Studies on curves on toric surfaces
トーリック曲面上の曲線の研究

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List of notation

$\mathbb{C}$: the complex number field,

$\mathbb{P}^n$: the n-dimensional projective space over $\mathbb{C}$,

Σ_a : the Hirzebruch surface of degree a,

$H^i(X, \mathcal{F})$: the i-th cohomology group of a sheaf $\mathcal{F}$,

$h^i(X, \mathcal{F})$: the dimension of $H^i(X, \mathcal{F})$,

$\chi(X, \mathcal{F})$: the Euler characteristic of a coherent sheaf $\mathcal{F}$,

$\mathrm{Pic}(X)$: the Picard group of a variety X,

$\mathrm{Div}(X)$: the Cartier divisor class group of a variety X,

$\mathcal{O}_X(D)$: the invertible sheaf associated to a divisor D on a variety X,

ω_X : the canonical sheaf of a nonsingular variety X,

K_X : the canonical divisor of a nonsingular variety X,

$D \sim D'$: linear equivalence of divisors,

$|D|$: the complete linear system of a divisor D,

$D.D'$: the intersection number of divisors,

D^2 : the self-intersection number of a divisor D,

g^1_k : a one-dimensional linear system (i.e. a pencil) of degree k on an algebraic curve,

T_N : the algebraic torus defined by a free $\mathbb{Z}$-module N,

$\square_D$: the lattice polytope of a divisor D on a toric surface (cf. § 2.4 in detail),

$\mathrm{Int}\,\square_D$: the interior of $\square_D$ (cf. § 4.2 in detail),

$\sharp A$: the cardinality of a set A.

Introduction

In this thesis, the author studies algebraic curves on toric surfaces and provides two results which are independent each other. Throughout this thesis, we assume that algebraic varieties are irreducible and defined over the complex number field.

In the study of algebraic curves, a computation of invariants such as intersection numbers and cohomology dimensions is one of the most essential operation what we have to do, although it is not so easy in general. On the other hand, the theory of toric varieties has been established at the beginning of the 1970's independently by Demazure, Mumford, Satake, Miyake, Oda and others. It revealed the close relation between algebraic geometry and the geometry of convex polytopes in real affine spaces, and produced various interesting applications. We can utilize this relation to reduce to elementary convex geometry a lot of problem in the study of toric varieties. In fact, for curves on toric surfaces, one can easily compute invariants mentioned above by investigating the properties of the associated lattice polytopes (see Section 2.4). Then it seems plausible to expect that we should understand such curves more precisely than other general curves. It is a consistent motivation in the author's works.

This thesis consists of four parts. The first chapter is devoted to review the classical and fundamental facts in the study of algebraic geometry. In the second chapter, we introduce toric surfaces which are the main stage of our consideration in this thesis. The third and fourth chapters contain the author's two main results Theorem 3.1.6 and 4.1.3 which are explained in detail below.

The first work is about the so-called *gonality conjecture*. The *gonality* is a significant invariant in the study of linear systems on curves, which is defined as the minimal degree of surjective morphisms from a complete nonsingular curve C to $\mathbb{P}^1$ and denoted by $\mathrm{gon}(C)$. Clearly, $\mathrm{gon}(C) = 1$ means that C is a rational curve. Besides, $\mathrm{gon}(C)$ is equal to two if and only if C is elliptic or hyperelliptic. The following classical result gives an upper bound of the gonality.

Theorem 0.0.1. *Let C be a nonsingular curve of genus g. Then*

$$\mathrm{gon}(C) \leq \frac{g+3}{2}.$$

For a nonsingular plane curve of degree $d\,(\geq 2)$, it is well known that its gonality is equal to $d-1$:

Theorem 0.0.2 ([26]). *Let C be a nonsingular plane curve of degree $d \geq 2$. Then* $\mathrm{gon}(C) = (d-1)$ *and any pencil of degree $d-1$ is cut out by lines passing through a fixed point on C.*

For singular plane curves, there are results of Coppens, Kato, Ohkouchi and Sakai (Theorem 3.1.1 and 3.1.2). On the other hand, Martens has computed the gonality of a nonsingular curve on a Hirzebruch surface (Theorem 3.1.3). Although there are many attempts to compute the gonality, in general, it is not so easy to determine it. Under such circumstances, the gonality conjecture proposes a new way to approach this problem. This conjecture predicts that one can read off the gonality of a curve from the minimal resolution of any one line bundle of sufficiently large degree (Conjecture 3.1.5). Aprodu has shown this conjecture holds for curves on Hirzebruch surfaces (cf. [1]). The author's first work is a natural continuation of this result, e.g., it treats the gonality conjecture for curves on compact nonsingular toric surfaces. Such a toric surface is obtained from a projective plane or a Hirzebruch surface by a finite succession of blowing-ups with T_N-fixed points (i.e. points which are invariant with respect to the action on a toric surface by the algebraic torus) as centers, and has finite $\mathbb{P}^1$-fibrations by toric morphisms. In Chapter 3, we will prove the conjecture affirmatively for nonsingular irrational curves on compact nonsingular toric surfaces which have only one toric morphism to $\mathbb{P}^1$ (Theorem 3.1.6). As we shall see in Chapter 2, a Hirzebruch surface is one of the simplest examples of such toric surfaces.

The second work deals with Weierstrass gap sequences. For a point P on a complete nonsingular algebraic curve C, a positive integer j is called a *gap value* at P if

$$h^0(C, jP) = h^0(C, (j-1)P).$$

The set of all gap values is called a *Weierstrass gap sequence* (or, simply, *gap sequence*) of C at P. By Riemann-Roch theorem, its cardinality is equal to the genus of C. The gap sequence $\{1, 2, \ldots, g\}$ is said to be trivial, and a point on a curve is called a *Weierstrass point* if its gap sequence is nontrivial. The following classical result is a basic tool in the study of gap sequences.

Theorem 0.0.3 (Weierstrass gap theorem). *Let C be a nonsingular projective curve of genus $g \geq 1$, and P a point on C. Then any gap value at P is less than $2g$.*

For a point on a hyperelliptic curve, there are two types of gap sequences:

Theorem 0.0.4. *Let P a point on a hyperelliptic curve C and $\Phi_{|K_C|} : C \to \mathbb{P}^1$ the holomorphic map associated to $|K_C|$.*

(i) *If P is a ramification point of $\Phi_{|K_C|}$, then the gap sequence of C at P is the set of odd numbers $\{1, 3, \ldots, 2g-1\}$.*
(ii) *If $\Phi_{|K_C|}$ is unramified at P, then the gap sequence of C at P is $\{1, 2, \ldots, g\}$.*

Namely, in the case of hyperelliptic curves, the notion of Weierstrass points coincides with that of ramification points of the canonical morphism. For trigonal curves, the list of the gap sequences has been obtained by Coppens and Kim (Theorem 4.1.1 and 4.1.2). Besides, Coppens and Kato studied inflection points of plane curves with δ ordinary nodes, and gave a list of all possible gap sequences for $\delta \leq 5$ ([7]). In Chapter 4, the author aims to establish the new technique to compute gap sequences on curves on toric surfaces which is based on the theory of toric surfaces. As a result, for certain points on a nonsingular nef curve on a compact nonsingular toric surface, we obtain a sufficient condition for a positive integer to be the gap value at these points (Theorem 4.1.3). Furthermore, under the suitable condition, it becomes the necessary and sufficient condition. This means that we can determine the gap sequence in its entirety. In the last section of Chapter 4, we will see some examples to which we can apply our new technique.

Chapter 1

Fundamentals of algebraic geometry

In this chapter, let X be a nonsingular projective variety of dimension n defined over the complex number field, and denote by g its genus in the case where $n = 1$. We collect some fundamentals needed in this thesis without the proofs. For the proofs of these facts, we refer the reader to [14].

The following two theorems are extremely important to compute the dimension of the cohomology group of a sheaf on X.

Theorem 1.0.1 (Serre duality theorem). *Let $\mathcal{F}$ be a locally free sheaf on X. Then $H^i(X, \mathcal{F})$ and $H^{n-i}(X, \omega_X \otimes \mathcal{F}^{-1})$ are dual each other for any integer $0 \leq i \leq n$.*

Theorem 1.0.2 (Riemann-Roch theorem). *Let D be a divisor on X.*
(i) *If $n = 1$, then $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$.*
(ii) *If $n = 2$, then $\chi(X, \mathcal{O}_X(D)) = \frac{1}{2}D.(D - K_X) + \chi(X, \mathcal{O}_X)$.*

Corollary 1.0.3. *If $\dim X = 1$, then $\deg K_X = 2g - 2$.*

Combining Corollary 1.0.3 and the following theorem, we obtain a quick method of computing the genus of a curve on a surface.

Theorem 1.0.4 (Adjunction formula). *Let D be a nonsingular prime divisor on X. Then $K_D = (D + K_X)|_D$.*

Besides, Theorem 1.0.2 gives elementary but useful criterions to know some properties of divisors on X :

Theorem 1.0.5 ([14, IV, Example. 1.3.4, Corollary 3.2])**.** *Assume $n = 1$ and let D be a divisor on X. Then the following hold :*
(i) *If $\deg D \geq 2g - 1$, then D is nonspecial.*
(ii) *If $\deg D \geq 2g$, then the complete linear system $|D|$ has no base points.*

The following Theorem 1.0.6 plays an essential part in the study of the linear systems.

Theorem 1.0.6 (Bertini's theorem). *Let D be a divisor on X. Then a general member of $|D|$ is nonsingular outside the set of base points of $|D|$. Besides, if $n \geq 2$, $\dim|D| \geq 2$ and $|D|$ has no fixed components, then a general member of $|D|$ is irreducible.*

Lastly, we see the identity between sheaves, line bundles and divisors on X.

Theorem 1.0.7 ([14, II, Ex. 5.18])**.** *There is a one-to-one correspondence between isomorphism classes of invertible sheaves on X and isomorphism classes of line bundles over X.*

Theorem 1.0.8 ([14, II, Proposition. 6.15])**.** *There is an isomorphism* $\mathrm{Div}(X)/\sim\ \simeq \mathrm{Pic}(X)$, *where '$\sim$' denotes linear equivalence.*

Because of Theorem 1.0.7 and 1.0.8, we often use the notion "invertible sheaf", "line bundle" and "Cartier divisor" interchangeably, if no confusion seems likely to result.

Proposition 1.0.9 ([14, II, Lemma 7.8])**.** *Let D be a divisor on X. Then the following are equivalent:*

(i) *$|D|$ has no base points.*

(ii) *$\mathcal{O}_X(D)$ is globally generated.*

Chapter 2

Toric varieties

As declared in the introduction, throughout this thesis, we consider curves on toric surfaces. Hence, first of all, we shall review the theory of toric varieties. As is well known, it has the close connection with the geometry of convex polytopes. This fact is the greatest advantage of considering curves on toric surfaces. Many basic properties of toric varieties and divisors on them can be interpreted in terms of the elementary geometry of fans. Many of the theoretical facts included in this chapter owe a lot to [15] and [28].

2.1 Cones and fans

We first define elementary objects in convex geometry called cones and fans. Let $N \simeq \mathbb{Z}^r$ be a free $\mathbb{Z}$-module of rank r and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual $\mathbb{Z}$-module. By scalar extension to the rational number field $\mathbb{R}$, we have r-dimensional $\mathbb{R}$-vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Then there is a canonical $\mathbb{Z}$-bilinear (resp. $\mathbb{R}$-bilinear) pairing $\langle\, ,\rangle : M \times N \to \mathbb{Z}$ (resp. $\langle\, ,\rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$).

Definition 2.1.1. A subset σ of $N_{\mathbb{R}}$ is called a *convex rational polyhedral cone* if there exist a finite number of elements $n_1, \ldots, n_s$ in N such that

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s.$$

Moreover, σ is said to be *strongly convex* if $\sigma \cap (-\sigma) = \{O\}$ holds. The dimension of σ is defined by $\dim \sigma = \dim_{\mathbb{R}}(\sigma + (-\sigma))$.

For a convex rational polyhedral cone σ, we define the subset $\sigma^{\vee}$ of $M_{\mathbb{R}}$ as

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, n\rangle \geq 0,\ {}^{\forall} n \in \sigma\}.$$

In fact, $\sigma^{\vee}$ becomes the convex rational polyhedral cone in $M_{\mathbb{R}}$ ([15, Theorem1.2.2]).

Definition 2.1.2. A subset τ of σ is called a *face* if there is an element $m_0 \in \sigma^\vee$ such that $\tau = \{n \in \sigma \mid \langle m_0, n\rangle = 0\}$, and is denoted $\tau \prec \sigma$.

We are now in a position to introduce the most fundamental notion in the toric theory.

Definition 2.1.3. A nonempty set Δ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ is called a *fan* in N if it satisfies the following properties:
(i) Every face of a cone in Δ is also a cone in Δ,
(ii) The intersection of any two cones in Δ is a face of each.

2.2 Toric varieties

In this section, we will see the process of constructing toric varieties from fans. Since a cone yields the normal integral domain naturally (Proposition 2.2.2), we can obtain the affine algebraic variety. We construct a toric variety by gluing together such affine toric varieties which are associated to cones contained in the fan. From now on, a *cone* will always means a strongly convex rational polyhedral cone.

Theorem 2.2.1 ([28, Proposition 1.1])**.** *Let σ be a cone in $N_\mathbb{R}$ and put $\mathcal{S}(\sigma) = M \cap \sigma^\vee$. Then $\mathcal{S}(\sigma)$ is a finitely generated additive subsemigroup of M.*

For a cone σ in $N_\mathbb{R}$, let $\mathbb{C}[\mathcal{S}(\sigma)] = \bigoplus_{m\in\mathcal{S}(\sigma)} \mathbb{C}\,\boldsymbol{e}(m)$ be the semigroup algebra of $\mathcal{S}(\sigma)$ over $\mathbb{C}$, where $\{\boldsymbol{e}(m) \mid m \in \mathcal{S}(\sigma)\}$ is the set of indeterminates and the ring multiplication is defined by

$$\boldsymbol{e}(m)\boldsymbol{e}(m') = \boldsymbol{e}(m+m'),$$
$$\Big(\sum_{m\in\mathcal{S}(\sigma)} c_m \boldsymbol{e}(m)\Big)\Big(\sum_{m'\in\mathcal{S}(\sigma)} c_{m'} \boldsymbol{e}(m')\Big) = \sum_{m''\in\mathcal{S}(\sigma)} \Big(\sum_{m+m'=m''} c_m d_{m'}\Big)\boldsymbol{e}(m'').$$

The quotient field of $\mathbb{C}[\mathcal{S}(\sigma)]$ is equal to that of $\mathbb{C}[M]$ ([15, Lemma5.5.2]).

Theorem 2.2.2 ([15, Theorem 5.5.1])**.** *Let σ be a cone in $N_\mathbb{R}$. Then the semigroup algebra $\mathbb{C}[\mathcal{S}(\sigma)]$ is finitely generated over $\mathbb{C}$, and is a normal integral domain.*

By Theorem 2.2.2, for a cone σ in $N_\mathbb{R}$, we can obtain an r-dimensional irreducible affine algebraic variety U_σ whose coordinate ring is $\mathbb{C}[\mathcal{S}(\sigma)]$, which is called an *affine toric variety* associated to the cone σ. Here we note that by the general theory of algebraic varieties, we can identify U_σ with the set of $\mathbb{C}$-valued points of $\mathbb{C}[\mathcal{S}(\sigma)]$, i.e., the set of $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[\mathcal{S}(\sigma)] \to \mathbb{C}$. Moreover, the map

$$\begin{aligned}\{\mathbb{C}\text{-valued points of } \mathbb{C}[\mathcal{S}(\sigma)]\} &\to \{u : \mathcal{S}(\sigma) \to \mathbb{C} \mid u(0) = 1, \\ &\qquad u(m+m') = u(m)u(m') \text{ for } {}^{\forall}m, m' \in \mathcal{S}(\sigma)\} \\ f &\mapsto u_f(m) := f(\boldsymbol{e}(m))\end{aligned}$$

is one-to-one. Consequently, U_σ can be canonically identified with the set of subsemigroup algebras from $\mathcal{S}(\sigma)$ to multiplicative semigroup $\mathbb{C}$ which map zero to one.

We are now ready to construct toric varieties. Our construction is carried out by gluing together affine toric varieties, where the following proposition plays an important role.

