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LOCAL LIMIT THEOREM FOR RANDOM WALK IN PERIODIC ENVIRONMENT

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1. Preliminaries and Results

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space on which all our random quantities will be defined. Let $\boldsymbol{Z}^d$ be the set of d-dimensional integer lattice. We consider Markov chain on $\boldsymbol{Z}^d$ with a transition function $P(x, y)$. We denote by $P_n(x, y)$ the n-th transition function of the Markov chain. We are interested in an asymptotic behaviour of $P_n(x, y)$ as $n \to \infty$, that is, a local limit theorem for the Markov chain. Spitzer showed a uniform estimate of a local limit theorem for random walk in $\boldsymbol{Z}^d$ (see, [10, Remark to P7.9 and P7.10]). The purpose of this paper is to extend his result to the Markov chain with the following assumptions.

Assumption 1.1. There exists $\boldsymbol{s} = (s_1, s_2, \dots, s_d) \in \boldsymbol{Z}^d$ with $s_l > 0$, $1 \leq l \leq d$, such that

$$P(x + s_l \boldsymbol{e}_l, y + s_l \boldsymbol{e}_l) = P(x, y)$$

for every x, $y \in \boldsymbol{Z}^d$ and l, $1 \leq l \leq d$. Here $\boldsymbol{e}_l$, $1 \leq l \leq d$, denotes the basis vector $(\underbrace{0, \dots, 0, 1}_{l}, 0, \dots, 0)$ in $\boldsymbol{Z}^d$.

We call a Markov chain with this assumption *a random walk in periodic environment* (*RWPE* for abbreviation), and the vector $\boldsymbol{s}$ *period of RWPE*.

Assumption 1.2. The Markov chain is irreducible and aperiodic, that is, for every x, $y \in \boldsymbol{Z}^d$, there exists a positive integer $n_0(x, y)$ such that $P_n(x, y) > 0$ for all $n \geq n_0(x, y)$.

We set

$$\Xi = \{(j_1, j_2, \dots, j_d) \in \boldsymbol{Z}^d \mid 0 \leq j_1 \leq s_1 - 1, \dots, 0 \leq j_d \leq s_d - 1\}.$$

For $x \in \boldsymbol{Z}$ and l, $1 \leq l \leq d$, we denote by $T_l(x)$ the remainder obtained when x is divided by s_l, and put $T(x) = (T_1(x_1), T_2(x_2), \dots, T_d(x_d))$ for $x = (x_1, x_2, \dots, x_d) \in \boldsymbol{Z}^d$.

Let k be a point in Ξ, then we say x in $\boldsymbol{Z}^d$ a point of *type* k if $T(x) = k$. Let $Q = (q_{jk})_{j,k\in\Xi}$ be a transition matrix, of which each component is given by

$$q_{jk} = \sum_{T(x)=k} P(j,x) \quad \text{for } j,\ k \in \Xi.$$

By Assumption 1.2, we see that the matrix Q is ergodic. Then Q has a stationary distribution $\boldsymbol{\pi} = (\pi_j)_{j\in\Xi}$, that is,

$$\lim_{n\to\infty} q_{jk}^{(n)} = \pi_k. \tag{1.1}$$

ASSUMPTION 1.3. For each $j \in \Xi$,

$$\sum_{x\in\boldsymbol{Z}^d} |x| P(j, j+x) < \infty \quad \text{and} \quad \sum_{j\in\Xi} \pi_j \sum_{x\in\boldsymbol{Z}^d} x P(j, j+x) = \boldsymbol{0}.$$

ASSUMPTION 1.4. The Markov chain has finite second moment, that is,

$$\sum_{x\in\boldsymbol{Z}^d} |x|^2 P(j, j+x) < \infty \quad \text{for each } j \in \Xi.$$

Let $j,\ k \in \Xi$ and $x \in \boldsymbol{Z}^d$. For $q_{jk} > 0$ we define

$$F_{jk}(x) = \begin{cases} \dfrac{1}{q_{jk}} P(j, j+x) & \text{if } T(j+x) = k \\ 0 & \text{otherwise,} \end{cases}$$

and for $q_{jk} = 0$, $F_{jk}(x) = 1$ if $x = k - j$ and 0 otherwise. Note that $F_{jk}(\cdot)$ is the jump size distribution of the *RWPE* under the condition that the transition from a point of type j to a point of type k occurs. Define the mean vectors $\mu_{jk} = (\mu_{jk;1}, \dots, \mu_{jk;d})$ and the covariance matrices $C_{jk} = (c_{jk;lm})_{1\le l,m\le d}$ of $F_{jk}(\cdot)$, $j,\ k \in \Xi$, that is,

$$\mu_{jk;l} = \sum_{x\in\boldsymbol{Z}^d} x_l F_{jk}(x) \quad \text{and} \quad c_{jk;lm} = \sum_{x\in\boldsymbol{Z}^d} (x_l - \mu_{jk;l})(x_m - \mu_{jk;m}) F_{jk}(x)$$

for $1 \le l,\ m \le d$. Note that by Assumption 1.3

$$\sum_{j,k\in\Xi} \pi_j q_{jk} \mu_{jk} = \boldsymbol{0}. \tag{1.2}$$

Put

$$Q(\theta) = \left(q_{jk} e^{i(\mu_{jk},\theta)}\right)_{j,k\in\Xi} \quad \text{and} \quad f(\theta, z) = |zI - Q(\theta)| \tag{1.3}$$

for $\theta \in \boldsymbol{R}^d$ and $z \in \boldsymbol{C}$. Note that $Q = Q(\mathbf{0})$. Since the matrix Q is ergodic, by Perron-Frobenius theorem, 1 is a simple root of the characteristic equation $f(\mathbf{0}, z) = 0$. See, e.g., Karlin [4]. Thus we see that

$$\frac{\partial f}{\partial z}(\mathbf{0}, 1) \neq 0. \tag{1.4}$$

Set

$$\begin{aligned} b_{lm} &= \frac{\partial^2 f}{\partial\theta_l \partial\theta_m}(\mathbf{0}, 1) \Big/ \frac{\partial f}{\partial z}(\mathbf{0}, 1) \quad \text{for } 1 \leq l,\ m \leq d, \\ B &= (b_{lm})_{1\leq l,m\leq d} \quad \text{and} \quad D = \sum_{j,k\in\Xi} \pi_j q_{jk} C_{jk} + B. \end{aligned} \tag{1.5}$$

In Lemma 6.9, we will show that the matrix D is positive definite if the transition function $P(x, y)$ satisfies Assumptions 1.1 through 1.4. Let $\#\Xi$ denote the cardinarity of the set Ξ. Now let us state our result.

Theorem 1.1. *Suppose that the transition function $P(x, y)$ satisfies* Assumptions 1.1 *through* 1.4. *Then*

$$\lim_{n\to\infty} \left((2\pi n)^{d/2} P_n(x, y) - (\#\Xi)|D|^{-1/2} \exp\left\{ -\frac{1}{2n}\big(y - x, D^{-1}(y - x)\big) \right\} \pi_{T(y)} \right) = 0 \tag{1.6}$$

uniformly for all $x,\ y \in \boldsymbol{Z}^d$.

Theorem 1.2. *Suppose that the transition function $P(x, y)$ satisfies* Assumptions 1.1 *through* 1.4. *Then*

$$\begin{aligned} \lim_{n\to\infty} &\frac{|y - x|^2}{n} \\ &\times \left((2\pi n)^{d/2} P_n(x, y) - (\#\Xi)|D|^{-1/2} \exp\left\{ -\frac{1}{2n}\big(y - x, D^{-1}(y - x)\big) \right\} \pi_{T(y)} \right) = 0 \end{aligned} \tag{1.7}$$

uniformly for all $x,\ y \in \boldsymbol{Z}^d$.

First we shall prove the relations (1.6) and (1.7) under additional assumptions given below and thereafter remove them.

ASSUMPTION 1.5. For some $j,\ k \in \Xi$ for which $q_{jk} > 0$, C_{jk} is positive definite.

Let $\phi_{jk}(\cdot)$, $j, k \in \Xi$, denote the characteristic function of $F_{jk}(\cdot)$, that is,

$$\phi_{jk}(\theta) = \sum_{x \in \mathbf{Z}^d} e^{i(\theta,x)} F_{jk}(x) \quad \text{for } j, \ k \in \Xi, \ \theta \in \boldsymbol{R}^d.$$

ASSUMPTION 1.6. On $[-\pi/s_1, \pi/s_1] \times \cdots \times [-\pi/s_d, \pi/s_d]$, $\prod_{j,k\in\Xi} |\phi_{jk}(\theta)|$ equals 1 if and only if $\theta = \mathbf{0}$.

Lemma 1.1. *Under* Assumptions 1.1 *through* 1.6, *the formula* (1.6) *holds.*

Lemma 1.2. *Under* Assumptions 1.1 *through* 1.6, *the formula* (1.7) *holds.*

For each $j, k \in \Xi$, let $\{Y_n^{jk}\}_{n\geq 1}$ be a family of independent identically distributed random vectors, and $\{\chi_n\}_{n\geq 0}$ be an ergodic Markov chain with a finite state space Ξ. Assume that $\{Y_n^{jk}\}_{n\geq 1}^{j,k\in\Xi}$ and $\{\chi_n\}_{n\geq 1}$ are mutually independent. Set $X_n = Y_1^{\chi_0\chi_1} + \cdots + Y_n^{\chi_{n-1}\chi_n}$. Then such a process may be called *a random walk defined on a finite Markov chain*. By Lemma 2.1 in Section 2, we will show that *RWPE* may be realized as such a process. In 1-dimensional case, Miller [8] studied an asymptotic behaviour of $\mathbf{P}\{X_n = x \mid \chi_n = j, \chi_0 = k\}$, and Keilson and Wishart [5] proved the central limit theorem of the process.

In [7] Kotani gave a Martingale approach to the central limit theorem and related problem for a class of periodic Markov chains. Kotani, Shirai and Sunada [6] considered local limit theorem for a class of Markov chains on an infinite graph satisfying a certain periodic condition. They treated the reversible Markov chain with the property that a particle at a given site can move to only finitely many sites in one unit of time.

In Section 2, in order to prove Lemma 1.1, we introduce the sequence of lemmas. In Section 3, we prove Lemma 1.1. In Section 4, we give some lemmas on which our proof of Lemma 1.2 is based. In Section 5, we prove Lemma 1.2. In Section 6, we give some lemmas for Theorems 1.1 and 1.2. In Section 7, we prove these theorems, extending Lemmas 1.1 and 1.2.

2. Some Lemmas for Lemma 1.1

In this section, we introduce some lemmas on which our proofs of Lemmas 1.1 and 1.2 are based.

Lemma 2.1. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumption 1.1. *Then for all* n, $n \geq 1$, *and* $x, y \in \mathbf{Z}^d$, *we have*

$$P_n(x, y) = \sum_{j_1,\ldots,j_{n-1}\in\Xi} q_{T(x)j_1} q_{j_1 j_2} \cdots q_{j_{n-1}T(y)} F_{T(x)j_1} * F_{j_1 j_2} * \cdots * F_{j_{n-1}T(y)}(y - x), \tag{2.1}$$

where $F * G$ is the convolution of F and G.

