Yamato, K.
Osaka J. Math.
16 (1979), 589-604

ON EXOTIC CHARACTERISTIC CLASSES OF CONFORMAL AND PROJECTIVE FOLIATIONS

KENJI YAMATO*)

(Received May 8, 1978)

Introduction

R. Bott [1] and A. Haefliger [3] defined exotic characteristic classes of foliations. In this paper, we shall study exotic characteristic classes of locally homogeneous conformal and projective foliations with trivialized normal bundles. Our purpose is to decide whether these exotic characteristic classes vanish always or not in general.

Let Γ be a pseudogroup acting *transitively* on a smooth manifold B of dimension n. A *locally homogeneous* Γ*-foliation* of condimension n on a manifold M is by definition a maximal family $\mathfrak{F}$ of submersions f_α: $U_\alpha \to B$ of open sets U_α in M such that the family $\{U_\alpha\}$ is an open covering and for each $x \in U_\alpha \cap U_\beta$ there exists and element $\gamma^x_{\alpha\beta} \in \Gamma$ with $f_\beta = \gamma^x_{\alpha\beta} \cdot f_\alpha$ in some neighbourhood of x. If the above Γ is consisting of locally conformal (resp. projective) transformations on B, $\mathfrak{F}$ is called *locally homogeneous conformal* (resp. *projective*) *foliation.*

Let $\mathfrak{F}$ be a foliation of codimension n on M with trivialized normal bundle and t the trivialization. Exotic characteristic classes of $(\mathfrak{F}, t)$ are defined as the images of the mapping

$$\lambda^*_{(\mathfrak{F},t)}\colon H^*(W_n) \to H^*_{DR}(M)$$

which depends only on $\mathfrak{F}$ and t ([1], [3]). The Vey-basis $\{Z_{(I,J)}\}$ of $H^*(W_n)$ is consisting of the following cohomology classes [4]

$$Z_{(I,J)} = [h_{j_0} \wedge h_{j_1} \wedge \cdots \wedge h_{j_k} \otimes (c_1)^{i_1} \cdots (c_n)^{i_n}]\,,$$

where $I=(i_1, \cdots, i_n)$ and $J=(j_0, \cdots, j_k)$ with

$$1 \leqq j_0 < j_1 < \cdots < j_k \leqq n,\ k \geqq 0,\ j_0 + \sum_{r=1}^{n} r i_r \geqq n+1,\ \sum_{r=1}^{n} r i_r \leqq n\,,$$

and $i_r = 0$ for $r < j_0$.

We devide these elements of Vey-basis into following three types;

(I) $\quad j_0 + \sum_r r i_r > n+1$ (*i.e.* rigid classes [4])

*) Supported by Grant-in-Aid for Scientific Research

(II. 1) $j_0+\sum_r ri_r=n+1$, and j_r is odd for some $r>0$.

(II. 2) $j_0+\sum_r ri_r=n+1$, and j_r are even for all $r>0$.

We have

Theorem. *Let $\mathfrak{F}$ be a locally homogeneous conformal (resp. projective) foliation of codimension $n(\geqq 3)$ on a manifold M with trivialized normal bundle. We have the following table.*

	Structure of $\mathfrak{F}$		
Types of Vey basis	Projective	conformal	
		n=odd	n=even
(I)	zero	zero	zero
(II. 1)	non-zero	zero	zero
(II. 2)	non-zero	non-zero	(*)

In case of (*), $Z_{(I,J)}$ with $J=\{j_0\}$ are zero at least.

REMARK. All of the exotic characteristic classes of riemannian foliations are always zero.

Recently, S. Morita [9] defined secondary characteristic classes for projective and conformal foliations and obtained the same results, without our assumption "locally homogeneous."

This paper is devided into 6 sections.

The example of the locally homogeneous conformal (resp. projective) foliations with non-trivial exotic characteristic classes is known as the typical example of conformal (resp. projective) foliations. In § 1, we review briefly these constructions following [5], [8] and [10], and these exotic characteristic classes will be calculated in § 4.

To perform generally the calculation of the exotic characteristic classes of our foliations, we stand on the Haefliger's definition of exotic characteristic classes ([3], for this, we require to assume "local homogeneity") and we use the method of calculation of F. Kamber and Ph. Tondeur [5]. Moreover, existence of normal Cartan connection plays an important role to show that the rigid classes are zero always. In § 2, we state the main Lemmas, one of these lemmas is concerning existence of "infinitesimal" normal Cartan connection and the other is a diagram for use their method of calculation. The proofs will be given in § 6. § 3 is devoted to prove vanishing of exotic characteristic classes of type (I) and for conformal case type (II.1). In § 5, we prove vanishing of exotic characteristic classes in case of (*).

1. Typical examples

Non-vanishing in our theorem is due to the following examples.

EXAMPLE 1 (locally homogeneous conformal foliation). Set

$$L = O(n+1, 1)$$
$$\cong \{X \in GL(n+2, \boldsymbol{R});\ {}^tXSX = S\}, \text{ where } S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$L_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ * & A & 0 \\ * & * & a^{-1} \end{pmatrix} \in L; A \in O(n),\ a \in \boldsymbol{R} \right\}.$$

Define subgroups $H \subset G$ of the Lie group L by

G=the identity component of L,

$H = L_0 \cap G$.

Then we have a foliation $\bar{\mathfrak{F}}$ on G whose leaves are fibres of fibration $G \to G/H$. Let D be a discrete subgroup of G such that $M = D\backslash G$ is a closed manifold ([2]). Since the foliation $\bar{\mathfrak{F}}$ is invariant under the left-action of G, we have a foliation $\mathfrak{F}$ defined on $M = D\backslash G$ of codimension n with trivial normal bundle, which is locally homogeneous conformal foliation.

EXAMPLE 2 (locally homogeneous projective foliation). Set

$$L = PGL(n+1, \boldsymbol{R}) \cong SL(n+1, \boldsymbol{R})/\text{center}\,,$$

$$L_0 = \left\{ \begin{pmatrix} A & 0 \\ * & a \end{pmatrix} \in L;\ A \in GL(n, \boldsymbol{R}),\ a \in \boldsymbol{R} \right\}.$$

Define subgroups $H \subset G$ of L by

G=the identity component of L,

$H = L_0 \cap G$.

Since there exists a discrete subgroup D of G such that $M = D\backslash G$ is a closed manifold, by the same method we have a locally homogeneous projective foliation $\mathfrak{F}$ on $M = D\backslash G$ of codimension n with trivial normal bundle.

For more detail, see [10].

2. Main lemmas

In this section, $\mathfrak{F}$ is as in Theorem, t denotes the trivialization and the base manifold B of dimension n is fixed.

