Watanabe, T.
Osaka J. Math.
15 (1978), 463-474

THE HOMOLOGY OF THE LOOP SPACE OF THE EXCEPTIONAL GROUP F_4

TAKASHI WATANABE

(Received May 26, 1977)

Let G be a compact, simply connected, simple Lie group and ΩG the space of loops on G. Bott [4] showed that $H_*(\Omega G)$ has no torsion and vanishing odd dimensional part. Since ΩG is a homotopy commutative H-space, $H_*(\Omega G)$ becomes a commutative Hopf algebra over the integers Z. Bott [5] also gave a general method for computing its Hopf algebra structure, and determined it explicitly for $G=SU(l+1)$, $\mathrm{Spin}(2l+1)$, $\mathrm{Spin}(2l)$ and G_2.

The object of this paper is to determine the Hopf algebra structure of $H_*(\Omega F_4)$, where F_4 is the compact exceptional Lie group of rank 4.

Let ψ denote the coproduct of $C=H_*(\Omega G)$ induced by the diagonal $\Omega G\to\Omega G\times\Omega G$. Since ψ is commutative, we may introduce a map $\tilde{\psi}\colon C\to C\otimes C$ satisfying

$$\psi(\sigma)-\sigma\otimes 1-1\otimes\sigma = \tilde{\psi}(\sigma)+T\tilde{\psi}(\sigma)$$

for all $\sigma\in C$, where $T\colon C\otimes C\to C\otimes C$ is defined by

$$T(\sigma\otimes\tau) = \begin{cases}\tau\otimes\sigma & \text{if } \sigma \neq \tau \\ 0 & \text{if } \sigma=\tau\,.\end{cases}$$

Then $\tilde{\psi}(\sigma)=0$ if and only if $\sigma\in P(C)$, where P denotes the primitive module functor.

We can now state our main result.

Theorem 1. *The Hopf algebra structure of* $H_*(\Omega F_4)$ *is given by*:

(i) $H_*(\Omega F_4)=Z[\sigma_1,\sigma_2,\sigma_3,\sigma_5,\sigma_7,\sigma_{11}]/(\sigma_1^2-2\sigma_2,\sigma_2\sigma_1-3\sigma_3)$ *where* $\deg\sigma_i=2i$.

(ii) *In suitable choice of generators* $\sigma_5,\sigma_7,\sigma_{11}$, *the coproduct is given by*

$$\psi(\sigma_k)=\sum_{i+j=k}\sigma_i\otimes\sigma_j \qquad (k=1,2,3)\,,$$

$$\tilde{\psi}(\sigma_5)=\sigma_4\otimes\sigma_1+2\sigma_3\otimes\sigma_2\,,$$

$$\tilde{\psi}(\sigma_7)=(\sigma_5\sigma_1-\sigma_6)\otimes\sigma_1+\sigma_5\otimes\sigma_2+\sigma_4\otimes\sigma_3\,,$$

$$\begin{aligned}\tilde{\psi}(\sigma_{11})= {} & 2(-\sigma_7\sigma_3+\sigma_5\sigma_4\sigma_1-\sigma_6\sigma_4)\otimes\sigma_1+2(-\sigma_7\sigma_2+3\sigma_5\sigma_4-\sigma_6\sigma_3)\otimes\sigma_2\\ & +2(-\sigma_7\sigma_1+3\sigma_5\sigma_3+\sigma_6\sigma_2)\otimes\sigma_3+(-\sigma_7+\sigma_5\sigma_2+8\sigma_6\sigma_1)\otimes\sigma_4\end{aligned}$$

$$+12\sigma_6\otimes\sigma_4\sigma_1$$

where $\sigma_4=\sigma_2^2-\sigma_3\sigma_1$ *and* $\sigma_6=\sigma_2^3-4\sigma_3^2$.

(iii) $PH_*(\Omega F_4)=Z\{\sigma_1,\sigma_5',\sigma_7',\sigma_{11}'\}$ *where*

$$\sigma_5' = 5\sigma_5-\sigma_4\sigma_1\,,$$
$$\sigma_7' = 7\sigma_7-14\sigma_5\sigma_2+10\sigma_6\sigma_1\,,$$
$$\sigma_{11}'= 11\sigma_{11}-33\sigma_5^2\sigma_1+11\sigma_7\sigma_4+22\sigma_5\sigma_6+6\sigma_6\sigma_4\sigma_1\,.$$

The paper is organized as follows. In §1 we prove part (i) by an easy spectral sequence argument. §2 is devoted to review Bott's work. In §3 we apply the argument in §2 to F_4. Finally in §4 we discuss parts (ii) and (iii).

1. The algebra structure of $H_*(\Omega F_4)$

It is well known that Spin(9)$\subset F_4$ and the quotient F_4/Spin(9) is the Cayley projective plane Π, whose cohomology is given by

$$H^*(\Pi) = Z[x]/(x^3)$$

where deg $x=8$.

