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WEGNER ESTIMATE AND LOCALIZATION FOR RANDOM MAGNETIC FIELDS

Dedicated to Professor Shin-ichi Kotani
on the occasion of his retirement from Osaka University

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Abstract

Inspired by a work of Hislop and Klopp, we prove precise Wegner estimates for three classes of Schrödinger operators, including Pauli Hamiltonians, with random magnetic fields. The support of the site vector potentials may be noncompact (long-range type random perturbation) and, for one class of the operators, the random vector potentials may be unbounded. In particular Gaussian random fields are also treated. Wegner estimates with correct volume dependence are applied to show Hölder estimates of the densities of states. We give also upper bounds on the infimum of the spectrum to show the existence of the Anderson localization near the infimum.

1. Introduction

In this paper, we consider the following three types of random Schrödinger operators with magnetic fields:

$$H^{\lambda\omega} = (i\nabla + A(x) + \lambda A^{\omega}(x))^2, \tag{1.1}$$

$$H^{\lambda\omega,g} = \left((i\nabla + \lambda A^{\omega}(x))^2 - \frac{g\lambda}{2} B^{\omega}(x)\right) \oplus \left((i\nabla + \lambda A^{\omega}(x))^2 + \frac{g\lambda}{2} B^{\omega}(x)\right) \tag{1.2}$$

and

$$H^{\lambda\omega,\mu\vartheta} = (i\nabla + A(x) + \lambda A^{\omega}(x))^2 + \mu v^{\vartheta}(x)|\lambda A^{\omega}(x)|^2, \tag{1.3}$$

where $\lambda,\ \mu > 0,\ g > 2,\ A(x)$ is an $\mathbb{R}^2$-valued smooth function, $A^{\omega}(x)$ is an $\mathbb{R}^2$-valued alloy type random vector potential $\sum_{a\in\mathbb{Z}^2}\omega(a)u(x-a)$ on $\mathbb{R}^2$, $B^{\omega}(x) = \partial_1 A_2^{\omega}(x) - \partial_2 A_1^{\omega}(x)$, and $v^{\vartheta}(x)$ is a real random field independent of $A^{\omega}(x)$ defined by a partition of unity. The operators $H^{\lambda\omega}$ and $H^{\lambda\omega,\mu\vartheta}$ are defined on $L^2(\mathbb{R}^2)$, and the operator $H^{\lambda\omega,g}$ is defined on $L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)$. For the details, see Sections 2, 3 and 4 below.

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In those sections, these operators are defined in general dimensional settings and we treat these general dimensional settings. We prove Wegner type estimates for these operators and apply them to obtain Hölder estimates of the integrated densities of states and to prove the Anderson localization.

The operator $H^{\lambda\omega}$ is the Schrödinger operator with magnetic field λB^{ω}. For this operator, Hislop and Klopp [12] gave a Wegner type estimate and state that this estimate is applied to prove the Anderson localization. However the Wegner type estimate holds under strict conditions and they did not give explicit conditions for the Anderson localization. In this paper we extend their method and obtain more precise Wegner estimates. Then we construct several models on which we can prove the Anderson localization rigorously. We use the same methods to consider Pauli Hamiltonian $H^{\lambda\omega,g}$, where g is the magnetic moment. Usually the case $g = 2$ is studied, but here we treat the case $g > 2$ only. This case is called anomalous (cf. [1]). Then the operator $H^{\lambda\omega,g}$ has negative energies. We are concerned with the spectrum of sufficiently low energy region. The condition for the Anderson localization for the operator $H^{\lambda\omega,g}$ is still restrictive since the random field A^{ω} must be bounded. On another model $H^{\lambda\omega,\mu\vartheta}$, we do not require the boundedness of the vector potential and we may choose, in particular, a Gaussian random field. The scalar potential $\mu v^{\vartheta}(x)|A^{\omega}(x)|^2$ is artificial. However, since μ can be taken arbitrarily small, the results for the operator $H^{\lambda\omega,\mu\vartheta}$ may suggest properties of the operator $H^{\lambda\omega}$.