Let $\underline{\mathfrak{l}}_B$ denote the Lie algebra of formal conformal (resp. projective) vector fields at $b_0 \in B$. Since $\underline{\mathfrak{l}}_B$ is a subalgebra of the Lie algebra $\mathfrak{a} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \cdots$ of formal vector fields at $b_0 \in B$, $\underline{\mathfrak{l}}_B$ has a filtered Lie algebra structure

$$\underline{\mathfrak{l}}_B = \underline{\mathfrak{l}}_B^{-1} \supset \underline{\mathfrak{l}}_B^0 \supset \underline{\mathfrak{l}}_B^1 \cdots$$

and for $\xi \in \underline{\mathfrak{l}}_B$ we have the following expression

$$\xi = \xi_{-1} + \xi_0 + \xi_1 + \cdots, \text{ where } \xi_j \in \mathfrak{a}_i .$$

If the conformal (resp. projective) structure of B is flat ([11]), $\underline{\mathfrak{l}}_B$ (resp. $\underline{\mathfrak{l}}_B^i$) is denoted by $\underline{\mathfrak{l}}$ (resp. $\underline{\mathfrak{l}}^i$). Remark that the Lie algebra $\underline{\mathfrak{l}}$ (resp. $\underline{\mathfrak{l}}^0$) is anti-isomorphic to the Lie algebra $\mathfrak{l}$ (resp. $\mathfrak{l}^0$) of the Lie group L (resp. L_0) of the examples in Section 1.

From Kobayashi and Nagano [6], we have the following. The proof will be given in Section 6

Proposition 2.1.
(1) $\underline{\mathfrak{l}}_B^k = 0$ *for* $k \geqq 2$, *and* $\dim \underline{\mathfrak{l}}_B \leqq \dim \underline{\mathfrak{l}} = \dim \mathfrak{l}$
(2) $\dim \underline{\mathfrak{l}}/\underline{\mathfrak{l}}_B^0 = n$ (*local homogeneity*).

The following two lemmas are proved in Section 6.

Lemma 1. *There exist an injective linear mapping* $\omega\colon \underline{\mathfrak{l}}_B \to \mathfrak{l}$ *and a linear mapping* $\theta_1\colon \underline{\mathfrak{l}}_B \to \mathfrak{l}^0$ *such that*
(1) $(\omega - \theta_1)|_{\underline{\mathfrak{l}}_B^0} = 0$

(2) $i(\xi)\Omega = 0$ *for* $\xi \in \underline{\mathfrak{l}}_B^0$, *where* $\Omega = d\omega + [\omega, \omega]/2$ (*that is,* $\omega([\xi, \eta]) = [\omega(\xi), \omega(\eta)]$ *for* $\xi \in \underline{\mathfrak{l}}_B^0$, $\eta \in \underline{\mathfrak{l}}_B$), *especially* $\Omega^k = 0$ *if* $k > [n/2]$.

(3) $\Omega_1(\xi, \eta) = 0$ *for* $\xi, \eta \in \underline{\mathfrak{l}}_B^0$ *where* $\Omega_1 = d\theta_1 + [\theta_1, \theta_1]/2$ (*that is,* $\theta_1|_{\underline{\mathfrak{l}}_B^0}\colon \underline{\mathfrak{l}}_B^0 \to \mathfrak{l}^0$ *is a homomorphism of Lie algebras*), *especially* $(\Omega_1)^k = 0$ *if* $k > n$.

Let

$$\lambda_{(\mathfrak{F}, t)}\colon \Lambda \mathfrak{a}^* \to A^*(M) \text{ (resp. } \lambda_{(\mathfrak{F}, t), \Gamma}\colon \Lambda \underline{\mathfrak{l}}_B^* \to A^*(M))$$

be the characteristic homomorphism of smooth (resp. locally homogeneous Γ-) foliations $(\mathfrak{F}, t)$ defined by Haefliger [3]. Following Kamber and Tondeur [5], we have a unique DGA-homomorphism

$$\Delta(\theta_1)\colon W(\mathfrak{l}^0)_n \to \Lambda \underline{\mathfrak{l}}_B^*$$

satisfying

$$\Delta(\theta_1)(\alpha) = \alpha \circ \theta_1 \quad \text{for } \alpha \in (\mathfrak{l}^0)^* = \Lambda^1(\mathfrak{l}^0)^* ,$$

where $W(\mathfrak{g})_n$ denotes the n-truncated Weil algebra of Lie algebra $\mathfrak{g}$.

Lemma 2. *The following diagram commutes*:

$$\begin{array}{ccccccc}
H^*(W_n) & \xrightarrow{\cong} & H^*(W(\mathfrak{gl}(n, \boldsymbol{R}))_n) & \xrightarrow{\cong} & H^*(\mathfrak{a}) & \xrightarrow{\lambda^*_{(\mathfrak{F},t)}} & H^*_{DR}(M) \\
 & & \Big\downarrow W(\rho^*) & & \Big\downarrow j^*_B & \nearrow \lambda^*_{(\mathfrak{F},t),\Gamma} & \\
 & & H^*(W(\mathfrak{l}^\circ)_n) & \xrightarrow[\Delta(\theta_1)]{} & H^*(\underline{\mathfrak{l}}_B) & &
\end{array}$$

where $j_B: \underline{\mathfrak{l}}_B \to \mathfrak{A}$ *is inclusion mapping,* $\rho: \mathfrak{l}^0 \to \mathfrak{gl}(\mathfrak{l}/\mathfrak{l}^0) = \mathfrak{gl}(n, \boldsymbol{R})$ *is the adjoint representation of* $\mathfrak{l}^0$ *and the above isomorphisms are as in* [3].

To show the vanishing of $\lambda^*_{(\mathfrak{F},t)}(Z_{(I,J)})$, we shall prove that

$$\Delta(\theta_1) \circ W(\rho^*)(Z_{(I,J)}) = 0 \qquad \text{in } H^*(\underline{\mathfrak{l}}_B).$$

3. Vanishing of exotic characteristic classes of type (I) and for conformal case type (II. 1)

In section 3–5, we use the following notations.

Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie algebras and a, b: $\mathfrak{h} \to \mathfrak{g}$ linear mappings. Set

$$I(\mathfrak{g}) = \{\mathfrak{g}\text{-invariant polynomials on } \mathfrak{g}\},$$
$$\Omega_a = da + [a, a]/2, \ \Omega_b = db + [b, b]/2,$$
$$\Omega_t(a, b) = t\Omega_a + (1-t)\Omega_b + t(t-1)[a-b, a-b]/2, \ t \in [0, 1],$$
$$\Omega_t(a) = \Omega_t(a, 0),$$
$$\int_0^1 f(a-b, \Omega_t(a, b))dt = \int_0^1 f(a-b, \Omega_t(a, b), \cdots, \Omega_t(a, b))dt \in \Lambda\mathfrak{h}^*,$$

where $f \in I(\mathfrak{g})$.

