{\Lambda_1 \otimes \Lambda_2}) \simeq Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2)$.
(*4) $Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2)$ is a semisimple ring.
(*5) $k_1 \otimes k_2$ is a semisimple ring for any $\overline{R}$-subalgebra k_i of $Z(\overline{\Lambda}_i)$.

Proof. (*) $\Rightarrow$ (*1) by 2.1 and 2.2; (*1) $\Rightarrow$ (*2) by 2.1; (*2) $\Rightarrow$ (*3) obvious; (*3) $\Rightarrow$ (*4) since $\overline{\Lambda_1 \otimes \Lambda_2}$ is semisimple; (*4) $\Rightarrow$ (*5) since $k_1 \otimes k_2$ cannot have nilpotent elements; (*5) $\Rightarrow$ (*4) obvious; (*4) $\Rightarrow$ (*): It obviously suffices to prove the claim when $\overline{\Lambda}_i$ are simple so that $k_i := Z(\overline{\Lambda}_i)$ are finite extension fields of $k := \overline{R}$. Assume (*4), so that $k_1 \otimes_k k_2 \simeq \oplus_{j=1}^t T_j$ by finite extension fields T_i. We have $\overline{\Lambda}_1 \otimes_k \overline{\Lambda}_2 = (\overline{\Lambda}_1 \otimes_{k_1} k_1) \otimes_k (k_2 \otimes_{k_2} \overline{\Lambda}_2) \simeq \oplus_j (\overline{\Lambda}_1 \otimes_{k_1} T_j \otimes_{k_2} \overline{\Lambda}_2)$. Since $\overline{\Lambda}_1$ is central simple over k_1, $\overline{\Lambda}_1 \otimes_{k_1} T_j$ is simple, which implies that $(\overline{\Lambda}_1 \otimes_{k_1} T_j) \otimes_{k_2} \overline{\Lambda}_2$ is also simple. □

2.4 If (Λ_1, Λ_2) satisfies the condition (*), then

$$e(\Lambda_1 \otimes \Lambda_2 | R) \leq e_1 + e_2 - 1.$$

Proof. By 2.3 (*1) and 2.2. □

2.5 Assume that Λ_1 is unramified, Λ_2 is hereditary and moreover the condition (*) is satisfied, then $\Lambda_1 \otimes \Lambda_2$ is hereditary.

Proof. By 2.3 (*1), $J(\Lambda_1 \otimes \Lambda_2) = J_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 = \pi\Lambda_1 \otimes \Lambda_2 + \Lambda_1 \otimes J_2 = \Lambda_1 \otimes \pi\Lambda_2 + \Lambda_1 \otimes J_2 = \Lambda_1 \otimes J_2$. Since Λ_2 is heredirary, we have $J_2 \oplus X \simeq \Lambda_2^{(\nu)}$, so that $(\Lambda_1 \otimes J_2) \oplus (\Lambda_1 \otimes X) \simeq \Lambda_1 \otimes (J_2 \oplus X) \simeq \Lambda_1 \otimes \Lambda_2^{(\nu)} \simeq (\Lambda_1 \otimes \Lambda_2)^{(\nu)}$, hence $J(\Lambda_1 \otimes \Lambda_2) = \Lambda_1 \otimes J_2$ is $\Lambda_1 \otimes \Lambda_2$-projective. □

2.6 ([4, Proposition 3]). If $\Lambda_1 \otimes \Lambda_2$ is hereditary, then both of Λ_i are hereditary.
Proof. Let M be a (left) ideal of Λ_2. Since Λ_1 is free over R, M is a direct summand of $\Lambda_1 \otimes M$. Since $\Lambda_1 \otimes \Lambda_2$ is hereditary, $\Lambda_1 \otimes M$ is $\Lambda_1 \otimes \Lambda_2$-projective, which implies, since Λ_1 is free over R, $\Lambda_1 \otimes M$ is Λ_2-projective so that M is Λ_2-projective. □

Main Lemma 2.7. *Let Λ_i $(i = 1, 2)$ be connected hereditary orders satisfying the condition (*). If $\Lambda_1 \otimes \Lambda_2$ is hereditary, then one of Λ_i is unramified.*

Proof. (I) First we assume that both of Λ_i are principal. Decompose $\Lambda_1 \otimes \Lambda_2$ into the connected components Γ_j $(1 \leq j \leq t)$, $\Lambda_1 \otimes \Lambda_2 = \oplus \Gamma_j$. Putting $f_i := [\overline{\Lambda}_i : \overline{R}]$, $e'_j := e(\Gamma j | R)$ and $f'_j := [\overline{\Gamma}_j : \overline{R}]$, we have