Let $\Lambda(\ )$ and $\Gamma[\ ]$ denote exterior and divided polynomial algebras over Z, respectively. Then we have

Lemma 2. (i) *As a Hopf algebra,*

$$H^*(\Omega\Pi) = \Lambda(a)\otimes\Gamma[b]$$

where deg $a=7$ *and* deg $b=22$.

(ii) *As a Hopf algebra,*

$$H_*(\Omega\Pi) = \Lambda(\alpha)\otimes Z[\beta]$$

where deg $\alpha=7$ *and* deg $\beta=22$.

Proof. It is sufficient to show (i), because (ii) is just the dual statement of (i). Consider the integral cohomology spectral sequence $\{E_r, d_r\}$ of the fibration

$$\Omega\Pi \to P\Pi \to \Pi\,,$$

so that $E_2^{p,q}=H^p(\Pi)\otimes H^q(\Omega\Pi)$ and $E_\infty^{p,q}=0$ except for $(p, q)=(0, 0)$. A routine spectral sequence argument shows that $H^*(\Omega\Pi)$ has an additive basis consisting of elements

$$\{b_0 = 1, a_0, b_1, a_1, b_2, a_2, \cdots\}$$

with deg $a_i=22i+7$ and deg $b_i=22i\ (i\geq 0)$ such that

$$d_8(1\otimes a_i) = x\otimes b_i \qquad \text{for } i\geq 0 ,$$
$$d_{16}(1\otimes b_i) = x^2\otimes a_{i-1} \qquad \text{for } i\geq 1 .$$

In terms of this basis we compute products a_ia_j, a_ib_j and b_ib_j. Clearly $a_ia_j=0$. Now $a_0b_i=a_i$ since $d_8(1\otimes a_0b_i)=x\otimes b_i$. Let $e_{i,j}$ be the integer such that $b_ib_j=e_{i,j}b_{i+j}$. Then $a_ib_j=a_0b_ib_j=e_{i,j}a_0b_{i+j}=e_{i,j}a_{i+j}$. Therefore

$$\begin{aligned} d_{16}(1\otimes b_ib_j) &= x^2\otimes a_{i-1}b_j+x^2\otimes b_ia_{j-1} \\ &= (e_{i-1,j}+e_{j-1,i})x^2\otimes a_{i+j-1} . \end{aligned}$$

Hence we get a relation $e_{i,j}=e_{i-1,j}+e_{j-1,i}$, which implies that $e_{i,j}=(i+j)!/i!j!$. Thus setting $a=a_0$ and $b=b_1$, we obtain the desired algebra structure. It remains to prove that a and b are primitive. But it is immediate from degree considerations. q.e.d.

Here we quote the following result from [5;Proposition 9.1]:

(1.1) $$H_*(\Omega\,\mathrm{Spin}(9)) = Z[\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_7]/(\sigma_1^2-2\sigma_2)$$

where $\deg \sigma_i=2i$.

Proof of Theorem 1 (i). Let $f\colon F_4\to K(Z,3)$ be a map which represents the generator of $H^3(F_4)=Z$. As seen from the table in [12; §1], $\Omega f_\sharp\colon \pi_j(\Omega F_4)\to \pi_j(K(Z,2))$ is an isomorphism for $j\leq 6$ and an epimorphism for $j=7$. So, by the Whitehead theorem, $\Omega f_*\colon H_j(\Omega F_4)\to H_j(K(Z,2))$ is an isomorphism for $j\leq 6$. Recall that $H_*(K(Z,2))=\Gamma[\gamma]$ with $\deg \gamma=2$. Let $\sigma_i=(\Omega f_*)^{-1}(\gamma_i)\in H_{2i}(\Omega F_4)$ for $i=1,2,3$ (where $\gamma_i=\gamma^i/i!$). Then we have

(1.2) $$H_*(\Omega F_4) = Z[\sigma_1, \sigma_2, \sigma_3]/(\sigma_1^2-2\sigma_2, \sigma_2\sigma_1-3\sigma_3) \quad \text{for dim.}\leq 6 .$$

(This observation is due to Bott and Samelson [6; Proposition 9.2].)