We use the following formulas. The proofs will be given at the end of this section.

(3.1) (1) $d(\deg(f) \cdot \int_0^1 f(a-b, \Omega_t(a, b))dt) = f(\Omega_a) - f(\Omega_b)$, $f \in I(\mathfrak{g})$.

(2) *If* $g(\Omega_b) = 0$,

$$\deg(f \cdot g) \cdot \int_0^1 (f \cdot g)(a-b, \Omega_t(a, b))dt$$
$$= (f(\Omega_a) \wedge (\deg(g) \int_0^1 g(a-b, \Omega_t(a, b))dt)) + \text{exact form},$$

where f, $g \in I(\mathfrak{g})$.

(3) Let c: $\mathfrak{h} \to \mathfrak{g}$ be another linear mapping. For $f \in I(\mathfrak{g})$, we have

$$\int_0^1 f(a-b, \Omega_t(a, b))dt - \int_0^1 f(a-c, \Omega_t(a, c))dt$$
$$+ \int_0^1 f(b-c, \Omega_t(b, c))dt = \text{exact form}.$$

Let $i: \mathfrak{l}^0 \to \mathfrak{l}$ be inclusion mapping and $\rho: \mathfrak{l}^0 \to \mathfrak{gl}(n, \boldsymbol{R})$ the adjoint representation. It is easy to see the following:

Proposition 3.1.

(1) *Set* $c_1^0 = \rho^* c_1 \in I^1(\mathfrak{l}^0)$. *For all* $k = 2, \cdots, n$, *there exist* $\bar{c}_k \in I^k(\mathfrak{l})$ *and* $f_k \in I^{k-1}(\mathfrak{l}^0)$ *such that*

$$\rho^* c_k = i^* \bar{c}_k + c_1^0 \cdot f_k .$$

When $\mathfrak{l} \cong \mathfrak{o}(n+1, 1)$ (ie. *locally homogeneous conformal case*), $\bar{c}_k = 0$ *for all odd* k.

(2) *For all* $I = (i_1, \cdots, i_n)$, f_j *of the above* $(1 < j \leq n)$ *and* $f_1 \in \boldsymbol{R}$,

$$f_j \cdot \rho^* c_I \equiv k (c_1^0)^{j-1+|I|}, \ \mathrm{mod}\ I(\mathfrak{l}^0) \cdot i^* I^+(\mathfrak{l}) ,$$

where $c_I = (c_1)^{i_1} \cdots (c_n)^{i_n}$, $|I| = \sum_{r=1}^{n} r i_r$, *and* $k \neq 0 \in \boldsymbol{R}$.

The notations in the following are as in Section 2.

Let $\mathcal{J}^k$ denote the ideal in $\Lambda \mathfrak{l}_B^*$ generated by $\Lambda^k(\mathfrak{l}_B/\mathfrak{l}_B^0)^*$.

For the elements of $I(\mathfrak{l}^0) \cdot i^* I^+(\mathfrak{l})$, we have the following formula. The proof will be given at the end of this section.

(3.2) For $f \cdot i^* g \in I(\mathfrak{l}^0) \cdot i^* I^+(\mathfrak{l})$ such that $\deg(f) + 2\deg(g) > n$,

$$(f \cdot i^* g)(\Omega_1) = d\beta \qquad \text{in } \Lambda \mathfrak{l}_B^* ,$$

where $\beta = f(\Omega_1) \wedge (\deg(g) \int_0^1 g(\theta_1 - \omega, \Omega_t(\theta_1, \omega)) dt$, and

$$\beta \in \mathcal{J}^p,\ p = \deg(f) + \deg(g) .$$

Moreover, by Lemma 1, we have

Proposition 3.2.

(1) $c_I^0(\Omega_1) \in \mathcal{J}^k$, *for* $c_I^0 \in I^k(\mathfrak{l}^0)$

(2) $\bar{c}_I(\Omega) \in \mathcal{J}^{2k}$ *and* $\int_0^1 \bar{c}_I(\theta_1 - \omega, \Omega_t(\theta_1, \omega)) dt \in \mathcal{J}^k$, *for* $\bar{c}_I \in I^k(\mathfrak{l})$.

(3) $\mathcal{J}^k = 0$, *for* $k > n = \dim(\mathfrak{l}_B/\mathfrak{l}_B^0)$.

By Proposition 3.1 and (3.1), we have

$$
\begin{aligned}
(3.3) \qquad \Delta(\theta_1) W(\rho^*)(h_j) &= j \int_0^1 (\rho^* c_j)(\theta_1, \Omega_t(\theta_1))\, dt \\
&= j \int_0^1 \bar{c}_j(\theta_1, \Omega_t(\theta_1)) dt + c_1^0(\theta_1) \wedge f_j(\Omega_1) + \text{exact form} \\
&= j \int_0^1 \bar{c}_j(\omega, \Omega_t(\omega)) dt + j \int_0^1 \bar{c}_j(\theta_1 - \omega, \Omega_t(\theta_1, \omega)) dt \\
&\quad + c_1^0(\theta_1) \wedge f_j(\Omega_1) + \text{exact form.}
\end{aligned}
$$

Therefore by Proposition 3.2, we have the following for all the elements $Z_{(I,J)} = [h_{j_0} \wedge h_{j_1} \wedge \cdots \wedge h_{j_k} \otimes c_I]$ of the Vey-basis.

(3.4) $$\Delta(\theta_1)W(\rho^*)(h_{j_0}\wedge h_{j_1}\wedge\cdots\wedge h_{j_k}\otimes c_I)$$
$$=\left(j_0\int_0^1 \bar{c}_{j_0}(\omega,\ \Omega_t(\omega))dt+c_1^0(\theta_1)\wedge f_{j_0}(\Omega_1)\right)\wedge$$
$$\left(j_1\int_0^1 \bar{c}_{j_0}(\omega,\ \Omega_t(\omega))dt\right)\wedge\cdots\wedge\left(j_k\int_0^1 \bar{c}_{j_k}(\omega,\ \Omega_t(\omega))dt\right)\wedge(\rho^*c_I)(\Omega_1)$$
$$+\text{exact form.}$$

Now, by Proposition 3.1. (1),

(3.5) $$(\rho^*c_I)(\Omega_1)=\bar{c}_I(\Omega_1)+c_1^0(\Omega_1)\wedge F(\Omega_1)$$
$$=\bar{c}_I(\Omega_1)+d(c_1^0(\theta_1)\wedge F(\Omega_1))\,,$$

where $F\in I^{k-1}(\mathfrak{l}^0)$ and $\bar{c}_I\in I^k(\mathfrak{l})(k=|I|)$.