1) $$\sum f'_j = \sum [\overline{\Gamma}_j : \overline{R}] = [\overline{\Lambda}_1 : \overline{R}][\overline{\Lambda}_2 : \overline{R}] = f_1 f_2.$$

Since Γ_j's are hereditary and Λ_i's are principal, by 1.2, we have

$$2)\qquad \sum e_j' f_j' \geq \sum [\Gamma_j : R] = [\Lambda_1 \otimes \Lambda_2 : R] = [\Lambda_1 : R][\Lambda_2 : R] = f_1 e_1 f_2 e_2.$$

Combining 1) and 2), we get

$$3)\qquad \sum (e_j' - e_1 e_2) f_j' \geq 0.$$

From $e(\oplus \Gamma_j | R) = \max e(\Gamma_j | R) \geq e_j'$, using 2.4, we get $e_1 + e_2 - 1 \geq e(\Lambda_1 \otimes \Lambda_2 | R) \geq e_j'$, so that

$$-(e_1 - 1)(e_2 - 1) \sum f_j' = \sum (e_1 + e_2 - 1 - e_1 e_2) f_j' \geq \sum (e_j' - e_1 e_2) f_j' \geq 0,$$

where the last inequality is by 3). Since $e_i \geq 1$, one of $e_i = 1$.

(II) Let Λ_i' be a basic (hence principal) hereditary order which is Morita equivalent with Λ_i. We shall show tha $\Lambda_1' \otimes \Lambda_2'$ is Morita equivalent with $\Lambda_1 \otimes \Lambda_2$ (hence is also hereditary). Indeed, Λ_2' has the form $\Lambda_2' \simeq \mathrm{Hom}_{\Lambda_2}(P, P)$ by some progenerator P, Λ_2 is free (hence flat) over R, and P is finitely presented as Λ_2-module, so that we have

$$\Lambda_1' \otimes \Lambda_2' \simeq \Lambda_1' \otimes \mathrm{Hom}_{\Lambda_2}(P, P) \simeq \mathrm{Hom}_{\Lambda_1' \otimes \Lambda_2}(\Lambda_1' \otimes P, \Lambda_1' \otimes P).$$

Since $\Lambda_1' \otimes P$ is a progenerator for $\Lambda_1' \otimes \Lambda_2$, $\Lambda_1' \otimes \Lambda_2'$ is Morita equivalent with $\Lambda_1' \otimes \Lambda_2$. By the same reason, $\Lambda_1' \otimes \Lambda_2$ is Morita equivalent with $\Lambda_1 \otimes \Lambda_2$. □

2.8 Proof of Theorem (A): 'If part' is by 2.5. 'Only if part' is easily derived from 2.7.

(B): By 1.3, $\Lambda_1 \otimes \Lambda_2$ is maximal if and only if $\Lambda_1 \otimes \Lambda_2$ is hereditary and $s(\overline{Z(\Lambda_1 \otimes \Lambda_2)}) = s(Z(\overline{\Lambda_1 \otimes \Lambda_2}))$. By 2.3 (*3), we have $Z(\overline{\Lambda_1 \otimes \Lambda_2}) = Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2)$. Since $\overline{Z(\Lambda_i)}$ is an R-subalgebra of $Z(\overline{\Lambda}_i)$, by 2.3 (*5), $\overline{Z(\Lambda_1)} \otimes \overline{Z(\Lambda_2)}$ is semisimple, hence by 2.3, $\overline{Z(\Lambda_1) \otimes Z(\Lambda_2)} = \overline{Z(\Lambda_1)} \otimes \overline{Z(\Lambda_2)}$. Thus $\Lambda_1 \otimes \Lambda_2$ is maximal if and only if $s(\overline{Z(\Lambda_1)} \otimes \overline{Z(\Lambda_2)}) = s(Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2))$.