Consider the integral homology spectral sequence $\{E^r, d^r\}$ of the fibration

$$\Omega\,\mathrm{Spin}(9) \to \Omega F_4 \to \Omega\Pi ,$$

so that $E^2_{p,q}=H_p(\Omega\Pi)\otimes H_q(\Omega\,\mathrm{Spin}(9))$ and $E^\infty_{p,q}=GrH_{p+q}(\Omega F_4)$. Note that this spectral sequence is multiplicative with respect to the Pontrjagin product in the usual sense (see [13; §1]). Using Lemma 2 (ii), we see that $E^2=E^7$ and $\alpha\in E^2_{7,0}$ is transgressive. Comparing (1.1) with (1.2) shows that the only element of $E^2_{0,6}$ which must be killed in E^r (for some r) is $\sigma_2\sigma_1-3\sigma_3$. We therefore have $d^7(\alpha\otimes 1)=1\otimes(\sigma_2\sigma_1-3\sigma_3)$, which gives

$$E^8 = Z[\beta]\otimes Z[\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_7]/(\sigma_1^2-2\sigma_2, \sigma_2\sigma_1-3\sigma_3) .$$

It follows from dimensional reasons that $d^r=0$ for $r\geq 8$. Hence $E^8=E^\infty$. Since $H_*(\Omega F_4)$ is commutative, no extension problem can occur and the result follows. q.e.d.

2. Review of Bott's work

In this section we collect some results concerning the cohomology of ΩG and related spaces. For details and proofs see [2], [3] and [5].

Suppose G is simple and simply connected as before. Then the rational cohomology ring of ΩG is given by

$$H^*(\Omega G; Q) = Q[u_1, u_2, \cdots, u_l]$$

where l=rank G and deg $u_i=2k_i$ with $1=k_1<k_2<\cdots<k_l$. (This last condition is not satisfied for $G=\mathrm{Spin}(2l)$; we shall omit it in the sequel.) Moreover, each u_i can be chosen to be primitive. These facts imply that in $H^{2k_i}(\Omega G)$ there exists only one primitive element p_i which is not divisible (where we do not mind the sign), and further that

(2.1) $$PH^*(\Omega G) = Z\{p_1, p_2, \cdots, p_l\} .$$

Suppose given a homomorphism s: $S^1 \to G$ of the circle into G, whose image is denoted by T^1. Let T be a maximal torus of G containing T^1, and C_s be the centralizer of T^1 in G. Then we have inclusions $T\subset C_s \subset G$ and a fibration

$$C_s/T \to G/T \xrightarrow{\tau_s} G/C_s .$$

Since $H^*(C_s/T)$, $H^*(G/T)$ and $H^*(G/C_s)$ are all torsion free and even-dimensional [4], it follows that

(2.2) τ_s^*: $H^*(G/C_s) \to H^*(G/T)$ is a split monomorphism.

Consider next the fibration

$$G/T \xrightarrow{\iota} BT \xrightarrow{\rho} BG$$

where BT and BG are the classifying spaces for T and G respectively. The following isomorphisms are elementary:

$$\mathrm{Hom}(T, S^1) \cong H^1(T) \cong H^2(BT) \cong H^2(G/T) .$$

By identifying these, we may view the roots or weights as elements of $H^1(T)$ etc. In particular for the fundamental weights ω_i $(1\leq i\leq l)$, we have

$$H^*(BT) = Z[\omega_1, \omega_2, \cdots, \omega_l]$$

on which the Weyl group $\Phi(G)$ acts in a natural way. Then ι induces an isomorphism

(2.3) $$H^*(BT; Q)/I_G \cong H^*(G/T; Q)$$

where I_G denotes the ideal generated in $H^*(BT; Q)$ by homogeneous invariants of $\Phi(G)$ having strictly positive degrees.

Suppose given a representation $\lambda\colon G\to U(n)$ with weights $\mu_1, \mu_2, \cdots, \mu_n\in H^2(BT)$. Its k-th Chern class $c_k(\lambda)$ is defined to be the k-th elementary symmetric function in the μ_j: $c_k(\lambda)=\sigma_k(\mu_1, \mu_2, \cdots, \mu_n)$. Let $I_k(\lambda)=\mu_1^k+\mu_2^k+\cdots+\mu_n^k$. $c_k(\lambda)$ and $I_k(\lambda)$ are related with each other by the Newton formula:

$$I_k(\lambda)=\sum_{1\leq j<k}(-1)^{j-1}c_j(\lambda)I_{k-j}(\lambda)+(-1)^{k-1}kc_k(\lambda)\,. \tag{2.4}$$

With (an arbitrary homomorphism) $s\colon S^1\to G$, we associate the following two maps. Let

$$f_s\colon G/C_s\to \Omega G$$

be defined by

$$f_s(q)\,(t)=g\cdot s(t)\cdot g^{-1}$$

for $q=gC_s\in G/C_s$ and $t\in S^1$. On the other hand, by the dual isomorphisms

$$\mathrm{Hom}(S^1, T)\cong H_1(T)\cong H_2(BT)\cong H_2(G/T)\,,$$

s (whose image is contained in T) may be considered as an element of $H_1(T)$ etc. Using this convention, we define

$$\theta_s\colon H^{q+1}(BT)\to H^{q-1}(BT)$$

to be the derivation which extends the assignment $\omega\to\langle\omega, s\rangle$, for $\omega\in H^2(BT)$, where $\langle\ ,\ \rangle$ stands for the Kronecker index.