Lemma 2.1 is suggested by Shiga. See Chapter 7 of his book [9].

By Lemma 2.1 and the inversion formula for Fourier transform, $P_n(x, y)$ equals

(2.2)
$$\sum_{j_1,\dots,j_{n-1}\in\Xi} q_{T(x)j_1}\cdots q_{j_{n-1}T(y)} \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-i(\theta,y-x)}\phi_{T(x)j_1}(\theta)\cdots\phi_{j_{n-1}T(y)}(\theta)\,d\theta.$$

Then by Assumption 1.1, we have the following lemma.

Lemma 2.2. *Suppose that the transition function $P(x, y)$ satisfies* Assumption 1.1. *Then for all $n \geq 1$ and $x,\ y \in \boldsymbol{Z}^d$ we have*

$$P_n(x, y) = \sum_{j_1,\dots,j_{n-1}\in\Xi} q_{T(x)j_1}\cdots q_{j_{n-1}T(y)} \times \frac{(\#\Xi)}{(2\pi)^d}\int_{[-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}]} e^{-i(\theta,y-x)}\phi_{T(x)j_1}(\theta)\cdots\phi_{j_{n-1}T(y)}(\theta)\,d\theta. \tag{2.3}$$

Proof. By Assumption 1.1, we have $\phi_{jk}(\theta+(2\pi/s_l)\boldsymbol{e}_l) = \exp\{i(2\pi/s_l)(k_l - j_l)\} \times \phi_{jk}(\theta)$. By applying this formula to (2.2), we obtain (2.3). □

Denote by $\{\xi_n\}_{n\geq 0}$ the Markov chain on Ξ with the transition matrix Q. Set

$$N_n^{jk} = \#\{1 \leq m \leq n \mid \xi_{m-1} = j, \xi_m = k\},$$
$$M_{n;l} = \sum_{j,k\in\Xi} \mu_{jk;l} N_n^{jk} \quad \text{and} \quad M_n = (M_{n;1}, M_{n;2}, \dots, M_{n;d}).$$

Put $\psi_{jk}(\theta) = \phi_{jk}(\theta)e^{-i(\theta,\mu_{jk})}$ for $j,\ k \in \Xi$. Then we have

$$P_n(x, y) = \frac{(\#\Xi)}{(2\pi)^d}\int_{[-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}]} \exp\{-i(\theta, y-x)\} \times \boldsymbol{E}\left[\prod_{j,k\in\Xi}\psi_{jk}(\theta)^{N_n^{jk}}\exp\{i(\theta, M_n)\}; \xi_n = T(y)\ \middle|\ \xi_0 = T(x)\right] d\theta. \tag{2.4}$$

It follows from the weak law of large numbers for ergodic Markov chains that, for j, $k \in \Xi$,

$$\frac{N_n^{jk}}{n} \to \pi_j q_{jk} \tag{2.5}$$

in probability as $n \to \infty$. Moreover we have the following large deviation type estimate. Set

$$A_{n\zeta} = \bigcap_{j,k\in\Xi} \left\{ \left| \frac{N_n^{jk}}{n} - \pi_j q_{jk} \right| < \zeta \right\}. \tag{2.6}$$

Lemma 2.3. *Suppose that the transition function $P(x,y)$ satisfies* Assumptions 1.1 *and* 1.2*. Then for all $\zeta > 0$, we have*

$$\boldsymbol{P}\{A_{n\zeta}^c\} \leq Ke^{-Ln} \quad \text{for } j,\ k \in \Xi, \tag{2.7}$$

where K and L are positive constants depending on ζ but not on n.

See, e.g., Dembo and Zeitouni [2, p. 64].

Recall Assumption 1.3 and (2.5). We have the following central limit theorem for M_n.

Lemma 2.4. *Suppose that the transition function $P(x,y)$ satisfies* Assumptions 1.1 *through* 1.3*. Then*

$$\begin{aligned} \lim_{n\to\infty} \boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = k \,\middle|\, \xi_0 = j\right] \\ = \exp\left\{-\frac{1}{2}(w, Bw)\right\}\pi_k \quad \text{for } j,\ k \in \Xi, \end{aligned} \tag{2.8}$$

where the matrix B is non-negative definite.

Proof. For a proof in the 1-dimensional case, see Hatori and Mori [3, p. 124]. We will show (2.8) in the multi-dimensional case.

We denote by $q_{jk}^{(n)}(\theta)$, $\theta \in \boldsymbol{R}^d$, the component of the matrix $Q(\theta)^n$. Note that we have

$$q_{jk}^{(n)}(\theta) = \boldsymbol{E}\left[\exp\{i(\theta, M_n)\};\ \xi_n = k \mid \xi_0 = j\right]. \tag{2.9}$$

We will show $\lim_{n\to\infty} q_{jk}^{(n)}(w/\sqrt{n}) = \exp\{(-1/2)(w, Bw)\}\pi_k$. For every z with $|z| < 1$, $z \in \boldsymbol{C}$, we have $\sum_{n=0}^{\infty} Q(\theta)^n z^n = (I - zQ(\theta))^{-1}$. We denote by $R(\theta, z) = (r_{jk}(\theta, z))_{j,k\in\Xi}$ the co-factor matrix of $I - zQ(\theta)$, so that

$$\sum_{n=0}^{\infty} q_{jk}^{(n)}(\theta) z^n = \frac{r_{jk}(\theta, z)}{|I - zQ(\theta)|}. \tag{2.10}$$

Let $\kappa_\nu(\theta)$, $\nu = 1, 2, \ldots, \#\Xi$, denote the eigenvalues of $Q(\theta)$. Since $Q(\mathbf{0}) = Q$, we may take $\kappa_1(\mathbf{0}) = 1$ and $\max_{2\leq\nu\leq\Xi} |\kappa_\nu(\mathbf{0})| < 1$. Moreover there exist a neibourhood $\boldsymbol{U}$

of $\theta = \mathbf{0}$ and constant ρ, $0 < \rho < 1$, such that $\kappa_1(\theta)$ is analytic in $\boldsymbol{U}$ (see, Bochner and Martin [1, p. 39]) and $\kappa_\nu(\theta)$, $1 \le \nu \le \#\Xi$, are continuous in $\boldsymbol{R}^d$ (see, Takagi [11, p. 56]) and

$$\inf_{\theta\in U} |\kappa_1(\theta)| > \rho \quad \text{and} \quad \sup_{\theta\in U} |\kappa_\nu(\theta)| < \rho \quad \text{for } \nu = 2, \dots, \#\Xi. \tag{2.11}$$

Since $f(\theta, \kappa_\nu(\theta)) = 0$, $\nu = 1, 2, \dots, \#\Xi$, we may write

$$|\boldsymbol{I} - z\boldsymbol{Q}(\theta)| = z^{(\#\Xi)} f\left(\theta, \frac{1}{z}\right) = \big(1 - \kappa_1(\theta)z\big) g(\theta, z),$$

where $g(\theta, z)$ is a polynomial of degree $\#\Xi - 1$ in z, and $g(\theta, 1/\kappa_\nu(\theta)) = 0$ if $\kappa_\nu(\theta) \neq 0$, $\nu = 2, \dots, \#\Xi$. Thus we have

$$\frac{r_{jk}(\theta, z)}{|\boldsymbol{I} - z\boldsymbol{Q}(\theta)|} = \frac{r_{jk}(\theta, z)}{(1 - \kappa_1(\theta)z)g(\theta, z)} = \frac{\sigma_{jk}(\theta)}{1 - \kappa_1(\theta)z} + \frac{\tau_{jk}(\theta, z)}{g(\theta, z)}, \tag{2.12}$$

where $\sigma_{jk}(\theta) = r_{jk}(\theta, 1/\kappa_1(\theta))/g(\theta, 1/\kappa_1(\theta))$ and $\tau_{jk}(\theta, z)$ is a polynomial of degree $\#\Xi - 2$ in z. Put

$$u(\theta) = \max_{2\le\nu\le\#\Xi} |\kappa_\nu(\theta)|. \tag{2.13}$$

Then $\tau_{jk}(\theta, z)/g(\theta, z)$ is analytic in z for $|z| < 1/u(\theta)$. Thus we may write

$$\frac{\tau_{jk}(\theta, z)}{g(\theta, z)} = \sum_{m=0}^{\infty} c_m^{jk}(\theta) z^m, \quad \text{where} \quad c_m^{jk}(\theta) = \frac{1}{m!} \frac{\partial^m}{\partial z^m} \left\{ \frac{\tau_{jk}(\theta, z)}{g(\theta, z)} \right\}_{z=0}.$$

Since this series has the radius $1/u(\theta)$ of convergence, we have

$$u(\theta) = \limsup_{m\to\infty} |c_m^{jk}(\theta)|^{1/m}.$$

Therefore by (2.10), (2.11), (2.12) and (2.13), we have

$$q_{jk}^{(n)}(\theta) = \sigma_{jk}(\theta)\kappa_1(\theta)^n + c_n^{jk}(\theta) \quad \text{and} \quad c_n^{jk}(\theta) = o\big(\kappa_1(\theta)^n\big). \tag{2.14}$$

Since $(\partial f/\partial z)(\mathbf{0}, 1) \neq 0$ by (1.4), we may use the implicit function theorem for $\kappa_1(\theta)$ to have

$$\kappa_1(\theta) = 1 + \sum_{l=1}^{d} \frac{\partial \kappa_1}{\partial \theta_l}(\mathbf{0})\theta_l + \frac{1}{2} \sum_{l,m=1}^{d} \frac{\partial^2 \kappa_1}{\partial \theta_l \partial \theta_m}(\mathbf{0})\theta_l\theta_m + o(|\theta|^2), \tag{2.15}$$

where

$$\frac{\partial \kappa_1}{\partial \theta_l}(\mathbf{0}) = -\frac{\partial f}{\partial \theta_l}(\mathbf{0}, 1) \bigg/ \frac{\partial f}{\partial z}(\mathbf{0}, 1) \tag{2.16}$$

and

$$\frac{\partial^2\kappa_1}{\partial\theta_l\partial\theta_m}(\mathbf{0}) = -\frac{\frac{\partial^2 f}{\partial\theta_l\partial\theta_m}(\mathbf{0},1)}{\frac{\partial f}{\partial z}(\mathbf{0},1)} + \frac{\frac{\partial f}{\partial\theta_l}(\mathbf{0},1)\frac{\partial^2 f}{\partial\theta_m\partial z}(\mathbf{0},1)}{(\frac{\partial f}{\partial z}(\mathbf{0},1))^2} + \frac{\frac{\partial^2 f}{\partial\theta_l\partial z}(\mathbf{0},1)\frac{\partial f}{\partial\theta_m}(\mathbf{0},1)}{(\frac{\partial f}{\partial z}(\mathbf{0},1))^2} - \frac{\frac{\partial f}{\partial\theta_l}(\mathbf{0},1)\frac{\partial f}{\partial\theta_l}(\mathbf{0},1)\frac{\partial^2 f}{\partial z^2}(\mathbf{0},1)}{(\frac{\partial f}{\partial z}(\mathbf{0},1))^3}. \tag{2.17}$$