One difficulty of the proof of the Wegner type estimate for random vector potentials is that the quadratic form $(\varphi, H^{\lambda\omega}\varphi)$ associated to the Schrödinger operator is not monotone as a function of the random variables ω for each function φ. Thus we cannot use the method of Wegner [32] to transform the variation of the energy to that of the random variables. The same difficulty appears also in the corresponding problem for the Schrödinger operator $-\Delta + V^{\omega}(x)$ with the alloy type scalar potential $V^{\omega}(x) = \sum_{a\in\mathbb{Z}^2} \omega(a)u(x - a)$ whose single site potential $u(x)$ is nonsign definite (cf. [4], [12], [16], [18] and [31]). In such situations, Klopp's method [16] using a vector field $\mathbb{A} = \sum_{a\in\mathbb{Z}^2} \omega(a)\partial/(\partial\omega(a))$ on a probability space is the effective method. This vector field is a number operator in the context of the quantum field theory, and its eigenfunctions are homogeneous polynomials of ω. Therefore his method is effective when the objective operator is a homogeneous polynomial of ω. In [16] Klopp applied the Birman-Schwinger principle to reduce the problem to the that for the operator $(-\Delta - E)^{-1/2}V^{\omega}(x)(-\Delta - E)^{-1/2}$, where E is in the resolvent set of the unperturbed operator $-\Delta$. This operator is linear in ω. However any magnetic Schrödinger operators with random vector potentials are not reduced to operators homogeneous in ω. In [12] Hislop and Klopp reduced to the problem of the operator $H^{\lambda\omega}$ to the same problem of the operator consist of the sum of the linear term $((i\nabla+A)^2-E)^{-1/2}\lambda\{(i\nabla+A)\cdot A^{\omega}(x) + A^{\omega}(x)\cdot(i\nabla + A)\}((i\nabla + A)^2 - E)^{-1/2}$ and the quadratic term $((i\nabla + A)^2 - E)^{-1/2}\lambda^2|A^{\omega}(x)|^2((i\nabla + A)^2 - E)^{-1/2}$. Then they applied Klopp's method [16] by neglecting the effect from the quadratic term under the condition that the vector potential λA^{ω} is small enough. Since λ is small, the spectrum is not necessarily broad enough to imply the localization in their paper. Here we address this problem, and show the existence of the Anderson localization under certain situations. On the other hand, the operator $H^{\lambda\omega,g}$ consists of the 0 order term $-\Delta$, the linear term $\lambda\{(i\nabla\cdot A^{\omega}(x)+A^{\omega}(x)\cdot i\nabla+gB^{\omega}(x)/2)\oplus(i\nabla\cdot A^{\omega}(x)+A^{\omega}(x)\cdot i\nabla-gB^{\omega}(x)/2)\}$ and the quadratic term $\lambda^2|A^{\omega}(x)|^2$. We now apply Klopp's method directly to this operator by neglecting the effects from the 0-order and the quadratic terms under the same condition that the vector potential λA^{ω} is small enough and the condition that the magnetic moment g is large enough. For this case, the problem of the existence of the energies treated by the Wegner type estimates becomes simpler since we concentrate only on sufficiently low energies for which the main contribution comes from the scalar potential. The contribution can be estimated accurately in general. Therefore we can give more general examples where the Anderson localization occurs. On the other hand, for the operator $H^{\lambda\omega,\mu\vartheta}$, we use the same reduction to the operator consists of the sum of the linear term $\Gamma_1=((i\nabla+A)^2-E)^{-1/2}\lambda(\nabla\cdot A^{\omega}(x)+A^{\omega}(x)\cdot\nabla)((i\nabla+A)^2-E)^{-1/2}$ and the quadratic term $\Gamma_2=((i\nabla+A)^2-E)^{-1/2}\lambda^2(1+\mu v^{\vartheta}(x))|A^{\omega}(x)|^2((i\nabla+A)^2-E)^{-1/2}$ with respect to the random variable ω. In this case, there is another random variable ϑ and only Γ_2 depends on it. Then, by changing the variables ϑ appropriately in the expectation with respect to this variable, we can compensate the dispersion by the inhomogeneity with respect to ω. Therefore we can apply Klopp's method without neglecting any parts.