Moreover, noticing that $|I|\geqq(n+1)/2$ by the conditions for the Vey-basis, we have the following by Lemma 1(2), (3.1) (1) and Proposition (3.2) (2).

(3.6) $$\bar{c}_I(\Omega_1)=\bar{c}_I(\Omega_1)-\bar{c}_I(\Omega)=d\alpha,\ \text{where}$$
$$\alpha=|I|\int_0^1 \bar{c}_I(\theta_1-\omega,\ \Omega_t(\theta_1,\ \omega))dt\in\mathcal{J}^{|I|}\,.$$

Hence

$$(\rho^*c_I)(\Omega_1)=d(A+\alpha),\ \text{and}\ A\in\mathcal{J}^{|I|-1},\ \alpha\in\mathcal{J}^{|I|}\,.$$

Therefore, noticing that $2j_0+(|I|-1)>n$ and $j_0+|I|>n$, we have the following (3.7) from (3.4) by Proposition 3.2, and the following (3.8) is obtained from (3.7) by using Proposition 3.1 (2) and (3.2).

(3.7) $$\Delta(\theta_1)W(\rho^*)(h_{j_0}\wedge h_{j_1}\wedge\cdots\wedge h_{k}\otimes c_I)$$
$$=a\,c_1^0(\theta_1)\wedge f_{j_0}(\Omega_1)\wedge(\rho^*c_I)(\Omega_1)\wedge$$
$$\left(j_1\int_0^1 \bar{c}_{j_1}(\omega,\ \Omega_t(\omega))dt\right)\wedge\cdots\wedge\left(j_k\int_0^1 \bar{c}_{j_k}(\omega,\ \Omega_t(\omega))dt\right)$$
$$+\text{exact form, where } a\neq 0\in\boldsymbol{R}\,.$$

(3.8) $$=b\,c_1^0(\theta_1)\wedge c_1^0(\Omega_1)^{j_0-1+|I|}\wedge$$
$$\left(j_1\int_0^1 \bar{c}_{j_1}(\omega,\ \Omega_t(\omega))dt\right)\wedge\cdots\wedge\left(j_k\int_0^1 \bar{c}_{j_k}(\omega,\ \Omega_t(\omega))dt\right)$$
$$+\text{exact form, where } b\neq 0\in\boldsymbol{R}\,.$$

Now, in view of Proposition 3.1 (1), it is trivial that

$$\Delta(\theta_1)W(\rho^*)(h_{j_0}\wedge\cdots\wedge h_{j_k}\otimes c_I)=\text{exact form}$$

in the following cases (i) and (ii);

(i) $j_0+|I|>n+1$ (ie. type (I)).

(ii) $\mathfrak{l}\cong\mathfrak{o}(n+1, 1)$ and j_r is odd integer for some $r>0$ (ie. type (II.1) for conformal case).

This completes proof of vanishing theorem in this section.

Proofs of (3.1) and (3.2),

We shall prove (3.1). (3.1)(1) is well-known as Chern-Weil theory (that is, $\frac{d}{dt}f(\Omega_t(a, b))=d(\deg(f)\, f(a-b, \Omega_t(a, b)))$ and (3.1) (3) is also known. See for example Bott [1] (p. 64–65). We shall prove (3.1) (2). Set $i=\deg(f)$, $j=\deg(g)$. We have

$$\begin{aligned}(i+j)&\int_0^1(f\cdot g)(a-b, \Omega_t(a, b))dt\\ &= i\int_0^1 f(a-b, \Omega_t(a, b))\wedge g(\Omega_t(a, b))dt\\ &\quad +j\int_0^1 f(\Omega_t(a, b))\wedge g(a-b, \Omega_t(a, b))dt\,.\end{aligned}$$

On the other hand, we have the following by $g(\Omega_b)=0$.

$$\begin{aligned}g(\Omega_t(a, b)) &= g(\Omega_t(a, b))-g(\Omega_b)\\ &= d\Big(j\int_0^t g(a-b, \Omega_s(a, b))ds\Big)\,.\end{aligned}$$

Hence

$$\begin{aligned}&i\int_0^1 f(a-b, \Omega_t(a, b))\wedge g(\Omega_t(a, b))dt\\ &=\int_0^1 i\, f(a-b, \Omega_t(a, b))\wedge\Big(d\Big(j\int_0^t g(a-b, \Omega_s(a, b))ds\Big)\Big)dt\\ &= \text{exact form}\\ &\quad +\int_0^1 (d(i\, f(a-b, \Omega_t(a, b)))\wedge\Big(j\int_0^t g(a-b, \Omega_s(a, b)ds)\Big)dt\\ &= \text{exact form}\\ &\quad +\int_0^1\Big(\frac{d}{dt}f(\Omega_t(a, b))\Big)\wedge\Big(j\int_0^t g(a-b, \Omega_s(a, b)ds)\Big)dt\\ &= \text{exact form}\\ &\quad +\Big[f(\Omega_t(a, b))\wedge j\int_0^t g(a-b, \Omega_s(a, b))ds\Big]_0^1\\ &\quad -\int_0^1 f(\Omega_t(a, b))\wedge j\, g(a-b, \Omega_t(a, b))dt\\ &= \text{exact form}\\ &\quad + f(\Omega_a)\wedge j\int_0^1 g(a-b, \Omega_t(a, b)dt\\ &\quad -\int_0^1 f(\Omega_t(a, b))\wedge j\; g(a-b, \Omega_t(a, b))dt\,.\end{aligned}$$

Therefore we have (3.1) (2). (q.e.d.)

We shall prove (3.2). By (3.1), we have

$$(f\cdot i^*g)(\Omega_1)=d\Big(p\int_0^1(f\cdot i^*g)(\theta_1,\ \Omega_t(\theta_1))dt\Big)$$
$$=d\Big(f(\Omega_1)\wedge\deg(g)\int_0^1 i^*g(\theta_1,\ \Omega_t(\theta_1))dt\Big)$$

and

$$\int_0^1 i^*g(\theta_1,\ \Omega_t(\theta_1))dt$$
$$=\int_0^1 g(\theta_1-\omega,\ \Omega_t(\theta_1,\ \omega))dt+\int_0^1 g(\omega,\ \Omega_t(\omega))dt$$
$$+\text{exact form.}$$

Hence we have

$$(f\cdot i^*g)(\Omega_1)=d\beta+f(\Omega_1)\wedge g(\Omega)\,.$$

Since $\deg(f)+2\deg(g)>n$, $f(\Omega_1)\wedge g(\Omega)=0$ by Lemma 1. Therefore we have (3.2). (q.e.d.)

4. Calculation of the exotic characteristic classes of the typical examples

In this section, we prove non-triviality in our theorem.

Let $(M,\ \mathfrak{F})$ be as in Section 1 and the trivialization t natural one. The following is known.