By (2.14), (2.15) and (2.16),

$$\lim_{n\to\infty} q_{jk}^{(n)}\left(\frac{w}{n}\right) = \sigma_{jk}(\mathbf{0}) \exp\left\{-\sum_{l=1}\left(\frac{\partial f}{\partial\theta_l}(\mathbf{0},1)\Big/\frac{\partial f}{\partial z}(\mathbf{0},1)\right) w_l\right\} \tag{2.18}$$

for every $w \in \boldsymbol{R}^d$. Set $w = \mathbf{0}$ in (2.18), then by (1.1) we obtain

$$\sigma_{jk}(\mathbf{0}) = \pi_k. \tag{2.19}$$

We show $(\partial f/\partial\theta_l)(\mathbf{0}, 1) = 0$ for all l, $1 \le l \le d$. Note that

$$\lim_{n\to\infty} \frac{M_{n;l}}{n} = \sum_{j,k\in\Xi} \mu_{jk;l}\pi_j q_{jk} \quad \text{in probability}, \ 1 \le l \le d. \tag{2.20}$$

By (1.2), the right hand side of (2.20) equals 0. Therefore

$$\lim_{n\to\infty} \boldsymbol{E}\left[\exp\left\{\boldsymbol{i}\left(w, \frac{M_n}{n}\right)\right\}; \ \xi_n = k \,\Big|\, \xi_0 = j\right] = \pi_k \quad \text{for all } w \in \boldsymbol{R}^d.$$

By (2.18) and (2.19),

$$\exp\left\{-\sum_{l=1}^{d}\left(\frac{\partial f}{\partial\theta_l}(\mathbf{0},1)\Big/\frac{\partial f}{\partial z}(\mathbf{0},1)\right) w_l\right\} = 1 \quad \text{for all } w \in \boldsymbol{R}^d,$$

so that $(\partial f/\partial\theta_l)(\mathbf{0}, 1) = 0$ for all l, $1 \le l \le d$. Thus we have from (2.15), (2.16) and (2.17)

$$\kappa_1(\theta) = 1 - \frac{1}{2}(\theta, B\theta) + o(|\theta|^2). \tag{2.21}$$

Substitute (2.21) to (2.14) and set $\theta = w/\sqrt{n}$, then we obtain (2.8). □

It follows from the central limit theorem for sums of i.i.d. random variables that

$$\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}} \to \exp\left\{-\frac{1}{2}(w, \pi_j q_{jk} C_{jk} w)\right\}. \tag{2.22}$$

By (2.8) and (2.22), we have the following lemma.

Lemma 2.5. *If the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.4, *then*

$$\lim_{n\to\infty} \prod_{j,k\in\Xi} \psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}} \boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = k' \,\middle|\, \xi_0 = j'\right]$$
$$= \exp\left\{-\frac{1}{2}(w, Dw)\right\}\pi_{k'}$$

for $w \in \boldsymbol{R}^d,\ j',\ k' \in \Xi$.

Using P 7.4 and P 7.7 of Spitzer [10], we have the following lemma.

Lemma 2.6. *If the transition function* $P(x, y)$ *satisfies* Assumptions 1.1, 1.4 *and* 1.5, *then there exist* $j, k \in \Xi$ *and positive constants* δ *and* λ *such that* $|\psi_{jk}(\theta)| \le e^{-\lambda|\theta|^2}$ *when* $|\theta| < \delta$.

By Assumption 1.4 and Maclaurin expansion for $\psi_{jk}(\theta)$, we have the following lemma.

Lemma 2.7. *Suppose that the transition function* $P(x,y)$ *satisfies Assumptions* 1.1, 1.2 *and* 1.4. *There exist positive constants* δ *and* a *such that for every* ζ, $0 < \zeta < 1$,

$$\left|1 - \prod_{j,k\in\Xi} \psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}(\omega) - n\pi_j q_{jk}}\right| \le a\zeta|w|^2 \exp\{a\zeta|w|^2\} \tag{2.23}$$

when $|w/\sqrt{n}| < \delta$ *and* $\omega \in \boldsymbol{A}_{n\zeta}$.

3. Proof of Lemma 1.1

We will prove Lemma 1.1. Suppose that $P(x, y)$ satisfies Assumptions 1.1 through 1.6 in this section. Then by Assumption 1.5 and Lemma 2.4, D is positive definite. We will show the formula (1.6). Take $0 < \alpha,\ \zeta < \infty$. Let δ be positive constant satisfying Lemmas 2.6 and 2.7. Set $w = \sqrt{n}\theta$. We may write

$$\frac{(2\pi n)^{d/2}}{(\#\Xi)} P_n(x, y)$$
$$= \frac{1}{(2\pi)^{d/2}} \int_{\sqrt{n}([-\frac{\pi}{s_1}, \frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d}, \frac{\pi}{s_d}])} \exp\left\{-i\frac{1}{\sqrt{n}}(w, y - x)\right\}$$

$$
\begin{aligned}
&\times \boldsymbol{E}\left[\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}}\right)\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=T(y)\ \middle|\ \xi_0=T(x)\right]dw\\
={}& I_0(n)+I_1(n,\alpha)+I_2(n,\alpha)+I_3(n,\alpha,\delta)+I_4(n,\delta,\zeta)\\
&+I_5(n,\delta,\zeta)+I_6(n,\delta,\zeta)+I_7(n,\delta,\zeta),
\end{aligned}
$$

where

$$
\begin{aligned}
I_0(n) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\exp\left\{-\frac{1}{2}(w,Dw)\right\}\pi_{T(y)}\,dw,\\
I_1(n,\alpha) &= -\frac{1}{(2\pi)^{d/2}}\int_{|w|>\alpha}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\exp\left\{-\frac{1}{2}(w,Dw)\right\}\pi_{T(y)}\,dw,\\
I_2(n,\alpha) &= \frac{1}{(2\pi)^{d/2}}\int_{|w|\le\alpha}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\left(\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\right.\\
&\qquad\times\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=T(y)\ \middle|\ \xi_0=T(x)\right]\\
&\qquad\left.-\exp\left\{-\frac{1}{2}(w,Dw)\right\}\pi_{T(y)}\right)dw,\\
I_3(n,\alpha,\delta) &= \frac{1}{(2\pi)^{d/2}}\int_{\alpha<|w|\le\sqrt{n}\delta}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\\
&\qquad\times\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=T(y)\ \middle|\ \xi_0=T(x)\right]dw,\\
I_4(n,\delta,\zeta) &= -\frac{1}{(2\pi)^{d/2}}\int_{|w|\le\sqrt{n}\delta}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\\
&\qquad\times\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=T(y),\boldsymbol{A}_{n\zeta}^c\ \middle|\ \xi_0=T(x)\right]dw,\\
I_5(n,\delta,\zeta) &= -\frac{1}{(2\pi)^{d/2}}\int_{|w|\le\sqrt{n}\delta}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\\
&\qquad\times\boldsymbol{E}\left[\left(1-\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}-n\pi_j q_{jk}}\right)\right)\right.\\
&\qquad\times\left.\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=T(y),\boldsymbol{A}_{n\zeta}\ \middle|\ \xi_0=T(x)\right]dw,\\
I_6(n,\delta,\zeta) &= \frac{1}{(2\pi)^{d/2}}\int_{|w|>\sqrt{n}\delta;\sqrt{n}([-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}])}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\\
&\qquad\times\boldsymbol{E}\left[\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}}\right)\right.
\end{aligned}
$$

$$\times \exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = T(y), \boldsymbol{A}_{n\zeta}\ \Big|\ \xi_0 = T(x)\Big]\, dw$$

and

$$I_7(n,\delta,\zeta) = \frac{1}{(2\pi)^{d/2}} \int_{\sqrt{n}([-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}])} \exp\left\{-i\frac{1}{\sqrt{n}}(w, y-x)\right\}$$
$$\times \boldsymbol{E}\Bigg[\Bigg(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}}\Bigg)$$
$$\times \exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = T(y), \boldsymbol{A}^c_{n\zeta}\ \Big|\ \xi_0 = T(x)\Bigg]\, dw.$$

A direct calculation shows that

$$I_0(n) = |D|^{-1/2}\exp\left\{-\frac{1}{2n}\big(y-x, D^{-1}(y-x)\big)\right\}\pi_{T(y)}.$$

It remains to show that the terms $I_1, I_2, \ldots, I_7$ go to zero uniformly in x, y as $n \to \infty$. We have

$$|I_1(n,\alpha)| \le (2\pi)^{-d/2}\int_{|w|>\alpha} \exp\left\{-\frac{1}{2}(w, Dw)\right\}\pi_{T(y)}\, dw,$$

which can be made arbitary small by taking α sufficiently large.

By Lemma 2.5 and Lebesgue's dominated convergence theorem, for every α

$$|I_2(n,\alpha)| = (2\pi)^{-d/2}\int_{|w|\le\alpha}\Bigg|\Bigg(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\Bigg)$$
$$\times \boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = T(y)\ \Big|\ \xi_0 = T(x)\right]$$
$$-\exp\left\{-\frac{1}{2}(w, Dw)\right\}\pi_{T(y)}\Bigg|\, dw \to 0 \quad \text{as } n\to\infty.$$

By Lemma 2.6, $|I_3(n,\alpha,\delta)| \le \int_{\alpha<|w|} e^{-\lambda|w|^2}dw$, which can be made arbitary small by taking α sufficientry large. By Lemma 2.3, $|I_4(n,\delta,\zeta)| \le K(2\delta\sqrt{n})^d e^{-Ln}$, where K and L are positive constants in Lemma 2.3. Note Lemmas 2.6 and 2.7. Then we have

$$|I_5(n,\delta,\zeta)| \le (2\pi)^{-d/2} a\zeta \int_{\boldsymbol{R}^d} |w|^2 e^{-(\lambda - a\zeta)|w|^2}\, dw,$$

which can be made arbitary small by taking ζ sufficiently small.

Let $\beta = \min\{\pi_j q_{jk} \mid q_{jk} > 0,\ \ j,\ \ k \in \Xi\}$ and choose ζ, $0 < \zeta < \beta/2$. Then by Assumption 1.6 there exists a positive constant γ, $0 < \gamma < 1$, such that

$$\prod_{j,k\in\Xi} \left|\phi_{jk}\left(\frac{w}{\sqrt{n}}\right)\right|^{N_n^{jk}(\omega)} < (1-\gamma)^{n\beta/2}$$

when

$$w \in \sqrt{n}\left(\left[-\frac{\pi}{s_1}, \frac{\pi}{s_1}\right] \times \cdots \times \left[-\frac{\pi}{s_d}, \frac{\pi}{s_d}\right]\right), \quad \left|\frac{w}{\sqrt{n}}\right| > \delta \quad \text{and} \quad \omega \in \boldsymbol{A}_{n\zeta}.$$

Hence $|I_6(n,\delta,\zeta)| \le (2\pi n)^{d/2}(1-\gamma)^{n\beta/2}$. By analougous way to $I_4(n,\delta)$, $|I_7(n,\delta,\zeta)| \le (2\pi n)^{d/2} K e^{-Ln}$.