Theorem 2.2.3 ([28, Proposition 1.3]). *Let σ be a cone in $N_\mathbb{R}$ and τ a face of σ. Then there exists $m_0 \in \mathcal{S}(\sigma)$ such that $\tau = \{n \in \sigma \mid \langle m_0, n\rangle = 0\}$. Hence τ is also a cone in $N_\mathbb{R}$. Moreover, the equalities $\mathcal{S}(\tau) = \mathcal{S}(\sigma) + \mathbb{Z}_{\geq 0}(-m_0)$ and $U_\tau = \{u \in U_\sigma \mid u(m_0) \neq 0\}$ hold.*

By Theorem 2.2.3, for a cone σ in $N_\mathbb{R}$, an affine toric variety associated to a face of σ becomes an open subset of U_σ. For a fan Δ in N and σ, $\tau \in \Delta$, the intersection $\sigma \cap \tau \in \Delta$ is clearly a face of both σ and τ by the definition of a fan. Thus by Theorem 2.2.3, $U_{\sigma\cap\tau}$ is naturally an open subset of both U_σ and U_τ. Therefore we can naturally glue $\{U_\sigma \mid \sigma \in \Delta\}$ together to obtain an r-dimensional irreducible normal algebraic variety $T_N(\Delta)$, which is called a *toric variety* associated to the fan Δ. For $T_N(\Delta)$, its nonsingularity and compactness can be dealt with by means of the properties of the fan Δ.

Theorem 2.2.4 ([28, Theorem 1.10]). *A toric variety $T_N(\Delta)$ is nonsingular if and only if each $\sigma \in \Delta$ is generated by a part of a $\mathbb{Z}$-basis of N.*

Theorem 2.2.5 ([28, Theorem 1.11]). *A toric variety $T_N(\Delta)$ is compact if and only if $\sigma \in \Delta$ is finite set and its support $|\Delta| = \bigcup_{\sigma\in\Delta} \sigma$ is the hole space $N_\mathbb{R}$.*

2.3 A torus action

In this section, let $X = T_N(\Delta)$ be a toric variety associated to a fan Δ in $N \simeq \mathbb{Z}^r$. We will see an action of an algebraic torus on X.

A cone $\{O\}$ is contained in Δ, and is a face of every $\sigma \in \Delta$. Obviously, $\mathcal{S}(\{O\}) = M$ and $U_{\{O\}}$ is an r-dimensional algebraic torus $T_N \simeq (\mathbb{C}^\times)^r$. Thus we have that T_N is an open subset of U_σ for any $\sigma \in \Delta$ by Theorem 2.2.3. Consequently, X contains T_N as an open subset. For $t \in T_N$ and $u \in U_\sigma$, we define $tu : \mathcal{S}(\sigma) \to \mathbb{C}$ by $tu(m) := t(m)u(m)$ for $m \in \mathcal{S}(\sigma)$. Since tu is an element of U_σ, this gives an action of T_N to U_σ. In the case of X, we can obtain an action of T_N on X called a *torus action* by natural gluing of the above action.

A prime divisor on X is called a T_N*-invariant divisor* if it is invariant with respect to the torus action, and a group consisting of T_N-invariant divisors is denoted by $T_N\mathrm{Div}(X)$. In order to examine their properties of in detail, we next consider orbits of toric varieties. For a cone σ, we define $\sigma^\perp = \{m \in M_\mathbb{R} \mid \langle m, n\rangle = 0, {}^\forall n \in \sigma\}$ and

$$\mathrm{orb}(\sigma) = \{u : M \cap \sigma^\perp \to \mathbb{C}^\times \text{ group homomorphism}\}.$$

Theorem 2.3.1 ([28, Proposition 1.6]). *For any $\sigma \in \Delta$, we can regard orb(σ) as a T_N-orbit in X by an embedding*

$$\begin{aligned} \mathrm{orb}(\sigma) &\hookrightarrow U_\sigma \\ u &\mapsto \tilde{u}(m) = \begin{cases} u(m) & (m \in M \cap \sigma^\perp), \\ 0 & (m \notin M \cap \sigma^\perp). \end{cases} \end{aligned}$$

Moreover, every T_N-orbit is of this form, and in this way, Δ is in one-to-one correspondence with the set of T_N-orbits in X.

Corollary 2.3.2 ([28, Corollary 1.7]). *Assume that X is compact. Then, for $\sigma \in \Delta$, a closure $V(\sigma)$ of orb(σ) in X becomes an $(r - \dim \sigma)$-dimensional normal closed subvariety of X.*

We define $\Delta(a) = \{\sigma \in \Delta \mid \dim \sigma = a\}$ for a non-negative integer a. Then a set $\{V(\sigma) \mid \sigma \in \Delta(1)\}$ is a $\mathbb{Z}$-basis of $T_N\mathrm{Div}(X)$. For $\sigma \in \Delta(1)$, $n \in N \cap \sigma$ is called a *primitive element* of σ if there are no elements of N on the line segment from the origin to n. Consider a compact nonsingular toric variety X. Then by Theorem 2.2.4 and 2.2.5, we can take a subset of $\Delta(1)$ such that their primitive elements compose a $\mathbb{Z}$-basis of N. For any such subset $\{\sigma_1, \ldots, \sigma_r\}$, there is a group isomorphism

$$\mathrm{Pic}(X) \simeq T_N\mathrm{Div}(X) = \bigoplus_{\sigma \in \Delta(1)} \mathbb{Z}V(\sigma) \Big/ \bigoplus_{i=1}^{r} \mathbb{Z}V(\sigma_i)$$

([28, Corollary2.5]). Hence the Picard number of X is equal to $\sharp\Delta(1) - r$. Besides, a set of T_N*-fixed points* (i.e. points which are invariant with respect to the torus action) on X coincides with a set of subvarieties defined by r-dimensional cones.

2.4 Toric surfaces

From now on, we restrict our interest to two-dimensional toric varieties, i.e., toric surfaces. Namely, we consider toric varieties associated to fans in $N_{\mathbb{R}}$ such that $N \simeq \mathbb{Z}^2$. Let $S = T_N(\Delta)$ be a toric surface associated to a fan Δ.

In Subsection 2.4.2, for divisors on toric surfaces, we will define the associated lattice polytopes. Then one can read off basic invariants (e.g. the intersection numbers and the cohomology dimensions) of such divisors from the information of the lattice polytopes.

2.4.1 Classification of toric surfaces

We first construct a holomorphic map between two toric surfaces. A fan Δ' in N is called a subdivision of Δ if for every cone σ' in Δ', there is a cone σ in Δ including σ'. It is obvious that $\mathcal{S}(\sigma) \subset \mathcal{S}(\sigma')$ in this case. Hence we naturally obtain a surjective holomorphic map $U_{\sigma'} \to U_\sigma$. Obviously, this map is equivariant with respect to the torus action. Hence we obtain an equivariant holomorphic map $T_N(\Delta') \to S$ by gluing affine pieces together ([28, Theorem1.13]). Moreover, this map is proper, that is, the inverse image of each compact subset is also compact ([28, Theorem1.15]). For instance, a blowing-up of S with center a T_N-fixed point can be described as a subdivision of Δ as follows:

Theorem 2.4.1 ([28, Proposition 1.26])**.** *Let $S = T_N(\Delta)$ be a nonsingular toric surface and $P = V(\sigma)$ a T_N-fixed point defined by a cone $\sigma = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 \in \Delta(2)$. Put $\sigma_i = \mathbb{R}_{\geq 0}n_i + \mathbb{R}_{\geq 0}(n_1 + n_2)$ for $i = 1, 2$, and define the subdivision*

$$\Delta^*(\sigma) = (\Delta \setminus \sigma) \cup \{\text{the faces of } \sigma_i \mid i = 1, 2\}$$

of Δ. Then the equivariant holomorphic map $T_N(\Delta^*(\sigma)) \to S$ coincides with the blowing-up of S with center P.

For a toric surface, a composite of a finite succession of blowing-ups with T_N-fixed points as centers is called a *toric morphism*. In the study of toric surfaces, the following theorem is one of the most fundamental results.

Theorem 2.4.2 ([28, Theorem 1.28])**.** *Any compact nonsingular toric surface is isomorphic to the surface obtained from the following* (i) *or* (ii) *by a toric morphism:*
(i) *the complex projective plane $\mathbb{P}^2$,*
(ii) *the Hirzebruch surfaces Σ_a of degree $a \geq 0$.*

Let (n_1, n_2) be a $\mathbb{Z}$-basis of N. Then the fans which define $\mathbb{P}^2$ and Σ_a are

$$\mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}(-n_1 - n_2),$$
$$\mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}(-n_1 + an_2) + \mathbb{R}_{\geq 0}(-n_2),$$

respectively (Fig. 2.1). Hence any toric surface is defined by a fan which is a subdivision

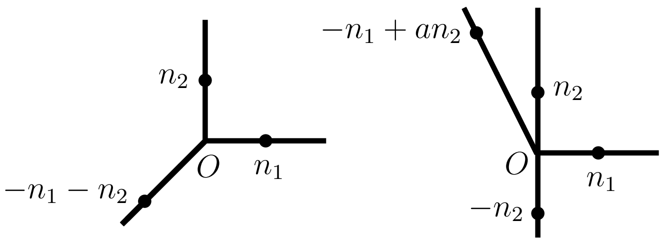

Figure 2.1.

of either of these fans.

2.4.2 Divisors on toric surfaces

Here we collect several basic properties of divisors on toric surfaces. In the remaining part of this section, we assume that S is compact and nonsingular, and let D be a divisor on S. In the case where the complete linear system $|D|$ has no base points, we have the following two results.

Theorem 2.4.3 ([28, Theorem 2.7]). *If $|D|$ has no base points, then $h^i(S, D) = 0$ for any positive integer i.*

Theorem 2.4.4 ([25, Theorem 3.1]). *The following are equivalent:*

(i) *$|D|$ has no base points.*
(ii) *D has a non-negative intersection number with every T_N-invariant divisor on S.*

By the fact mentioned in the last paragraph of the previous section, $T_N\mathrm{Div}(S)$ is in one-to-one correspondence with $\Delta(1)$, i.e., the set of half-lines starting from the origin. We denote by $D_1, \ldots, D_d$ the T_N-invariant divisors on S, by σ_i a one-dimensional cone corresponding to D_i and by (x_i, y_i) its primitive element. Since $\bigcup_{i=1}^d D_i$ is a simple chain of nonsingular rational curves, we can assume the following properties:

$$D_i.D_j = \begin{cases} 1 & (j = i-1, i+1), \\ 0 & (\text{otherwise}), \end{cases}$$

where we formally set

$$D_0 = D_d,\ D_{d+1} = D_1.$$

There are essentially two ways to label the T_N-invariant divisors according to the value of $x_i y_{i-1} - y_i x_{i-1}$ is one or minus one. In this thesis, we adopt the former, that is, we assume the equality

$$x_i y_{i-1} - y_i x_{i-1} = 1$$

for each integer $1 \le i \le d$. This means that the one-dimensional cones $\sigma_1, \dots, \sigma_d$ are arranged clockwise (Fig. 2.2). The self-intersection numbers of T_N-invariant divisors

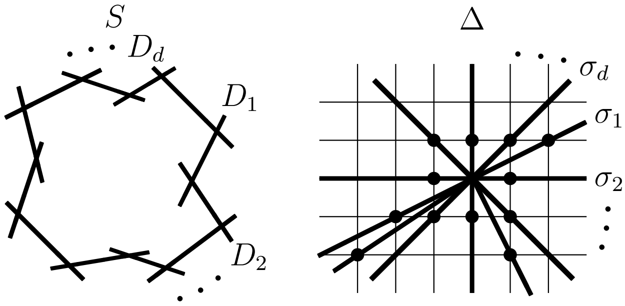

Figure 2.2.

are computed by the following formula.

Theorem 2.4.5 ([28, Proposition 1.19]). *For any integer $0 \le i \le d$ and $1 \le j \le d$,*

$$x_i D_i^2 = -x_{i-1} - x_{i+1},$$
$$y_i D_i^2 = -y_{i-1} - y_{i+1}.$$

The Picard group of S is generated (not freely) by the classes of T_N-invariant divisors $D_1, \dots, D_d$. Hence we can write the linear equivalence class of D as the sum of them with integral coefficients. For example, the canonical divisor K_S of S is

$$K_S \sim -\sum_{i=1}^{d} D_i, \tag{2.1}$$

where the symbol "$\sim$" means linear equivalence. There is the following relation between the coefficients of the linear equivalence class of D and the primitive elements of the cones:

Proposition 2.4.6. *Let $D \sim \sum_{i=1}^{d} a_i D_i$ be a divisor on S. Then, for each integer $1 \le k \le d$,*

$$a_k = x_k(y_d a_1 - y_1 a_d) - y_k(x_d a_1 - x_1 a_d) + \sum_{i=1}^{k-1} (x_k y_i - y_k x_i) C.D_i.$$

Proof. By Theorem 2.4.5, an easy computation shows that

$$x_1(y_d a_1 - y_1 a_d) - y_1(x_d a_1 - x_1 a_d) = x_1 y_d a_1 - y_1 x_d a_1 = a_1,$$
$$x_2(y_d a_1 - y_1 a_d) - y_2(x_d a_1 - x_1 a_d) + (x_2 y_1 - y_2 x_1) C.D_1 = -a_1 D_1^2 - a_d + C.D_1 = a_2.$$

We proceed by induction on integer k. By computing, we have

$$\begin{aligned}
&\sum_{i=1}^{k-1}(x_k y_i - y_k x_i)C.D_i\\
&= x_k \sum_{i=1}^{k-1} y_i(a_{i-1} + a_i D_i^2 + a_{i+1}) - y_k \sum_{i=1}^{k-1} x_i(a_{i-1} + a_i D_i^2 + a_{i+1})\\
&= x_k \sum_{i=1}^{k-1}(y_i a_{i-1} - (y_{i-1} + y_{i+1})a_i + y_i a_{i+1}) - y_k \sum_{i=1}^{k-1}(x_i a_{i-1} - (x_{i-1} + x_{i+1})a_i + x_i a_{i+1})\\
&= x_k(y_1 a_d - y_d a_1 - y_k a_{k-1} + y_{k-1}a_k) - y_k(x_1 a_d - x_d a_1 - x_k a_{k-1} + x_{k-1}a_k)\\
&= x_k(y_1 a_d - y_d a_1) - y_k(x_1 a_d - x_d a_1) + a_k.
\end{aligned}$$

□

We next introduce the notion of lattice polytopes, which is extremely important objects in the study of divisors on toric surfaces.

Definition 2.4.7. For a divisor $D \sim \sum_{i=1}^{d} a_i D_i$ on S, the *lattice polytope* $\square_D \subset \mathbb{R}^2$ is defined by

$$\square_D = \{(z, w) \in \mathbb{R}^2 \mid x_i z + y_i w \leq a_i \text{ for } 1 \leq i \leq d\}.$$

Though $\square_D$ can change according to how we describe the linear equivalence class of D, those differences induce only parallel translations of $\square_D$. Hence the shape of the lattice polytope is determined uniquely. The lattice polytopes has many information about a divisor. For example, the dimension of cohomology group of D can be read off the lattice points contained in $\square_D$:

Theorem 2.4.8 ([28, Lemma 2.3]). *The equation $h^0(S, D) = \sharp(\square_D \cap \mathbb{Z}^2)$ holds.*

Several useful facts follow immediately from Theorem 2.4.8. First, the irregularity of S vanishes. If D is effective, then we have $h^0(S, -D) = 0$, especially $h^0(S, K_S) = 0$. Besides, for a nonsingular curve C on S, its genus is computed by the formula $g = \sharp(\square_{C+K_S} \cap \mathbb{Z}^2)$.

Proposition 2.4.9. *Let C be a nonsingular curve on S. If C is irrational, then C is nef.*

Proof. It is sufficient to show $C^2 \geq 0$. Note that since T_N-invariant divisors are rational curve, $C.D_i \geq 0$ for each integer $1 \leq i \leq d$. Moreover, by Corollary 1.0.3 and Theorem 1.0.4, we have

$$g = \frac{1}{2}C.(C + K_S) + 1 \geq 1,$$

which implies that $C^2 \geq \sum_{i=1}^{d} C.D_i \geq 0$. □

In fact, the Picard number of S is $d-2$. The Picard group of S is freely generated by the classes of T_N-invariant divisors except two adjacent divisors (e.g. $D_2, \dots, D_{d-1}$). Hence, for a divisor D on S, we can write the linear equivalence class of D as

$$D \sim \sum_{i=1}^{d} a_i D_i \quad (a_i \in \mathbb{Z},\ a_1 = a_d = 0) \tag{2.2}$$

without loss of generality.

Proposition 2.4.10. *If $|D|$ has no base points, then in the form (2.2), a_k is non-negative for any integer $2 \leq k \leq d-1$.*

Proof. In this proof, we admit Proposition 2.4.13 in advance, which will be shown in the last of this section. Note that Theorem 2.4.4 implies that $D.D_i \geq 0$ for any integer $1 \leq i \leq d$. If $x_k y_1 - y_k x_1 \geq 0$, then by Proposition 2.4.13, we have

$$x_k y_i - y_k x_i \geq 1$$

for any integer $2 \leq i \leq k-1$. This means that $a_k \geq 0$.

