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## ON KRULL-SCHMIDT'S THEOREM AND THE INDECOMPOSABILITY OF AMALGAMATED SUM

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Let  $R$  be a ring and  $M$  an  $R$ -module. If  $M$  has direct decompositions  $M = \bigoplus_{i \in I} L_i = \bigoplus_{j \in J} M_j$  into completely indecomposable  $R$ -modules  $L_i$  and  $M_j$ . Then, by Krull-Remak-Schmidt-Azumaya's theorem ([1]), it holds that for any finite subset  $J'$  of  $J$ , there exists a finite subset  $I'$  of  $I$  such that (1)  $M = \bigoplus_{i \in I'} L_i \oplus (\bigoplus_{j \in J - J'} M_j)$ . Then however it is not necessarily satisfied (2)  $M = \bigoplus_{j \in J'} M_j \oplus (\bigoplus_{i \in I - I'} L_i)$ . In § 1, we shall show that (1) and (2) hold simultaneously for suitable subset  $I'$  of  $I$ . The assertion is first showed in case  $M$  is semi-simple (in any completely reducible Grothendieck category). Then it is valid in general case, using the method of Harada and Sai [3, Corollary 1, p. 334]. But when the index set  $I$  is finite, we give an elementary proof for it.

Next, we consider finitely generated indecomposable modules over right artinian rings. In [4] and [5], Tachikawa investigated algebras of right local type (i.e. every indecomposable right module has the simple top) and of local or colocal type. To prove his assertions, he constructed indecomposable modules which were obtained by amalgamated sums (they were called interlacings there). In § 2, we shall slightly generalize his method, and give sufficient conditions for amalgamated sums to be indecomposable.

The authors wish to express their appreciation to Professor M. Harada and Mr. T. Inoue. The former suggested them that Theorem 1.3 (Theorem 1.7) is obtained from Lemma 1.1 (Lemma 1.6) using the method in [3], and the latter simplified their proof of Lemma 1.1 by his own method. As its proof, we take his own.

Throughout this note,  $R$  denotes a ring with unity and  $R$ -modules are (unital) right  $R$ -modules unless otherwise stated. For  $R$ -modules  $L_i (i \in I)$ , we use a notation  $\bigoplus_I L_i$  instead of  $\bigoplus_{i \in I} L_i$ . If  $f: L \rightarrow M$  is a homomorphism and  $L'$  is a submodule of  $L$ , we also denote the restriction map to  $L'$  of  $f: L \rightarrow M$  by  $f: L' \rightarrow M$ . Let  $I$  be a set and  $I_j$  a subset of  $I$  for each  $j=1, \dots, n$ .

If  $I = I_1 \cup \dots \cup I_n$  and  $I_j \cap I_k = \emptyset$  for all  $j$  and  $k (j \neq k)$ , we say that the union  $I_1 \cup \dots \cup I_n$  is a *partition* (of  $I$ ), and denote it by  $I = I_1 \amalg \dots \amalg I_n$ .

**1. Krull-Remak-Schmidt-Azumaya's theorem**

In this section, we study a generalization of Krull-Remak-Schmidt-Azumaya's theorem. The following lemma is basic for our results.

**Lemma 1.1.** *Let  $M = L_1 \oplus \dots \oplus L_n$  be a simple decomposition of a semi-simple  $R$ -module  $M$ . Then for any direct decomposition  $M = M_1 \oplus M_2$ , there exists a partition  $\{1, \dots, n\} = I_1 \amalg I_2$  such that*