Now we consider the case of $SU(n+1)$. As is well known,

$$H^*(BSU(n+1))=Z[c_2, c_3, \cdots, c_{n+1}]$$

where c_{j+1} ($\deg c_{j+1}=2j+2$) is the $(j+1)$-th universal Chern class for $j=1,2,\cdots,n$. Set $G'=SU(n+1)$. Let

$$\sigma_E^*\colon H^{q+1}(BG')\to H^q(G')$$

and

$$\sigma_P^*\colon H^q(G')\to H^{q-1}(\Omega G')$$

be the cohomology suspensions associated with the fibrations

$$G'\to EG'\to BG'$$

and

$$\Omega G'\to PG'\to G'$$

respectively. Then we have

Lemma 3. *For $j=1, 2, \cdots, n$, the element $p'_j=\sigma_P^*\sigma_E^*(c_{j+1})$ is primitive and not divisible in $H^{2j}(\Omega G')$. That is,*

$$PH^*(\Omega G') = Z\{p'_1, p'_2, \cdots, p'_n\} .$$

Proof. Recall first the following results:

$$H^*(G') = \Lambda(x_3, x_5, \cdots, x_{2n+1})$$

with deg $x_{2j+1}=2j+1$ and

$$H_*(\Omega G') = Z[\sigma_1, \sigma_2, \cdots, \sigma_n]$$

with $\deg \sigma_j=2j$. By Borel's transgression theorem [1; Théorèm 19.1], $\sigma_E^*(c_{j+1})=x_{2j+1}$ and so each x_{2j+1} is primitive. Thus the problem reduces to showing that the map s_P^*: $QH^*(G')\to PH^*(\Omega G')$ induced by σ_P^* is split monic. It is then enough to verify that the dual map s_*^P: $QH_*(\Omega G')\to PH_*(G')$ is epic. But this is an exercise of the homology Eilenberg-Moore spectral sequence (see [8; §4]). q.e.d.

Hereafter we simply write λ for the composite

$$G \to U(n)\subset SU(n+1) = G' .$$

Let s' be the composite λs: $S^1\to G'$, T' a maximal torus of G' containing $\lambda(T)$, and $C_{s'}$ the centralizer of $\lambda(T^1)$ in G'. A similar treatment holds for the pair (G', s'). Specifically we have, with the obvious notation,

$$\tau_{s'}^* f_{s'}^* \sigma_P^* \sigma_E^* = \iota^* \theta_{s'} \rho^* . \tag{2.5}$$

This key formula was established in [5; §7].

Proposition 4. *Let $k=k_i$ for $i=1, 2, \cdots, l$. Then $\iota^*\theta_s(c_{k+1}(\lambda))$ is an integer multiple of $\tau_s^* f_s^*(p_i)$ in $H^{2k}(G/T)$.*

Proof. The homomorphism λ induces a homomorphism $\tilde{\lambda}$: $T\to T'$, maps $\bar{\lambda}$: $G/T\to G'/T'$ and $\bar{\bar{\lambda}}$: $G/C_s\to G'/C_{s'}$ so that appropriate diagrams can be (homotopy) commutative. We first show that $\iota^*\theta_s B\tilde{\lambda}^*=\bar{\lambda}^*\iota^*\theta_{s'}$. By the naturality of the Kronecker index, $\langle B\tilde{\lambda}^*(\omega), s\rangle=\langle \omega, B\tilde{\lambda}_*(s)\rangle=\langle \omega, s'\rangle$ for $\omega\in H^2(BT')$. Then it follows that $\theta_s B\tilde{\lambda}^*=B\tilde{\lambda}^*\theta_{s'}$ and hence $\iota^*\theta_s B\tilde{\lambda}^*=\iota^* B\tilde{\lambda}^*\theta_{s'} =\bar{\lambda}^*\iota^*\theta_{s'}$.

Now since $\Omega\lambda^*$: $H^*(\Omega G')\to H^*(\Omega G)$ is a homomorphism of Hopf algebras over Z, we have

$$\Omega\lambda^*(p'_k) = a\cdot p_i$$

for some $a\in Z$. But $\tau_s^* f_s^* \Omega\lambda^*(p'_k)=\tau_s^*\bar{\bar{\lambda}}^* f_{s'}^*(p'_k)=\bar{\lambda}^*\tau_{s'}^* f_{s'}^*(p'_k)=\bar{\lambda}^*\tau_{s'}^* f_{s'}^*\sigma_P^*\sigma_E^*(c_{k+1})$, which equals $\bar{\lambda}^*\iota^*\theta_{s'}\rho^*(c_{k+1})$ by (2.5). On the other hand, since $c_{k+1}(\lambda)$

$=B\tilde{\lambda}^*\rho^*(c_{k+1})$, it follows that $\iota^*\theta_s(c_{k+1}(\lambda))=\iota^*\theta_s B\tilde{\lambda}^*\rho^*(c_{k+1})=\bar{\lambda}^*\iota^*\theta_{s'}\rho^*(c_{k+1})$. Combining these, it follows that $\iota^*\theta_s(c_{k+1}(\lambda))=a\cdot\tau_s^*f_s^*(p_i)$. q.e.d.