Assume that $x_k y_1 - y_k x_1 \leq -1$. An easy computation gives the equation

$$\sum_{i=1}^{d} x_i D.D_i = \sum_{i=1}^{d} y_i D.D_i = 0.$$

Namely, we have

$$a_k = -x_k \sum_{i=k+1}^{d} y_i D.D_i + y_k \sum_{i=k+1}^{d} x_i D.D_i = \sum_{i=k+1}^{d} (x_i y_k - y_i x_k) D.D_i. \tag{2.3}$$

On the other hand, Proposition 2.4.13 implies that

$$x_k y_i - y_k x_i \leq -1$$

for any integer $k+1 \leq i \leq d$. Hence the inequality $a_k \geq 0$ follows from (2.3). □

Our last aim in this section is to show Proposition 2.4.13, which plays an active role especially in Chapter 4. Before proceeding to its claim and proof, we show the following two lemmas.

Lemma 2.4.11. *Let (z, w), (z_1, w_1) and (z_2, w_2) be lattice points such that $z_1 w_2 - w_1 z_2 \neq 0$. Then there is a unique pair of real numbers (α, β) such that*

$$(z, w) = \alpha(z_1, w_1) + \beta(z_2, w_2).$$

In particular, if $z_1 w_2 - w_1 z_2 = \pm 1$, then α and β are integers.

Lemma 2.4.12. *Let i and j be integers with $0 \leq i \leq d$ and $1 \leq j \leq d$. Let α and β be integers such that*

$$(x_j, y_j) = \alpha(x_i, y_i) + \beta(x_{i+1}, y_{i+1}).$$

Then at least one of α and β is not positive. Furthermore, if either α or β is zero, then the other is one or minus one.

Proposition 2.4.13. *Let i and j be distinct integers with $0 \leq i \leq d$ and $1 \leq j \leq d+1$. If $x_i y_j - y_i x_j \leq 0$, then*

$$x_{i+1} y_j - y_{i+1} x_j \leq 0,$$
$$x_i y_{j-1} - y_i x_{j-1} \leq 0.$$

The equalities hold if and only if $j = i + 1$.

Proof. We will show only the first inequality. One can similarly verify the second one. By Lemma 2.4.11, we can write

$$(x_j, y_j) = \alpha(x_i, y_i) + \beta(x_{i+1}, y_{i+1})$$

with integers α and β. Then we have

$$0 \leq x_j y_i - y_j x_i = \beta(x_{i+1} y_i - y_{i+1} x_i) = \beta.$$

Recall that $j \neq i$. In the case of $\beta = 0$, by Lemma 2.4.12, we have $(x_j, y_j) = -(x_i, y_i)$. Hence

$$x_{i+1} y_j - y_{i+1} x_j = -x_{i+1} y_i + y_{i+1} x_i = -1.$$

In the case of $\beta \geq 1$, we have $\alpha \leq 0$ by Lemma 2.4.12. Hence

$$x_{i+1} y_j - y_{i+1} x_j = \alpha(x_{i+1} y_i - y_{i+1} x_i) = \alpha \leq 0.$$

If $x_{i+1} y_j - y_{i+1} x_j = 0$, then we have $\alpha = 0$. Hence, by Lemma 2.4.12, we have $\beta = 1$, which means $j = i + 1$. $\square$

Chapter 3

The gonality conjecture for curves on toric surfaces

This chapter contains the author's first result. Concretely, we will prove that the gonality conjecture affirmatively for curves on toric surfaces which have only one toric morphism to the projective line $\mathbb{P}^1$. In this chapter, a *curve* will always mean a nonsingular projective curve unless otherwise stated. Note that, as mentioned in Chapter 1, we often identify the notions of invertible sheaves, line bundles and divisors.

3.1 Preliminaries and the main result

3.1.1 Gonalities of curves

For a curve C, the *gonality* is defined as the minimal degree of surjective morphisms from C to $\mathbb{P}^1$:

$$\begin{aligned} \mathrm{gon}(C) &= \min\{\deg\varphi \mid \varphi : C \to \mathbb{P}^1 \text{ surjective morphism}\} \\ &= \min\{k \mid C \text{ has a } g^1_k\}, \end{aligned}$$

where g^1_k denotes a one-dimensional linear system of degree k on C. A curve is said to be k-gonal if its gonality is k. By definition, C is one-gonal (resp. two-gonal) if and only if it is isomorphic to $\mathbb{P}^1$ (resp. elliptic or hyperelliptic).

Let us review the developments of the study of gonalities roughly. First of all, it is classically well known that a nonsingular plane curve of degree d is $(d-1)$-gonal (cf. Theorem 0.0.2). Coppens and Kato generalized this result to the case of singular plane curves. They computed the gonality of its normalization under certain numerical conditions on the degree d and the number of singular points:

Theorem 3.1.1 ([5]). *Let C be a singular plane curve of degree d with δ ordinary nodes or cusps. If there is a positive integer n such that $d \geq 2(n+1)$ and $\delta \leq nd-(n+1)^2+2$, then the gonality of the normalization of C is $d-2$.*

Ohkouchi and Sakai studied more general cases :

Theorem 3.1.2 ([29]). *Let C be a singular plane curve of geometric genus g and of degree d, n the number of singular points (including also infinitely near singular points) and $m_1, \ldots, m_n$ their multiplicities. Put $\nu = \max\{m_i \mid i = 1, \ldots, n\}$ and $a(\nu) = (2 - \sqrt{1-2/\nu}\,)^2$. Define $\delta = (d-1)(d-2)/2 - g$, $\eta = \sum_{i=1}^n (m_i/\nu)^2$ and*

$$R(\nu, \delta, i) = \frac{\nu^2 + (\nu-2)i}{2\nu(\nu-1)} + \sqrt{\frac{\delta-\nu}{\nu-1} + \left(\frac{\nu-2+i}{2\nu-2}\right)^2},$$

where we let $d \equiv i \pmod{\nu}$. Then the gonality of the normalization of C is $d-\nu$ in the following cases :

(i) $\dfrac{d}{\nu} > R(\nu, \delta, i)$.

(ii) $\nu \geq 3$ *and* $\dfrac{d}{\nu} \begin{cases} > \dfrac{\eta+1}{2} & (5 \leq \eta < a(\nu)), \\ > 2\sqrt{\eta} - \left(1 + \dfrac{1}{\nu}\right) & (a(\nu) \leq \eta < 4), \\ \geq 3 & (4 \leq \eta < 5). \end{cases}$

On the other hand, Martens determined the gonalities of nonsingular curves on Hirzebruch surfaces with some trivial exceptions.

Theorem 3.1.3 ([24]). *Let Σ_a be a Hirzebruch surface of degree $a \geq 0$ with the ruling $\pi : \Sigma_a \to \mathbb{P}^1$, and C a nonsingular curve on Σ_a. Denote by Δ_0 and F the minimal section and a fiber of π, respectively. Assume that $C \not\sim F$. Then the gonality of C is $C.F$ unless $a = 1$ and $C \sim \alpha(\Delta_0 + F)$ with $\alpha \geq 2$ in which case C is isomorphic to a plane curve of degree α.*

3.1.2 The gonality conjecture

The gonality is one of important invariants in the study of linear systems on curves, although, in general, it is often difficult to determine it for a given curve. One of the central problems around the gonality is the so-called *gonality conjecture* (Conjecture 3.1.5 below) posed by Green and Lazarsfeld in [13]. In order to give its precise statement, we introduce the following vanishing property (M_k) (we use the notation of Koszul cohomology defined in Section 3.2).

Definition 3.1.4 ([13])**.** Let L be a line bundle over a curve C, and k a non-negative integer. We say that the pair (C, L) satisfies the property (M_k) (or, simply, L satisfies the property (M_k)) if a Koszul cohomology $K_{p,1}(C, L)$ vanishes for any integer $p \geq h^0(C, L) - k - 1$.

If C is a k-gonal curve of genus g, then it is well known that a line bundle over C does not satisfy (M_k) if its degree is greater than or equal to $2g + k$. The gonality conjecture predicts the converse of this fact.

Conjecture 3.1.5 ([13, **The gonality conjecture**])**.** *Let C be a curve of genus g and k a positive integer. If the property (M_k) fails for any line bundle L over C with* $\deg L \gg 2g$*, then C has a g^1_k.*

Hence, if this conjecture is true, then we can read off the gonality of a curve from any one line bundle of sufficiently large degree over it. Green has shown this conjecture affirmatively for $k = 1, 2$ in [11]. The case where $k = 3$ has been done by Ehbauer in [2]. As for curves on the Hirzebruch surfaces, we have not only Martens' result (Theorem 3.1.3) but also an affirmative answer to the gonality conjecture. This work was done by Aprodu in [1]. So it is a natural question to extend their results to curves on more general surfaces, e.g., toric surfaces. In this chapter, we restrict ourselves to a class of toric surfaces admitting a unique toric morphism to $\mathbb{P}^1$. Our aim is to determine the gonality of curves on such surfaces and also show that the gonality conjecture is valid for them. Namely, we shall show the following:

Theorem 3.1.6. *Let S be a toric surface which has a unique toric morphism to $\mathbb{P}^1$ and denote its fiber by F. Let C be a nonsingular irrational curve on S and put $C.F = k$. Then one of the following holds:*
(i) *C is isomorphic to a plane curve of degree k,*
(ii) *C is k-gonal, and the gonality conjecture is valid for C.*

The proof owes much to [1] and will go with the induction on the sum of k and the Picard number of S.

3.2 Koszul cohomology

In this section, we will introduce the notion of Koszul cohomology of a line bundle over a projective variety, and review several previous results. See [1] and [11] for further details.

Let V be a finite dimensional complex vector space, SV the symmetric algebra of V, and $B = \bigoplus_{q\in\mathbb{Z}} B_q$ a graded SV-module. Then there is a natural map between vector spaces

$$\begin{aligned} d_{p,q}: \quad \bigwedge^p V \otimes B_q \quad &\to \bigwedge^{p-1} V \otimes B_{q+1} \\ (e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes b &\mapsto \sum_{j=1}^{p} (-1)^{j-1} (e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_p}) \otimes (e_{i_j} b), \end{aligned}$$

which yields a *Koszul complex*

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} V \otimes B_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} V \otimes B_{q+1} \to \cdots .$$

The *Koszul cohomology* is defined by

$$K_{p,q}(B, V) = \operatorname{Ker} d_{p,q} \big/ \operatorname{Im} d_{p+1,q-1}.$$

It is a well-known fact (the so-called *Syzygy theorem*) that a complex vector space $K_{p,q}$ is isomorphic to the syzygy of order p and weight $p+q$ for B (cf. [11, Theorem 1.b.4]). Besides, it is also essential that a morphism of graded SV-modules canonically induces linear maps of Koszul cohomologies:

Theorem 3.2.1 ([11, Corollary 1.d.4])**.** *Let $0 \to P \to Q \to R \to 0$ be a short exact sequence of SV-modules with maps preserving the gradings. Then, for any integer p, there is a Koszul cohomology long exact sequence*

$$\begin{aligned} \cdots \to K_{p+1,0}(R, V) \to K_{p,1}(P, V) \to K_{p,1}(Q, V) \to K_{p,1}(R, V) \to K_{p-1,2}(P, V) \to \\ \cdots \to K_{0,p+1}(P, V) \to K_{0,p+1}(Q, V) \to K_{0,p+1}(R, V) \to 0. \end{aligned}$$

Remark 3.2.2 ([1, Remark 1.1])**.** In the above situation, if $R_0 = R_1 = 0$, then $K_{p,1}(P, V) \simeq K_{p,1}(Q, V)$ for any integer p.

For a projective variety X, a line bundle L over X and a vector bundle E over X, we define

$$\begin{aligned} K_{p,q}(X, E, L) &= K_{p,q}\Bigg(\bigoplus_{i\in\mathbb{Z}} H^0(X, E \otimes iL)\,, H^0(X, L) \Bigg), \\ K_{p,q}(X, L) &= K_{p,q}\Bigg(\bigoplus_{i\in\mathbb{Z}} H^0(X, iL)\,, H^0(X, L) \Bigg). \end{aligned}$$

For a restriction of the Koszul cohomology to the hypersurface, we have the following isomorphism:

Theorem 3.2.3 ([1, Remark 1.3]). *Let X be a nonsingular projective variety, L a line bundle over X and $Y \in |L|$ an irreducible divisor on X. If the irregularity of X is zero, then $K_{p,1}(X, L) \simeq K_{p,1}(Y, L|_Y)$ for any integer p.*

Here we shall see two vanishing theorems.

Theorem 3.2.4 ([11, Theorem 3.a.1]). *Let X be a projective variety, L a line bundle over X and E a vector bundle over X. Then $K_{p,q}(X, E, L) = 0$ for any integer $p \geq h^0(X, E \otimes qL)$.*

Theorem 3.2.5 ([11, Theorem 3.c.1]). *Let L be a line bundle over a curve C. Then $K_{p,1}(C, L) = 0$ for any integer $p \geq h^0(C, L) - 1$.*

Lastly, we see Aprodu's results which played a central role in his work and also in the proof of Theorem 3.1.6.

Theorem 3.2.6 ([1, Theorem 1]). *Let C be an irrational curve, L a nonspecial and globally generated line bundle over C, and k a non-negative integer such that L satisfies (M_k). Then, for any effective divisor D on C, $L + D$ also satisfies (M_k).*

This proposition gives us a simple criterion for verifying the gonality conjecture, which reduces it to the problem of finding a single line bundle with the property (M_{k-1}) over C.

Corollary 3.2.7 ([1, Corollary 2]). *Let C be an irrational curve which has a g^1_k. If there is a nonspecial and globally generated line bundle over C satisfying (M_{k-1}), then C is k-gonal, and the gonality conjecture is valid for C.*

3.3 Proof of Theorem 3.1.6

3.3.1 Toric surfaces with a unique ruling to $\mathbb{P}^1$

Let us see some properties of surfaces dealt with in this chapter. We keep the notation introduced in Section 2.4.

Let S be a toric surface associated to the fan Δ composed by d cones, which has a unique toric morphism φ to $\mathbb{P}^1$. We denote its general fiber by F. In terms of the fan, this condition means that there is only one cone $\sigma \in \Delta$ such that $-\sigma$ is also contained in Δ. We put $\sigma_1 = \sigma$, $\sigma_{d_0} = -\sigma$ and label T_N-invariant divisors in the way defined in Section 2.4, that is, we assume the equality

$$x_i y_{i-1} - y_i x_{i-1} = 1$$

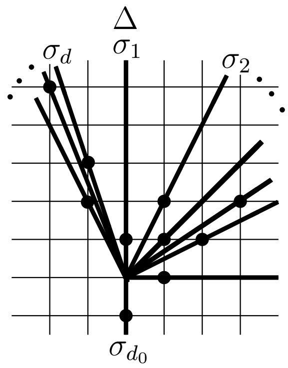

Figure 3.1.

for each integer $1 \leq i \leq d$. Hence we can draw Δ as in Fig. 3.1. Remark that, in this case, we can classify the primitive elements of the cones in Δ roughly as follows:

$$x_i \begin{cases} = 0 & (i = 1, d_0), \\ = 1 & (i = 2, d_0 - 1), \\ = -1 & (i = d_0 + 1, d), \\ \geq 1 & (3 \leq i \leq d_0 - 2), \\ \leq -1 & (d_0 + 2 \leq i \leq d - 1), \end{cases} \qquad y_i \begin{cases} = 1 & (i = 1), \\ = 0 & (i = d_0 - 1), \\ = -1 & (i = d_0), \\ = D_{d_0}^2 \geq 1 & (i = d_0 + 1), \\ \geq 1 & (2 \leq i \leq d_0 - 2), \\ \geq -x_i + 1 & (d_0 + 2 \leq i \leq d). \end{cases} \tag{3.1}$$

Moreover, we have

$$D_{d_0} \sim \sum_{i=1}^{d_0-1} y_i D_i + \sum_{i=d_0+1}^{d} y_i D_i. \tag{3.2}$$

The linear equivalence class of F is written as

$$F \sim \sum_{i=2}^{d_0-1} x_i D_i \sim -\sum_{i=d_0+1}^{d} x_i D_i. \tag{3.3}$$

Hence, by (3.1) and Theorem 2.4.5, we have

$$F.D_i = \begin{cases} 1 & (i = 1, d_0), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.4}$$

3.3.2 Several lemmas

In this subsection, we prove several lemmas needed in the proof of Lemma 3.3.8 which is a key to proving Theorem 3.1.6. We keep the notation in the previous section. Let C be a curve of genus g on S and put $k = C.F$. As mentioned in Section 2.4, we can write the linear equivalence class of C as

$$C \sim \sum_{i=1}^{d} p_i D_i \quad (p_i \in \mathbb{Z},\ p_1 = p_d = 0).$$

Note that $p_{d_0} = k$ follows from (3.4). We first consider the case where C is isomorphic to a plane curve.