Theorem 1 (Kamber and Tondeur [5] and Morita [8]).
Let $(M,\ \mathfrak{F})$ *be a foliation of Example* 2 *in Section* 1. *Then* $\lambda^*_{(\mathfrak{F},t)}(Z_{(I,J)})\neq 0$ *in* $H^*_{DR}(M)$, *for all* $Z_{(I,J)}$ *of type* (II.1) *and* (II.2).

In the following, we consider the foliation $\mathfrak{F}$ on $M=D\backslash G$ of Example 1. Then we have,

Theorem 2. *If n is an odd* (*resp. even*) *integer,*

$$\lambda^*_{(\mathfrak{F},t)}(Z_{(I,J)})\neq 0\ (resp.\ =0)\ \ in\ \ H^*_{DR}(M)\,,$$

for all $z_{(I,J)}$ *of type* (II.2).

We shall prove Theorem 2.

It is well-known that the Lie algebra $\mathfrak{l}=\mathfrak{o}(n+1,\ 1)$ has a graded Lie algebra structure $\mathfrak{l}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ such that $\mathfrak{l}^0=\mathfrak{g}_0\oplus\mathfrak{g}_1$ and $\mathfrak{g}_0=\boldsymbol{R}e\oplus\mathfrak{so}(n)$, where e is an element of $\mathfrak{g}_0$ satisfying $ad(e)|_{\mathfrak{g}_k}=k\cdot id_{\mathfrak{g}_k}$ for $k=-1,\ 0,\ 1$. Hence, we have an $\mathfrak{so}(n)$-equivariant splitting

$$\theta : \mathfrak{l} \to \mathfrak{l}^0$$

of the exact sequence $0 \to \mathfrak{l}^0 \to \mathfrak{l} \to \mathfrak{l}/\mathfrak{l}^0 = \boldsymbol{R}^n \to 0$, by $\theta|_{\mathfrak{l}^0} = id_{\mathfrak{l}^0}$ and $\theta|_{\mathfrak{g}_{-1}} = 0$.

Following Kamber and Tondeur [5], we have the following commutative diagram.

$$(4.1)\qquad \begin{array}{ccc} H^*(W(\mathfrak{gl}(n, \boldsymbol{R})_n) & & \\ \downarrow W(\rho^*) & \searrow & \\ H^*(W(\mathfrak{l}^0)_n) & & \lambda^*_{(\mathfrak{F}, t)} \\ \downarrow \Delta(\theta)_* & & \searrow \\ H^*(\mathfrak{l}) & \xrightarrow{\ \gamma_*\ } & H^*_{DR}(M) \end{array}$$

where γ denotes the canonical inclusion. (Recall that $M = D\backslash G$ and G was the identity component of $L = O(n+1, 1)$.)

Let $\bar{c}_2, \bar{c}_4, \cdots, \bar{c}_{2m} \in I(O(n+1, 1))$ be the restrictions of Chern-polynomials $C_2, C_4, \cdots, C_{2m}$ of $I(GL(n+2, \boldsymbol{R}))$, where $n = 2m$ or $n = 2m+1$. The primitive element corresponding to $\bar{c}_{2j}$ is denoted by $\bar{y}_{2j} \in \Lambda^{4j-1}\mathfrak{l}^*$. It is a closed form which represents a non-zero cohomology class of $H^*(\mathfrak{l})$. Let $\bar{y}_{n+1}$ denote a non-zero element of one-dimensional vector space $(\Lambda^{2n+1}\mathfrak{l}^*)_{\mathfrak{so}(n)}$, which is a closed form in $\Lambda\mathfrak{l}^*$.

We have the following.

Proposition 4.1.

(1) $\Delta(\theta) \circ W(\rho^*)(h_1 \otimes c_1^n) = c_1^0(\theta) \wedge c_1^0(\Omega_\theta)^n = a\bar{y}_{n+1}$

(2) $\Delta(\theta) \circ W(\rho^*)(h_{j_0} \wedge h_{2j_1} \wedge \cdots \wedge h_{2j_k} \otimes c_I) = b\bar{y}_{2j_1} \wedge \cdots \wedge \bar{y}_{2j_k} \wedge \bar{y}_{n+1} +$ exact form, *for* $k \geqq 0$ *where* $a, b \in \boldsymbol{R}$ *are non-zero.*

Moreover, using the method of calculation of Kamber and Tondeur [5], we have

Proposition 4.2.

(1) *If* $n = 2m+1$, $\gamma(\bar{y}_2 \wedge \bar{y}_4 \wedge \cdots \wedge \bar{y}_{2m} \wedge \bar{y}_{n+1})$ *is a volume form on* $M = D\backslash G$,.

(2) *If* $n = 2m$, *the closed form* $\bar{y}_{n+1}$ *is an exact form in* $\Lambda\mathfrak{l}^*$.

Now, by the above propositions, we have Theorem 2 from the diagram (4.1).

Proof of Proposition 4.1. It is easy to see the following.

$$(4.2)\qquad \Delta(\theta) \circ W(\rho^*)(h_1 \otimes c_1^n) = c_1^0(\theta) \wedge c_1^0(\Omega_\theta)^n \neq 0\,.$$

By the definitions of c_1^0 and θ, we have (1) easily from (4.2). We prove (2). Let I denote id: $\mathfrak{l}\to\mathfrak{l}$. Then by the definition of primitive elements we have

$$(4.3) \qquad \bar{y}_{2j} = 2j\int_0^1 \bar{c}_{2j}(I,\ \Omega_t(I))dt\,, \qquad j=1,\cdots,m\,.$$

On the other hand, calculations in Section 3 admit us to obtain (3.8) replaced θ_1 by θ, Ω_1 by Ω_θ, ω by I respectively. Therefore, by (4.3) and Proposition (4.1) (1), we have (2) of Proposition (4.1). (q.e.d.)