From now on we assume that G has trivial center. Then the simple roots $\alpha_i(1\leq i\leq l)$ constitute a base for $H^1(T)$. According to Bott [5; §§1 and 5], if $s\in H_1(T)$ is dual to a long root, then (s becomes a generating circle and) f_s has the property that the image of f_{s*}: $H_*(G/C_s)\rightarrow H_*(\Omega G)$ generates the algebra $H_*(\Omega G)$. Dualization then gives

(2.6) f_s^*: $H^*(\Omega G)\rightarrow H^*(G/C_s)$ is a split monomorphism when restricted to $PH^*(\Omega G)$.

To use this fact we shall take such an s.

We can now characterize the generators p_i in (2.1).

Proposition 5. *Under the hypotheses and notations as above, if $k=k_i$ for $i=1,2,\cdots,l$ and $\bar{q}_k\in H^{2k}(G/T)$ is a unique element such that $\bar{q}_k$ is not divisible and*

$$\iota^*\theta_s(c_{k+1}(\lambda)) = \bar{a}\cdot\bar{q}_k$$

for some $\bar{a}\in Z$, then

(i) *The following properties of a primitive element $\bar{p}_k\in H^{2k}(\Omega G)$ are equivalent:*

(1) *$\bar{p}_k$ is not divisible, i.e., $\bar{p}_k=p_i$,*

(2) *$f_s^*(\bar{p}_k)$ is not divisible,*

(3) *$\tau_s^*f_s^*(\bar{p}_k)$ is not divisible,*

(4) *$\tau_s^*f_s^*(\bar{p}_k)=\bar{q}_k$.*

(ii) *There is a unique element $q_k\in H^{2k}(G/C_s)$ such that $\tau_s^*(q_k)=\bar{q}_k$. Then q_k is not divisible, and p_i is uniquely determined by q_k via $f_s^*(p_i)=q_k$.*

Proof. By (2.6), (1) is equivalent to (2). By (2.2), (2) is equivalent to (3). Clearly (4) implies (3). Conversely, suppose (3) (and so (1)) is given. By Proposition 4 and the definition of $\bar{q}_k$, $a\cdot\tau_s^*f_s^*(\bar{p}_k)=\iota^*\theta_s(c_{k+1}(\lambda))=\bar{a}\cdot\bar{q}_k$. But by uniqueness, $\tau_s^*f_s^*(\bar{p}_k)=\bar{q}_k$ (and $a=\bar{a}$). This completes the proof of (i). (ii) is only a corollary of (i). q.e.d.

Therefore we conclude:

(2.7) In order to characterize p_i, we must find q_k in $H^{2k}(G/C_s)$ by computing $\iota^*\theta_s(c_{k+1}(\lambda))$ for suitable s and λ, where $k=k_i$ $(1\leq i\leq l)$.

Lemma 6. $\iota^*\theta_s(I_k(\lambda)) = (-1)^{k-1}k\cdot\iota^*\theta_k(c_k(\lambda))$.

Proof. Since the set $\{\mu_1, \mu_2, \cdots, \mu_n\}$ is invariant under the action of $\Phi(G)$, it follows from (2.3) that $\iota^*(c_j(\lambda))=\iota^*(I_j(\lambda))=0$. Then the lemma follows from (2.4) and the derivativity of θ_s. q.e.d.

3. The primitive elements in $H^*(\Omega F_4)$

Since F_4 has trivial center, the argument developed in the previous section can be applied to F_4. In this case, let us carry the project (2.7) into practice.