As relating works, Klopp, Nakamura, Nakano and Nomura [17] gave the same results for a certain discrete model corresponding to $H^{\lambda\omega}$. On the other hand the author [30] also give same results for a class of Schrödinger operators

$$(i\nabla+A^{\omega}(x))^2+V^{\omega}(x)$$

where the random vector potential $A^{\omega}(x)$ and the random scalar potential $V^{\omega}(x)$ are correlated. However this class does not include any operators treated above, since the scalar potential $V^{\omega}(x)$ was assumed to be unbounded below: by the strong effect of $V^{\omega}(x)$, the effect of $A^{\omega}(x)$ is dominated.

The organization of this paper is as follows. In Section 2 we extend the method of the paper by Hislop and Klopp [12] for the operator $H^{\lambda\omega}$ and make explicit the condition for the Wegner type estimate and the Anderson localization. In Section 3 we study the same problem for the operator $H^{\lambda\omega,g}$. In Sections 4 we study the same problem for the operator $H^{\lambda\omega,\mu\vartheta}$. In Section 5 we study the same problem for the operator $H^{\lambda\omega,\mu\vartheta}$ in the case that the vector potential is a Gaussian random field.

2. Random magnetic Schrödinger operators without scalar potentials

In this section we extend the theory [12] by Hislop and Klopp on random magnetic Schrödinger operators. We treat the operator

$$H^{\lambda\omega}=\sum_{j=1}^{d}(i\partial_j+A_j(x)+\lambda A_j^{\omega}(x))^2 \tag{2.1}$$

on $L^2(\mathbb{R}^d)$, where $d = 2d'$, $\lambda > 0$, $A(x)$ is a C^1 vector potential such that the magnetic field $B_{jk}(x) = \partial_j A^k(x) - \partial_k A^j(x)$ is $\mathbb{Z}^d$-periodic, and $A^\omega(x)$ is the alloy type random vector potential

$$\sum_{a\in\mathbb{Z}^d} \omega(a)u(x-a). \tag{2.2}$$

$u(x)$ is a nonzero C^1 vector field satisfying $\lim_{|x|\to\infty} u(x) = 0$ and

$$|\nabla u(x)| \le (1+|x|)^{-\alpha} \tag{2.3}$$

for some $\alpha > d+1$. $\omega = (\omega(a))_{a\in\mathbb{Z}^d}$ is a family of independently and identically distributed real random variables whose distribution has a C^1 density $h(s)$ such that $\operatorname{supp} h \subset [-1, 1]$. We set

$$\mathcal{H}' = 1 + \int |sh'(s)|\, ds.$$

The operator $H^{\lambda\omega}$ with the domain $C_0^\infty(\mathbb{R}^d)$ is known to be essentially self-adjoint on $L^2(\mathbb{R}^d)$ (cf. [19]). Accordingly we take the unique self-adjoint extension, and denote it by the same symbol. In [12], the internal gaps of the spectrum $\sigma(H^0)$ of H^0 are treated, where H^0 is $H^{\lambda\omega}$ with $\lambda = 0$. In this paper, we assume

$$B_0 := \inf \sigma(H^0) > 0$$

and concentrate only on the spectrum less than B_0 for the simplicity. The periodicity of B_{jk} is also assumed for the simplicity and also for assuring the existence of the density of states. For each $L > 0$, let $H_L^{\lambda\omega}$ be the restriction to $L^2(\Lambda_L)$ with the Dirichlet boundary condition, where $\Lambda_L := (-L/2, L/2)^d$. The spectrum of this operator is purely discrete. For any self-adjoint operator A and any interval J in $\mathbb{R}$, let $N(J : A)$ be the number of eigenvalues of A in the interval J. Then the following is our extension of the Wegner type estimate given by Hislop and Klopp [12]:

Theorem 1 (Wegner type estimate). (i) *We assume* $\operatorname{supp} u$ *is compact. We denote its diameter by* $\operatorname{diam}\operatorname{supp} u$. *Then there exists a finite constant c depending only on d and* $\operatorname{diam}\operatorname{supp} u$ *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega})] \le cW_d(E, \delta, \lambda, B_0, \mathcal{H}')\eta L^{2d} \tag{2.4}$$

for any $L \ge 1$, $0 < \delta < 1$, $0 \le E \le B_0 - (\lambda\bar{A})^2/(1-\delta)$ *and* $0 < \eta \le \delta(B_0 - E)/4$, *where*

$$W_d(E, \delta, \lambda, B_0, \mathcal{H}') = \frac{\lambda^{\hat{d}} B_0^{(d+\hat{d})/2}\mathcal{H}'}{\delta(B_0 - E)^{\hat{d}+1}},$$

$\hat{d} = \min\{(2\mathbb{N}) \cap (d, \infty)\}$ *and* $\bar{A} = \sup\{|A^{\omega}(x)| : x \in \mathbb{R}^d,\ \omega \in [-1, 1]^{\mathbb{Z}^d}\}$.

(ii) *For any* $q \geq \hat{d}$ *and* $l \in \mathbb{N}$ *satisfying* $q < l < (\alpha - 1)q/d$, *there exists a finite constant* c *depending only on* d, q, l *and* α *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega})] \leq c\mathcal{W}_{d,q,l}(E, \delta, \lambda, B_0, \mathcal{H}')\eta^{1-q/l}L^{2d} \tag{2.5}$$

for any $L \geq 1$, $0 < \delta < 1$, $0 \leq E \leq B_0 - (\lambda\bar{A})^2/(1-\delta)$ *and* $0 < \eta \leq \delta(B_0 - E)/4$, *where*

$$\mathcal{W}_{d,q,l}(E, \delta, \lambda, B_0, \mathcal{H}') = \frac{\lambda^q B_0^{d/2}(\lambda + \sqrt{B_0})^q \mathcal{H}'}{\delta(B_0 - E)^{q+1-q/l}}.$$

(iii) *We assume* $\operatorname{supp} u$ *is compact. Then, for any* $q \geq \hat{d}$ *and* $q < l \in \mathbb{N}$, *there exists a finite constant* c *depending only on* d, q, l *and* $\operatorname{diam}\operatorname{supp} u$ *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega})] \leq cW_{d,q,l}(E, \delta, \lambda, B_0, \mathcal{H}')\eta^{1-q/l}L^{d} \tag{2.6}$$

for any $0 < \delta < 1$, $L \geq 1$, $0 \leq E \leq B_0 - (\lambda\bar{A})^2/(1-\delta)$ *and* $0 < \eta \leq \delta(B_0 - E)/4$, *where*

$$W_{d,q,l}(E, \delta, \lambda, B_0, \mathcal{H}') = \frac{\lambda^q B_0^{d/2}(\lambda + \sqrt{B_0})^q \mathcal{H}'}{\delta(B_0 - E)^{q+1-q/l}} \left(\frac{\sqrt{B_0}}{B_0 - E} \vee 1\right)^{q(1-1/l)}.$$

In (iii), the bound is linear in the volume of the domain Λ_L. Therefore, as in the original paper by Wegner [32], we can obtain a Hölder estimate of the density of states $N(B)$, $B \in \mathcal{B}(\mathbb{R})$, of the Schrödinger operator $H^{\lambda\omega}$ defined as a deterministic Borel measure such that the Borel measures $L^{-d}N(\,\cdot\, : H_L^{\lambda\omega})$ on $\mathbb{R}$ converges vaguely to $N(\,\cdot\,)$ as $L \to \infty$ for almost all ω:

Corollary. *Under the situation of* Theorem 1 (iii), *the density of states* $N(\,\cdot\,)$ *of the operator* $H^{\lambda\omega}$ *satisfies*

$$N([E-\eta, E+\eta])] \leq cW_{d,q,l}(E, \delta, \lambda, B_0, \mathcal{H}')\eta^{1-q/l} \tag{2.7}$$

for any $0 < \delta < 1$, $0 \leq E \leq B_0 - (\lambda\bar{A})^2/(1-\delta)$ *and* $0 < \eta \leq \delta(B_0 - E)/4$.