Proof of Proposition 4.2. We shall prove (1). Let $c_2', c_4', \cdots, c_{2m}'\in I(SO(n))$ denote the restrictions of $\bar{c}_2, \bar{c}_4, \cdots, \bar{c}_{2m}\in I(O(n+1,\ 1))$. Since $n=2m+1$, they are generators of $I(SO(n))$. Therefore, primitive element y_{2j}' corresponding to c_{2j}' is the restriction of $\bar{y}_{2j}$ to $\Lambda\mathfrak{so}(n)^*$, and $y_2'\wedge y_4'\wedge\cdots\wedge y_{2m}'$ is non-zero form with degree equal to dim $\mathfrak{so}(n)$. Recall that $\bar{y}_{n+1}\in(\Lambda^{2n+1}\mathfrak{l}^*)_{\mathfrak{so}(n)}$ and $\dim(\mathfrak{l}/\mathfrak{so}(n))=2n+1$, then we have (1). We shall prove (2). We can define $\bar{y}_{n+1}$ explicitly as follows. The Lie algebra $\mathfrak{l}=\mathfrak{o}(n+1,\ 1)$ is consisting of the following elements

$$X=\begin{pmatrix} a & u & 0\\ {}^tv & A & {}^tu\\ 0 & v & -a\end{pmatrix}\in\mathfrak{gl}(n+2,\ \boldsymbol{R})$$

with $a\in\boldsymbol{R}$, $A\in\mathfrak{so}(n)$ and $u=(u_1,\ \cdots,\ u_n)$, $v=(v_1,\ \cdots,\ v_n)\in\boldsymbol{R}^n$. Then we have an injective Lie algebra homomorphism

$$\psi\colon \mathfrak{l}\to\mathfrak{so}(n+2)^C$$

by

$$\psi(X)=\begin{pmatrix} 0 & -a\sqrt{-1} & \xi\\ a\sqrt{-1} & 0 & \eta\\ -{}^t\xi & -{}^t\eta & A\end{pmatrix},\ \text{where } \begin{matrix}\xi=\sqrt{-1}(v+u)/\sqrt{2}\\ \eta=(v-u)/\sqrt{2}\end{matrix}$$

Let $E_{m+1}\in I^{m+1}(\mathfrak{so}(n+2))\subset I^{m+1}(\mathfrak{so}(n+2)^C)$ and $e_m\in I^m(\mathfrak{so}(n))$ denote the (normalized) Pfaffian polynomiasl ([5], P. 147), where $n=2m$. Then $\psi^*E_{m+1}\in I^{m+1}(\mathfrak{o}(n+1,\ 1)^C)$ satisfies the following.

(4.4) There is an element $\bar{e}_{m+1}\in I^{m+1}(\mathfrak{o}(n+1,\ 1))$ with

$$\psi^*E_{m+1}=\sqrt{-1}\,\bar{e}_{m+1}\,.$$

Moreover, we have

(4.5) (1) $i^*\bar{e}_{m+1}=kc_1^0e_m$ in $I(\mathfrak{l}^0)$, for some $k\neq 0\in\boldsymbol{R}$.
(2) $j^*\bar{e}_{m+1}=0$ in $I(\mathfrak{so}(n))$

where $i\colon \mathfrak{l}^0\to\mathfrak{l}$, $j\colon \mathfrak{so}(n)\to\mathfrak{l}$ are inclusion mappings and $e_m\in I(\mathfrak{so}(n))\subset I(\mathfrak{l}^0)$.

Let $\theta_0\colon \mathfrak{l}\to\mathfrak{so}(n)$ be an $\mathfrak{so}(n)$-equivariant splitting of the exact sequence $0\to\mathfrak{so}(n)\to\mathfrak{l}\to\mathfrak{l}/\mathfrak{so}(n)\to 0$. Define $\sigma(\bar{e}_{m+1})\in\Lambda^{n+1}\mathfrak{l}^*$ by

$$\sigma(\bar{e}_{m+1}) = (m+1)\int_0^1 \bar{e}_{m+1}(I-\theta_0,\ \Omega_t(I,\ \theta_0))dt\,.$$

Since θ_0 is $\mathfrak{so}(n)$-equivariant and $j^*\bar{e}_{m+1}=0$ in $I(\mathfrak{so}(n))$, we have

(4.6) (1) $\sigma(\bar{e}_{m+1})\in(\Lambda^{n+1}\mathfrak{l}^*)_{\mathfrak{so}(n)}$, which is a closed form in $\Lambda\mathfrak{l}^*$.

(2) $e_m(\Omega_{\theta_0})\in(\Lambda^n\mathfrak{l}^*)_{\mathfrak{so}(n)}$, which is an exact form in $\Lambda\mathfrak{l}^*$, that is

$$e_m(\Omega_{\theta_0}) = d\left(m\int_0^1 e_m(\theta_0,\ \Omega_t(\theta_0))dt\right) \text{ in } \Lambda\mathfrak{l}^*\,.$$

(3) $\sigma(\bar{e}_{m+1})\wedge e_m(\Omega_{\theta_0})$ is a non-zero element of $(\Lambda^{2n+1}\mathfrak{l}^*)_{\mathfrak{so}(n)}$.

Now, we obtained $\bar{y}_{n+1}$ explicitly and (4.6) implies (2) of Proposition 4.2. (q.e.d.)

5. Vanishing of some exotic characteristic classes of type (II.2) for locally homogeneous conformal foliations

In this section, we consider locally homogeneous conformal foliations $\mathfrak{F}$ with trivialized normal bundles of codimension $n=2m$ and prove that these exotic characteristic classes corresponding to elements $Z_{(I,J)}=[h_{j_0}\otimes c_I]$ of Vey-basis are always zero.

From (3.8), we have

(5.1) $\Delta(\theta_1)\circ W(\rho^*)(h_{j_0}\otimes c_I) = b\, c_1^0(\theta_1)\wedge c_1^0(\Omega_1)^n+$exact form, where $b\neq 0\in\boldsymbol{R}$.

Define $c_n^0\in I(\mathfrak{so}(n))$ by $c_n^0(X)=\det(X)$ for $X\in\mathfrak{so}(n)$. Then we know ([5], P 148)

$$c_n^0 = \alpha(e_m)^2 \qquad \text{for some } \alpha\neq 0\in\boldsymbol{R}\,.$$

And it is easy to see

$$i^*\bar{c}_n\equiv\beta c_n^0+\beta'(c_1^0)^n, \text{ mod } I(\mathfrak{l}^0)\cdot i^*I^+(\mathfrak{l})$$

for some $\beta,\beta'\neq 0\in\boldsymbol{R}$.
Therefore we have

(5.2) (1) $(c_1^0)^n\equiv\gamma(e_m)^2$, mod $I(\mathfrak{l}^0)i^*I^+(\mathfrak{l})$, where $\gamma\neq 0\in\boldsymbol{R}$.

(2) $i^*\bar{e}_{m+1}=kc_1^0e_m$, where $k\neq 0\in\boldsymbol{R}$ (by (4.5) (1)).

Then, by applying (3.1) and (3.2) to (5.2), we have the following from (5.1).

(5.3) (1) $\Delta(\theta_1)W(\rho^*)(h_{j_0}\otimes c_I)$
$= pc_1^0(\theta_1)\wedge e_m(\Omega_1)^2+$exact form, where $p\neq 0\in\boldsymbol{R}$.

(2) $c_1^0(\theta_1)\wedge e_m(\Omega_1)=\sigma+\lambda+$exact form, where σ is a closed form which is defined as follows

$$\sigma = (m+1)\int_0^1 \bar{e}_{m+1}(\omega,\ \Omega_t(\omega))dt\,,$$

and

$$\lambda = (m+1)\int_0^1 \bar{e}_{m+1}(\theta_1-\omega,\ \Omega_t(\theta_1,\ \omega))dt\in\mathcal{J}^{m+1}\,.$$

Now, noticing that $e_m(\Omega_1)\in\mathcal{J}^m$ is an exact form, we have our result.