$$M = (\oplus_{I_1} L_i) \oplus M_2 = M_1 \oplus (\oplus_{I_2} L_i).$$

*Proof.* We prove the assertion by induction on  $n$ . Let  $L'_1$  and  $L''_1$  denote the images of  $L_1$  under the projections  $M \rightarrow M_1$  and  $M \rightarrow M_2$ , respectively. Then it holds either  $M = L'_1 \oplus L_2 \oplus \dots \oplus L_n$  or  $M = L''_1 \oplus L_2 \oplus \dots \oplus L_n$ . We may assume  $M = L'_1 \oplus L_2 \oplus \dots \oplus L_n$ . Under the canonical homomorphism  $M \rightarrow M/L'_1$ , we denote the image of  $N$  by  $\bar{N}$  for every submodule  $N$  of  $M$ . Then  $\bar{M} = L_2 \oplus \dots \oplus L_n = \bar{M}_1 \oplus \bar{M}_2$ . By inductive assumption, there exists a partition  $\{2, \dots, n\} = I'_1 \amalg I'_2$  such that  $\bar{M} = (\oplus_{I'_1} L_i) \oplus \bar{M}_2 = \bar{M}_1 \oplus (\oplus_{I'_2} L_i)$ . Hence we have  $M = L'_1 + (\oplus_{I'_1} L_i) + M_2 = M_1 + (\oplus_{I'_2} L_i)$ . Comparing the composition lengths of the three terms, we see the sums are direct. Then it follows  $M = L_1 \oplus (\oplus_{I'_1} L_i) \oplus M_2$  and hence our assertion is satisfied for  $I_1 = \{1\} \cup I'_1$  and  $I_2 = I'_2$ .

From Lemma 1.1, we can obtain Theorem 1.3 using the argument in the proof of Harada and Sai [3, Corollary 1, p. 334]. We shall however give an elementary proof of it. (See [3, Lemma 2] for the following Lemma 1.2 and Remark.)

**Lemma 1.2.** *Let  $e$  and  $f$  be idempotents of a ring  $R$ . If  $\bar{R} = \bar{e}\bar{R} \oplus \bar{f}\bar{R}$ , then we have  $R = eR \oplus fR$ , where  $\bar{R} = R/J$ ,  $J$  the Jacobson radical of  $R$  and  $\bar{e} = e + J$ ,  $\bar{f} = f + J \in R/J$ .*

*Proof.* Assume  $\bar{R} = \bar{e}\bar{R} \oplus \bar{f}\bar{R}$ . Then  $R = eR + fR$  since  $J_R$  is small in  $R_R$ . Hence  $(1-f)R = (1-f)eR$ , which implies that a left multiplication map  $eR \rightarrow (1-f)R (ea \mapsto (1-f)ea)$  by  $(1-f)$  is an epimorphism. Since  $(1-f)R$  is projective, we have a split exact sequence

$$0 \rightarrow eR \cap fR \rightarrow eR \rightarrow (1-f)R \rightarrow 0.$$

By the assumption, however,  $eR \cap fR \subset eJ$  and hence  $eR \cap fR$  is small in  $eR$ . Therefore  $eR \cap fR = 0$  and so  $R = eR \oplus fR$ .

REMARK. Lemma 1.2 holds more generally. Let  $e_1, \dots, e_n$  be idempotents

of  $R$ . Using the notation of Lemma 1.2, if  $\bar{R} = \bar{e}_1 \bar{R} \oplus \cdots \oplus \bar{e}_n \bar{R}$ , then  $R = e_1 R \oplus \cdots \oplus e_n R$ . In fact, put  $P_i = e_i R$  and consider an external direct sum  $\bigoplus_{i=1}^n P_i$ . Then we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^n P_i & \xrightarrow{\tau} & \bigoplus_{i=1}^n \bar{e}_i \bar{R} \\ \downarrow \rho & & \parallel \\ R & \xrightarrow{\sigma} & \bar{R} \end{array}$$

with canonical maps. Since  $\sigma$  and  $\tau$  are the projective covers of  $\bar{R} = \bigoplus_{i=1}^n \bar{e}_i \bar{R}$ ,  $\rho$  is an isomorphism. This shows  $R = e_1 R \oplus \cdots \oplus e_n R$ .

Recall an  $R$ -module  $M$  is *completely indecomposable* provided its endomorphism ring  $\text{End}_R(M)$  is a local ring. A direct decomposition  $M = \bigoplus_I L_i$  is called a *completely indecomposable decomposition* if  $L_i$  is completely indecomposable for each  $i \in I$ .