Definition 3.3.1. We say that the pair (S, C) satisfies the property ($\sharp$) (or, simply, C satisfies ($\sharp$)) if $D_{d_0}^2 = 1$ and $C \sim kD_{d_0}$.

Lemma 3.3.2. *If (S, C) satisfies ($\sharp$), then C is isomorphic to a nonsingular plane curve of degree k.*

Proof. In this case, an easy computation shows that

$$C.D_i = \begin{cases} k & (i = d_0 - 1, d_0, d_0 + 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Considering the construction of S, if $d \geq 5$, then there is at least one T_N-invariant divisor D_i with self-intersection number -1 such that $i \neq 1, d_0 - 1, d_0, d_0 + 1$. Hence, by a finite succession of blowing-downs along such divisors, we can obtain an embedding of C in Σ_1. In particular, the image of D_1 becomes the minimal section of the ruling

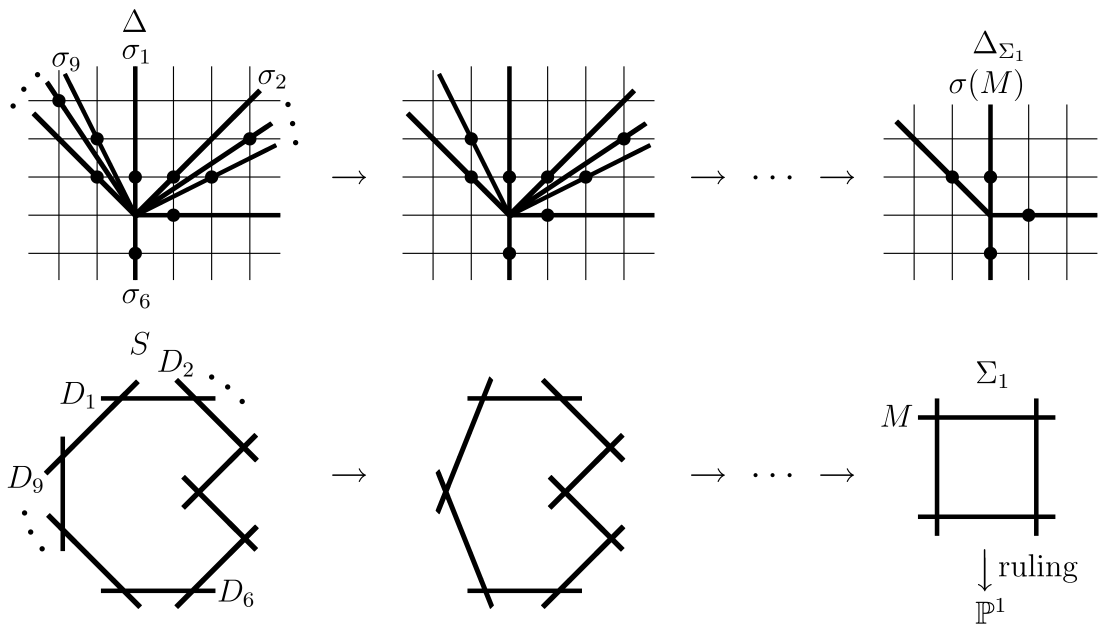

Figure 3.2.

map of Σ_1 (Fig. 3.2). We denote it by M. Since M has self-intersection number -1 and does not meet C, by blowing it down, C can be embedded in the projective plane as a curve of degree k. $\square$

By the following lemma, we can clarify the case where C is rational, which is a special case of Lemma 3.3.2.

Lemma 3.3.3. *Assume that $k \geq 2$ and C is nef. Then C is rational if and only if $k = 2$ and (S, C) satisfies $(\sharp)$.*

Proof. The sufficiency is easy: If $D_{d_0}^2 = 1$ and $C \sim 2D_{d_0}$, then we have $C.(C + K_S) = -2$ by computing. Then $g = \frac{1}{2}C.(C + K_S) + 1 = 0$.

To prove the necessity, we assume $g = 0$. By noting (3.2) and $p_{d_0} = k$, we have

$$C + K_S \sim (k-2)D_1 + \sum_{i=2}^{d_0-1}(p_i + (k-1)y_i - 1)D_i + \sum_{i=d_0+1}^{d}(p_i + (k-1)y_i - 1)D_i.$$

Here we note that Proposition 2.4.10 implies that p_i is non-negative for $2 \leq i \leq d-1$. Hence, by (3.1), we have $p_i + (k-1)y_i - 1 \geq 0$ except for $i = 1, d_0 - 1, d_0$. On the other hand, since $h^0(S, K_S) = h^0(C, K_C) = 0$, we have $h^0(S, C + K_S) = 0$. This means that $C + K_S$ is not linearly equivalent to an effective divisor. Hence the coefficient $p_{d_0-1} + (k-1)y_{d_0-1} - 1$ must be negative, which implies that $p_{d_0-1} = 0$. We thus have $C.D_{d_0-1} = p_{d_0-2} + k \geq k$. Considering the equation $C.F = C.\big(\sum_{i=2}^{d_0-1} x_i D_i\big) = k$ and (3.1), we can conclude

$$C.D_i = \begin{cases} 0 & (2 \leq i \leq d_0 - 2), \\ k & (i = d_0 - 1). \end{cases}$$

We next write the linear equivalence class of $C + K_S$ as

$$\begin{aligned} C + K_S = C + K_S + F - F &\sim C + K_S + \sum_{i=2}^{d_0-1} x_i D_i + \sum_{i=d_0+1}^{d} x_i D_i \\ &\sim (k-2)D_1 + \sum_{i=2}^{d_0-1}(p_i + x_i + (k-1)y_i - 1)D_i \\ &\qquad + \sum_{i=d_0+1}^{d}(p_i + x_i + (k-1)y_i - 1)D_i. \end{aligned}$$

Since $h^0(S, C+K_S) = 0$, the coefficient $p_{d_0+1}+x_{d_0+1}+(k-1)y_{d_0+1}-1$ must be negative. This implies that $k = 2$, $D_{d_0}^2 = 1$ and $p_{d_0+1} = 0$. We thus have $C.D_{d_0+1} = k+p_{d_0+2} \geq 2$. Considering the equation $C.F = C.\big(-\sum_{i=d_0+1}^{d} x_i D_i\big) = 2$ and (3.1), we can conclude

$$C.D_i = \begin{cases} 2 & (i = d_0 + 1), \\ 0 & (d_0 + 2 \leq i \leq d). \end{cases}$$

Moreover, we have $C.D_{d_0} = p_{d_0-1} + kD_{d_0}^2 + p_{d_0+1} = 2$, and

$$C.D_1 = C.\left(D_{d_0} - \sum_{i=2}^{d_0-1} y_i D_i - \sum_{i=d_0+1}^{d} y_i D_i\right) = C.D_{d_0} - y_{d_0+1}C.D_{d_0+1} = 0.$$

Consequently, if C is rational, then $D_{d_0}^2 = 1$, $k = 2$ and C is numerically equivalent to $2D_{d_0}$. Since S is simply connected, we also have $C \sim 2D_{d_0}$. $\square$

We next see the properties of the Koszul cohomology of a divisor obtained by subtracting an effective divisor from C.

Lemma 3.3.4. *Let I be a nonzero effective divisor on S and put $H = C - I$. If $H^1(S, -I) = 0$, then $K_{p,1}(S, H) \simeq K_{p,1}(C, H|_C)$ for any integer $p \geq h^0(S, H - I) + 1$.*

Proof. The short exact sequence of sheaves $0 \to \mathcal{O}_S(-I) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H) \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, -I) \to H^0(S, H) \to H^0(C, H|_C) \to H^1(S, -I) \to \cdots .$$

Since $H^0(S, -I) = H^1(S, -I) = 0$, we have $H^0(S, H) \simeq H^0(C, H|_C)$. We put $V = H^0(S, H)$, $B = \bigoplus_{q \geq 0} H^0(S, qH)$, $B' = \bigoplus_{q \geq 0} H^0(S, qH - C)$ and $Q = B/B'$. Considering the short exact sequence of graded SV-modules $0 \to B' \to B \to Q \to 0$, we obtain the Koszul cohomology long exact sequence

$$\cdots \to K_{p,1}(B', V) \to K_{p,1}(B, V) \to K_{p,1}(Q, V) \to K_{p-1,2}(B', V) \to \cdots .$$

Then Theorem 3.2.4 shows that

$$\begin{aligned} K_{p,1}(B', V) = 0 \ \text{ for } p \geq h^0(S, H - C) = 0, \\ K_{p-1,2}(B', V) = 0 \ \text{ for } p \geq h^0(S, 2H - C) + 1 = h^0(S, H - I) + 1. \end{aligned}$$

We thus have $K_{p,1}(S, H) \simeq K_{p,1}(Q, V)$ for any integer $p \geq h^0(S, H - I) + 1$.

Next, let us consider the short exact sequence of graded SV-modules

$$0 \to Q \to \bigoplus_{q \geq 0} H^0(C, qH|_C) \to R := \left(\bigoplus_{q \geq 0} H^0(C, qH|_C) \right) \Big/ Q \to 0.$$

The isomorphisms $Q_0 \simeq \mathbb{C}$ and $Q_1 \simeq H^0(C, H|_C)$ imply $R_0 = R_1 = 0$. Hence we can apply Remark 3.2.2 to obtain

$$K_{p,1}(Q, V) \simeq K_{p,1}\left(\bigoplus_{q \geq 0} H^0(C, qH|_C), H^0(C, H|_C) \right) = K_{p,1}(C, H|_C)$$

for any integer p. $\square$

Lemma 3.3.5. *Assume C is irrational. Let I be a nonzero effective divisor on S and put $H = C - I$. If all of the following* (i)–(v) *hold, then $\mathcal{O}_C(C)$ satisfies* (M_1).

(i) *$\mathcal{O}_S(H)$ is globally generated,*
(ii) $H^2 > 0$,
(iii) *$H|_C$ is nonspecial,*
(iv) $h^0(S, H) - h^0(S, H - I) \geq 3$,
(v) $H^1(S, -I) = 0$.

Proof. We can take a nonsingular irreducible curve $Y \in |H|$ by (i), (ii) and Bertini's theorem. Then Theorem 3.2.5 shows that $K_{p,1}(Y, H|_Y) = 0$ for any integer $p \geq h^0(Y, H|_Y) - 1$. On the other hand, the short exact sequence of sheaves $0 \to \mathcal{O}_S \to \mathcal{O}_S(H) \to \mathcal{O}_Y(H) \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, H) \to H^0(Y, H|_Y) \to H^1(S, \mathcal{O}_S) \to \cdots .$$

Since $H^0(S, \mathcal{O}_S) = \mathbb{C}$ and $H^1(S, \mathcal{O}_S) = 0$, we obtain $h^0(Y, H|_Y) = h^0(S, H) - 1$. Consequently, we have that

$$K_{p,1}(Y, H|_Y) = 0 \tag{3.5}$$

for any integer $p \geq h^0(S, H) - 2$.

By Theorem 3.2.3, we have $K_{p,1}(Y, H|_Y) \simeq K_{p,1}(S, H)$ for any integer p. Besides, Lemma 3.3.4 gives the isomorphism $K_{p,1}(S, H) \simeq K_{p,1}(C, H|_C)$ for any integer $p \geq h^0(S, H - I) + 1$. Hence, by combining these facts with (3.5) and (iv), we have

$$K_{p,1}(C, H|_C) = 0$$

for any integer $p \geq h^0(S, H) - 2$.

We next consider the short exact sequence $0 \to \mathcal{O}_S(-I) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H) \to 0$. It induces the cohomology long exact sequence

$$0 \to H^0(S, -I) \to H^0(S, H) \to H^0(C, H|_C) \to H^1(S, -I) \to \cdots .$$

Then the equalities $H^0(S, -I) = H^1(S, -I) = 0$ implies $h^0(S, H) = h^0(C, H|_C)$. In sum, we can conclude

$$K_{p,1}(C, H|_C) = 0$$

for any integer $p \geq h^0(C, H|_C) - 2$, that is, $H|_C$ satisfies (M_1). Recall the condition (iii) and note that $\mathcal{O}_C(H)$ is globally generated. Therefore, by Theorem 3.2.6, $\mathcal{O}_C(C)$ also satisfies (M_1). $\square$

In the rest of this section, we define

$$\begin{gathered} d_1 = \min \{ i \geq 2 \mid D_i^2 \geq -1 \} , \\ d_2 = \max \{ i \leq d \mid D_i^2 \geq -1 \} , \\ I = \sum_{i=1}^{d_1 - 1} D_i + \sum_{i=d_2+1}^{d} D_i + F, \\ H = C - I. \end{gathered}$$

Considering the construction of Δ, it is obvious that $d_1 \leq d_0 - 1$ and $d_2 \geq d_0 + 1$. The short exact sequence of sheaves $0 \to \mathcal{O}_S(-I) \to \mathcal{O}_S \to \mathcal{O}_I \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, -I) \to H^0(S, \mathcal{O}_S) \to H^0(I, \mathcal{O}_I) \to H^1(S, -I) \to H^1(S, \mathcal{O}_S) \to \cdots .$$

Since $H^0(S, -I) = H^1(S, \mathcal{O}_S) = 0$ and $H^0(S, \mathcal{O}_S) = H^0(I, \mathcal{O}_I) = \mathbb{C}$, we have

$$H^1(S, -I) = 0.$$

Lemma 3.3.6. *Assume that $k \geq 2$ and C is nef. If $C.D_{d_1} \geq 1$ and $C.D_{d_2} \geq 1$, then the following* (i)–(iii) *hold:*

(i) *$\mathcal{O}_S(H)$ is globally generated,*
(ii) *$H^2 > 0$,*
(iii) *$H|_C$ is nonspecial.*

Proof. By Theorem 2.4.4, it is sufficient for (i) to verify that H has non-negative intersection numbers with D_i for each $1 \leq i \leq d$. First, for $2 \leq i \leq d_1 - 2$, we have

$$H.D_i = C.D_i - I.D_i = C.D_i - D_i^2 - 2 \geq C.D_i \geq 0.$$

In the case where $d_1 \geq 3$, we have $H.D_{d_1-1} = C.D_{d_1-1} - D_{d_1-1}^2 - 1 \geq -D_{d_1-1}^2 - 1 \geq 1$. Besides, we have $H.D_{d_1} = C.D_{d_1} - 1 \geq 0$ and $H.D_i = C.D_i \geq 0$ for $d_1 + 1 \leq i \leq d_0 - 1$. A similar argument can be adapted for integers $d_0 + 1 \leq i \leq d$. In sum, we obtain

$$H.D_i \geq \begin{cases} 0 & (i \neq 1,\ d_1 - 1,\ d_0,\ d_2 + 1), \\ 1 & (i = d_1 - 1 \text{ if } d_1 \geq 3), \\ 1 & (i = d_2 + 1 \text{ if } d_2 \leq d - 1). \end{cases}$$

For D_{d_0}, we have

$$H.D_{d_0} = C.D_{d_0} - I.D_{d_0} = p_{d_0-1} + kD_{d_0}^2 + p_{d_0+1} - 1 \geq k - 1 \geq 1.$$

It remains to check that $H.D_1$ is non-negative. Since

$$I.D_1 = D_1^2 + \begin{cases} 1 & (d_1 = 2,\ d_2 = d), \\ 3 & (d_1 \geq 3,\ d_2 \leq d - 1), \\ 2 & (\text{otherwise}), \end{cases} \quad D_1^2 \leq \begin{cases} -1 & (d_1 = 2,\ d_2 = d), \\ -3 & (d_1 \geq 3,\ d_2 \leq d - 1), \\ -2 & (\text{otherwise}), \end{cases}$$

we obtain $I.D_1 \leq 0$. We thus have $H.D_1 \geq C.D_1 \geq 0$.

(ii) Since $\mathcal{O}_S(H)$ is globally generated, by Proposition 2.4.10, we can write the linear equivalence class of H as

$$H \sim \sum_{i=2}^{d-1} b_i D_i$$

with non-negative integers b_i. The equation $H.F = k-1$ and (3.4) implies that $p_{d_0} = k-1$. Then we have $H^2 \geq (k-1)H.D_{d_0} \geq k-1 \geq 1$.