The proof of Lemma 1.1 is complete. □

4. Some Lemmas for Lemma 1.2

We introduce the d-dimensional Laplacian $\Delta_\theta = \sum_{l=1}^d (\partial^2/\partial\theta_l^2)$.

Lemma 4.1. *Suppose that the transition function* $P(x,y)$ *satisfies* Assumptions 1.1 *and* 1.4*. Then we have*

$$\begin{aligned} |x-y|^2 P_n(x,y) = &-\frac{(\#\Xi)}{(2\pi)^d}\int_{[-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}]} e^{-i(\theta,y-x)} \\ &\times\Delta_\theta\left\{\boldsymbol{E}\left[\left(\prod_{j,k\in\Xi}\psi_{jk}(\theta)^{N_n^{jk}}\right)e^{i(\theta,M_n)};\ \xi_n = T(y) \,\middle|\, \xi_0 = T(x)\right]\right\} d\theta \end{aligned}$$

for all $n \ge 1$ *and* $x,\ y \in \boldsymbol{Z}^d$.

Proof. Using the formula for integration by parts and Assumptions 1.1 and 1.4, we have

$$\begin{aligned} &(y_l - x_l)^2 \int_{-\pi/s_l}^{\pi/s_l} e^{-i(\theta,y-x)} \phi_{T(x)j_1}(\theta)\phi_{j_1j_2}(\theta)\cdots\phi_{j_{n-1}T(y)}(\theta)\, d\theta_l \\ &= -\int_{-\pi/s_l}^{\pi/s_l} e^{-i(\theta,y-x)} \frac{\partial^2}{\partial\theta_l^2}\{\phi_{T(x)j_1}(\theta)\phi_{j_1j_2}(\theta)\cdots\phi_{j_{n-1}T(y)}(\theta)\}\, d\theta_l \end{aligned}$$

for l, $1 \le l \le d$. Thus we obtain the relation of the lemma. □

Lemma 4.2. *Suppose that the transition function* $P(x,y)$ *satisfies* Assumptions 1.1 *through* 1.3. *Then*

$$\lim_{n\to\infty}\frac{\partial}{\partial w_l}\left\{\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=k\ \Big|\ \xi_0=j\right]\right\} = \frac{\partial}{\partial w_l}\left\{\exp\left\{-\frac{1}{2}(w,Bw)\right\}\pi_k\right\} \tag{4.1}$$

and

$$\lim_{n\to\infty}\frac{\partial^2}{\partial w_m\partial w_l}\left\{\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=k\ \Big|\ \xi_0=j\right]\right\} = \frac{\partial^2}{\partial w_m\partial w_l}\left\{\exp\left\{-\frac{1}{2}(w,Bw)\right\}\pi_k\right\}. \tag{4.2}$$

for $w\in\boldsymbol{R}^d$, $j,\ k\in\Xi$ *and* $1\le l,\ m\le d$.

Proof. Differentiate each side of (2.10). Thus by arguments similar to that made for the proof of Lemma 2.4, we obtain (4.1) and (4.2). □

As in Spitzer [10, p. 80], we have

$$\lim_{n\to\infty}\frac{\partial}{\partial w_l}\left\{\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right\} = \frac{\partial}{\partial w_l}\left\{\exp\left\{-\frac{1}{2}(w,\pi_j q_{jk}C_{jk}w)\right\}\right\} \tag{4.3}$$

and

$$\lim_{n\to\infty}\frac{\partial^2}{\partial w_l^2}\left\{\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right\} = \frac{\partial^2}{\partial w_l^2}\left\{\exp\left\{-\frac{1}{2}(w,\pi_j q_{jk}C_{jk}w)\right\}\right\}, \tag{4.4}$$

$1\le l\le d$. By Lemma 4.2, (4.3) and (4.4), we have the following lemma.

Lemma 4.3. *Suppose that the transition function* $P(x,y)$ *satisfies* Assumptions 1.1 *through* 1.4. *Then*

$$\lim_{n\to\infty}\Delta_w\left\{\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n=k'\ \Big|\ \xi_0=j'\right]\right\} = \Delta_w\left\{\exp\left\{-\frac{1}{2}(w,Dw)\right\}\pi_{k'}\right\}$$

for $w\in\boldsymbol{R}^d$, $j',\ k'\in\Xi$.

Lemma 4.4. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.5. *There exist positive constants* δ, λ *and* b_1 *such that*

$$\left| \frac{\partial}{\partial w_l} \left\{ \prod_{j,k\in\Xi} \psi_{jk} \left(\frac{w}{\sqrt{n}} \right)^{n\pi_j q_{jk}} \right\} \right| \le b_1 |w| \exp\{-\lambda |w|^2\} \tag{4.5}$$

and

$$\left| \frac{\partial^2}{\partial w_l^2} \left\{ \prod_{j,k\in\Xi} \psi_{jk} \left(\frac{w}{\sqrt{n}} \right)^{n\pi_j q_{jk}} \right\} \right| \le b_1 (1 + |w|^2) \exp\{-\lambda |w|^2\} \tag{4.6}$$

for all l, $1 \le l \le d$, *when* $|w/\sqrt{n}| < \delta$.

Proof. See, for a proof, Spitzer [10, p. 81].

As in the proof of Lemma 2.7, we have the following lemma.

Lemma 4.5. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1, 1.2 *and* 1.4. *There exist positive constans* δ *and* b_2 *such that for every* ζ, $0 < \zeta < 1$, *we have*

$$\left| \frac{\partial}{\partial w_l} \left\{ 1 - \prod_{j,k\in\Xi} \psi_{jk} \left(\frac{w}{\sqrt{n}} \right)^{N_n^{jk}(\omega) - n\pi_j q_{jk}} \right\} \right| \le b_2 \zeta |w| \exp\{b_2 \zeta |w|^2\} \tag{4.7}$$

and

$$\left| \frac{\partial^2}{\partial w_l^2} \left\{ 1 - \prod_{j,k\in\Xi} \psi_{jk} \left(\frac{w}{\sqrt{n}} \right)^{N_n^{jk}(\omega) - n\pi_j q_{jk}} \right\} \right| \le b_2 \zeta (1 + |w|^2) \exp\{b_2 \zeta |w|^2\} \tag{4.8}$$

when $|w/\sqrt{n}| < \delta$ *and* $\omega \in \boldsymbol{A}_{n\zeta}$.

By analogous way to Lemma 4.2, we have the following lemma.

Lemma 4.6. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.3. *There exist positive constants* δ *and* b_3 *such that for* $1 \le l \le d$, j, $k \in \Xi$,

$$\left| \frac{\partial}{\partial w_l} \left\{ \boldsymbol{E} \left[\exp\left\{ i \frac{1}{\sqrt{n}} (w, M_n) \right\} ;\ \xi_n = k \;\middle|\; \xi_0 = j \right] \right\} \right| \le b_3 (|w| + 1) \tag{4.9}$$

and

$$\left| \frac{\partial^2}{\partial w_l^2} \left\{ \boldsymbol{E} \left[\exp\left\{ i \frac{1}{\sqrt{n}} (w, M_n) \right\} ;\ \xi_n = k \;\middle|\; \xi_0 = j \right] \right\} \right| \le b_3 (|w|^2 + 1) \tag{4.10}$$

when $|w/\sqrt{n}| < \delta$.

Lemma 4.7. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *and* 1.2. *Then there exists positive constant* b_4 *such that for every* $\zeta > 0$

$$\left|\frac{\partial}{\partial w_l}\left\{\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = k, \boldsymbol{A}^c_{n\zeta}\ \middle|\ \xi_0 = j\right]\right\}\right| \le b_4 K\sqrt{n}\exp\{-Ln\} \tag{4.11}$$

and

$$\left|\frac{\partial^2}{\partial w_l^2}\left\{\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = k, \boldsymbol{A}^c_{n\zeta}\ \middle|\ \xi_0 = j\right]\right\}\right| \le b_4 K n\exp\{-Ln\}, \tag{4.12}$$

where K *and* L *are positive constants given in* Lemma 2.3.

Proof. Note that $|M_{n;l}| \le \text{constant} \times n$ for all $\omega \in \Omega$ and l, $l \le l \le d$. Thus by Lemma 2.3, we obtain (4.11) and (4.12). □

Lemma 4.8. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1, 1.2, 1.4 *and* 1.6. *Let* $\beta = \min\{\pi_j q_{jk} \mid q_{jk} > 0,\ \ j,\ k \in \Xi\}$ *and* $\zeta < \beta/2$. *Then for every* δ, $0 < \delta < \min_{1\le l\le d} \pi/s_l$, *there exist positive constants* b_5 *and* γ, $0 < \gamma < 1$, *such that*

$$\left|\frac{\partial^2}{\partial w_l^2}\left\{\prod_{j,k\in\Xi}\phi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}(\omega)}\right\}\right| \le b_5 n(1-\gamma)^{(1/4)\beta n} \tag{4.13}$$

when $|w| \ge \delta\sqrt{n}$, $w \in \sqrt{n}([-\pi/s_1, \pi/s_1] \times \cdots \times [-\pi/s_d, \pi/s_d])$ *and* $\omega \in \boldsymbol{A}_{n\zeta}$.

See, for a proof, Spitzer [10, p. 81].