**Theorem 1.3.** *Let  $M = L_1 \oplus \cdots \oplus L_n$  be a completely indecomposable decomposition of an  $R$ -module  $M$ . Then for any direct decomposition  $M = M_1 \oplus M_2$  there exists a partition  $\{1, \dots, n\} = I_1 \amalg I_2$  such that*

$$M = (\bigoplus_{I_1} L_i) \oplus M_2 = M_1 \oplus (\bigoplus_{I_2} L_i).$$

*Proof.* Let  $e_i$  denote the composition map of a projection  $M \rightarrow L_i$  and an injection  $L_i \rightarrow M$ , and  $g_i$  also the composition map  $M \rightarrow M_j \rightarrow M$  of canonical maps. Put  $S = \text{End}_R(M)$  and  $\bar{S} = S/J(S)$ , and denote  $x + J(S) \in S/J(S)$  by  $\bar{x}$  for  $x \in S$ , where  $J(S)$  is the Jacobson radical of  $S$ . Then, since  $S$  is a semi-perfect ring and  $e_1, \dots, e_n$  are mutually orthogonal primitive idempotents, and  $g_1, g_2$  are orthogonal idempotents, we have  $\bar{S} = \bar{e}_1 \bar{S} \oplus \cdots \oplus \bar{e}_n \bar{S} = \bar{g}_1 \bar{S} \oplus \bar{g}_2 \bar{S}$  and each  $\bar{e}_i \bar{S}$  is a simple  $\bar{S}$ -module. Therefore by Lemma 1.1 there exists a partition  $\{1, \dots, n\} = I_1 \amalg I_2$  such that  $\bar{S} = \bar{f}_1 \bar{S} \oplus \bar{g}_2 \bar{S} = \bar{g}_1 \bar{S} \oplus \bar{f}_2 \bar{S}$ , where  $f_1 = \sum_{I_1} e_i$ ,  $f_2 = \sum_{I_2} e_i$ . Hence by Lemma 1.2,  $S = f_1 S \oplus g_2 S = g_1 S \oplus f_2 S$ . Then it is easy to see that  $M = (\bigoplus_{I_1} L_i) \oplus M_2 = M_1 \oplus (\bigoplus_{I_2} L_i)$ .

Let  $M = M_1 \oplus M_2$  be a decomposition and let  $g_1: M \rightarrow M_1$  denote the projection. For a submodule  $L$  of  $M$  the restriction map  $g_1: L \rightarrow M_1$  is isomorphic if and only if  $M = L \oplus M_2$ . Therefore putting Theorem 1.3 in this way, we have

**Lemma 1.4** *Let  $L = \bigoplus_{i=1}^n L_i$  and  $M = M_1 \oplus M_2$  be decompositions of  $R$ -modules such that each  $L_i$  is completely indecomposable, and let  $g_j: M \rightarrow M_j$  denote the projection. If  $f: L \rightarrow M$  is an isomorphism, then there exists a partition  $\{1, \dots, n\} = I_1 \amalg I_2$  such that restriction map  $g_j f: \bigoplus_{I_j} L_i \rightarrow M_j$  is isomorphic for each  $j = 1, 2$ .*

The following corollary is a generalization of Krull-Remak-Schmidt's theorem.

**Corollary 1.5.** *Let  $M = \bigoplus_{i=1}^n L_i = \bigoplus_{j=1}^r M_j$  be direct decompositions of an  $R$ -module  $M$  with completely indecomposable modules  $L_i$ . Then there exists a partition  $\{1, \dots, n\} = I_1 \amalg \dots \amalg I_r$  such that the induced map  $g_j: N_j \rightarrow M_j$  is isomorphic and  $M = M_1 \oplus \dots \oplus M_j \oplus N_{j+1} \oplus \dots \oplus N_r$  for each  $j=1, \dots, r$ , where  $N_j = \bigoplus_{I_j} L_i$  and  $g_j: M \rightarrow M_j$  is a projection.*

*Proof.* Regard  $\bigoplus_{j=k}^r M_j$  as  $M_k \oplus (\bigoplus_{j=k+1}^r M_j)$ ,  $k=1, \dots, r-1$ . Then we get the assertion applying Lemma 1.4 inductively from case  $k=1$  and  $f$  is the identity map.