(B1): Assume that $Z(\overline{\Lambda}_1) = \overline{R}$. Then $\overline{Z(\Lambda_1)} \otimes \overline{Z(\Lambda_2)} \simeq \overline{Z(\Lambda_2)}$ and $Z(\overline{\Lambda}_1) \otimes Z(\overline{\Lambda}_2) \simeq Z(\overline{\Lambda}_2)$. $\Lambda_1 \otimes \Lambda_2$: maximal $\Leftrightarrow s(\overline{Z(\Lambda_2)}) = s(Z(\overline{\Lambda}_2)) \Leftrightarrow \Lambda_2$: maximal (by 1.3). □

3. Proof of Corollary (B2)

3. Let Λ_i ($i = 1, 2$) be connected maximal R-orders satisfying (*). Put $k := \overline{R}$, $k_i := \overline{Z(\Lambda_i)}$ and $k_i' := Z(\overline{\Lambda}_i)$. Then k_i' is an extension field of k containing k_i, and $k_1 \otimes k_2 = \oplus_{j=1}^{t} T_j$ is a direct sum of extension fields T_j of k. Obviously the following two conditions are equivalent:

$$(**)\qquad s(k_1' \otimes k_2') = s(k_1 \otimes k_2),$$

$$(**1) \qquad k_1' \otimes_{k_1} T_j \otimes_{k_2} k_2' \ \text{ is a field for any } \ j \ (1 \le j \le t).$$

3.0 Assume that Λ_1 is unramified and residually separable over R. Then, by (c) 0.3 (or more elementary Hilfssatz 3 of [5]), we have

$$k_1' = Z(\overline{\Lambda}_1) = \overline{Z(\Lambda_1)} = k_1.$$

Being separable over k, k_1 has the form $k_1 = k[x]/fk[x]$ by a separable polynomial f in $k[x]$. The decomposition $k_1 \otimes k_2 = \oplus T_j$ corresponds to the decomposition of $f = \Pi f_j$ as irreducible factors in $k_2[x]$. Thus $(**1)$ is equivalent with

$$(**2) \qquad f_j \ \text{ is irreducible in } \ k_2'[x] \ \text{ for any } \ j \ (1 \le j \le t).$$

3.1 Further assume that Λ_2 is also residually separable over R, so that k_2' is separable over k, and moreover k_2'/k_2 is a (cyclic) Galois extension by (c) 0.3. Since the contition $(**2)$ depends only on the k-algebra structure of k_2', we consider that k_2' and k_1 are contained in a fixed separable closure k_{sep} of k, and apply Galois theory.

Let $G := \mathrm{Gal}(k_{sep}/k)$ and $G(L) := \{\sigma \in G : \sigma|_L = id_L\}$ for $L \subset k_{sep}$. The decomposition of f in $k_2[x]$ (respectively $k_2'[x]$) corresponds to the double cosets decomposition $G(k_2)\backslash G/G(k_1)$ (respectively $G(k_2')\backslash G/G(k_1)$), so that $(**2)$ is equivalent with

$$(**3) \qquad G(k_2) \subset G(k_2')\sigma G(k_1)\sigma^{-1} = G(k_2')G(\sigma(k_1)) \ \text{ for any } \ \sigma \in G.$$

Proposition 3.2. *Let Λ_i $(i = 1, 2)$ be connected residually separable maximal orders and Λ_1 be unramified over R. Then*:
(i) $\Lambda_1 \otimes \Lambda_2$ *is maximal if and only if*

$$k_2 = k_2' \cap \sigma(k_1)k_2 \ \textit{ for any } \ \sigma \in \mathrm{Gal}(k_{sep}/k).$$

(ii) *If further, one of k_1 or k_2' is Galois over k, then*:
$\Lambda_1 \otimes \Lambda_2$ *is maximal if and only if*

$$k_2 \cap \sigma(k_1) = k_2' \cap \sigma(k_1) \ \textit{ for any } \ \sigma \in \mathrm{Gal}(k_{sep}/k).$$

Proof. (i) Since $G(k_2')$ is a normal subgroup of $G(k_2)$:

$$G(k_2) \subset G(k_2')G(\sigma(k_1)) \Leftrightarrow G(k_2) = G(k_2')(G(\sigma(k_1)) \cap G(k_2)) = G(k_2')G(\sigma(k_1)k_2)$$
$$\Leftrightarrow k_2 = k_2' \cap \sigma(k_1)k_2.$$

(ii) $G(k_2) \subset G(k'_2)G(\sigma(k_1)) \Rightarrow G(k_2) \subset < G(k'_2), G(\sigma(k_1)) > \Leftrightarrow k_2 \supset k'_2 \cap \sigma(k_1) \Leftrightarrow k_2 \cap \sigma(k_1) = k'_2 \cap \sigma(k_1)$.

At the first implication, the converse holds if one of k'_2 or k_1 is Galois over k. □

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H. Hijikata
Department of Mathematics
Kyoto University
Kyoto, 606-8502, Japan

K. Nishida
Department of Mathematical Sciences
Shinshu University
Matsumoto, 390-8621, Japan
e-mail: kenisida@math.sinshu-u.ac.jp