5. Proof of Lemma 1.2

Suppose that the transition function $P(x, y)$ satisfies Assumption 1.1 through 1.6. Set $w = \sqrt{n}\,\theta$ in Lemma 4.1, then we have

(5.1)

$$\begin{aligned}
&\frac{|y-x|^2}{n}\frac{(2\pi n)^{d/2}}{(\#\Xi)}P_n(x, y)\\
&= -\frac{1}{(2\pi)^{d/2}}\int_{\sqrt{n}([-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}])}\exp\left\{-i\frac{1}{\sqrt{n}}(w, y-x)\right\}\\
&\times\Delta_w\left\{\boldsymbol{E}\left[\left(\prod_{j,k\in\Xi}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}}\right)\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = T(y)\ \middle|\ \xi_0 = T(x)\right]\right\}dw.
\end{aligned}$$

Take $0 < \alpha,\ \zeta < \infty$ and $0 < \delta < \min_{1\le l\le d} \pi/s_l$. Decompose the right hand side of (5.1) as follows:

$$\frac{|y-x|^2}{n}\frac{(2\pi n)^{d/2}}{(\#\boldsymbol{\Xi})}P_n(x,y) = J_0(n) + J_1(n,\alpha) + J_2(n,\alpha) + J_3(n,\alpha,\delta) + J_4(n,\delta,\zeta) + J_5(n,\delta,\zeta) + J_6(n,\delta,\zeta) + J_7(n,\delta,\zeta),$$

where

$$J_0(n) = -\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\Delta_w\left\{\exp\left\{-\frac{1}{2}(w,Dw)\right\}\right\}\pi_{T(y)}\,dw,$$

$$J_1(n,\alpha) = \frac{1}{(2\pi)^{d/2}}\int_{|w|>\alpha}\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\Delta_w\left\{\exp\left\{-\frac{1}{2}(w,Dw)\right\}\right\}\pi_{T(y)}\,dw,$$

$$\begin{aligned}J_2(n,\alpha) = -\frac{1}{(2\pi)^{d/2}}\int_{|w|\le\alpha}&\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\left(\Delta_w\left\{\left(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\right.\right.\\&\times\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n = T(y)\ \middle|\ \xi_0 = T(x)\right]\\&\left.\left.-\exp\left\{-\frac{1}{2}(w,Dw)\right\}\pi_{T(y)}\right\}\right)dw,\end{aligned}$$

$$\begin{aligned}J_3(n,\alpha,\delta) = -\frac{1}{(2\pi)^{d/2}}\int_{\alpha<|w|\le\sqrt{n}\delta}&\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\\&\times\Delta_w\left\{\left(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\right.\\&\left.\times\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n = T(y)\ \middle|\ \xi_0 = T(x)\right]\right\}dw,\end{aligned}$$

$$\begin{aligned}J_4(n,\delta,\zeta) = \frac{1}{(2\pi)^{d/2}}\int_{|w|\le\sqrt{n}\delta}&\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\\&\times\Delta_w\left\{\left(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\right.\\&\left.\times\boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w,M_n)\right\};\ \xi_n = T(y),\boldsymbol{A}_{n\zeta}^c\ \middle|\ \xi_0 = T(x)\right]\right\}dw,\end{aligned}$$

$$\begin{aligned}J_5(n,\delta,\zeta) = \frac{1}{(2\pi)^{d/2}}\int_{|w|\le\sqrt{n}\delta}&\exp\left\{-i\frac{1}{\sqrt{n}}(w,y-x)\right\}\\&\times\Delta_w\left\{\left(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\right)\right.\\&\times\boldsymbol{E}\left[\left(1-\left(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}-n\pi_j q_{jk}}\right)\right)\right.\end{aligned}$$

$$\times \exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = T(y), \boldsymbol{A}_{n\zeta}\ \Bigg|\ \xi_0 = T(x)\Bigg]\Bigg\}\, dw,$$

$$J_6(n,\delta,\zeta) = -\frac{1}{(2\pi)^{d/2}}\int_{|w|>\sqrt{n}\delta;\sqrt{n}\,([-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}])} \exp\left\{-i\frac{1}{\sqrt{n}}(w, y-x)\right\}$$

$$\times \Delta_w\Bigg\{\boldsymbol{E}\Bigg[\Bigg(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}}\Bigg)\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};$$

$$\xi_n = T(y), \boldsymbol{A}_{n\zeta}\ \Bigg|\ \xi_0 = T(x)\Bigg]\Bigg\}\, dw$$

and

$$J_7(n,\delta,\zeta) = -\frac{1}{(2\pi)^{d/2}}\int_{\sqrt{n}([-\frac{\pi}{s_1},\frac{\pi}{s_1}]\times\cdots\times[-\frac{\pi}{s_d},\frac{\pi}{s_d}])} \exp\left\{-i\frac{1}{\sqrt{n}}(w, y-x)\right\}$$

$$\times \Delta_w\Bigg\{\boldsymbol{E}\Bigg[\Bigg(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{N_n^{jk}}\Bigg)\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};$$

$$\xi_n = T(y), \boldsymbol{A}_{n\zeta}^c\ \Bigg|\ \xi_0 = T(x)\Bigg]\Bigg\}\, dw.$$

A direct calculation shows that

$$J_0(n) = \frac{|y-x|^2}{n}\exp\left\{-\frac{1}{2n}\big(y-x, \boldsymbol{D}^{-1}(y-x)\big)\right\}\pi_{T(y)}.$$

Let us estimate remaining terms J_1 through J_7. We have

$$|J_1(n,\alpha)| \le \frac{1}{(2\pi)^{d/2}}\int_{|w|>\alpha}\left|\Delta_w\left\{\exp\left\{-\frac{1}{2}(w, Dw)\right\}\right\}\right|\pi_{T(y)}\, dw,$$

which can be made arbitary small by taking α sufficiently large. We apply Lemmas 4.3, 4.4 and 4.6 to get an estimate of $J_2(n,\alpha)$:

$$|J_2(n,\alpha)| \le (2\pi)^{-d/2}\int_{|w|\le\alpha}\Bigg|\Delta_w\Bigg\{\Bigg(\prod_{j,k\in\boldsymbol{\Xi}}\psi_{jk}\left(\frac{w}{\sqrt{n}}\right)^{n\pi_j q_{jk}}\Bigg)$$

$$\times \boldsymbol{E}\left[\exp\left\{i\frac{1}{\sqrt{n}}(w, M_n)\right\};\ \xi_n = T(y)\ \Bigg|\ \xi_0 = T(x)\right]$$

$$-\exp\left\{-\frac{1}{2}(w, Dw)\right\}\pi_{T(y)}\Bigg\}\Bigg|\, dw \to 0 \quad \text{as} \quad n\to\infty.$$

By Lemmas 2.6, 4.4 and 4.6, we may choose a positive constant c_1 so that

$$|J_3(n,\alpha,\delta)| \le (2\pi)^{-d/2}c_1\int_{\alpha<|w|}\big(1+|w|^2\big)e^{-\lambda|w|^2}dw \to 0 \quad \text{as } \alpha\to\infty.$$

By Lemmas 4.4 and 4.7, there exists a positive constant c_2 such that $|J_4(n,\delta,\zeta)| \leq c_2 K n e^{-Ln}$.

Using Lemmas 2.6, 2.7, 4.4, 4.5 and 4.6, there exists a positive constant c_3 such that

$$|J_5(n,\delta,\zeta)| \leq (2\pi)^{-d/2} c_3 \zeta \int_{\boldsymbol{R}^d} \left(1+|w|^4\right) e^{-(1/2)\lambda|w|^2} dw \to 0 \quad \text{as } \zeta \to +0.$$

Take β and ζ as in Lemma 4.8. Then by Lemma 4.8, there exist positive constants c_4 and γ, $0<\gamma<1$, such that

$$|J_6(n,\delta,\zeta)| \leq c_4 (2\pi n)^{d/2} n (1-\gamma)^{n\beta/4}.$$

By Lemma 2.3, there exists a positive constant c_5 such that

$$|J_7(n,\delta,\zeta)| \leq c_5 K (2\pi n)^{d/2} n e^{-Ln}.$$

We see from the estimates given above that J_k $(1 \leq k \leq 7)$ tend to zero as $n \to \infty$ uniformly for x, y. This completes the proof of Lemma 1.2. □

6. Some Lemmas for Theorems 1.1 and 1.2

Let t be a positive integer. Set $Q^t = (q_{jk}^{(t)})_{j,k\in\Xi}$. Note that $q_{jk}^{(t)} = \sum_{T(x)=k} P_t(j,x)$. In a similar way to $F_{jk}(\cdot)$ we define, for $q_{jk}^{(t)} > 0$

$$F_{jk}^{(t)}(x) = \begin{cases} \dfrac{1}{q_{jk}^{(t)}} P_t(j, j+x) & \text{if } T(j+x)=k \\ 0 & \text{otherwise,} \end{cases}$$

and for $q_{jk}^{(t)} = 0$, $F_{jk}^{(t)}(x) = 1$ if $x = k-j$ and 0 otherwise.

Suppose that the transition function $P(x,y)$ satisfies Assumptions 1.1 and 1.3. Then we may set $\mu_{jk;l}^{(t)} = \sum_{x\in\boldsymbol{Z}} x_l F_{jk}^{(t)}(x)$ and $G_l^{(t)} = (q_{jk}^{(t)} \mu_{jk;l}^{(t)})_{j,k\in\Xi}$ for l, $1 \leq l \leq d$. Let $G_l^{(0)}$, $1 \leq l \leq d$, be the null matrices.

Lemma 6.1. *Suppose that the transition function* $P(x,y)$ *satisfies* Assumptions 1.1 *and* 1.3. *Then, for every positive integer* t, *we have*

$$G_l^{(t)} = \sum_{n=0}^{t-1} Q^n G_l^{(1)} Q^{t-1-n}. \tag{6.1}$$

Proof. The lemma is trivial for $t=1$. Let us consider for $t>1$. By the definition of $G_l^{(t)}$,

$$G_l^{(t)} = G_l^{(t-1)} Q + Q^{t-1} G_l^{(1)}. \tag{6.2}$$

Indeed,

$$\begin{aligned} q^{(t)}_{jk}\mu^{(t)}_{jk;l} &= \sum_{j'\in\Xi}\sum_{T(j+x')=j'} x'_l P_{t-1}\big(j, j+x'\big)\sum_{T(j+x)=k} P\big(j+x', j+x\big) \\ &\quad + \sum_{j'\in\Xi}\sum_{T(j+x')=j'} P_{t-1}\big(j, j+x'\big)\sum_{T(j+x)=k}(x_l - x'_l)P\big(j+x', j+x\big) \\ &= \sum_{j'\in\Xi} q^{(t-1)}_{jj'}\mu^{(t-1)}_{jj';l}q_{j'k} + \sum_{j'\in\Xi} q^{(t-1)}_{jj'}q_{j'k}\mu_{j'k;l}. \end{aligned}$$

Suppose that the lemma is true for $t = t' - 1$. Then by (6.2) and the induction hypothesis, we have

$$G^{(t')}_l = \sum_{n=0}^{t'-2} Q^n G^{(1)}_l Q^{t'-1-n} + Q^{t'-1}G^{(1)}_l = \sum_{n=0}^{t'-1} Q^n G^{(1)}_l Q^{t'-1-n}.$$

The proof now follows by mathematical induction. □

Suppose that the transition function $P(x, y)$ satisfies Assumptions 1.1, 1.2 and 1.4. Denote by $\mathbf{1}$ the $(\#\Xi)$-dimensional column vector with all the components equal to 1. Then, by multiplying both sides (6.1) on the left by $\boldsymbol{\pi}$ and on the right by $\mathbf{1}$, we have

$$\sum_{j,k\in\Xi} \pi_j q^{(t)}_{jk}\mu^{(t)}_{jk;l} = t\sum_{j,k\in\Xi}\pi_j q_{jk}\mu_{jk;l} \quad \text{for } 1 \le l \le d. \tag{6.3}$$

By Assumption 1.4, we may set $h^{(t)}_{jk;lm} = \sum_{x\in\mathbf{Z}} x_l x_m F^{(t)}_{jk}(x)$, $1 \le l,\ m \le d$. Put $H^{(t)}_{lm} = (q^{(t)}_{jk}h^{(t)}_{jk;lm})_{j,k\in\Xi}$, $G^{(t)}_{lm} = (q^{(t)}_{jk}\mu^{(t)}_{jk;l}\mu^{(t)}_{jk;m})_{j,k\in\Xi}$ and

$$c^{(t)}_{jk;lm} = \sum_{x\in\mathbf{Z}}\big(x_l - \mu^{(t)}_{jk;l}\big)\big(x_m - \mu^{(t)}_{jk;m}\big)F^{(t)}_{jk}(x)$$

for $1 \le l,\ m \le d$. Then we have

$$\sum_{j,k\in\Xi}\pi_j q^{(t)}_{jk}c^{(t)}_{jk;lm} = \boldsymbol{\pi} H^{(t)}_{lm}\mathbf{1} - \boldsymbol{\pi} G^{(t)}_{lm}\mathbf{1}. \tag{6.4}$$

Let $H^{(0)}_{lm}$ and $G^{(0)}_{lm}$, $1 \le l,\ m \le d$, be the null matrices.