(iii) Recall Corollary 1.0.3, the adjunction formula (Theorem 1.0.4) and Theorem 1.0.5. Then the last claim can be verified by a simple computation:

$$\begin{aligned} \deg H|_C - 2g &= C.(-I-K_S) - 2 = C.\left(\sum_{i=d_1}^{d_2} D_i - F\right) - 2 \\ &\geq C.(D_{d_1} + D_{d_0} + D_{d_2} - F) - 2 \geq C.(D_{d_0} - F) \\ &= p_{d_0-1} + kD_{d_0}^2 + p_{d_0+1} - k \geq 0. \end{aligned}$$ □

Lemma 3.3.7. *Assume that $k \geq 2$ and C is nef and does not satisfy ($\sharp$). If $C.D_{d_1} \geq 1$ and $C.D_{d_2} \geq 1$, then $h^0(S,H) - h^0(S,H-I) \geq k+1$.*

Proof. By Lemma 3.3.6, $\mathcal{O}_S(H)$ is globally generated and $H^2 > 0$. Then by Bertini's theorem, we can take a nonsingular irreducible curve $Y \in |H|$. We denote by g_Y its genus. As we saw in the proof of Lemma 3.3.5, we have $h^0(S,H) = h^0(Y,H|_Y)+1$, and $h^0(S,H-I) = h^0(Y,(H-I)|_Y)$. Hence it is sufficient for the proof to verify $h^0(Y,H|_Y) - h^0(Y,(H-I)|_Y) \geq k$. By Corollary 1.0.3 and the adjunction formula, we have

$$\deg H|_Y - 2g_Y = -Y.K_S - 2 = \sum_{i=1}^{d} H.D_i - 2 \geq H.D_{d_0} - 2 \geq -1.$$

Hence $H|_Y$ is nonspecial by Theorem 1.0.5. Similarly, we have

$$\begin{aligned} \deg(H-I)|_Y - 2g_Y &= Y.(-I-K_S) - 2 = H.\left(\sum_{i=d_1}^{d_2} D_i - F\right) - 2 \\ &\geq H.(D_{d_0} - F) - 2 = p_{d_0-1} + kD_{d_0}^2 + p_{d_0+1} - k - 2. \end{aligned}$$

If $p_{d_0-1} = p_{d_0+1} = 0$ and $D_{d_0}^2 = 1$, then we can show that C satisfies ($\sharp$) by the same argument as in the proof of Lemma 3.3.3. Hence at least one of the inequalities $p_{d_0-1} \geq 1$, $p_{d_0+1} \geq 1$ and $D_{d_0}^2 \geq 2$ holds. It follows that $\deg(H-I)|_Y - 2g_Y \geq -1$, which means that $(H-I)|_Y$ is also nonspecial. Hence, by Riemann-Roch theorem,

$$\begin{aligned} h^0(Y,H|_Y) - h^0(Y,(H-I)|_Y) &= \deg H|_Y + 1 - g_Y - (\deg(H-I)|_Y + 1 - g_Y) \\ &= Y.H - Y.(H-I) = H.I. \end{aligned}$$

Finally, we shall verify that $H.I \geq k$. In the case where $d_1 \geq 3$, as we saw in the proof of Lemma 3.3.6, the inequality $H.D_{d_1-1} \geq 1$ holds. We thus have

$$H.I = H.\left(\sum_{i=1}^{d_1-1} D_i + \sum_{i=d_2+1}^{d} D_i\right) + k - 1 \geq H.D_{d_1-1} + k - 1 \geq k.$$

Similarly, one can show $H.I \geq k$ in the case where $d_2 \leq d-1$. Let us assume $d_1 = 2$ and $d_2 = d$. Then we have $H.I = H.D_1 + k - 1 = p_2 - D_1^2 + k - 2$. If $p_2 \geq 1$ or $D_1^2 \leq -2$, then we obtain $H.I \geq k$. On the other hand, if $p_2 = 0$ and $D_1^2 = -1$, we have $y_d + y_2 = 1$ by Theorem 2.4.5 and (3.1). In this case, the type of Δ has only one possibility, which is a fan defining Hirzebruch surface Σ_1 of degree one (Fig. 3.3).

Δ

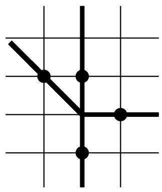

Figure 3.3.

Since $p_2 = 0$, we have $C \sim p_3 D_3$. This means that C satisfies ($\sharp$). Therefore, the equalities $d_1 = 2, d_2 = d, p_2 = 0$ and $D_1^2 = -1$ do not occur at the same time under the assumption of the lemma. □

3.3.3 Proof of Theorem 3.1.6

Combining the results in previous subsection, we obtain the following lemma which plays a central role in the proof of Theorem 3.1.6.

Lemma 3.3.8. *If $k \geq 2$ and C is nef and does not satisfy ($\sharp$), then $\mathcal{O}_C(C)$ satisfies (M_{k-1}).*

Proof. We have $g \geq 1$ by Lemma 3.3.3. We denote by $\rho(S)$ (≥ 2) the Picard number of S. We shall show the claim by induction on $k + \rho(S)$.

In the case where $k = \rho(S) = 2$, since $d = 4$, it is obvious that S is a Hirzebruch surface. Hence we have $d_1 = 2$, $d_2 = d$ and $F \sim D_2 \sim D_4$. Since $C.D_2 = C.D_4 = C.F = k = 2$, Lemma 3.3.6 and 3.3.7 allow us to apply Lemma 3.3.5 to C. Therefore, the claim is true in this case.

We next consider the case of $k + \rho(S) \geq 5$ under the following assumption: Let S' be a toric surface with a unique toric morphism to $\mathbb{P}^1$ and C' a nonsingular irrational curve on S'. We denote by k' the intersection number of C' and a fiber of the toric morphism of S'. We assume that if (S', C') does not satisfy ($\sharp$) and $k' + \rho(S') < k + \rho(S)$, then $(C', \mathcal{O}_{C'}(C'))$ satisfies ($M_{k'-1}$).

(i) Assume that $C.D_{d_1} \geq 1$ and $C.D_{d_2} \geq 1$. If $k = 2$, then the claim can be verified by Lemma 3.3.5. Assume that $k \geq 3$. By Lemma 3.3.6 and Bertini's theorem, we can take a nonsingular irreducible curve $Y \in |H|$. Note that Y is nef and $Y.F = k - 1$.

Now we suppose that Y satisfies ($\sharp$), that is, $D_{d_0}^2 = 1$ and $Y \sim (k-1)D_{d_0}$. Then we have

$$C \sim Y + I \sim (k-1)D_{d_0} + \sum_{i=1}^{d_1-1} D_i + \sum_{i=d_2+1}^{d} D_i + F.$$

If $d_1 \geq 3$, then $C.D_{d_1-1} = D_{d_1-1}^2 + 1 \leq -1$. This contradicts the fact that C is nef. Hence we have $d_1 = 2$. Similarly, one can obtain $d_2 = d$. Hence $C \sim (k-1)D_{d_0} + D_1 + F$, and the inequality $C.D_1 = D_1^2 + 1 \geq 0$ implies that $D_1^2 = -1$. Then, as we saw in the proof of Lemma 3.3.7, S is a Hirzebruch surface Σ_1. In this case, $F \sim D_d$ and (3.2) implies $D_{d_0} \sim D_1 + D_d$. We thus have $C \sim kD_{d_0}$, which contradicts the assumption that C does not satisfy ($\sharp$). Consequently, we have that (S, Y) does not satisfy ($\sharp$).

Since $Y.F + \rho(S) = k + \rho(S) - 1$, we have that $(Y, H|_Y)$ satisfies (M_{k-2}) by the hypothesis of the induction. Namely, for any integer $p \geq h^0(Y, H|_Y) - k + 1$,

$$K_{p,1}(Y, H|_Y) = 0.$$

As we saw in the proof of Lemma 3.3.5, $h^0(Y, H|_Y) = h^0(S, H) - 1 = h^0(C, H|_C) - 1$ hold. Moreover, by Theorem 3.2.3, we have $K_{p,1}(Y, H|_Y) \simeq K_{p,1}(S, H)$ for any integer p. Hence we have

$$K_{p,1}(S, H) = 0 \tag{3.6}$$

for any integer $p \geq h^0(C, H|_C) - k$. On the other hand, by Lemma 3.3.4, we have $K_{p,1}(S, H) \simeq K_{p,1}(C, H|_C)$ for any integer $p \geq h^0(S, H - I) + 1$. We remark that $h^0(S, H) - h^0(S, H - I) \geq k + 1$ holds by Lemma 3.3.7. Consequently, by combining these facts with (3.6), we obtain

$$K_{p,1}(C, H|_C) = 0$$

for any integer $p \geq h^0(C, H|_C) - k$, that is, $(C, H|_C)$ satisfies (M_{k-1}). Since $H|_C$ is nonspecial and globally generated by Lemma 3.3.6, we can apply Theorem 3.2.6 to conclude that $(C, \mathcal{O}_C(C))$ also satisfies (M_{k-1}).
(ii) Assume that $C.D_{d_1} = 0$. In this case, we have $d_0 \geq 4$. Indeed, if $d_0 = 3$, the fan Δ defining S is as in Fig. 3.4. Then we have $d_1 = 2$ and $F \sim D_2$, which yield

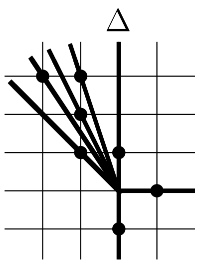

Figure 3.4.

a contradiction $C.F = 0$. The fact $d_0 \geq 4$ implies $D_{d_1}^2 = -1$. Let S' be a surface

obtained from S by blowing D_{d_1} down. Considering the relation between a blowing-up and a subdivision of a fan (cf. Theorem 2.4.1), S' also becomes a toric surface of Picard number $\rho(S') = \rho(S) - 1$ with a unique toric morphism to $\mathbb{P}^1$. We denote by F' its general fiber. If we regard C as a curve on S', then we have $C.F' = k$ obviously. Therefore, by the hypothesis of the induction, $(C, \mathcal{O}_C(C))$ satisfies (M_{k-1}).
(iii) In the case where $C.D_{d_2} = 0$, a similar argument to the case of (ii) goes through to show the claim. □

We are now in a position to prove Theorem 3.1.6.

Proof of Theorem 3.1.6. We first show that C is rational if $k \leq 1$. If $k = 0$, then C is contained in a fiber, that is, C is rational. Assume $k = 1$. In this case, the toric morphism of S induces a surjective morphism from C to $\mathbb{P}^1$ of degree one. Namely, C is rational. Hence we consider the case where $k \geq 2$.

If C satisfies ($\sharp$), then by Lemma 3.3.2, C is isomorphic to a nonsingular plane curve. Hence we may assume that C does not satisfy ($\sharp$). Hence Lemma 3.3.8 shows that $\mathcal{O}_C(C)$ satisfies (M_{k-1}). On the other hand, since

$$\deg \mathcal{O}_C(C) - 2g = -C.K_S - 2 \geq C.D_{d_0} - 2 = p_{d_0-1} + kD_{d_0}^2 + p_{d_0+1} - 2 \geq 0,$$

$\mathcal{O}_C(C)$ is nonspecial and globally generated by Theorem 1.0.5. Besides, C is irrational by Lemma 3.3.3. Therefore, it follows from Corollary 3.2.7 that C is k-gonal and the gonality conjecture is valid for C. □

Chapter 4

Weierstrass gap sequences on curves on toric surfaces

In this chapter, we see the author's second result dealing with Weierstrass gap sequences. We consider a curve on a toric surface and its intersection points with T_N-invariant divisors, and try to compute the gap sequences at such points. As a result of this attempt, we give a new technique to determine them by using the relation between certain lines and the lattice polytope associated to the curve. Similarly to the previous chapter, a *curve* will always mean a nonsingular projective curve unless otherwise stated.

4.1 Preliminaries and the main result

4.1.1 Weierstrass gap sequences

First we define Weierstrass gap sequences and review several previous results for them. Let C be a curve of genus g. For a point P on C, a positive integer j is called a *gap value* of C at P if

$$h^0(C, jP) = h^0(C, (j-1)P).$$

The set of all gap values is called a *Weierstrass gap sequence* (or, simply, *gap sequence*) of C at P. By Riemann-Roch theorem, its cardinality is equal to g. The classical result so-called Weierstrass gap theorem (cf. Theorem 0.0.3) is a basic tool in the study of gap sequences. As we saw in Theorem 0.0.4, there are two types of gap sequences at points on hyperelliptic curves.

For trigonal curves, Coppens has computed gap sequences at their ramification points.

Theorem 4.1.1 ([3, 4]). *Let C be a trigonal curve and $\varphi : C \to \mathbb{P}^1$ the trigonal morphism. A point P on C is called a total (resp. an ordinary) ramification point if the ramification index of φ at P is three (resp. two).*
(i) *The gap sequence at a total ramification point of φ is one of the following two types :*

$$\{1, 2, 4, \ldots, 3n-2, 3n-1, 3n+1, 3n+4, \ldots, 3(g-n-1)+1\},$$
$$\{1, 2, 4, \ldots, 3n-2, 3n-1, 3n+2, 3n+5, \ldots, 3(g-n-1)+2\}.$$

(ii) *The gap sequence at an ordinary ramification point of φ is one of the following two types :*

$$\{1, 2, 3, \ldots, 2n-1, 2n, 2n+1, 2n+3, \ldots, 2g-2n-1\},$$
$$\{1, 2, 3, \ldots, 2n-1, 2n, 2n+2, 2n+4, \ldots, 2g-2n\}.$$

Kato and Horiuchi [16] established a criterion for deciding the kinds of ramification points and their gap sequences. Besides, Kim studied unramified points and completed the classification of the gap sequences in the trigonal case.

Theorem 4.1.2 ([18]). *Let C and φ be as in Theorem 4.1.1, and denote by g the genus of C. Assume that $g \geq 5$, and define $j_0 = \max\{j \in \mathbb{N} \mid jP \text{ is special}\}$. If φ is unramified at a point P on C, then the gap sequence of C at P is of the form $\{1, 2, \ldots, g\}$ or*

$$\{1, 2, \ldots, n-1,\ n+j_0-g+1, n+j_0-g+2, \ldots, j_0+1\}$$

for some integer n with $[(j_0+1)/2]+1 \leq n \leq g$, where $[x]$ is the so-called Gauss' symbol, that is, the greatest integer not greater than x.

Actually, the notion of gap sequence was extended to singular points by Lax and Widland [23]. In [9], some methods were given by Gatto to compute gap sequences at singular points on a plane curve. They allowed to determine gap sequences at ordinary nodes on quartic curves or at cusps on quintic curves. Notari [27] has developed a technique to compute the gap sequence at a given point on a plane curve, either it is smooth or singular. Note that a projective plane is a typical example of a toric surface.

4.1.2 The technique to compute gap sequences

In general, however, it is not so easy to determine a gap sequence in its entirety at a given point. In this chapter, as mentioned before, we restrict ourselves to a curve C on a toric surface S and consider its intersection points with T_N-invariant divisors on S. Theorem 4.1.3 below provides a sufficient condition for a positive integer to be a gap value of C at such points. Moreover, as we will see in Corollary 4.1.4, it becomes the necessary and sufficient condition under the suitable condition. Namely, in such cases, one can detect all the gap values (i.e. the gap sequence). In Section 4.3, we will apply this technique to three examples. Concretely, we will consider singular plane curves

$$x^6y^3 + x^3y + y - 1 = 0,$$
$$x^5 + x^2y + xy^6 + y^6 = 0,$$
$$x^p + y^q + x^ry^s = 0 \quad (p \geq q \geq 1,\ r + s \geq 1)$$

and the nonsingular models of them. In these cases, we can determine the gap sequences at the infinitely near points of singularities.

We use the notation introduced in Section 2.4. In order to give a precise statement, we define a line $l_i(n) \subset \mathbb{R}^2$ by

$$l_i(n) = \{(z, w) \in \mathbb{R}^2 \mid x_iz + y_iw = n\}$$

for integers $1 \leq i \leq d$ and n. Then our main result in this chapter is stated as follows:

Theorem 4.1.3. *Let S be a complete nonsingular toric surface defined by a fan composed by d cones, and $C \sim \sum_{i=1}^{d} p_iD_i$ a nonsingular nef curve on S. Assume that C does not pass through any T_N-fixed point on S. For integers i_0 with $1 \leq i_0 \leq d$ and $j \geq 1$, if the line $l_{i_0}(p_{i_0} - j)$ has at least $C.D_{i_0}$ lattice points in the interior of $\Box_C$ (see Definition 2.4.7), then j is a gap value of C at the intersection points of C and D_{i_0}.*

Here we remark that it is not an essential assumption that C does not pass through any T_N-fixed point on S. Indeed, if there are T_N-fixed points lying on C, then by a succession of blowing-ups with those points as centers, we can obtain an embedding of C in a toric surface which satisfies the assumptions of Theorem 4.1.3.

As declared at the beginning of this subsection, under a suitable condition, Theorem 4.1.3 gives the necessary and sufficient condition for j to be a gap value at the intersection points of C and D_{i_0}. Concretely, the following corollary holds.