**Lemma 1.6.** *Let  $M = \bigoplus_I L_i$  be a simple decomposition of a semi-simple  $R$ -module  $M$ . Then for any direct decomposition  $M = M_1 \oplus M_2$  where  $M_1$  has a finite composition length, there exists a partition  $I = I_1 \amalg I_2$  such that  $M = (\bigoplus_{I_1} L_i) \oplus M_2 = M_1 \oplus (\bigoplus_{I_2} L_i)$ .*

*Proof.* Since  $M_1 \subset \bigoplus_I L_i$ , we have  $M_1 \subset \bigoplus_{I'} L_i$  for some finite subset  $I'$  of  $I$ . Put  $M' = \bigoplus_{I'} L_i$  and  $I'' = I - I'$ . Since  $M_1 \subset M'$ ,  $M' = M_1 \oplus M'_2$  ( $M'_2 = M' \cap M_2$ ) and  $M_2 = M'_2 \oplus M''_2$  for some submodules  $M'_2$  and  $M''_2$  of  $M_2$ . It is clear  $M = M' \oplus M''_2 = M' \oplus (\bigoplus_{I''} L_i)$ . Applying Lemma 1.1 to  $M' = \bigoplus_{I'} L_i = M_1 \oplus M'_2$ , there exists a partition  $I' = I_1 \amalg I'_2$  such that  $M' = \bigoplus_{I_1} L_i \oplus M'_2 = M_1 \oplus (\bigoplus_{I'_2} L_i)$ . Thus, for  $I_2 = I'_2 \cup I''$  it holds  $I = I_1 \amalg I_2$  and  $M = (\bigoplus_{I_1} L_i) \oplus M_2 = M_1 \oplus (\bigoplus_{I_2} L_i)$ .

**Theorem 1.7.** *Let  $M = \bigoplus_I L_i = \bigoplus_J M_j$  be completely indecomposable decompositions of an  $R$ -module  $M$ . Then for any finite subset  $J' = \{j_1, \dots, j_n\}$  of  $J$ , there exists a subset  $I' = \{i_1, \dots, i_n\}$  of  $I$  such that  $L_{i_k} \cong M_{j_k}$  for each  $k=1, \dots, n$  and*

$$M = (\bigoplus_{I'} L_i) \oplus (\bigoplus_{J-J'} M_j) = (\bigoplus_{J'} M_j) \oplus (\bigoplus_{I-I'} L_i).$$

*Proof.* We see easily that the proof of Lemma 1.6 is valid in any completely reducible and Grothendieck category. Hence by the method of Harada and Sai [3, Corollary 1, p. 334], the assertion holds.

**EXAMPLE.** Let  $M = \bigoplus_{i=1}^n L_i = \bigoplus_J M_j$  be completely indecomposable decompositions of  $M$  and  $J = J_1 \amalg J_2$  a partition of  $J$ . Then by Krull-Remak-Schmidt's theorem, for some subset  $I_1$  of  $I = \{1, \dots, n\}$ ,  $M = (\bigoplus_{I_1} L_i) \oplus (\bigoplus_{J_2} M_j)$ . But it is not necessarily satisfied that for  $I_2 = I - I_1$ ,  $M = (\bigoplus_{I_1} M_j) \oplus (\bigoplus_{I_2} L_i)$ .

Let  $R$  be a field and  $M$  a vector space with dimension 3 over  $R$ . That is  $M (=R^3) = \{(a_1, a_2, a_3)^t \mid a_i \in R\}$ , where  $(a_1, a_2, a_3)^t$  expresses the transposed matrix of  $(a_1, a_2, a_3)$ . Put

$$\begin{aligned} v_1 &= (1, 1, 1)^t, & v_2 &= (1, 1, 0)^t, & v_3 &= (1, 0, 1)^t, \\ u_1 &= (1, 0, 0)^t, & u_2 &= (0, 1, 0)^t, & u_3 &= (0, 0, 1)^t; \end{aligned}$$

and  $L_i = v_i R$ ,  $M_i = u_i R$ ,  $i = 1, 2, 3$ . Then  $M = L_1 \oplus L_2 \oplus L_3 = (M_1 \oplus M_2) \oplus M_3$  and moreover

$$\begin{aligned} M &= L_1 \oplus M_2 \oplus M_3 = M_1 \oplus L_2 \oplus L_3 \\ &= M_1 \oplus L_2 \oplus M_3 = (M_1 \oplus M_2) \oplus L_3. \end{aligned}$$

On the other hand,  $(L_1 \oplus L_2) + M_3 = (u_1 + u_2)R \oplus u_3 R (\neq M)$ .