First note that $l=4$ and $(k_1,k_2,k_3,k_4)=(1,5,7,11)$. We use the root system given in [7], where the fundamental weights ω_i are expressed in terms of the simple roots α_i as follows:

$$\begin{aligned}\omega_1 &= 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\,,\\ \omega_2 &= 3\alpha_1+6\alpha_2+8\alpha_3+4\alpha_4\,,\\ \omega_3 &= 2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4\,,\\ \omega_4 &= \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\,.\end{aligned} \tag{3.1}$$

Here long roots are α_1, α_2 and so forth. Hence we take

$$s = \text{the dual of } -\alpha_1\,.$$

Then C_s turns out to be $T^1\cdot Sp(3)$ with $T^1\cap Sp(3)=Z_2$. Set $V=F_4/T^1\cdot Sp(3)$. In [11] Ishitoya and Toda have computed the ring structure of $H^*(V)$. Their result is

$$H^*(V) = Z[t, u, v, w]/(t^3-2u,\ u^2-3t^2v+2w,\ 3v^2-t^2w,\ v^3-w^2) \tag{3.2}$$

where deg $t=2$, deg $u=6$, deg $v=8$ and deg $w=12$. Besides we need the following information on the generators t,u,v and w (see [11; §4]): Put

$$t = \omega_1,\ y_1 = \omega_2-\omega_3,\ y_2 = \omega_3-\omega_4 \text{ and } y_3 = \omega_4;$$

let $z_i=y_i(t-y_i)$ and let $q_i=\sigma_i(z_1, z_2, z_3)$ for $i=1, 2, 3$; then

$$q_1 = t^2,\ q_2 = 3v \quad\text{and}\quad q_3 = w \tag{3.3}$$

where these elements are regarded as those of $H^*(F_4/T; Q)=Q[t,y_1,y_2,y_3]/I_{F_4}$.

For convenience we introduce the notation:

$$x = \frac{1}{2}t \ \text{ and } \ x_i = x-y_i\ (i = 1, 2, 3)\,.$$

Then $H^*(BT; Q)=Q[x,x_1,x_2,x_3]$. In view of (3.1), the derivation associated with our s is represented by

$$\theta_s = -\frac{\partial}{\partial x}\colon Q[x, x_1, x_2, x_3] \to Q[x, x_1, x_2, x_3]\,. \tag{3.4}$$

Let $p_i=\sigma_i(x_1^2,x_2^2,x_3^2)$ $(i=1,2,3)$ and $s_n=x_1^n+x_2^n+x_3^n$ $(n\geq 0)$. We get again the Newton formula

$$s_{2n} = \sum_{1\leq i<n}(-1)^{i-1}p_i s_{2n-2i}+(-1)^{n-1}np_n \tag{3.5}$$

with the convention $p_n=0$ for $n>3$. By definition, $z_i=y_i(t-y_i)=(x-x_i)(x+x_i)=x^2-x_i^2$. Then

$$\begin{aligned}\sum p_i &= \prod(1+x_i^2) = \prod(1+x^2-z_i) = \sum(-1)^i q_i(1+x^2)^{3-i}\\ &= \sum(-1)^i q_i\left(\sum\binom{3-i}{j}x^{2j}\right)\end{aligned}$$

which gives a formula

$$p_k = \sum_{i+j=k}(-1)^i\binom{3-i}{j}q_i x^{2j}\,. \tag{3.6}$$

Next we take

$\lambda =$ the irreducible representation with highest weight ω_4.

By making use of 47.8 and 43.1.10 of [10], one can check that dim $\lambda=26$ and the set of weights of λ is given by

$$I = \{\pm x\pm x_i,\ \pm x_i\pm x_j(1\leq i<j\leq 3),\ 0,\ 0\}\,.$$

Put

$$\begin{aligned} J &= \{\pm x\pm x_i\}\,, \qquad & J_k &= \sum_{y\in J} y^k;\\ K &= \{\pm x_i\pm x_j\}\,, \qquad & K_k &= \sum_{y\in K} y^k\,.\end{aligned}$$

Since $I=J\cup K\cup\{0, 0\}$, it follows that $I_k(\lambda)=J_k+K_k$ for $k>0$. Then $\theta_s(I_k(\lambda))=\theta_s(J_k)$ by (3.4). Since

$$\begin{aligned}\sum J_k/k! &= \sum_{y\in J} e^y = (e^x+e^{-x})\cdot\sum(e^{x_i}+e^{-x_i})\\ &= (2\sum x^{2j}/(2j)!)\cdot(2\sum s_{2n}/(2n)!)\,,\end{aligned}$$

it follows that $J_{2k}=4\sum_{0\leq j\leq k}\binom{2k}{2j}s_{2k-2j}x^{2j}$ (and $J_{2k+1}=0$). Using these together with Lemma 6, we obtain a formula

$$\iota^*\theta_s(c_{2k}(\lambda)) = \frac{4}{k}\sum_{1\leq j\leq k} j\binom{2k}{2j}s_{2k-2j}x^{2j-1}\,. \tag{3.7}$$

The above discussion is summarized in the figure below.

$$\iota^*\theta_s(c_{2k}(\lambda)) \xrightarrow{(3.7)} s_{2n},\ x$$

$$s_{2n} \xrightarrow{(3.5)} p_i \xrightarrow{(3.6)} q_i,\ x$$

$$q_i \xrightarrow{(3.3)} t,\ v,\ w$$

where "$A\xrightarrow{X}B$" means that X expresses A in terms of B. A direct calculation following these arrows and using the relations in (3.2) yields:

(3.8)

k	$\iota^*\theta_s(c_{k+1}(\lambda))$	
1	$6\,t$	
5	$12\,b$	$b = t^2u-5tv$
7	$30\,c$	$c = 2uv-3tw$
11	$270\,d$	$d = 3tvw-2uv^2$

Observe that the elements t, b, c and d are not divisible in $H^{2k}(V)$ for $k=1, 5, 7$ and 11 respectively.