Lemma 6.2. *Suppose that the transition function $P(x, y)$ satisfies* Assumptions 1.1 *and* 1.4. *Then for every positive integer t,*

$$\begin{aligned} H^{(t)}_{lm} &= \sum_{n=0}^{t-1} Q^n H^{(1)}_{lm} Q^{t-1-n} \\ &\quad + \sum_{0\le n_1\le t-2}\ \sum_{0\le n_2\le t-2-n_1}\big(Q^{n_2}G^{(1)}_l Q^{t-2-n_1-n_2}G^{(1)}_m Q^{n_1} + Q^{n_2}G^{(1)}_m Q^{t-2-n_1-n_2}G^{(1)}_l Q^{n_1}\big). \end{aligned}$$

Proof. The lemma is trivial for $t = 1$. By analogous way to (6.2), we have

$$H_{lm}^{(t)} = H_{lm}^{(t-1)} Q + Q^{t-1} H_{lm}^{(1)} + G_l^{(t-1)} G_m^{(1)} + G_m^{(t-1)} G_l^{(1)} \tag{6.5}$$

for every positive integer t. Suppose that the lemma is true for $t = t'-1$. Then by (6.5) and the induction hypothesis, we have

$$\begin{aligned} H_{lm}^{(t')} &= \Bigg(\sum_{n=0}^{t'-2} Q^n H_{lm}^{(1)} Q^{t'-2-n} + \sum_{0\le n_1 \le t'-3} \sum_{0 \le n_2 \le t'-3-n_1} \Big(Q^{n_2} G_l^{(1)} Q^{t'-3-n_1-n_2} G_m^{(1)} Q^{n_1} \\ &\quad + Q^{n_2} G_m^{(1)} Q^{t'-3-n_1-n_2} G_l^{(1)} Q^{n_1} \Big) \Bigg) Q + Q^{t'-1} H_{lm}^{(1)} \\ &\quad + \left(\sum_{n=0}^{t'-2} Q^n G_l^{(1)} Q^{t'-2-n} \right) G_m^{(1)} + \left(\sum_{n=0}^{t'-2} Q^n G_m^{(1)} Q^{t'-2-n} \right) G_l^{(1)} \\ &= \sum_{n=0}^{t'-1} Q^n H_{lm}^{(1)} Q^{t'-1-n} + \sum_{n_1=0}^{t'-2} \sum_{n_2=0}^{t'-2-n_1} Q^{n_2} G_l^{(1)} Q^{t'-2-n_1-n_2} G_m^{(1)} Q^{n_1} \\ &\quad + \sum_{n_1=0}^{t'-2} \sum_{n_2=0}^{t'-2-n_1} Q^{n_2} G_m^{(1)} Q^{t'-2-n_1-n_2} G_l^{(1)} Q^{n_1}. \end{aligned}$$

The proof follows by mathematical induction. □

Suppose that the transition function $P(x, y)$ satisfies Assumptions 1.1, 1.2 and 1.4. Then, by (6.4) and Lemma 6.2, we have, for every positive integer t,

$$\begin{aligned} &\sum_{j,k\in\Xi} \pi_j q_{jk}^{(t)} c_{jk;lm}^{(t)} \\ &\quad = t\boldsymbol{\pi} H_{lm}^{(1)} \mathbf{1} + \boldsymbol{\pi} \sum_{0\le n\le t-2} (t-1-n) \big(G_l^{(1)} Q^n G_m^{(1)} + G_m^{(1)} Q^n G_l^{(1)} \big) \mathbf{1} - \boldsymbol{\pi} G_{lm}^{(t)} \mathbf{1}. \end{aligned} \tag{6.6}$$

Denote by Π a matrix of order $(\#\Xi)$ with all the row vectors equal to $\boldsymbol{\pi}$.

Lemma 6.3. *Suppose that Q is ergodic. Then $(I - Q + \Pi)$ has its inverse matrix, and $(I - Q + \Pi)^{-1} = \sum_{n=0}^{\infty} (Q - \Pi)^n$.*

See, for a proof, Hatori and Mori [3, p. 107].

Suppose that Q is ergodic. Let t be a positive integer. Then, by Lemma 6.3, we may define

$$Z^{(t)} = \big(z_{jk}^{(t)} \big)_{j,k\in\Xi} = \big(I - Q^t + \Pi \big)^{-1} = \sum_{n'=0}^{\infty} \big(Q^t - \Pi \big)^{n'}, \tag{6.7}$$

and we have

$$Z^{(t)}\Pi = \Pi Z^{(t)} = \Pi. \tag{6.8}$$

Define $Z = Z^{(1)}$. Set

$$\mu^{(t)}_{jk} = \left(\mu^{(t)}_{jk;1}, \ldots, \mu^{(t)}_{jk;d}\right), \ \ Q^{(t)}(\theta) = \left(q^{(t)}_{jk} e^{i(\mu^{(t)}_{jk},\theta)}\right)_{j,k\in\Xi} \quad \textit{and} \quad f^{(t)}(\theta, z) = |zI - Q^{(t)}(\theta)|$$

for $\theta \in \boldsymbol{R}^d$ and $z \in \boldsymbol{C}$. Note that $Q^{(t)}(\mathbf{0}) = Q^t$. By (1.4), $(\partial f^{(t)}/\partial z)(\mathbf{0}, 1) \neq 0$. Thus we may set

$$b^{(t)}_{lm} = \frac{\partial^2 f^{(t)}}{\partial\theta_l\partial\theta_m}(\mathbf{0}, 1) \Big/ \frac{\partial f^{(t)}}{\partial z}(\mathbf{0}, 1) \quad \text{and} \quad B^{(t)} = \left(b^{(t)}_{lm}\right)_{1\le l,m\le d}.$$

Lemma 6.4. *Suppose that the transition function* $P(x, y)$ *satisfies Assumptions* 1.1 *through* 1.4. *Then*

$$b^{(t)}_{lm} = \boldsymbol{\pi}\left(G^{(t)}_{lm} + G^{(t)}_{l} Z^{(t)} G^{(t)}_{m} + G^{(t)}_{m} Z^{(t)} G^{(t)}_{l}\right)\mathbf{1} \tag{6.9}$$

for every positive integer t.

Proof. Let $\{\xi^{(t)}_n\}_{n\ge 0}$ be a Markov chain on Ξ with the transition matrix Q^t. Set

$$N^{(t)jk}_n = \#\{1 \le n' \le n \mid \xi^{(t)}_{n'-1} = j,\ \xi^{(t)}_{n'} = k\} \quad \text{and} \quad M^{(t)}_{n;l} = \sum_{j,k\in\Xi} \mu^{(t)}_{jk;l} N^{(t)jk}_n. \tag{6.10}$$

Under Assumptions 1.1 and 1.2, Q^t is ergodic. Moreover, by (6.3) we have

$$\sum_{j,k\in\Xi} \pi_j q^{(t)}_{jk} \mu^{(t)}_{jk;l} = \mathbf{0} \quad \text{for } 1 \le l \le d.$$

Thus we may apply Lemma 4.2 to $\{\xi^{(t)}_n\}_{n\ge 0}$ to have

$$b^{(t)}_{lm} = \lim_{n\to\infty} \frac{1}{\pi_k n} \boldsymbol{E}\left[M^{(t)}_{n;l} M^{(t)}_{n;m};\ \xi^{(t)}_n = k \,\middle|\, \xi^{(t)}_0 = j\right], \quad 1 \le l,\ m \le d. \tag{6.11}$$

By analogous way to Hatori and Mori [3, p. 123], in which they treated the case that $d = 1$, we may see that the right hand side of (6.11) equals the right hand side of (6.9). Thus we obtain (6.9). □

Lemma 6.5. *If* Q *is ergodic, then* $ZQ^n = -\sum_{n'=0}^{n-1} Q^{n'} + n\Pi + Z$.

Proof. By (6.7) and (6.8), we have

$$ZQ = Z - I + \Pi. \tag{6.12}$$

Therefore the lemma is true for $n = 1$. Suppose that the lemma is true for $n = n' - 1$. Then

$$ZQ^{n'} = \left(ZQ^{n'-1}\right)Q = -\sum_{n''=0}^{n'-1} Q^{n''} + n'\Pi + Z.$$

The proof follows by mathematical induction. □

Lemma 6.6. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.4. *Then for every positive integer* t,

$$b_{lm}^{(t)} = \boldsymbol{\pi} G_{lm}^{(t)} \mathbf{1} - \boldsymbol{\pi} \sum_{0 \le n \le t-2} (t - 1 - n)\left(G_l^{(1)} Q^n G_m^{(1)} + G_m^{(1)} Q^n G_l^{(1)}\right)\mathbf{1} \\ + t\boldsymbol{\pi} G_l^{(1)} Z G_m^{(1)} \mathbf{1} + t\boldsymbol{\pi} G_m^{(1)} Z G_l^{(1)} \mathbf{1}. \tag{6.13}$$

Proof. By Lemma 6.1, we have

$$\boldsymbol{\pi} G_l^{(t)} Z^{(t)} G_m^{(t)} \mathbf{1} = \boldsymbol{\pi} G_l^{(1)} \left(\sum_{n'=0}^{t-1} Q^{n'}\right) Z^{(t)} \left(\sum_{n''=0}^{t-1} Q^{n''}\right) G_m^{(1)} \mathbf{1}. \tag{6.14}$$