Corollary 4.1.4. *Let S, C and i_0 be as in Theorem 4.1.3. Assume that $C.D_{i_0} = 1$ and the line $l_{i_0}(p_{i_0} - j)$ has at most one lattice point in the interior of $\Box_C$ for any integer j. Then j is a gap value of C at $P = C \cap D_{i_0}$ if and only if $l_{i_0}(p_{i_0} - j)$ has a lattice point in the interior of $\Box_C$.*

Indeed, under these assumptions, the gap values at P detected by Theorem 4.1.3 are in one-to-one correspondence with the lattice points contained in the interior of $\Box_C$. Since $\Box_C$ has g lattice points in its interior (cf. Theorem 2.4.8), this means that all the gap values at P are completely found by Theorem 4.1.3.

4.2 Proof of Theorem 4.1.3

In this section, let C be a curve of genus g on toric surface S. By renumbering of T_N-invariant divisors, we can assume $i_0 = 1$ in the Theorem 4.1.3. We thus focus exclusively on the case where $i_0 = 1$ henceforth. Furthermore, by (2.2) and the following Lemma 4.2.1, it is sufficient to consider the form of the linear equivalence class

$$C \sim \sum_{i=1}^{d} p_i D_i \quad (p_i \in \mathbb{Z},\ p_1 = p_d = 0). \tag{4.1}$$

We denote by $\operatorname{Int} \square_C$ the interior of $\square_C$, that is,

$$\operatorname{Int} \square_C = \{(z, w) \in \mathbb{R}^2 \mid x_i z + y_i w < p_i \text{ for } 1 \leq i \leq d\}.$$

Lemma 4.2.1. *For a curve $C \sim \sum_{i=1}^{d} m_i D_i$ on S and an integer j, the number of lattice points contained in $l_1(m_1 - j) \cap \operatorname{Int} \square_C$ does not depend on the form of the linear equivalence class of C.*

Proof. Assume $C \sim \sum_{i=1}^{d} m_i D_i \sim \sum_{i=1}^{d} n_i D_i$, and define maps f_1 and f_2 from $\mathbb{Z}$ to itself as

$$f_1(z) = z + y_d n_1 - y_1 n_d - y_d m_1 + y_1 m_d,$$
$$f_2(w) = w - x_d n_1 + x_1 n_d + x_d m_1 - x_1 m_d.$$

Then, for any integer $1 \leq k \leq d$ and $(z, w) \in \mathbb{Z}^2$, we have

$$x_k f_1(z) + y_k f_2(w) = x_k z + y_k w + n_k - m_k$$

by Proposition 2.4.6. Considering the definition of $\square_C$, the map $(f_1, f_2) : \mathbb{Z}^2 \to \mathbb{Z}^2$ gives a one-to-one correspondence between $l_1(m_1 - j) \cap \operatorname{Int} \square_C$ and $l_1(n_1 - j) \cap \operatorname{Int} \square_C$. □

4.2.1 Key lemma.

The aim of this subsection is to show Lemma 4.2.10 which is the key to proving Theorem 4.1.3. In this subsection we consider the linear equivalence class of C as (4.1), and assume C is nef and $C.D_1 \geq 1$. Let j be a positive integer such that $l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbb{Z}^2 \neq \emptyset$, and denote by (z_0, w_0) the lattice point in $l_1(-j) \cap \operatorname{Int} \square_C$ closest to the line $l_d(0)$. Since C is nef, $|C|$ has no base points by Theorem 2.4.4. Hence, by Proposition 2.4.6 and 2.4.10, we have

$$p_k = \sum_{i=1}^{k-1} (x_k y_i - y_k x_i) C.D_i \geq 0 \tag{4.2}$$

for any integer $2 \leq k \leq d - 1$. All the remaining lemmas in this subsection are closely related to the notion of lattice polytope. Hence, for a better understanding, we will argue together with the following example.

Example 4.2.2. Let S be a toric surface associated to the fan in Fig. 2.2, and

$$C_0 \sim 2D_2 + 6D_3 + 10D_4 + 5D_5 + 7D_6 + 16D_7 + 10D_8 + 4D_9 + 3D_{10}$$

a nonsingular nef curve on S. Then the lattice polytope $\square_{C_0}$ is drawn as in Fig. 4.1.

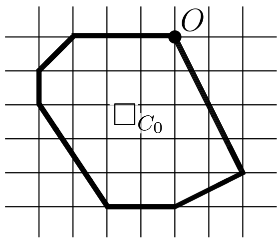

Figure 4.1.

We next define an effective divisor I which plays an important part in the proof of Theorem 4.1.3.

Definition 4.2.3. We define

$$\begin{gathered} a = \min\{i \geq 2 \mid x_i(z_0 - y_1) + y_i(w_0 + x_1) \geq 0\}, \\ b = \max\{i \leq d \mid x_i z_0 + y_i w_0 \geq 0\}, \\ q_i = \begin{cases} x_i(y_1 - z_0) - y_i(x_1 + w_0) & (1 \leq i \leq a-1), \\ -x_i z_0 - y_i w_0 & (b+1 \leq i \leq d), \\ 0 & (\text{otherwise}), \end{cases} \\ I = \sum_{i=1}^{d} q_i D_i. \end{gathered}$$

We remark that $b \leq d - 1$. Indeed, by the definition of (z_0, w_0), the inequality $x_d z_0 - y_d w_0 \leq p_d - 1 = -1$ holds. For instance, in the case of Example 4.2.2, for $j = 8$, we have $a = 5$, $b = 10$ and

$$I = 8D_1 + 4D_2 + 4D_3 + 4D_4 + 2D_{11} + 5D_{12}.$$

Then the line $l_1(-8)$ and $\square_I$ is as in Fig. 4.2. Note that the origin has changed.

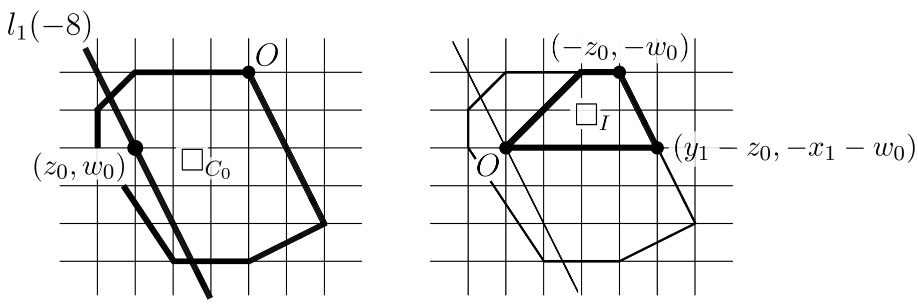

Figure 4.2.

Lemma 4.2.4. *For any integer $b+1 \leq k \leq d$, the inequality*

$$x_k y_1 - y_k x_1 \leq -1$$

holds. Moreover, if $a \geq 3$, then $x_m y_1 - y_m x_1 \geq 1$ for any integer $2 \leq m \leq a-1$.

Proof. Since $x_1 z_0 + y_1 w_0 = -j \neq 0$, we can write

$$(x_b, y_b) = \alpha_1(x_1, y_1) + \beta_1(w_0, -z_0),$$
$$(x_{b+1}, y_{b+1}) = \alpha_2(x_1, y_1) + \beta_2(w_0, -z_0)$$

with some real numbers. By the definition of b, we have

$$x_b z_0 + y_b w_0 = \alpha_1(x_1 z_0 + y_1 w_0) = -j\alpha_1 \geq 0,$$
$$x_{b+1} z_0 + y_{b+1} w_0 = \alpha_2(x_1 z_0 + y_1 w_0) = -j\alpha_2 < 0.$$

Hence we have $\alpha_1 \leq 0$ and $\alpha_2 > 0$. Now, we suppose that $x_{b+1} y_1 - y_{b+1} x_1 \geq 0$. Then Proposition 2.4.13 implies that $x_b y_1 - y_b x_1 \geq 0$. Hence we have

$$x_b y_1 - y_b x_1 = \beta_1(x_1 z_0 + y_1 w_0) = -j\beta_1 \geq 0,$$
$$x_{b+1} y_1 - y_{b+1} x_1 = \beta_2(x_1 z_0 + y_1 w_0) = -j\beta_2 \geq 0,$$

which imply $\beta_1 \leq 0$ and $\beta_2 \leq 0$. Then, by computing, we have

$$x_b y_{b+1} - y_{b+} x_{b+1} = j(\alpha_1 \beta_2 - \beta_1 \alpha_2) \geq 0.$$

This contradicts the fact that $x_{b+1} y_b - y_{b+1} x_b = 1$. We thus obtain that $x_{b+1} y_1 - y_{b+1} x_1 \leq -1$. Then by Proposition 2.4.13,

$$x_k y_1 - y_k x_1 \geq 1$$

for any integer $b+1 \leq k \leq d$. Similarly, by considering the descriptions of (x_{a-1}, y_{a-1}) and (x_a, y_a) as the sum of (x_1, y_1) and $(x_1 + w_0, y_1 - z_0)$ with real coefficients, one can show the second inequality in the lemma. □

Remark 4.2.5. The inequality $a \leq b+1$ immediately follows from Lemma 4.2.4. Indeed, if $a \geq b+2$, then we have

$$x_{a-1} y_1 - y_{a-1} x_1 \leq -1 \tag{4.3}$$

by Lemma 4.2.4. However, this contradicts the second statement in the lemma in the case where $a \geq 3$. It goes without saying that (4.3) is a contradiction in the case where $a = 2$.

Lemma 4.2.6. *The complete linear system $|I|$ has no base points.*

Proof. By Theorem 2.4.4, it is sufficient to verify $I.D_i \geq 0$ for each integer $1 \leq i \leq d$. Recall Theorem 2.4.5. Then we have

$$I.D_1 = q_d + q_1 D_1^2 + d_2 = -x_d z_0 - y_d w_0 - x_1 z_0 D_1^2 - y_1 w_0 D_1^2 + x_2 y_1 - y_2 x_1 - x_2 z_0 - y_2 w_0 = 1.$$

For integers $2 \leq k_1 \leq a-2$,

$$I.D_{k_1} = (x_{k_1-1} + x_{k_1} D_{k_1}^2 + x_{k_1+1})(y_1 - z_0) - (y_{k_1-1} + y_{k_1} D_{k_1}^2 + y_{k_1+1})(x_1 + w_0) = 0.$$

For integers $b+2 \leq k_2 \leq d$,

$$I.D_{k_2} = -(x_{k_2-1} + x_{k_2} D_{k_2}^2 + x_{k_2+1}) z_0 - (y_{k_2-1} + y_{k_2} D_{k_2}^2 + y_{k_2+1}) w_0 = 0.$$

Moreover, it is obvious that $I.D_{k_3} = 0$ for any integer $a+1 \leq k_3 \leq b-1$.

Let us check the remaining divisors D_{a-1}, D_a, D_b and D_{b+1}. Recall Lemma 4.2.4. Then we have

$$I.D_{a-1} = \begin{cases} x_a(z_0 - y_1) + y_a(w_0 + x_1) \geq 0 & (a \leq b), \\ -x_{b+1} y_1 + y_{b+1} x_1 \geq 1 & (a = b+1), \end{cases}$$

$$I.D_a = \begin{cases} -x_{a-1}(z_0 - y_1) - y_{a-1}(w_0 + x_1) \geq 1 & (a \leq b-1), \\ -x_{a-1}(z_0 - y_1) - y_{a-1}(w_0 + x_1) - x_{b+1} z_0 - y_{b+1} w_0 \geq 2 & (a = b), \\ x_{a-1} y_1 - y_{a-1} x_1 \geq 1 & (a = b+1). \end{cases}$$

Similarly, we have

$$I.D_b = \begin{cases} -x_{b+1} z_0 - y_{b+1} w_0 \geq 1 & (a \leq b-1), \\ -x_{a-1}(z_0 - y_1) - y_{a-1}(w_0 + x_1) - x_{b+1} z_0 - y_{b+1} w_0 \geq 2 & (a = b), \\ -x_{b+1} y_1 + y_{b+1} x_1 \geq 1 & (a = b+1), \end{cases}$$

$$I.D_{b+1} = \begin{cases} x_b z_0 + y_b w_0 \geq 0 & (a \leq b), \\ x_{a-1} y_1 - y_{a-1} x_1 \geq 1 & (a = b+1). \end{cases}$$ □

Very roughly speaking, Theorem 4.1.3 is verified by comparing the cohomology dimension $h^0(C, I|_C)$ with $h^0(C, (I - D_1)|_C)$. In fact, however, it is not enough for the proof to deal with only I. We need to introduce the following auxiliary divisor X and consider the divisor obtained by subtracting it from I. We define

$$X = \sum_{i=2}^{a-1} D_i + \sum_{i=b+1}^{d} D_i,$$
$$L_i(n) = \{(z, w) \in \mathbb{Z}^2 \mid x_i z + y_i w \leq n\}$$

for integers n and i with $1 \leq i \leq d$.

Lemma 4.2.7. *The vanishing* $h^1(S, I - X) = 0$ *holds.*

PROOF. Consider the cohomology long exact sequence

$$\begin{aligned}0 &\to H^0(S, I-X) \to H^0(S, I) \to H^0(X, I|_X)\\ &\to H^1(S, I-X) \to H^1(S, I) \to H^1(X, I|_X) \to H^2(S, I-X) \to \cdots .\end{aligned}$$

Lemma 4.2.6, Serre duality and Theorem 2.4.8 imply that $h^1(S, I) = 0$ and $h^2(S, I - X) = h^0(S, K_S + X - I) = 0$. Hence Riemann-Roch theorem yields the equality

$$h^0(X, I|_X) = \deg I|_X + 1 - \frac{1}{2}X.(X + K_S) - 1 = I.X - \frac{1}{2}X.(X + K_S).$$

We thus have

$$h^1(S, I - X) = h^0(S, I - X) - h^0(S, I) + I.X - \frac{1}{2}X.(X + K_S). \tag{4.4}$$

Since $I.D_i = 0$ for any integer $2 \le i \le a - 2$ or $b + 2 \le i \le d$, we have

$$I.X = \begin{cases} I.D_{a-1} + I.D_{b+1} & (a \ge 3),\\ I.D_{b+1} & (a = 2).\end{cases} \tag{4.5}$$

Moreover, by computing, we have

$$X.(X + K_S) = \begin{cases} -4 & (3 \le a \le b),\\ -2 & (\text{otherwise}).\end{cases} \tag{4.6}$$

In order to compute the value of $h^0(S, I) - h^0(S, I - X)$, we first verify the following inclusions:

$$\begin{aligned} L_1(q_1) \cap L_{a-1}(q_{a-1}) \subset \bigcap_{i=2}^{a-1} L_i(q_i) \text{ if } a \ge 3,\\ L_1(q_1) \cap L_{b+1}(q_{b+1}) \subset \bigcap_{i=b+1}^{d} L_i(q_i).\end{aligned} \tag{4.7}$$

Assume $a \ge 3$ and let (z_1, w_1) be a lattice point contained in $L_1(q_1) \cap L_{a-1}(q_{a-1})$. We write

$$(z_1, w_1) \ = \ (y_1 - z_0, -x_1 - w_0) + \alpha_1(y_1, -x_1) + \beta_1(y_{a-1}, -x_{a-1})$$

with real numbers α_1 and β_1. Then the inequalities

$$\begin{aligned} x_1 z_1 + y_1 w_1 = q_1 + \beta_1(x_1 y_{a-1} - y_1 x_{a-1}) \le q_1,\\ x_{a-1} z_1 + y_{a-1} w_1 = q_{a-1} + \alpha_1(x_{a-1} y_1 - y_{a-1} x_1) \le q_{a-1}\end{aligned}$$

implies $\alpha_1 \le 0$ and $\beta_1 \ge 0$, respectively. Let k_1 be an integer with $2 \le k_1 \le a-1$. Then Lemma 4.2.4 and Proposition 2.4.13 imply that $x_{k_1}y_1 - y_{k_1}x_1 \ge 1$ and $x_{a-1}y_{k_1} - y_{a-1}x_{k_1} \ge 0$. We thus have

$$x_{k_1}z_1 + y_{k_1}w_1 = q_{k_1} + \alpha_1(x_{k_1}y_1 - y_{k_1}x_1) + \beta_1(x_{k_1}y_{a-1} - y_{k_1}x_{a-1}) \le q_{k_1}.$$

Hence we obtain the first inclusion of (4.7). Similarly, for a point (z_2, w_2) contained in $L_1(q_1) \cap L_{b+1}(q_{b+1})$, we write

$$(z_2, w_2) = (-z_0, -w_0) + \alpha_2(y_1, -x_1) + \beta_2(y_{b+1}, -x_{b+1}).$$

Then one can show $\alpha_2 \ge 0$, $\beta_2 \le 0$ and the second inclusion of (4.7).