Proposition 7. *There exists a unique primitive element a_1 [resp. b_5, c_7 and d_{11}] of $H^*(\Omega F_4)$ such that $f_s^*(a_1)=t$ [resp. $f_s^*(b_5)=b$, $f_s^*(c_7)=c$ and $f_s^*(d_{11})=d$]. Then*

$$PH^*(\Omega F_4) = Z\{a_1, b_5, c_7, d_{11}\} .$$

This is a consequence of Proposition 5 (ii) and (3.8).

4. The coalgebra structure of $H_*(\Omega F_4)$

In this section we display our computation of the cohomology ring $H^*(\Omega F_4)$ for dim. ≤ 10, which gives a partial proof of parts (ii) and (iii) of Theorem 1. To prove the whole we need to determine it for dim. ≤ 22 (see Theorem 1 (i)). However, as will be seen, the remainder is no more than a tedious computation and is left to the reader.

We choose an additive basis of $H^*(V)$ for dim. ≤ 22 as follows (cf. [11; Corollary 4.5]):

(4.1)

deg =	0	2	4	6	8	10	12	14	16	18	20	22
	1	t	t^2	u	tu	b	t^2v	c	tuv	t^2uv	vw	d
					v	b'	w	c'	v^2	x	tx	d'

where $x=uw-tv^2$; b, c, d are given in (3.8); and b', c', d' are determined by the following equations:

$$B\cdot\begin{pmatrix} t^2u \\ tv \end{pmatrix} = \begin{pmatrix} b \\ b' \end{pmatrix},\; C\cdot\begin{pmatrix} uv \\ tw \end{pmatrix} = \begin{pmatrix} c \\ c' \end{pmatrix},\; D\cdot\begin{pmatrix} tvw \\ t^2x \end{pmatrix} = \begin{pmatrix} d \\ d' \end{pmatrix}$$

where B, C, D are 2×2 matrices over Z whose determinant is 1; for example, $B=\begin{pmatrix} 1 & -5 \\ k & l \end{pmatrix}$ with $k, l \in Z$ such that $5k+l=1$, and then $b=k\,t^2u+l\,tv$.

With respect to this basis, let α, β, γ and δ be the duals of t, b, c and d respectively. Then we may set

$$\sigma_1 = f_{s*}(\alpha),\; \sigma_5 = f_{s*}(\beta),\; \sigma_7 = f_{s*}(\gamma) \text{ and } \sigma_{11} = f_{s*}(\delta)\,,$$

for this notation fits in with that used in Theorem 1 (i). In fact, Proposition 7 assures us that $\sigma_1, \sigma_5, \sigma_7$ and σ_{11} are indecomposable and not divisible in $H_*(\Omega F_4)$.

Next, by Theorem 1(i), we choose an additive basis of $H_*(\Omega F_4)$ for dim. ≤ 22 as follows:

deg = 0	2	4	6	8	10	12	14	16	18	20	22
1	σ_1	σ_2	σ_3	σ_4	$\sigma_4\sigma_1$	σ_6	$\sigma_6\sigma_1$	$\sigma_6\sigma_2$	$\sigma_6\sigma_3$	$\sigma_6\sigma_4$	$\sigma_6\sigma_4\sigma_1$
					σ_5	$\sigma_5\sigma_1$	$\sigma_5\sigma_2$	$\sigma_5\sigma_3$	$\sigma_5\sigma_4$	$\sigma_5\sigma_4\sigma_1$	$\sigma_5\sigma_6$
							σ_7	$\sigma_7\sigma_1$	$\sigma_7\sigma_2$	$\sigma_7\sigma_3$	$\sigma_7\sigma_4$
										σ_5^2	$\sigma_5^2\sigma_1$
											σ_{11}
1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
					b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
							c_7	c_8	c_9	c_{10}	c_{11}
										b'_{10}	b'_{11}
											d_{11}

where $\sigma_4=\sigma_2^2-\sigma_3\sigma_1$ and $\sigma_6=\sigma_2^3-4\sigma_3^2$; the reader should notice that the relations $\sigma_2=\sigma_1^2/2$, $\sigma_3=\sigma_1^3/6$, $\sigma_4=\sigma_1^4/12$ and $\sigma_6=\sigma_1^6/72$ hold in $H_*(\Omega F_4; Q)$. The lower table indicates the corresponding dual basis.