By (6.7),

$$\left(Z^{(t)}\right)^{-1} = Z^{-1} + Q - Q^t. \tag{6.15}$$

Multiply both sides (6.15) on the left by Z, and on the right by $Z^{(t)}$. Thus we obtain

$$Z = Z^{(t)} + \sum_{1 \le n \le t-1} Z(I - Q)Q^n Z^{(t)}. \tag{6.16}$$

By applying (6.8) and (6.12) to (6.16),

$$Z = \left(\sum_{n=0}^{t-1} Q^n\right) Z^{(t)} - (t - 1)\Pi. \tag{6.17}$$

By multiplying both sides (6.17) on the right by $\sum_{n=0}^{t-1} Q^n$, we obtain

$$\left(\sum_{n=0}^{t-1} Q^n\right) Z^{(t)} \left(\sum_{n=0}^{t-1} Q^n\right) = \sum_{n=0}^{t-1} ZQ^n + t(t - 1)\Pi.$$

By Lemma 6.5, we have

$$\left(\sum_{n=0}^{t-1} Q^n\right) Z^{(t)} \left(\sum_{n=0}^{t-1} Q^n\right) = -\sum_{0 \le n \le t-2} (t - n - 1)Q^n + \frac{3}{2}t(t - 1)\Pi + tZ. \tag{6.18}$$

Substitute (6.18) to (6.14), and note Assumption 1.3. Thus we have

$$\boldsymbol{\pi} G_l^{(t)} Z^{(t)} G_m^{(t)} \mathbf{1} = -\boldsymbol{\pi} \sum_{0 \le n \le t-2} (t-n-1) G_l^{(1)} Q^n G_m^{(1)} \mathbf{1} + t \boldsymbol{\pi} G_l^{(1)} Z G_m^{(1)} \mathbf{1}.$$

By Lemma 6.4, we obtain (6.13). The proof is complete. □

Set $C_{jk}^{(t)} = (c_{jk;lm}^{(t)})_{1 \le l,m \le d}$ and $D^{(t)} = \sum_{j,k \in \boldsymbol{\Xi}} \pi_j q_{jk}^{(t)} C_{jk}^{(t)} + B^{(t)}$. Thus by (6.6) and Lemma 6.6, we have the following lemma.

Lemma 6.7. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.4. *Then for every positive integer* t, $D^{(t)} = tD$.

Lemma 6.8. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.4. *Put*

$$t_0 = \max\{n_0(\mathbf{0}, \mathbf{0}), n_0(\mathbf{0}, \pm s_l \boldsymbol{e}_l),\ 1 \le l \le d\}.$$

Then the transition function defined by $P'(x, y) = P_{t_0}(x, y)$ *satisfies* Assumptions 1.1 *through* 1.6.

The proof is omitted.

Lemma 6.9. *Suppose that the transition function* $P(x, y)$ *satisfies* Assumptions 1.1 *through* 1.4. *Let* t_0 *be as in* Lemma 6.8. *Then we have*

$$\lim_{n\to\infty} \Big((2\pi n t_0)^{d/2} P_{nt_0}(x, y) - (\#\boldsymbol{\Xi}) |D|^{-1/2} \exp\Big\{ -\frac{1}{2nt_0} \big(y - x, D^{-1}(y - x) \big) \Big\} \pi_{T(y)} \Big) = 0 \tag{6.19}$$

and

$$\lim_{n\to\infty} \frac{|y-x|^2}{nt_0} \Big((2\pi n t_0)^{d/2} P_{nt_0}(x, y) - (\#\boldsymbol{\Xi}) |D|^{-1/2} \exp\Big\{ -\frac{1}{2nt_0} \big(y - x, D^{-1}(y - x) \big) \Big\} \pi_{T(y)} \Big) = 0 \tag{6.20}$$

uniformly for $x,\ y \in \boldsymbol{Z}^d$, *where* D *is positive definite.*

Proof. Since $P'(x, y)$ in Lemma 6.8 satisfies Assumptions 1.1 through 1.6, we have from Lemma 1.1 $(2\pi n)^{d/2} P'_n(x, y)$ converges to

$$(\#\Xi)|D^{(t_0)}|^{-1/2} \exp\left\{-\frac{1}{2n}\big(y - x, (D^{(t_0)})^{-1}(y - x)\big)\right\} \pi_{T(y)}$$

uniformly for $x,\ y \in \boldsymbol{Z}^d$ and $D^{(t_0)}$ is positive definite. By Lemma 6.7, D is positive definite and (6.19) holds. Similary, by Lemma 1.2, we obtain (6.20). □

7. Proof of Theorems 1.1 and 1.2

Suppose that the transition function $P(x, y)$ satisfies Assumption 1.1 through 1.4. Let t_0 be the positive integer given in Lemma 6.8. In order to prove Theorem 1.1, it suffices to show that

$$\begin{aligned}\lim_{n\to\infty} \sup_{x,y\in\boldsymbol{Z}^d} \Bigg| &(2\pi(nt_0 + n'))^{d/2} P_{nt_0+n'}(x, y) \\ &- (\#\Xi)|D|^{-1/2} \exp\left\{-\frac{1}{2(nt_0 + n')}\big(y - x, D^{-1}(y - x)\big)\right\} \pi_{T(y)}\Bigg| = 0\end{aligned}$$

for every n', $0 \le n' \le t_0 - 1$. By Lemma 6.9, we may write

$$\begin{aligned}&\big(2\pi(nt_0 + n')\big)^{d/2} P_{nt_0+n'}(x, y) \\ &\quad = (\#\Xi)|D|^{-1/2} \exp\left\{-\frac{1}{2(nt_0 + n')}\big(y - x, D^{-1}(y - x)\big)\right\} \pi_{T(y)} \\ &\qquad + I'_1(n) + I'_2(n) + o(1),\end{aligned}$$

where

$$\begin{aligned}I'_1(n) = {}& (\#\Xi)|D|^{-1/2} \exp\left\{-\frac{1}{2nt_0}\big(y - x, D^{-1}(y - x)\big)\right\} \pi_{T(y)} \\ &- (\#\Xi)|D|^{-1/2} \exp\left\{-\frac{1}{2(nt_0 + n')}\big(y - x, D^{-1}(y - x)\big)\right\} \pi_{T(y)}\end{aligned}$$

and

$$\begin{aligned}I'_2(n) = {}& (\#\Xi)|D|^{-1/2} \sum_{x'\in\boldsymbol{Z}^d} P_{n'}\big(x, x'\big) \exp\left\{-\frac{1}{2nt_0}\big(y - x', D^{-1}(y - x')\big)\right\} \pi_{T(y)} \\ &- (\#\Xi)|D|^{-1/2} \exp\left\{-\frac{1}{2nt_0}\big(y - x, D^{-1}(y - x)\big)\right\} \pi_{T(y)},\end{aligned}$$

and $o(1)$ tends to zero as $n \to \infty$ uniformly for $x,\ y$. We will show that the terms I'_1 and I'_2 go to zero uniformly in $x,\ y$ as $n \to \infty$.

It follows from the inequality $|e^{-y} - e^{-x}| \le e^{-x}|y-x|$ for $y > x > 0$ that $|I_1'(n)| \le a_1'/n$, where a_1' is a positive constant.

Since D is a symmetric matrix and positive definite, all its eigenvalues $\{\lambda_l\}_{1\le l\le d}$ are positive and there exists an orthogonal matrix $U = (u_{lm})_{1\le l,m\le d}$ such that

$$D = U\Lambda U^*, \tag{7.1}$$

where $\Lambda = \mathrm{diag}\{\lambda_1 \dots \lambda_d\}$ and U^* is the transposed matrix of U. Then we have

$$\begin{aligned}
&\exp\left\{-\frac{1}{2nt_0}\big(y-x', D^{-1}(y-x')\big)\right\} - \exp\left\{-\frac{1}{2nt_0}\big(y-x, D^{-1}(y-x)\big)\right\}\\
&= \sum_{l'=1}^{d}\left(\exp\left\{-\frac{1}{2nt_0\lambda_{l'}}\left(\sum_{m=1}^{d}u_{ml'}\big(y_m - x_m'\big)\right)^2\right\}\right.\\
&\qquad\left. - \exp\left\{-\frac{1}{2nt_0\lambda_{l'}}\left(\sum_{m=1}^{d}u_{ml'}\big(y_m - x_m\big)\right)^2\right\}\right)\\
&\quad\times \exp\left\{-\frac{1}{2nt_0}\sum_{1\le l\le l'-1}\frac{1}{\lambda_l}\left(\sum_{m=1}^{d}u_{ml}\big(y_m - x_m'\big)\right)^2\right.\\
&\qquad\left. - \frac{1}{2nt_0}\sum_{l'+1\le l\le d}\frac{1}{\lambda_l}\left(\sum_{m=1}^{d}u_{ml}(y_m - x_m)\right)^2\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
|I_2'(n)| \le (\#\Xi)|D|^{-1/2}\sum_{x'\in\mathbf{Z}^d}P_{n'}\big(x,x'\big)\sum_{l'=1}^{d}&\left|\exp\left\{-\frac{1}{2nt_0\lambda_{l'}}\left(\sum_{m=1}^{d}u_{ml'}\big(y_m - x_m'\big)\right)^2\right\}\right.\\
&\left. - \exp\left\{-\frac{1}{2nt_0\lambda_{l'}}\left(\sum_{m=1}^{d}u_{ml'}(y_m - x_m)\right)^2\right\}\right|\pi_{T(y)}.
\end{aligned}$$

Note that

$$\left|e^{-y^2/n} - e^{x^2/n}\right| = \left|\int_y^x 2\frac{v}{n}e^{-v^2/n}\,dv\right| \le a_2'\frac{1}{\sqrt{n}}|y-x| \tag{7.2}$$

where a_2' is positive constant. Thus, by Assumption 1.3, there exists positive constant a_3' such that $|I_2'(n)| \le a_3'/\sqrt{n}$. The proof is complete. □