### 2. Indecomposability of amalgamated sums

In this section, we assume a ring  $R$  is always right artinian and  $R$ -modules mean finitely generated right  $R$ -modules except for Proposition 2.2.

Let  $(E) 0 \rightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \rightarrow 0$  be an exact sequence of  $R$ -modules. We consider the following condition:

(\*) If  $X$  is a (non-zero)  $R$ -module and  $\varphi: L \rightarrow X$  is a retraction (i.e. split epimorphism) then there is no homomorphism  $\psi: M \rightarrow X$  such that  $\varphi = \psi\beta$ .

If an exact sequence (E) satisfies the condition (\*), we say that (E) is a (\*)-sequence. Consider the following commutative diagrams of  $R$ -modules such that  $\sigma$  is a retraction and  $\tau'$  is a section (i.e. split monomorphism):

$$(D) \quad \begin{array}{ccc} X & \xrightarrow{\rho} & Y \\ \sigma \searrow & & \nearrow \tau \\ & Z & \end{array} \quad (D') \quad \begin{array}{ccc} X & \xrightarrow{\rho} & Y \\ \sigma' \nearrow & & \nwarrow \tau' \\ & Z & \end{array}$$

In (D), let  $\sigma'$  be a right inverse of  $\sigma$  and put  $\tau' = \rho\sigma'$ . Then  $\tau'$  is a right inverse of  $\tau$ , and so we get the diagram (D'). Conversely, we can get (D) from (D') and hence the condition (\*) is equivalent to the following condition.

(\*)' If  $X$  is a (non-zero)  $R$ -module and  $\psi': X \rightarrow M$  is a section, then there is no homomorphism  $\varphi': X \rightarrow L$  such that  $\psi' = \beta\varphi'$ .

REMARK 1. It is easy to see that (E) is a (\*)-sequence if  $M$  has no direct summand which is isomorphic to a direct summand of  $L$ . In particular, if (E) does not split and  $M$  is indecomposable, then (E) is a (\*)-sequence.

2. For an exact sequence (E), we can consider the following dual condition (\*\*) of (\*), and show the duals of all the results in Section 2 except for Propositions 2.2 and 2.3.

(\*\*) If  $X_1$  is a (non-zero)  $R$ -module and  $\varphi_1: X_1 \rightarrow L$  is a section, then there is no homomorphism  $\psi_1: X_1 \rightarrow K$  with  $\varphi_1 = \alpha\psi_1$ .

Let  $L = \bigoplus_{i=1}^n L_i$  be a direct decomposition of  $L$  into indecomposable modules

$L_i, 1 \leq i \leq n$ . Put  $S = \text{End}_R(L)$  and denote by  $J(S)$  the Jacobson radical of  $S$ . Then every element  $\varphi$  in  $S$  is expressed by a matrix with coefficients  $\varphi_{ij}: L_j \rightarrow L_i; \varphi = (\varphi_{ij})$ . As is well known,  $\varphi \in J(S)$  if and only if each  $\varphi_{ij}$  is not an isomorphism  $1 \leq i, j \leq n$ . Moreover, let  $\alpha = (\alpha_1, \dots, \alpha_n)^t: K \rightarrow \bigoplus_{i=1}^n L_i$  be a matrix expression of  $\alpha: K \rightarrow L$ . Then the following proposition is immediate.

**Proposition 2.1.** *Let  $(E) 0 \rightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \rightarrow 0$  be an exact sequence and  $L = \bigoplus_{i=1}^n L_i$  a decomposition with indecomposable modules  $L_i, 1 \leq i \leq n$ . Then the following conditions are equivalent.*

- (a)  $(E)$  is a  $(*)$ -sequence.
- (b)  $\{\varphi \in S \mid \varphi\alpha = 0\} \subset J(S)$ .
- (c) For  $\varphi = (\varphi_{ij})$  in  $S$ , each  $\varphi_{ij}$  is not isomorphic, whenever  $\varphi_{i1}\alpha_1 + \dots + \varphi_{in}\alpha_n = 0 (i=1, \dots, n)$ .