6. Proofs of Main lemmas

Let L, L_0 be as in Section 1 and $J^2(B)$ the bundle of frames of order 2 on B with group $G_0^2(n)$. Let $\pi\colon P\to B$ denote the principal subbundle of $J^2(B)$ with group $L_0\subset G_0^2(n)$ which is the conformal (resp. projective) structure on B. For each element $\xi=\xi_{-1}+\xi_0+\xi_1+\cdots$ of $\mathfrak{A}$, there is a local vector field $\bar{\xi}$ around $b_0\in B$ such that $j_{b_0}^2(\bar{\xi})=\xi_{-1}+\xi_0+\xi_1$, and the natural lifting $\bar{\bar{\xi}}$ of $\bar{\xi}$ to $J^2(B)$. In case of $\xi\in\underline{\mathfrak{l}}_B$, we can choose as $\bar{\xi}$ a local conformal (resp. projective (vector field and the natural lifting $\bar{\bar{\xi}}$ is a vector field on P.

We identify $\xi_{-1}+\xi_0+\xi_1$ and $(\bar{\bar{\xi}})_{\bar{b}_0}\in T_{\bar{b}_0}(J^2(B))$ through the local diffeomorphism $\varphi\colon(\boldsymbol{R}^n,\ 0)\to(B,\ b_0)$ with $\bar{b}_0=j_{b_0}^2(\varphi)\in P\subset J^2(B)$. And $\bar{b}_0\in J^2(B)$ will be fixed.

Proof of Proposition 2.1. Under the above identification, we have

(6.1) For $\xi=\xi_0+\xi_1+\cdots\in\underline{\mathfrak{l}}_B^0$, $\xi_0+\xi_1\in\underline{\mathfrak{l}}^0\subset\mathfrak{g}_0^2$, where $\mathfrak{g}_0^2(=\mathfrak{A}_0\oplus\mathfrak{A}_1)$ is the Lie algebra of $G_0^2(n)$.

On the other hand, the graded Lie algebra $\underline{\mathfrak{l}}^0=\mathfrak{g}_0\oplus\mathfrak{g}_1$ satisfies that $\mathfrak{g}_0\subset\mathfrak{A}_0$ and $\mathfrak{g}_1\subset\mathfrak{A}_1$. And the results of Kobayashi and Nagano [6] (P. 683–687) show the following.

(6.2) Let $\mathfrak{g}_k(k\geqq 2)$ be defined inductively as follows:

$$\mathfrak{g}_k = \{\eta\in\mathfrak{A}_k\,|\,[\partial/\partial x_i,\ \eta]\in\mathfrak{g}_{k-1},\ \text{for } i=1,\ \cdots,\ n\}\,,\ \text{then } \mathfrak{g}_k=0 \text{ for } k\geqq 2\,.$$

Now, by local homogeneity there are elements X_i $(i=1,\cdots,n)$ of $\underline{\mathfrak{l}}_B$ satisfying

$$X_i = \partial/\partial x_i+(X_i)_0+(X_i)_1+\cdots\,,$$

where $(X_i)_0\in\mathfrak{A}_0$, $(X_i)_1\in\mathfrak{A}_1$ etc.

We shall prove $\underline{\mathfrak{l}}_B^2/\underline{\mathfrak{l}}_B^3=0$. For $\xi=\xi'+\xi''\in\underline{\mathfrak{l}}_B^2$ with $\xi'\in\mathfrak{A}_2$ and $\xi''''\in\mathfrak{A}_3\oplus\mathfrak{A}_4\oplus\cdots$, we have the following for $i=1,\ \cdots,\ n$.

$$[X_i,\ \xi]\in\underline{\mathfrak{l}}_B^1\,,$$

and

$$[X_i,\ \xi] = [\partial/\partial x_i,\ \xi']+\lambda\,,$$

where $[\partial/\partial x_i, \xi']\in\mathfrak{A}_1$ and $\lambda\in\mathfrak{A}_2\oplus\mathfrak{A}_3\oplus\cdots$.
That is, $[\partial/\partial x_i, \xi']\in\mathfrak{g}_1$ for $i=1, \cdots, n$. Theeefore, by (6.2), we have $\xi'=0$ and $\underline{\mathfrak{l}}_B^2/\underline{\mathfrak{l}}_B^3=0$.

By the same argument, we have $\underline{\mathfrak{l}}_B^k/\underline{\mathfrak{l}}_B^{k+1}=0$ for $k\geqq 2$ inductively. Therefore we have $\underline{\mathfrak{l}}_B^k=0$ for $k\geqq 2$. Now, it is clear that

$$\dim \underline{\mathfrak{l}}_B \leqq n+\dim \underline{\mathfrak{l}}^0 = \dim \underline{\mathfrak{l}}\,. \quad \text{(q.e.d.)}$$

Proof of Lemma 1. It is known that there exists a unique Cartan connection $\bar{\omega}$ on P called normal conformal (resp. projective) connection if dim $B=n\geqq 3$ ([7], [11]). Let $\bar{\bar{\theta}}_1$ be a connection on P. Then we have well-defined linear mappings

$$\omega\colon \underline{\mathfrak{l}}_B \to \mathfrak{l}$$

and

$$\theta_1\colon \underline{\mathfrak{l}}_B \to \mathfrak{l}^0$$

as follows.

$$\begin{aligned} \omega(\xi) &= -\bar{\omega}(\bar{\bar{\xi}})_{\bar{\bar{b}}_0}\,, \\ \theta_1(\xi) &= -\bar{\bar{\theta}}_1(\bar{\bar{\xi}})_{\bar{\bar{b}}_0}\,, \qquad \text{for } \xi\in\underline{\mathfrak{l}}_B\,. \end{aligned} \tag{6.3}$$

Injectivity of ω is due to injectivity of $\bar{\omega}$ and (1) is clear.