Then the aspect of our computation is described by the following table:

deg	coproduct	relation	base	f_s^*-image
2	$\tilde{\psi}(\sigma_1)=0$		a_1	t
4	$\tilde{\psi}(\sigma_2)=\sigma_1\otimes\sigma_1$	$a_1^2=a_2$	$a_2=a_1^2$	t^2
6	$\tilde{\psi}(\sigma_3)=\sigma_2\otimes\sigma_1$	$a_2a_1=a_3$	$a_3=a_2a_1$	$2u$
8	$\tilde{\psi}(\sigma_4)=2\sigma_3\otimes\sigma_1+2\sigma_2\otimes\sigma_2$	$a_3a_1=2a_4,\ a_2^2=2a_4$	$a_4=\frac{1}{2}a_3a_1$	tu

Now we confront the case of degree 10. A base for $H_{10}(\Omega F_4)$ is given by $\{\sigma_4\sigma_1, \sigma_5\}$. Since $\sigma_4\sigma_1=\sigma_1^5/12$, it follows that $\tilde{\psi}(\sigma_4\sigma_1)=5\sigma_4\otimes\sigma_1+10\sigma_3\otimes\sigma_2$. Suppose that $\tilde{\psi}(\sigma_5)=m\sigma_4\otimes\sigma_1+\cdots$, for some $m\in Z$. Then $a_4a_1=5a_5+mb_5$ and hence $5f_s^*(a_5)=f_s^*(a_4a_1-mb_5)=t^2u-mb=(1-m)t^2u+5mtv$. On the other hand, since $\langle f_s^*(a_5), \beta\rangle=\langle a_5, f_{s*}(\beta)\rangle=\langle a_5, \sigma_5\rangle=0$, it follows that $f_s^*(a_5)=nb'$ for some $n\in Z$. Conbining these gives

$$(1-m)t^2u+5mtv = 5n(kt^2u+ltv)\,.$$

Since $\{t^2u, tv\}$ is a base, we have

$$1-m = 5kn \quad\text{and}\quad m = ln\,.$$

But since $5k+l=1$, it follows that $n=1$. For simplicity we may take $m=1$;

simultaneously $k=0$ and $l=1$. Thus we have shown:

deg	coporduct	relation	base	f_s^*-image
10	$\tilde{\psi}(\sigma_4\sigma_1)=5\sigma_4\otimes\sigma_1+10\sigma_3\otimes\sigma_2$	$a_4a_1=5a_5+b_5$	$a_5=\frac{1}{5}a_4a_1-b_5$	$b'=tv$
	$\tilde{\psi}(\sigma_5)=\sigma_4\otimes\sigma_1+2\sigma_3\otimes\sigma_2$	$a_3a_2=10a_5+2b_5$	b_5	b

In this way we can determine the cohomology ring $H^*(\Omega F_4)$ so as to realize the situation (2.6). In practice, we have settled

$$c'=uv-tw \quad\text{and}\quad d'=-tvw+t^2x$$

in (4.1).

Note. There is a misprint in Bott's result on $H_*(\Omega G_2)$ [5;p.60]. The coproduct formula for $w\in H_{10}(\Omega G_2)$ is an error. It is corrected by exchanging 2 for 3. In this connection see also [9;Note on p.17].

OSAKA CITY UNIVERSITY

References

[1] A. Borel: *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

[2] ———: *Topology of Lie groups and characteristic classes*, Bull. Amer. Math. Soc. **61** (1955). 397–432.

[3] ——— and F. Hirzebruch: *Characteristic classes and homogeneous spaces*, I, Amer. J. Math. **80** (1958), 458–538.

[4] R. Bott: *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281.

[5] ———: *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.

[6] ——— and H. Samelson: *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029.

[7] N. Bourbaki: Groupes et algèbre de Lie, IV-VI, Hermann, Paris, 1968.

[8] A. Clark: *Homotopy commutativity and the Moore spectral sequence*, Pacific J. Math. **15** (1965), 65–74.

[9] F. Clarke: *On the K-theory of the loop space of a Lie group*, Proc. Camb. Phil. Soc. **76** (1974), 1–20.

[10] H. Freudenthal and H. de Vries: Linear Lie groups, Academic Press, London and New York, 1969.

[11] K. Ishitoya and H. Toda: *On the cohomology of irreducible symmetric spaces of exceptional type*, to appear in J. Math. Kyoto Univ.

[12] M. Mimura: *The homotopy groups of Lie groups of low rank*, J. Math. Kyoto Univ. **6** (1967), 131–176.

[13] J.C. Moore: *The double suspension and p-primary components of the homotopy groups of spheres*, Bol. Soc. Mat. Mexicana **1** (1956), 28–37.