REMARK 3. Monomorphisms  $\alpha$  with the property (b) in Proposition 2.1 were investigated by Dickson and Kelly [2].

Let  $e$  and  $f$  be primitive idempotents of  $R$  and  $u_i$  an element of  $fJe$  such that  $u_i \notin fJ^2e$  for  $i=1, \dots, n$ , where  $J$  is the Jacobson radical of  $R$ . If an element  $m$  of a module  $M$  and  $N$  is a submodule of  $M$ , a notation " $\bar{m} \in M/N$ " means  $\bar{m} = m + N (\in M/N)$ . Then for  $\bar{u}_i \in Je/J^2e, R\bar{u}_i \cong Rf/Jf$  (simple). Moreover, put  $L_i = fR/fJ^2, i=1, \dots, n$  and  $K = eR/eJ$ . Consider (external) direct sum  $L = \bigoplus_{i=1}^n L_i$  and define a map  $\alpha: K \rightarrow L$  by  $\alpha(\bar{ea}) = \sum_{i=1}^n \bar{u}_i \bar{ea}$ , where  $\bar{ea} \in eR/eJ = K$  and  $\bar{u}_i \bar{ea} \in fR/fJ^2 = L_i$ . Now put  $M = \text{Coker } \alpha$ . Then the following proposition is immediate from the definition of  $(*)$ -sequences. (See [5, Propositions 3.3 and 3.5] for Propositions 2.2 and 2.3.)

**Proposition 2.2.** *Under the above notation, the following conditions are equivalent.*

- (a) For  $\bar{u}_i \in Je/J^2e, 1 \leq i \leq n$ , we have  $R\bar{u}_1 \oplus \dots \oplus R\bar{u}_n \subset Je/J^2e$ .
- (b) For  $\bar{u}_i \in fJe/fJ^2e, 1 \leq i \leq n, \bar{u}_1, \dots, \bar{u}_n$  are linearly independent over a division ring  $fRf/fJf$  (considering  $fJe/fJ^2e$  as a left module).
- (c) The exact sequence  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  is a  $(*)$ -sequence.

Next, let  $L_1, \dots, L_r$  be indecomposable  $R$ -modules such that every homomorphism  $\varphi: L_i \rightarrow L_j$  vanishes the socle of  $L_i$  for each pair  $i$  and  $j (i \neq j)$ , and let  $K$  be a simple  $R$ -module. Put  $N_i = L_i^{(k_i)}$  ( $k_i$ -times direct sum of copies of  $L_i$ ) and  $L = \bigoplus_{i=1}^r N_i$ . Let  $\alpha'_i: K \rightarrow N_i (1 \leq i \leq r)$ , and  $\alpha: K \rightarrow L$  be maps with  $\alpha = (\alpha'_1, \dots, \alpha'_r)^t$ , and put  $M_i = \text{Coker } \alpha'_i$  and  $M = \text{Coker } \alpha$ . Then the following proposition is immediate.

**Proposition 2.3.** *Under the above notation, the following conditions are equivalent.*

- (a) *The exact sequence  $0 \rightarrow K \rightarrow N_i \rightarrow M_i \rightarrow 0$  is a (\*)-sequence for each  $i = 1, \dots, r$ .*
- (b) *The exact sequence  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  is a (\*)-sequence.*

Let  $M$  be an  $R$ -module. Recall that  $M$  is *local* (resp. *colocal*) if  $M$  has a unique maximal (resp. minimal) submodule. We denote the composition length of  $M$  by  $|M|$ .