The same argument can be applied to show the inclusions

$$\begin{aligned} L_1(q_1) \cap L_{a-1}(q_{a-1}-1) &\subset \bigcap_{i=2}^{a-1} L_i(q_i - 1) \text{ if } a \ge 3, \\ L_1(q_1) \cap L_{b+1}(q_{b+1}-1) &\subset \bigcap_{i=b+1}^{d} L_i(q_i - 1). \end{aligned} \tag{4.8}$$

Recall the notation $l_i(n)$ defined in Subsection 4.1.2. Then by (4.7) and (4.8), in the case where $a \ge 3$, we have

$$\begin{aligned} & h^0(S, I) - h^0(S, I - X) \\ = & \sharp\Big(\bigcap_{i=1}^{d} L_i(q_i)\Big) - \sharp\Big(L_1(q_1) \cap \bigcap_{i=2}^{a-1} L_i(q_i - 1) \cap \bigcap_{i=a}^{b} L_i(q_i) \cap \bigcap_{i=b+1}^{d} L_i(q_i-1)\Big) \\ = & \sharp\Big(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i)\Big) - \sharp\Big(L_1(q_1) \cap L_{a-1}(q_{a-1}-1) \cap \bigcap_{i=a}^{b} L_i(q_i) \cap L_{b+1}(q_{b+1}-1)\Big) \\ = & \sharp\Big(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i) \setminus \big(L_{a-1}(q_{a-1}-1) \cap L_{b+1}(q_{b+1}-1)\big)\Big) \\ = & \sharp\Big(\Big(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i) \setminus L_{a-1}(q_{a-1}-1)\Big) \cup \Big(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i) \setminus L_{b+1}(q_{b+1}-1)\Big)\Big) \\ = & \sharp\Big(\Big(L_1(q_1) \cap l_{a-1}(q_{a-1}) \cap \bigcap_{i=a}^{b+1} L_i(q_i)\Big) \cup \Big(L_1(q_1) \cap \bigcap_{i=a-1}^{b} L_i(q_i) \cap l_{b+1}(q_{b+1})\Big)\Big). \end{aligned}$$

Similarly, if $a = 2$, one can obtain

$$h^0(S, I) - h^0(S, I - X) = \sharp\Big(\bigcap_{i=1}^{b} L_i(q_i) \cap l_{b+1}(q_{b+1})\Big).$$

We define

$$\begin{aligned} M &= L_1(q_1) \cap l_{a-1}(q_{a-1}) \cap \bigcap_{i=a}^{b+1} L_i(q_i), \\ N &= L_1(q_1) \cap \bigcap_{i=a-1}^{b} L_i(q_i) \cap l_{b+1}(q_{b+1}). \end{aligned}$$

Then we have

$$h^0(S,I) - h^0(S,I-X) = \begin{cases} \sharp M + \sharp N - \sharp(M\cap N) & (a \geq 3), \\ \sharp N & (a=2). \end{cases}$$

Here let us see the case of Example 4.2.2. As we saw after Definition 4.2.3, in this example, we have $a = 5$ and $b = 10$ for $j = 8$. Hence M and N are the sets of lattice points contained in $l_4(4) \cap A$ and $l_{11}(2) \cap B$, respectively (Fig. 4.3), where $A = L_1(8) \cap \bigcap_{i=5}^{11} L_i(q_i)$ and $B = L_1(8) \cap \bigcap_{i=4}^{10} L_i(q_i)$.

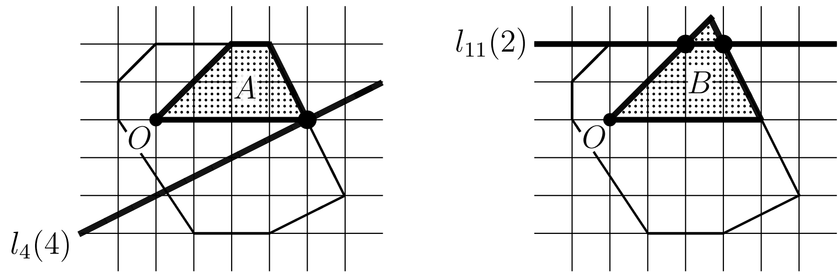

Figure 4.3.

We shall examine $\sharp M$. Let (u,v) be a lattice point contained in M. Since both (u,v) and $(y_1 - z_0, -x_1 - w_0)$ are contained in $l_{a-1}(q_{a-1})$, we can write

$$(u,v) = (y_1 - z_0, -x_1 - w_0) + \gamma(y_{a-1}, -x_{a-1})$$

with some integer γ. We obtain $\gamma \geq 0$ by Lemma 4.2.4 and the inequality

$$x_1 u + y_1 v = q_1 + \gamma(x_1 y_{a-1} - y_1 x_{a-1}) \leq q_1.$$

Since (u,v) is contained in $L_a(q_a)$, we have

$$q_a \geq x_a u + y_a v = x_a(y_1 - z_0) - y_a(x_1 + w_0) + \gamma = \begin{cases} -I.D_{a-1} + \gamma & (a \leq b), \\ -I.D_{a-1} + q_a + \gamma & (a = b+1). \end{cases}$$

Since $q_a = 0$ in the case where $a \leq b$, we obtain $\gamma \leq I.D_{a-1}$.

Conversely, let us show that for any integer $0 \leq \gamma' \leq I.D_{a-1}$, the lattice point

$$(u',v') = (y_1 - z_0, -x_1 - w_0) + \gamma'(y_{a-1}, -x_{a-1})$$

is contained in M. Since (u',v') is clearly contained in $L_1(q_1) \cap l_{a-1}(q_{a-1})$, it is sufficient to verify that it is contained in $\bigcap_{i=a}^{b+1} L_i(q_i)$. We remark the equality

$$\begin{aligned} & I.D_{a-1}(x_{a-1}, y_{a-1}) \\ = \ & (q_{a-2} + q_{a-1} D_{a-1}^2 + q_a)(x_{a-1}, y_{a-1}) \\ = \ & (-x_a(y_1 - z_0) + y_a(x_1 + w_0) + q_a)(x_{a-1}, y_{a-1}) \\ = \ & (-w_0 - x_1, z_0 - y_1) + (x_{a-1}(z_0 - y_1) + y_{a-1}(w_0 + x_1))(x_a, y_a) + q_a(x_{a-1}, y_{a-1}). \end{aligned} \tag{4.9}$$

We first show that (u', v') is contained in $L_{b+1}(q_{b+1})$.
(i) If $x_{b+1}y_{a-1} - y_{b+1}x_{a-1} \le 0$, then $x_{b+1}y_1 - y_{b+1}x_1 \le 0$ by Proposition 2.4.13. We thus have

$$\begin{aligned} x_{b+1}u' + y_{b+1}v' &= x_{b+1}y_1 - y_{b+1}x_1 - x_{b+1}z_0 - y_{b+1}w_0 + \gamma'(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\le -x_{b+1}z_0 - y_{b+1}w_0 = q_{b+1}. \end{aligned}$$

(ii) If $x_{b+1}y_{a-1} - y_{b+1}x_{a-1} \ge 1$, then $x_{b+1}y_a - y_{b+1}x_a \ge 0$ by Proposition 2.4.13. By the equation (4.9), we have

$$\begin{aligned} & I.D_{a-1}(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ = \ & x_{b+1}(z_0 - y_1) + y_{b+1}(w_0 + x_1) + (x_{a-1}(z_0 - y_1) + y_{a-1}(w_0 + x_1))(x_{b+1}y_a - y_{b+1}x_a) \\ & + q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ \le \ & x_{b+1}(z_0 - y_1) + y_{b+1}(w_0 + x_1) + q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}). \end{aligned}$$

Hence we have

$$\begin{aligned} x_{b+1}u' + y_{b+1}v' &= x_{b+1}(y_1 - z_0) + y_{b+1}(-x_1 - w_0) + \gamma'(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\le x_{b+1}(y_1 - z_0) + y_{b+1}(-x_1 - w_0) + I.D_{a-1}(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\le q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}). \end{aligned}$$

If $a \le b$, then $q_a = 0$ and we have $x_{b+1}u' + y_{b+1}v' \le 0 < q_{b+1}$. If $a = b + 1$, then we have

$$q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) = q_{b+1}(x_{b+1}y_b - y_{b+1}x_b) = q_{b+1}.$$

Hence we can conclude that (u', v') is contained in $L_{b+1}(q_{b+1})$.

In the case where $a = b + 1$, the above argument is enough to show that (u', v') is contained in $\bigcap_{i=a}^{b+1} L_i(q_i)$. On the other hand, in the case where $a \le b$, we have to check that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(q_i)$, too. Assume $a \le b$ and let m be an integer with $a \le m \le b$. Note that $q_m = 0$ in this case.
(i) If $x_m y_{a-1} - y_m x_{a-1} \ge 0$, then we have $x_m y_a - y_m x_a \ge 0$ by Proposition 2.4.13. Then by the equation (4.9), we have

$$\begin{aligned} & x_m u' + y_m v' \\ = \ & (\gamma' - I.D_{a-1})(x_m y_{a-1} - y_m x_{a-1}) + (x_{a-1}(z_0 - y_1) + y_{a-1}(w_0 + x_1))(x_m y_a - y_m y_a) \\ \le \ & 0 = q_m. \end{aligned}$$

(ii) If $x_m y_{a-1} - y_m x_{a-1} \le -1$, then Proposition 2.4.13 yields the inequalities $x_m y_1 - y_m x_1 \le -1$, $x_b y_1 - y_b x_1 \le -1$ and $x_m y_b - y_m x_b \le 0$. Thus we can write

$$(x_m, y_m) = \delta(x_1, y_1) + \varepsilon(x_b, y_b)$$

with real numbers $\delta \leq 0$ and $\varepsilon > 0$. Recall that (z_0, w_0) lies on $l_1(-j)$. Then we have

$$x_m z_0 + y_m w_0 = \delta(x_1 z_0 + y_1 w_0) + \varepsilon(x_b z_0 + y_b w_0) \geq 0,$$
$$x_m u' + y_m v' = x_m y_1 - y_m x_1 - x_m z_0 - y_m w_0 + \gamma'(x_m y_{a-1} - y_m x_{a-1}) < 0 = q_m.$$

Hence we have that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(q_i)$.

In sum, we can conclude that

$$M = \{(y_1 - z_0, -x_1 - w_0) + \gamma(y_{a-1}, -x_{a-1}) \mid 0 \leq \gamma \leq I.D_{a-1}\}.$$

A similar argument can be applied to show that

$$N = \{(-z_0, -w_0) - \zeta(y_{b+1}, -x_{b+1}) \mid 0 \leq \zeta \leq I.D_{b+1}\}.$$

Next we examine $M \cap N$ under the assumption that $a \geq 3$. By the definition of M and N, the intersection $M \cap N$ includes at most one point $l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$.
(i) In the case where $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} = 0$, we have $(x_{b+1}, y_{b+1}) = -(x_{a-1}, y_{a-1})$. Let (u_1, v_1) be a lattice point on $l_{a-1}(q_{a-1})$. Then by Lemma 4.2.4, we have

$$\begin{aligned} x_{b+1}u_1 + y_{b+1}v_1 &= -x_{a-1}u_1 - y_{a-1}v_1 = -q_{a-1} = x_1 y_{a-1} - y_1 x_{a-1} + x_{a-1}z_0 + y_{a-1}w_0 \\ &\leq x_{a-1}z_0 + y_{a-1}w_0 - 1 = -x_{b+1}z_0 - y_{b+1}w_0 - 1 = q_{b+1} - 1. \end{aligned}$$

Hence (u_1, v_1) does not lie on $l_{b+1}(q_{b+1})$. This means $M \cap N = \emptyset$.

Assume $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} \neq 0$. In this case, the intersection $l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$ clearly consists of only one lattice point. We denote it by (u_0, v_0).
(ii) Consider the case where $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} \geq 1$. Since both (u_0, v_0) and $(-z_0, -w_0)$ lie on $l_{b+1}(q_{b+1})$, we can write

$$(u_0, v_0) = (-z_0, -w_0) + \eta(y_{b+1}, -x_{b+1})$$

with some integer η. Then the inequality

$$\begin{aligned} &-x_{a-1}z_0 - y_{a-1}w_0 + \eta(x_{a-1}y_{b+1} - y_{a-1}x_{b+1}) \\ &= x_{a-1}u_0 + y_{a-1}v_0 = q_{a-1} = x_{a-1}(y_1 - z_0) - y_{a-1}(x_1 + w_0) \geq -x_{a-1}z_0 - y_{a-1}w_0 + 1 \end{aligned}$$

implies $\eta \geq 1$. Hence we have

$$x_1 u_0 + y_1 v_0 = q_1 + \eta(x_1 y_{b+1} - y_1 x_{b+1}) \geq q_1 + 1.$$

This means that (u_0, v_0) is not contained in $L_1(q_1)$, that is, $M \cap N = \emptyset$.
(iii) Consider the case where $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} \leq -1$. We write

$$(u_0, v_0) = \theta(y_{a-1}, -x_{a-1}) + \iota(y_{b+1}, -x_{b+1})$$

with real numbers θ and ι. Since (u_0, v_0) is contained in $l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$, we have $\theta > 0$ and $\iota < 0$.
(iii)-(a) If $a \leq b$, then $q_b = 0$. Since Proposition 2.4.13 implies that $x_{a-1}y_b - y_{a-1}x_b \leq -1$, we have

$$x_b u_0 + y_b v_0 = \theta(x_b y_{a-1} - y_b x_{a-1}) - \iota \geq \theta - \iota > 0 = q_b.$$

This means that (u_0, v_0) is not contained in $L_b(q_b)$, that is, $M \cap N = \emptyset$.
(iii)-(b) If $a = b + 1$, then $M \cap N = L_1(q_1) \cap l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$. Since $q_1 = -x_1 z_0 - y_1 w_0 = j \geq 1$, we have

$$x_1 u_0 + y_1 v_0 = \theta(x_1 y_{a-1} - y_1 x_{a-1}) + \iota(x_1 y_{b+1} - y_1 x_{b+1}) \leq -\theta + \iota < 0 \leq q_1 - 1.$$

Hence, in this case, (u_0, v_0) is contained in $L_1(q_1)$ and we have $M \cap N = \{(u_0, v_0)\}$.

Here we note that $a \leq b$ in the case of (i) and (ii). Indeed, if $a = b + 1$, then $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} = -1$. We thus conclude that

$$\sharp(M \cap N) = \begin{cases} 0 & (3 \leq a \leq b), \\ 1 & (3 \leq a = b + 1). \end{cases}$$

In sum, we have

$$h^0(S, I) - h^0(S, I - X) = \begin{cases} I.D_{a-1} + I.D_{b+1} + 2 & (3 \leq a \leq b), \\ I.D_{a-1} + I.D_{b+1} + 1 & (3 \leq a = b + 1), \\ I.D_{b+1} + 1 & (a = 2). \end{cases} \tag{4.10}$$

Therefore, combining (4.4), (4.5), (4.6) and (4.10), we can obtain $h^1(S, I - X) = 0$. □

In order to compute the difference between the dimensions of global sections of $(I - X)|_C$ and $(I - X - D_1)|_C$, we examine their cohomologies of higher order in Lemma 4.2.9 below.

Lemma 4.2.8. *If $\sharp(l_1(-j) \cap \operatorname{Int} \Box_C \cap \mathbb{Z}^2) \geq C.D_1$, then $a \geq 3$.*

PROOF. We put $c = C.D_1$. Let (z, w) be a lattice point contained in $l_1(-j) \cap \operatorname{Int} \Box_C$. Then we can write

$$(z, w) = (z_0, w_0) + \alpha(y_1, -x_1)$$

with some integer α. Since (z_0, w_0) is the lattice point in $l_1(-j) \cap \operatorname{Int} \Box_C$ closest to $l_d(0)$, we have $\alpha \geq 0$. Hence, by assumption, the point $(z_0, w_0) + (c - 1)(y_1, -x_1)$ have to be contained in $\operatorname{Int} \Box_C$. We thus have

$$x_2(z_0 + (c - 1)y_1) + y_2(w_0 - (c - 1)x_1) = x_2(z_0 - y_1) + y_2(w_0 + x_1) + c < p_2 = c,$$

where the last equality follows from (4.2). Hence we have $x_2(z_0 - y_1) + y_2(w_0 + x_1) < 0$, which means $a \geq 3$. □

Lemma 4.2.9. *If* $\sharp(l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbb{Z}^2) \geq C.D_1$, *then*

$$h^0(S, K_S + C - I + X + D_1) = h^0(S, K_S + C - I + X) + C.D_1.$$

Proof. We put $c = C.D_1$. Recall that $p_1 = 0$ and $q_1 = j$. Then by Theorem 2.4.8, we have

$$\begin{aligned} & h^0(S, K_S + C - I + X + D_1) - h^0(S, K_S + C - I + X) \\ = & \sharp\left(L_1(-j) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i)\right) \\ & -\sharp\left(L_1(-j-1) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i)\right) \\ = & \sharp\left(l_1(-j) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i)\right). \end{aligned}$$

We define

$$K = l_1(-j) \cap \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i).$$

Now our purpose is to show that $\sharp K = c$. Let (u, v) be a lattice point contained in K. Since both (z_0, w_0) and (u, v) lie on $l_1(-j)$, we can write

$$(u, v) = (z_0, w_0) + \alpha(y_1, -x_1)$$

with some integer α. Since $p_d = 0$, we have

$$x_d u + y_d v = -q_d + \alpha(x_d y_1 - y_d x_1) \leq p_d - q_d = -q_d,$$

which implies $\alpha \geq 0$. On the other hand, since $a \geq 3$ by Lemma 4.2.8, (u, v) is contained in $L_2(p_2 - q_2)$. Hence we have

$$x_2 u + y_2 v = x_2 z_0 + y_2 w_0 + \alpha \leq p_2 - q_2 = c + x_2 z_0 + y_2 w_0 - 1,$$

that is, $\alpha \leq c - 1$.