In order to prove the Theorem 1.2, it suffices to show that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{nt_0+n'}\sup_{x,y\in\mathbf{Z}^d}|y-x|^2&\left|\big(2\pi(nt_0+n')\big)^{d/2}P_{nt_0+n'}(x,y)\right.\\
&\left.- (\#\Xi)|D|^{-1/2}\exp\left\{-\frac{1}{2(nt_0+n')}\big(y-x, D^{-1}(y-x)\big)\right\}\pi_{T(y)}\right| = 0
\end{aligned}$$

for every n', $0 \le n' \le t_0 - 1$. We may write

$$\begin{aligned}
&\frac{|y-x|^2}{nt_0+n'}\Big(\big(2\pi(nt_0+n')\big)^{d/2}P_{nt_0+n'}(x,y)\\
&\qquad -(\#\Xi)|D|^{-1/2}\exp\Big\{-\frac{1}{2(nt_0+n')}\big(y-x,D^{-1}(y-x)\big)\Big\}\pi_{T(y)}\Big)\\
&\quad = J_1'(n)+J_2'(n)+J_3'(n)+J_4'(n)
\end{aligned}$$

where

$$\begin{aligned}
J_1'(n) &= \frac{|y-x|^2}{nt_0+n'}\left(\frac{nt_0+n'}{nt_0}\right)^{d/2}\sum_{x'\in\mathbf{Z}^d}P_{n'}(x,x')\Big((2\pi nt_0)^{d/2}P_{nt_0}(x',y)\\
&\quad -(\#\Xi)|D|^{-1/2}\exp\Big\{-\frac{1}{2nt_0}\big(y-x',D^{-1}(y-x')\big)\Big\}\pi_{T(y)}\Big),\\
J_2'(n) &= \frac{|y-x|^2}{nt_0+n'}(\#\Xi)|D|^{-1/2}\sum_{x'\in\mathbf{Z}^d}P_{n'}(x,x')\exp\Big\{-\frac{1}{2nt_0}\big(y-x',D^{-1}(y-x')\big)\Big\}\pi_{T(y)}\\
&\quad -\frac{|y-x|^2}{nt_0+n'}(\#\Xi)|D|^{-1/2}\exp\Big\{-\frac{1}{2nt_0}\big(y-x,D^{-1}(y-x)\big)\Big\}\pi_{T(y)},\\
J_3'(n) &= (\#\Xi)|D|^{-1/2}\frac{|y-x|^2}{nt_0+n'}\exp\Big\{-\frac{1}{2nt_0}\big(y-x,D^{-1}(y-x)\big)\Big\}\pi_{T(y)}\\
&\quad -(\#\Xi)|D|^{-1/2}\frac{|y-x|^2}{nt_0+n'}\exp\Big\{-\frac{1}{2(nt_0+n')}\big(y-x,D^{-1}(y-x)\big)\Big\}\pi_{T(y)},\\
J_4'(n) &= \left(\left(\frac{nt_0+n'}{nt_0}\right)^{d/2}-1\right)(\#\Xi)|D|^{-1/2}\frac{|y-x|^2}{nt_0+n'}\\
&\quad \times\sum_{x'\in\mathbf{Z}^d}P_{n'}(x,x')\exp\Big\{-\frac{1}{2nt_0}\big(y-x',D^{-1}(y-x')\big)\Big\}\pi_{T(y)}.
\end{aligned}$$

We will show that the terms J_1', J_2', J_3' and J_4' go to zero uniformly for x, y as $n\to\infty$.

Note that

$$|y-x|^2 \le 2|y-x'|^2+2|x'-x|^2. \tag{7.3}$$

Thus, by Lemma 6.9 and Assumption 1.4, there exist positive constants b_1' and b_2' such that

$$\begin{aligned}
|J_1'(n)| \le b_1'\frac{1}{nt_0}\sup_{y,x'\in\mathbf{Z}^d}|y-x'|^2\Big|&(2\pi nt_0)^{d/2}P_{nt_0}(x',y)\\
&-(\#\Xi)|D|^{-1/2}\exp\Big\{-\frac{1}{2nt_0}\big(y-x',D^{-1}(y-x')\big)\Big\}\pi_{T(y)}\Big|
\end{aligned}$$

$$+ b_2' \frac{1}{nt_0} \sup_{y,x' \in \mathbf{Z}^d} \Big| (2\pi n t_0)^{d/2} P_{nt_0}(x', y) - (\#\Xi)|D|^{-1/2} \exp\left\{ -\frac{1}{2nt_0} \left(y - x', D^{-1}(y - x')\right) \right\} \pi_{T(y)} \Big| \to 0 \quad \text{as } n \to \infty.$$

Since $|y - x|^2 = |y - x'|^2 + (|y - x|^2 - |y - x'|^2)$, we have $J_2'(n) \le J_{21}'(n) + J_{22}'(n)$, where

$$J_{21}'(n) = (\#\Xi)|D|^{-1/2} \sum_{x' \in \mathbf{Z}^d} P_{n'}(x, x') \frac{|\, |y - x|^2 - |y - x'|^2 \,|}{nt_0 + n'} \times \exp\left\{ -\frac{1}{2nt_0} \left(y - x', D^{-1}(y - x')\right) \right\} \pi_{T(y)},$$

$$J_{22}'(n) = (\#\Xi)|D|^{-1/2} \times \sum_{x' \in \mathbf{Z}^d} P_{n'}(x, x') \frac{1}{nt_0 + n'} \Big| |y - x'|^2 \exp\left\{ -\frac{1}{2nt_0} \left(y - x', D^{-1}(y - x')\right) \right\} - |y - x|^2 \exp\left\{ -\frac{1}{2nt_0} \left(y - x, D^{-1}(y - x)\right) \right\} \Big| \pi_{T(y)}.$$

Note that the inequality $||y - x|^2 - |y - x'|^2| \le |x' - x|^2 + 2|y - x'||x' - x|$. Thus, by Assumption 1.4, there exists a positive constant b_3' such that $J_{21}'(n) \le b_3' n^{-1/2}$.

By (7.1), the right hand side of $J_{22}'(n)$ equals

$$(\#\Xi)|D|^{-1/2} \sum_{x' \in \mathbf{Z}^d} P_{n'}\left(x, x'\right) \frac{1}{nt_0 + n'} \times \Bigg| \left(\sum_{l=1}^{d} \left(\sum_{m=1}^{d} u_{ml} \left(y_m - x_m'\right) \right)^2 \right) \exp\left\{ -\frac{1}{2nt_0} \sum_{l=1}^{d} \frac{1}{\lambda_l} \left(\sum_{m=1}^{d} u_{ml} \left(y_m - x_m'\right) \right)^2 \right\} - \left(\sum_{l=1}^{d} \left(\sum_{m=1}^{d} u_{ml} \left(y_m - x_m\right) \right)^2 \right) \exp\left\{ -\frac{1}{2nt_0} \sum_{l=1}^{d} \frac{1}{\lambda_l} \left(\sum_{m=1}^{d} u_{ml} \left(y_m - x_m\right) \right)^2 \right\} \Bigg|.$$

Note that

$$\left(\sum_{l=1}^{d} x_l^2 \right) \exp\left\{ -\sum_{l=1}^{d} \frac{1}{\lambda_l} x_l^2 \right\} - \left(\sum_{l=1}^{d} y_l^2 \right) \exp\left\{ -\sum_{l=1}^{d} \frac{1}{\lambda_l} y_l^2 \right\} = \sum_{l_1=1}^{d} x_{l_1}^2 \exp\left\{ -\frac{1}{\lambda_{l_1}} x_{l_1}^2 \right\} \sum_{\substack{1 \le l_2 \le d \\ l_2 \ne l_1}} \left(\exp\left\{ -\frac{1}{\lambda_{l_2}} x_{l_2}^2 \right\} - \exp\left\{ -\frac{1}{\lambda_{l_2}} y_{l_2}^2 \right\} \right)$$

$$\times \exp\left\{-\sum_{\substack{1\le l_3\le l_2-1\\ l_3\ne l_1}}\frac{1}{\lambda_{l_3}}x_{l_3}^2-\sum_{\substack{l_2+1\le l_4\le d\\ l_4\ne l_1}}\frac{1}{\lambda_{l_4}}y_{l_4}^2\right\}$$

$$+\sum_{m_1=1}^{d}\left(x_{m_1}^2\exp\left\{-\frac{1}{\lambda_{m_1}}x_{m_1}^2\right\}-y_{m_1}^2\exp\left\{-\frac{1}{\lambda_{m_1}}y_{m_1}^2\right\}\right)\exp\left\{-\sum_{\substack{1\le m_2\le d\\ m_2\ne m_1}}\frac{1}{\lambda_{m_2}}y_{m_2}^2\right\}.$$

Thus we have

$$J'_{22}(n)\le(\#\Xi)|D|^{-1/2}\frac{1}{nt_0+n'}\sum_{x'\in\boldsymbol{Z}^d}P_{n'}\left(x,x'\right)$$
$$\times\sum_{l_1=1}^{d}\left(\sum_{m=1}^{d}u_{ml_1}\left(y_m-x'_m\right)\right)^2\exp\left\{-\frac{1}{2nt_0\lambda_{l_1}}\left(\sum_{m=1}^{d}u_{ml_1}\left(y_m-x'_m\right)\right)^2\right\}$$
$$\times\sum_{\substack{1\le l_2\le d\\ l_2\ne l_1}}\left|\exp\left\{-\frac{1}{2nt_0\lambda_{l_2}}\left(\sum_{m=1}^{d}u_{ml_2}\left(y_m-x'_m\right)\right)^2\right\}\right.$$
$$\left.-\exp\left\{-\frac{1}{2nt_0\lambda_{l_2}}\left(\sum_{m=1}^{d}u_{ml_2}\left(y_m-x_m\right)\right)^2\right\}\right|\pi_{T(y)}$$
$$+(\#\Xi)|D|^{-1/2}\frac{1}{nt_0+n'}\sum_{x'\in\boldsymbol{Z}^d}P_{n'}(x,x')$$
$$\times\sum_{l_3=1}^{d}\left|\left(\sum_{m=1}^{d}u_{ml_3}\left(y_m-x'_m\right)\right)^2\exp\left\{-\frac{1}{2nt_0\lambda_{l_3}}\left(\sum_{m=1}^{d}u_{ml_3}\left(y_m-x'_m\right)\right)^2\right\}\right.$$
$$\left.-\left(\sum_{m=1}^{d}u_{ml_3}\left(y_m-x_m\right)\right)^2\exp\left\{-\frac{1}{2nt_0\lambda_{l_3}}\left(\sum_{m=1}^{d}u_{ml_3}\left(y_m-x_m\right)\right)^2\right\}\right|\pi_{T(y)}.$$

Noting (7.2) and that $|y^2e^{-(1/n)y^2}-x^2e^{-(1/n)x^2}|\le b'_4\sqrt{n}\,|y-x|$, where b'_4 is a positive constant, we see that there exists a positive constant b'_5 such that $J'_{22}(n)\le b'_5/\sqrt{n}$.

By analogous way to $I'_1(n)$ there exists a positive constant b'_6 such that $|J'_3(n)|\le b'_6/n$.

Since $x^2e^{-x^2}$ is bounded, by (7.3) there exists a positive constant b'_7 such that $|J'_4(n)|\le b'_7/n$. The proof is complete. □

REMARK. Consider a *RWPE* on $\boldsymbol{Z}^d$ with the transition function $P(x,y)$, where $P(x,y)>0$ if $y=x\pm\boldsymbol{e}_l$, $1\le l\le d$, and $P(x,y)=0$ otherwise. Such a *RWPE* does not satisfy Assumption 1.2. Nevertheless, for 1 and 2-dimensional case, by modifying the transition function we may apply Theorems 1.1 and 1.2 to the *RWPE*. For 1-dimensional case, we may assume that its period s is even. Set $P'(x,y)=$

$P_2(2x, 2y)$. Then $P'(x, y)$ is a transition function on $\boldsymbol{Z}$ with period $s/2$ and satisfies Assumptions 1.1 through 1.4. For 2-dimensional case, we may assume that $\boldsymbol{s} = (s, s)$. Set $S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $P'(x, y) = P_2(Sx, Sy)$. Then $P'(x, y)$ is the transition function on $\boldsymbol{Z}^2$ with period $\boldsymbol{s}$ and satisfies Assumptions 1.1 through 1.4.

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