**Lemma 2.4.** *Let  $L = \bigoplus_{i=1}^n L_i$  and  $M = M_1 \oplus M_2$  be decompositions of  $R$ -modules  $L$  and  $M$ , where  $L_i$ 's are local (resp. colocal), and  $\pi_j: M \rightarrow M_j$  a projection for  $j=1, 2$ . If  $\beta: L \rightarrow M$  is an epimorphism (resp. monomorphism), then there exists a partition  $\{1, \dots, n\} = I_1 \amalg I_2$  such that each  $\pi_j \beta: L \rightarrow M_j$  induces an epimorphism (resp. monomorphism)  $\pi_j \beta: \bigoplus_{I_j} L_i \rightarrow M_j$  for  $j=1, 2$ .*

*Proof.* We shall only show the assertion in case  $L_i$ 's are local, because we can similarly do it in the other case. Let  $\bar{M}$  denote the top  $M/MJ$  of  $M$  and put  $\bar{N} = \sigma(N)$  for every submodule  $N$  of  $M$ , where  $\sigma$  is the canonical homomorphism  $M \rightarrow \bar{M}$ . Then for some subset  $I'$  of  $\{1, \dots, n\}$ , we have  $\bar{M} = \bigoplus_{I'} \overline{\beta(L_i)} = \bar{M}_1 \oplus \bar{M}_2$ . Using Lemma 1.1, as easily seen, there exists a partition  $I' = I'_1 \amalg I'_2$  such that  $M = \sum_{I'_1} \beta(L_i) + M_2 = \sum_{I'_2} \beta(L_i) + M_1$ . Then the assertion is immediate from  $\pi_1(M_2) = \pi_2(M_1) = 0$ .

**Theorem 2.5.** *Let  $(E) 0 \rightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \rightarrow 0$  be an exact sequence of  $R$ -modules such that  $L = \bigoplus_{i=1}^n L_i$ ,  $L_i$  is local but is not simple and  $K$  is simple. Then the following conditions are equivalent.*

- (a)  *$M$  is indecomposable.*
- (b)  *$M$  has no direct summand which is isomorphic to  $L_i$  for some  $i=1, \dots, n$ .*
- (c) *The exact sequence  $(E)$  is a (\*)-sequence.*

*Proof.* We shall only prove that (c) implies (a), for the others are clear (see Remark 1). Assume  $M$  is decomposable, say  $M = M_1 \oplus M_2$ . By Lemma 2.4, there exists a partition  $\{1, \dots, n\} = I_1 \amalg I_2$  such that the restriction map  $\psi_j: \bigoplus_{I_j} L_i \rightarrow M_j$  of  $\pi_j \beta: \bigoplus_{i=1}^n L_i \rightarrow M_j$  is an epimorphism,  $j=1, 2$ . But  $|L| = |M| + |K| = |M| + 1$ . Hence  $\psi_j$  is an isomorphism for some  $j$  ( $j=1$  or  $2$ ), say  $j=1$ . Put  $\varphi = \psi_1^{-1} \pi_1 \beta$  and let  $\kappa: \bigoplus_{I_1} L_i \rightarrow \bigoplus_{i=1}^n L_i$  denote the canonical monomorphism. Then  $\varphi \kappa$  is clearly the identity map of  $\bigoplus_{I_1} L_i$  and so (E) is not a (\*)-sequence.

**REMARK 4.** Theorem 2.5 is essentially due to Tachikawa [5, Lemma 1.1] under Lemma 1.1. By Theorem 2.5 and Propositions 2.2 and 2.3, we can give



simple proofs of Propositions 2.4, 3.1, 3.3, 3.5 in Tachikawa [4].

**Proposition 2.6.** *Let  $L = \bigoplus_{i=1}^n L_i$  be an indecomposable decomposition and  $(E) 0 \rightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \rightarrow 0$  be a  $(*)$ -sequence such that the  $n$ -th coordinate map  $\alpha_n: K \rightarrow L_n$  of  $\alpha$  is monomorphic. If  $L_1, \dots, L_n$  are colocal and  $\text{Coker } \alpha_n$  is simple, then  $M$  is indecomposable.*

Proof. Put  $L' = \bigoplus_{i=1}^{n-1} L_i$  and  $K_n = \alpha_n(K)$ . Consider the following diagram with injection  $\kappa'$  and projection  $\pi_n$ .