Conversely, let us verify that, for an integer α' with $0 \leq \alpha' \leq c - 1$, the point

$$(u', v') = (z_0, w_0) + \alpha'(y_1, -x_1)$$

is contained in K. Let k_1 be an integer with $2 \leq k_1 \leq a - 1$. By Lemma 4.2.4 and Proposition 2.4.13, we have

$$x_{k_1} y_m - y_{k_1} x_m \geq 1$$

for any integer $1 \le m \le k_1 - 1$. Hence we have $p_{k_1} \ge c(x_{k_1}y_1 - y_{k_1}x_1)$ by (4.2) and

$$\begin{aligned} x_{k_1}u' + y_{k_1}v' &= x_{k_1}(z_0 - y_1) + y_{k_1}(w_0 + x_1) + (\alpha' + 1)(x_{k_1}y_1 - y_{k_1}x_1) \\ &\le -q_{k_1} + c(x_{k_1}y_1 - y_{k_1}x_1) \le p_{k_1} - q_{k_1}. \end{aligned}$$

For integers $b + 1 \le k_2 \le d$, we have

$$x_{k_2}u' + y_{k_2}v' = x_{k_2}z_0 + y_{k_2}w_0 + \alpha'(x_{k_2}y_1 - y_{k_2}x_1) \le -q_{k_2} \le p_{k_2} - q_{k_2}.$$

Finally, we shall check that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(p_i - 1)$. Since (z_0, w_0) is the lattice point in $l_1(-j) \cap \operatorname{Int} \square_C$ closest to $l_d(0)$, we have that $(z_0, w_0) + \beta(y_1, -x_1)$ is not contained in $\operatorname{Int} \square_C$ if $\beta \le -1$. On the other hand, by the assumption of the lemma, $l_1(-j)$ has at least c lattice points in $\operatorname{Int} \square_C$. We thus have that (u', v') is contained in $\operatorname{Int} \square_C$ (in particular $\bigcap_{i=a}^{b} L_i(p_i - 1)$) for $0 \le \alpha' \le c - 1$. In sum, we can conclude that (u', v') is contained in K. It follows that $\sharp K = c$. $\square$

By using Lemma 4.2.7 and 4.2.9 in cohomology long exact sequences, we can obtain the following equality:

Lemma 4.2.10. *If* $\sharp(l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbb{Z}^2) \ge C.D_1$, *then*

$$h^0(C, (I - X)|_C) = h^0(C, (I - X - D_1)|_C).$$

Proof. It is sufficient to verify the inequality $h^0(C, (I - X)|_C) \le h^0(C, (I - X - D_1)|_C)$. By Lemma 4.2.7, we have the cohomology long exact sequence

$$0 \to H^1(C, (I - X)|_C) \to H^2(S, I - X - C) \to H^2(S, I - X) \to \cdots.$$

By Serre duality and Theorem 2.4.8, we have

$$\begin{aligned} h^2(S, I - X - C) &= h^0(S, K_S + C - I + X), \\ h^2(S, I - X) &= h^0(S, -I - D_1 - \textstyle\sum_{i=a}^{b} D_i) = 0. \end{aligned}$$

Hence, by Riemann-Roch theorem, we have

$$\begin{aligned} h^0(C, (I - X)|_C) &= h^1(C, (I - X)|_C) + \deg(I - X)|_C + 1 - g \\ &= h^0(S, K_S + C - I + X) + (I - X).C + 1 - g. \end{aligned}$$

On the other hand, the cohomology long exact sequence

$$\cdots \to H^1(C, (I - X - D_1)|_C) \to H^2(S, I - X - D_1 - C) \to H^2(S, I - X - D_1) \to \cdots$$

and the vanishings $h^2(S, I - X - D_1) = h^0(S, -I - \sum_{i=a}^{b} D_i) = 0$ lead the inequality

$$h^1(C, (I - X - D_1)|_C) \ge h^0(S, K_S + C - I + X + D_1).$$

Hence, by Riemann-Roch theorem and Lemma 4.2.9, we have

$$\begin{aligned} h^0(C, (I - X - D_1)|_C) &\ge h^0(S, K_S + C - I + X + D_1) + (I - X - D_1).C + 1 - g \\ &= h^0(C, (I - X)|_C). \end{aligned}$$ $\square$

4.2.2 Proof of Theorem 4.1.3

Finally, let us show Theorem 4.1.3.

Proof of Theorem 4.1.3. We first consider the case where $g = 0$. In this case, the gap sequence at P is empty. Indeed, the equation $h^0(C, jP) = j - 1$ holds for any positive integer j. On the other hand, by Theorem 2.4.8, there are no lattice points in $\operatorname{Int} \square_C$. Hence the statement is obviously true.

We assume that $g \geq 1$ and put $D_{i_0}|_C = \{P_1, \ldots, P_c\}$. Lemma 4.2.10 implies that

$$h^0(C, (I - X)|_C) = h^0(C, (I - X)|_C - P_1).$$

Namely, P_1 is the base point of $\big|(I - X)|_C\big|$. Note that $q_1 = j$. We define

$$I' = I - jD_1 - X = \sum_{i=2}^{a-1}(q_i - 1)D_i + \sum_{i=a}^{b} q_i D_i + \sum_{i=b+1}^{d}(q_i - 1)D_i.$$

It is clear that I' is effective by Definition 4.2.3. Besides, since P_1 lies on neither D_2 nor D_d by assumption, $I'|_C$ does not contain P_1. Therefore, P_1 is also the base point of the complete linear system

$$\big|(I - X)|_C - I'|_C - jP_2 - \cdots - jP_c\big| = |jP_1|$$

on C, that is, $h^0(C, jP_1) = h^0(C, (j-1)P_1)$. The same argument as above goes through for the points $P_2, \ldots, P_c$. $\square$

4.3 Examples.

In this section, we shall apply Corollary 4.1.4 to concrete examples in practice. Our attempt is to compute the gap sequences at the infinitely near points of a (possibly singular) point on a plane curve. Let Q be a point on plane curve C', and consider the resolution of singularities of C' by a succession of blowing-ups. Then, for some cases, we can determine the gap sequences of the nonsingular model of C' at the infinitely near points of Q by Corollary 4.1.4.

Let $\mathbb{P}^2(X_0 : X_1 : X_2)$ be the projective plane. We denote $x = X_1/X_0$, $y = X_2/X_0$ the local coordinates on the affine open subset $U_0 = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 \mid x_0 \neq 0\}$.

Example 4.3.1. Let C' be a plane curve defined by the local equation

$$x^6y^3 + x^3y + y - 1 = 0.$$

One can obtain a toric morphism $\varphi : S \to \mathbb{P}^2$ such that S is a nonsingular compact toric surface and the proper transform $C := \varphi_*^{-1}(C')$ is a nonsingular nef curve on S of genus 3. The fan Δ_S and $\square_C$ associated to the surface S and the curve C are as in

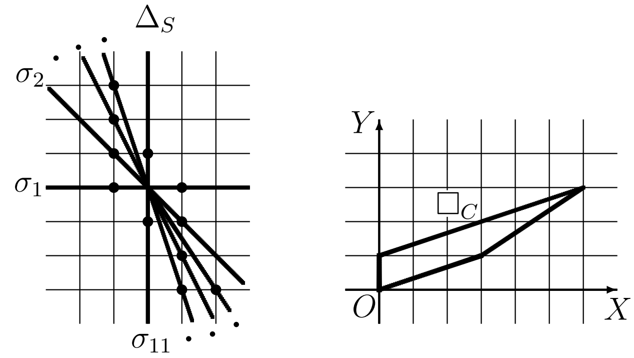

Figure 4.4.

Fig. 4.4. The linear equivalence class of C is written as

$$C \sim D_2 + 2D_3 + 3D_4 + 3D_5 + 6D_6 + 3D_7 + 3D_8 + D_9.$$

Consider the point $Q = (0, 1)$ on $C' \cap U_0$. Then Q has only one infinitely near point P on C, which is in fact the intersection point $C \cap D_1$. The cone σ_1 corresponding to D_1 has the primitive element $(-1, 0)$. Since the line $X = j$ has at most one lattice point in $\mathrm{Int}\,\square_C$ for any integer j, by Corollary 4.1.4, the gap sequence of C at P is

$$\{j \in \mathbb{N} \mid \text{the line } X = j \text{ has a lattice point in } \mathrm{Int}\,\square_C\} = \{1, 2, 4\}.$$

Example 4.3.2. Let C' be a plane curve defined by the local equation

$$x^5 + x^2y + xy^6 + y^6 = 0,$$

and $\varphi : S \to \mathbb{P}^2$ a toric morphism such that $C := \varphi_*^{-1}(C')$ is a nonsingular curve on S. Then the genus of C is equal to 8 and the fan Δ_S and the lattice polytope $\square_C$ are as

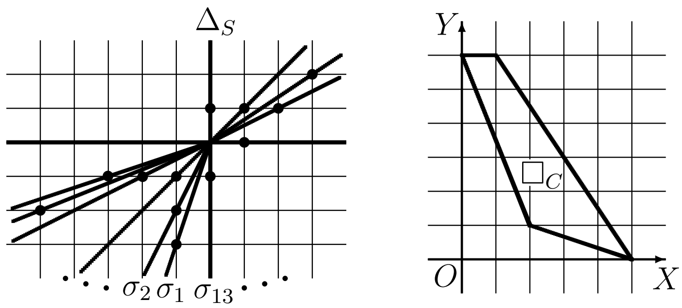

Figure 4.5.

in Fig. 4.5. The linear equivalence class of C is written as

$$C \sim -5D_1 - 4D_2 - 3D_3 - 5D_4 - 12D_5 - 6D_6 + 6D_8 + 7D_9 + 15D_{10} + 10D_{11} + 5D_{12}.$$

Consider the origin $O = (0,0)$ on $C' \cap U_0$, which is a singular point on C'. Then the infinitely near points of O on C are $P_1 = C \cap D_1$ and $P_2 = C \cap D_5$. The primitive elements of σ_1 and σ_5 are $(-1,-3)$ and $(-5,-2)$, respectively.

It is obvious that the lines $X + 3Y = k$ and $5X + 2Y = l$ have at most one lattice point in $\operatorname{Int}\square_C$ for any integer k and l. Hence, by Corollary 4.1.4, the gap sequences of C at P_1 and P_2 are

$$\{j \in \mathbb{N} \mid \text{the line } X+3Y=j+5 \text{ has a lattice point in } \operatorname{Int}\square_C\} = \{1,2,3,4,6,8,9,11\},$$
$$\{j \in \mathbb{N} \mid \text{the line } 5X+2Y=j+12 \text{ has a lattice point in } \operatorname{Int}\square_C\} = \{1,2,3,4,5,6,7,9\},$$

respectively.

Before proceeding to the last example, we define the following function.

Definition 4.3.3. For a positive integer m and a non-negative integer n, we define a function f as

$$f(m,n) = \begin{cases} \gcd(m,n) & (n \geq 1), \\ m & (n = 0). \end{cases}$$

Example 4.3.4. Let C' be a plane curve defined by the local equation of the form

$$x^p + y^q + x^r y^s = 0,$$

where $p \geq q \geq 1$ and $r + s \geq 1$. One can obtain a toric morphism $\varphi : S \to \mathbb{P}^2$ such that $C := \varphi_*^{-1}(C')$ is nonsingular. We write the linear equivalence class of C as $C \sim \sum_{i=1}^{d} p_i D_i$. The genus of C can be computed by the formula

$$g = \begin{cases} \frac{1}{2}(|pq - rq - sp| - f(p,p-q) - f(p-r,s) - f(q-s,r)) + 1 & (pq - rq - sp \neq 0), \\ 0 & (pq - rq - sp = 0). \end{cases}$$

In this case, the lattice polytope $\square_C$ become a triangle and we can place it such that its vertices are $(p,0)$, $(0,q)$ and (r,s). Then, by Corollary 4.1.4, we can compute the gap sequence of C at the infinitely near points of the origin $O = (0,0)$ on C' in the following cases:

(i) $pq - rq - sp = 0$,
(ii) $pq - rq - sp < 0$ and $f(p,p-q) = 1$,
(iii) $pq - rq - sp > 0$ and $f(p-r,s) = f(q-s,r) = 1$.

The case (i) do not require Corollary 4.1.4. Since $g = 0$, the gap sequence is empty at every point on C.

In the case (ii), the fan Δ_S is as in Fig. 4.6. The point Q has one infinitely near

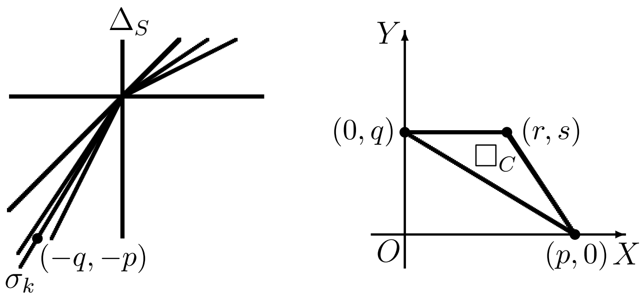

Figure 4.6.

point P on C, which is the intersection point $C \cap D_k$. The primitive element of σ_k is $(-q,-p)$ and $p_k = -pq$. Hence, by Corollary 4.1.4, the gap sequence of C at P is

$$\{j \in \mathbb{N} \mid \text{the line } qX + pY = pq + j \text{ has lattice points in } \operatorname{Int} \square_C\}.$$

In the case (iii), the fan Δ_S and the lattice polytope $\square_C$ are as in Fig. 4.7. The

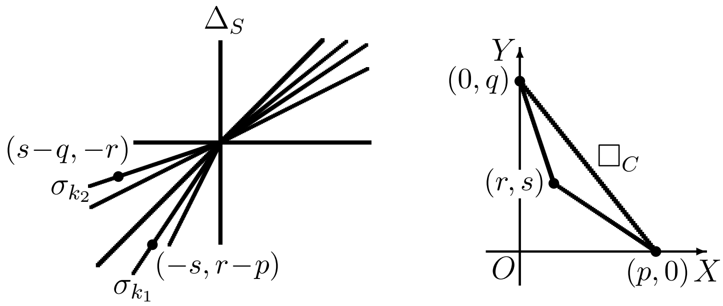

Figure 4.7.

infinitely near points of Q on C are $P_1 = C \cap D_{k_1}$ and $P_2 = C \cap D_{k_2}$. The primitive elements of σ_{k_1} and σ_{k_2} are $(-s, r-p)$ and $(s-q, -r)$, respectively. Since $p_{k_1} = -sp$ and $p_{k_2} = -rq$, by Corollary 4.1.4, we see that the gap sequences of C at P_1 and P_2 are

$$\{j \in \mathbb{N} \mid \text{the line } sX + (p-r)Y = sp + j \text{ has lattice points in } \operatorname{Int} \square_C\},$$
$$\{j \in \mathbb{N} \mid \text{the line } (q-s)X + rY = rq + j \text{ has lattice points in } \operatorname{Int} \square_C\},$$

respectively.

Acknowledgement

The author would like to express his sincere gratitude to Professor Kazuhiro Konno for taking notice of progress of his work continuously and giving accurate advice. He is grateful to Professors Tadashi Ashikaga, Jiryo Komeda, Akira Ohbuchi and Sampei Usui for their helpful advice and suggestions. He also thanks Doctors Atsushi Ikeda, Shinya Kitagawa and Takeshi Harui for uncountable discussions. Lastly, it is also necessary to mention that he has been encouraged and supported by his parents for a long time.

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