Since the normal conformal (resp. projective) connection $\bar{\omega}$ is invariant under conformal (resp. projective) transformations, the curvature form $\bar{\bar{\Omega}}$ of $\bar{\omega}$ satisfies

$$\bar{\bar{\Omega}}(\bar{\bar{\xi}}, \bar{\bar{\eta}}) = \bar{\omega}([\bar{\bar{\xi}}, \bar{\bar{\eta}}])+[\bar{\omega}(\bar{\bar{\xi}}), \bar{\omega}(\bar{\bar{\eta}})]\,, \qquad \text{for } \xi, \eta\in\underline{\mathfrak{l}}_B\,.$$

On the other hand, it is not difficult to see that

$$[\bar{\bar{\xi}}, \bar{\bar{\eta}}] = \overline{\overline{[\xi, \eta]}} \ \text{ at } \bar{\bar{b}}_0\,, \qquad \text{for } \xi, \eta\in\underline{\mathfrak{l}}_B\,.$$

Therefore, we have

$$\bar{\bar{\Omega}}(\bar{\bar{\xi}}, \bar{\bar{\eta}})_{\bar{\bar{b}}_0} = \Omega(\xi, \eta)\,, \qquad \text{for } \xi, \eta\in\underline{\mathfrak{l}}_B\,. \tag{6.4}$$

Moreover, it it is known ([11]) that

$$i(A^*)\bar{\bar{\Omega}} = 0\,, \qquad \text{for } A\in\mathfrak{l}^0\,. \tag{6.5}$$

Therefore, we have (2) of Lemma 1 form (6.4) and (6.5). Now, by Lemma 1 (1), (3) of Lemma 1 is clear. (q.e.d.)

Proof of Lemma 2. Commutativity of right-hand side is known ([3]). Set. Set $\bar{\bar{\Theta}}$ the local canonical flat connection in $J^2(B)$ around $\bar{\bar{b}}_0$ defined by the local diffeomorphism φ. We have a well-defined linear mapping

$$\Theta\colon \mathfrak{A} \to \mathfrak{g}_0^2(=\mathfrak{A}_0\oplus\mathfrak{A}_1)$$

by $\Theta(\xi)=-\bar{\bar{\Theta}}(\bar{\bar{\xi}})_{\bar{\mathfrak{b}}_0}$ for $\xi\in\mathfrak{A}$.

Let $\rho\colon \mathfrak{g}_0^2\to\mathfrak{gl}(n, \boldsymbol{R})$ denote the adjoint representation of $\mathfrak{g}_0^2$, where $\mathfrak{gl}(n, \boldsymbol{R})=\mathfrak{gl}(\mathfrak{A}/(\mathfrak{A}_0\oplus\mathfrak{A}_1\oplus\cdots))$.

We denote not only $\Theta|_{\mathfrak{l}_B}$ by Θ but $\rho|_{\mathfrak{l}^0}$ by ρ.

Since $(\Omega_\Theta)^k=0$ for $k>n$, we have the following diagram.

$$
\text{(6.6)}\qquad
\begin{array}{ccccc}
W(\mathfrak{gl}(n, \boldsymbol{R})_n) & \xrightarrow{\;W(\rho^*)\;} & W(\mathfrak{g}_0^2)_n & \xrightarrow{\;\Delta(\Theta)\;} & \Lambda\mathfrak{A}^* \\
 & {\scriptstyle W(\rho^*)}\searrow & & {\scriptstyle \Delta(\Theta)}\searrow \quad \swarrow{\scriptstyle j_B^*} & \\
 & W(\mathfrak{l}^0)_n & \xrightarrow{\;\Delta(\theta_1)\;} & \Lambda\mathfrak{l}_B^* &
\end{array}
$$

It is clear that $j_B^*\circ\Delta(\Theta)=\Delta(\Theta)$. And the isomorphism of $H^*(W(\mathfrak{gl}(n, \boldsymbol{R})_n)$ onto $H^*(\mathfrak{A})$ in Lemma 2 is induced by $\Delta(\Theta)\circ W(\rho^*)$.

We shall prove that $\Delta(\theta_1)\circ W(\rho^*)$ and $\Delta(\Theta)\circ W(\rho^*)$ induce the same mapping in cohomologies. Since ρ is a homomorphism of Lie algebras, we have

$$
\text{(6.7)}\qquad \Delta(\theta_1)\circ W(\rho^*) = \Delta(\rho\circ\theta_1) = \Delta(\hat{\theta}_1)
$$

and
$$
\Delta(\Theta)\circ W(\rho^*) = \Delta(\rho\circ\Theta) = \Delta(\hat{\Theta}),
$$

where we denoted $\rho\circ\theta_1$(resp. $\rho\circ\Theta$) by $\hat{\theta}_1$ (resp. $\hat{\Theta}$).

By (3.1) and definitions of $\Delta(\hat{\theta}_1)$ and $\Delta(\hat{\Theta})$, we have

(6.8) (1) $\Delta(\hat{\theta}_1)(h_j)-\Delta(\hat{\Theta})(h_j) = \Delta(c_j)+\text{exact form, for } j=1, \cdots, n,$

where $\Delta(c_j)=j\int_0^1 c_j(\theta_1-\hat{\Theta}, \Omega_t(\hat{\theta}_1, \hat{\Theta}))dt\in\mathcal{I}^j$.

(2) $\Delta(\hat{\theta}_1)(c_j)-\Delta(\hat{\Theta}_j)(c)=d(\Delta(c_j))$, for $j=1, \cdots, n$,

and
$$
\Delta(\hat{\theta}_1)(c_j)\,,\ \Delta(\hat{\Theta})(c_j)\in\mathcal{I}^j\,.
$$

Therefore we have easily $\Delta(\hat{\theta}_1)_*(Z_{(I,J)})=\Delta(\hat{\Theta})_*(Z_{(I,J)})$ for all $Z_{(I,J)}$ of Vey-basis. (q.e.d)

OSAKA UNIVERSITY AND INSTITUT DE RECHERCHE MATHEMATIQUE AVANCEE, STRASBOURG

References

[1] R. Bott: *Lectures on characteristic classes and foliations*, Lectures on algebraic and differential tnpology, Lecture Notes in Math., 279, Springer, 1975, 1–94.

[2] A. Borel: *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122.

[3] A. Haefliger: *Sur les classes caractéristiques des feuilletages*, Séminaire Bourbaki, 24e année 1971/72, n° 412.

[4] J. Heitsch: *Deformations of secondary characteristic classes*, Topology **12** (1973), 381–388.

[5] F. Kamber and Ph. Tondeur: Foliated bundles and characteristic classes, Lecture Notes in Math., 493, Springer, 1975.

[6] S. Kobayashi and T. Nagano: *On filtered Lie algebras and geomatric structures* III, J. Math. Mech. **14** (1965), 679–706.

[7] S. Kobayashi: Transformation groups in differential geometry, Springer, 1972.

[8] S. Morita: *Non-triviality of some secondary characteristic classes of foliations*. (preprint)

[9] S. Morita: *On characteristic classes of conformal and projective foliations*. (preprint)

[10] S. Nishikawa and H. Sato: *On characteristic classes of riemannian, conformal and projective foliations*, J. Math. Soc. Japan **28** (1976), 223–241.

[11] T. Ochiai: *Geometry associated with semisimple flat homogeneous spaces*, Trans. Amer. Math. Soc. **152** (1970), 159–193.