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M \rightarrow 0 \\ & & & & \parallel & & \\ 0 & \rightarrow & L' & \xrightarrow{\kappa'} & L & \xrightarrow{\pi_n} & L_n \rightarrow 0. \end{array}$$

Then, since  $\alpha_n$  (i.e.  $\pi_n \alpha$ ) is monomorphic, the restriction map  $\beta': L' \rightarrow M$  (i.e.  $\beta \kappa'$ ) of  $\beta: L \rightarrow M$  is also monomorphic, and we have an exact sequence  $0 \rightarrow L' \rightarrow M \rightarrow N \rightarrow 0$ , where  $N = \text{Coker } \beta'$ . It is easy to see that  $\beta(L') \cap \beta(L_n) = \beta(K_n)$  and so  $N = (\beta(L') + \beta(L_n)) / \beta(L') \cong \beta(L_n) / \beta(K_n)$ . Therefore  $N$  is simple (or zero) since  $L_n / K_n$  is simple by the assumption. Suppose  $M$  is decomposable, say  $M = M_1 \oplus M_2$ , and let  $\pi_j: M \rightarrow M_j$  denote the projection,  $j = 1, 2$ . Then by Lemma 2.4, there exists a partition  $\{1, \dots, n-1\} = I_1 \amalg I_2$  such that the restriction map  $\psi_k: \bigoplus_{i \in I_k} L_i \rightarrow M_k$  of  $\pi_k \beta': L' \rightarrow M$  is monomorphic,  $k = 1, 2$ . Since  $|M| = |L'| + |N| \leq |L'| + 1$ , we may assume the monomorphism  $\psi_1: \bigoplus_{i \in I_1} L_i \rightarrow M_1$  is isomorphic. Then by the same way as in the proof of Theorem 2.5, we see that (E) is not a  $(*)$ -sequence. This verifies the assertion.

**Proposition 2.7.** *Let  $L = \bigoplus_{i=1}^n L_i$  and (E) be as in the above proposition. If  $L_1, \dots, L_n$  are local and colocal and  $|L_n|$  divides  $|L_i|$  for each  $i$ , ( $1 \leq i \leq n$ ), then  $M$  is indecomposable.*

Proof. Suppose  $M$  is decomposable, say  $M = M_1 \oplus M_2$ . Let  $\pi_k: M \rightarrow M_k$  denote the projection,  $k = 1, 2$ . As in the proof of Theorem 2.5, for some partition  $\{1, \dots, n\} = J_1 \amalg J_2$ , the restriction map  $\pi_k \beta: \bigoplus_{i \in J_k} L_i \rightarrow M_k$  of  $\pi_k \beta: L \rightarrow M_k$  is epimorphic for each  $k = 1, 2$ . On the other hand, as in the proof of Proposition 2.6, for some partition  $\{1, \dots, n-1\} = I_1 \amalg I_2$ , the restriction map  $\pi_k \beta: \bigoplus_{i \in I_k} L_i \rightarrow M_k$  of  $\pi_k \beta: \bigoplus_{i=1}^{n-1} L_i \rightarrow M_k$  is monomorphic. Put  $c_i = |L_i|$ . Then the above fact implies  $\sum_{i \in J_k} c_i \leq |M_k| \leq \sum_{i \in J_k} c_i$ , and hence  $(\sum_{i \in J_k} c_i) / c_n \leq (\sum_{i \in J_k} c_i) / c_n$  for  $k = 1, 2$ . But  $c_n$  divides  $c_i$  for each  $i = 1, \dots, n$  and  $(\sum_{i \in I_1} c_i) / c_n + (\sum_{i \in I_2} c_i) / c_n + 1 = (\sum_{i=1}^n c_i) / c_n = (\sum_{i \in J_1} c_i) / c_n + (\sum_{i \in J_2} c_i) / c_n$ . This shows  $(\sum_{i \in I_k} c_i) / c_n = (\sum_{i \in J_k} c_i) / c_n$  for some  $k = 1$  or  $2$ , say  $k = 1$ . Hence  $\sum_{i \in I_1} c_i = |M_1| = \sum_{i \in J_1} c_i$  and the monomorphism

$\pi_1\beta: \bigoplus_{I_1} L_i \rightarrow M_1$  is isomorphic. Thus (E) is not a (\*)-sequence. This verifies the assertion.

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