From (i), we have the following result on the Anderson localization by applying the theory by Germinet and Klein [11]:

Theorem 2. *We assume* $\operatorname{supp} u$ *is compact. Then, for any positive number* ϵ, *there exist finite positive constants* c_1, c_2 *and* c_3 *depending only on* d, ϵ *and* diam supp u *such that* $0 < \delta < 1$ *and*

$$\mathbb{P}\left(|\omega(0)| \geq \frac{1}{\lambda \bar{A}}\left(\sqrt{B_0} - \sqrt{B_0 - \frac{(\lambda \bar{A})^2}{1-\delta}} - \epsilon\right)\right) \tag{2.8}$$
$$< \{c_1(\lambda^2\delta)^{\epsilon}\} \wedge \frac{c_2}{(1+B_0)^{\epsilon}} \wedge \left\{c_3\left(\frac{\lambda^{\hat{d}+2}\delta}{B_0^{(d+\hat{d})/2}\mathcal{H}'}\right)^{\epsilon}\right\}$$

imply the following on the interval $I = [0, B_0 - (\lambda \bar{A})^2/(1-\delta)]$:

(i) (*Strong dynamical localization*) *For any* $0 < \zeta < 1$, *there exists a finite constant* C_ζ *such that*

$$\mathbb{E}\left[\sup_{f \in \mathcal{B}_1(\mathbb{R})} |||\chi_x f(H^{\lambda\omega}) E(I : H^{\lambda\omega})\chi_y|||_2^2\right] \leq C_\zeta \exp(-|x-y|_\infty^\zeta)$$

for any $x, y \in \mathbb{Z}^d$, *where* $\mathcal{B}_1(\mathbb{R})$ *is the set of all real valued Borel functions* f *on* $\mathbb{R}$ *with* $\sup|f| \leq 1$ *and* χ_x *is the operator of the multiplication of the characteristic function of the box* $x + \Lambda_1$.

(ii) (*Semi uniformly localized eigenfunction*) *For any* $\varepsilon > 0$, *there exists* $m_\varepsilon > 0$ *such that the following holds for a.e.* ω: $\sigma_c(H^{\lambda\omega}) \cap I = \emptyset$ *and, if* $\{\phi_j^{\lambda\omega}\}_{j\in\mathbb{N}}$ *is the normalized eigenfunctions of* $H^{\lambda\omega}$ *with energy* $E_j^{\lambda\omega}$ *in* I, *then, for any* $\nu > 1/2$, *there exist* $C_\varepsilon^{\lambda\omega}$, $\widetilde{C^{\lambda\omega}} \in (0,\infty)$ *and* $\{x_j^{\lambda\omega}\}_{j\in\mathbb{N}} \subset \mathbb{Z}^d$ *such that*

$$\left\|\chi_x \phi_j^{\lambda\omega}\right\| \leq C_\varepsilon^{\lambda\omega} \exp\left(m_\varepsilon\left(\log\left|x_j^{\lambda\omega}\right|_\infty\right)_+^{1+\varepsilon} - m_\varepsilon\left|x - x_j^{\lambda\omega}\right|_\infty\right)$$

and

$$\left|x_j^{\lambda\omega}\right|_\infty \geq \widetilde{C^{\lambda\omega}}\, j^{1/(4\nu)}$$

for any $j \in \mathbb{N}$ *and* $x \in \mathbb{Z}^d$, *where, for any self-adjoint operator* A, $\sigma_c(A)$ *is its continuous spectrum.*

From (ii) of Theorem 1, we have the following result on the Anderson localization for a long range case by applying the theory of Kirsch, Stollmann and Stolz [15]:

Theorem 3. *We assume* (2.3) *for some* $\alpha > (5+\sqrt{21})d/2+1$. *Then, for any positive number* δ *and* ϵ, *there exist finite positive constants* c_1, c_2, c_3 *and* c_4 *depending* *only on d, α and ϵ such that* $0 < \delta < 1$ *and*

$$\mathbb{P}\left(|\omega(0)| \geq \frac{1}{\lambda \bar{A}}\left(\sqrt{B_0} - \sqrt{B_0 - \frac{(\lambda\bar{A})^2}{1-\delta}} - \epsilon\right)\right)$$
$$< \left\{c_1\left(\frac{\sqrt{B_0 - (\lambda\bar{A})^2/(1-\delta)}}{\sqrt{B_0}\big(1 \vee \log(1+B_0) \vee (\log(\lambda(\lambda+\sqrt{B_0})))_-\big)_1}\right)^{d+\epsilon}\right\}$$
$$\wedge \left\{c_2\left(\frac{\lambda\delta}{\lambda+\sqrt{B_0}} \wedge \frac{1}{\lambda}\right)^{(d+\epsilon)/(\alpha-1-d)}\right\}$$
$$\wedge \left\{c_3\left(\frac{\delta\lambda^{\hat{d}}}{B_0^{d+\hat{d}/2}\mathcal{H}'}\right)^{(d+\epsilon)/\{(\alpha-1-d)^2/(\alpha-1)-3d\}}\left(\frac{\lambda}{\lambda+\sqrt{B_0}}\right)^{c_4}\right\} \tag{2.9}$$

imply the exponential localization on the interval $I = [0, B_0 - (\lambda\bar{A})^2/(1-\delta)]$: *for a.e.* ω, *the spectrum of* $H^{\lambda\omega}$ *is pure point on the interval* I *with exponentially decaying eigenfunctions. In* (2.9), $(\,\cdot\,)_-$ *means the negative part.*

Unless the spectrum $\sigma(H^{\lambda\omega})$ intersects with the interval where these results hold, they would be meaningless. In order to address this problem, we give an upper estimate of the infimum of the spectrum in a simple 2-dimensional example as follows: we take $A(x)$ as the vector potential $(-Bx_2/2,\, Bx_1/2)$ for the uniform magnetic filed $B > 0$, and the single site potential $u(x)$ as $(0, \zeta(x_1)\zeta(x_2))$, where $\zeta(t) = 0$ for $|t| \geq 3/2$, $\zeta(t) = 1$ for $|t| \leq 1/2$, and $\zeta(t) = (3-2|t|)/2$ for $1/2 \leq |t| \leq 3/2$. Then $B_0 = B$ and $\bar{A} = 4$, since

$$\sum_{a_1 \in \mathbb{Z}} \zeta(x_1 - a_1) = 2. \tag{2.10}$$

The estimate of the spectrum is the following:

Proposition 2.1. *Let* $\omega_0 = (\omega_0(a_1, a_2))_{a_1, a_2 \in \mathbb{Z}}$ *be the element of the probability space* $[-1, 1]^{\mathbb{Z}^2}$ *defined by* $\omega_0(a_1, a_2) = 1$ *for* $a_1 \in 4\mathbb{Z} - 1$, $\omega_0(a_1, a_2) = -1$ *for* $a_1 \in 4\mathbb{Z} + 1$ *and* $\omega_0(a_1, a_2) = 0$ *for* $a_1 \in 2\mathbb{Z}$. *Then it holds that*

$$\inf \sigma(H^{\lambda\omega_0}) \leq B(\lambda), \tag{2.11}$$

where

$$B(\lambda) = B - 4\lambda e^{-6\lambda}(1 - e^{-B/8}) - \frac{\lambda e^{-6\lambda - 81B/4}}{\sqrt{2}} + \lambda e^{6\lambda - 9B/8}\left(1 + 8\sqrt{\frac{B}{2\pi}}\right).$$

By this proposition and the characterization of the almost sure spectral set by periodic operators (e.g. Theorem (5.33) in [23], §1.4 in [28]), we have

$$[B(\lambda) \wedge B,\ B] \subset \sigma(H^{\lambda\omega}) \tag{2.12}$$

if $\operatorname{supp} h = [-1, 1]$. Therefore, for the above theorems to be meaningful, it is enough that

$$B(\lambda) < B - (4\lambda)^2.$$

This holds if λ is small and B is large. Then, for small $\delta > 0$ so that the interval $[B(\lambda), B - (4\lambda)^2/(1-\delta)]$ is not empty, we can take the probability density h so that the Anderson localization occurs on this energy interval as follows:

EXAMPLE 2.1. For any $p, l \in (0, 1/2]$, there exists a smooth probability density function h such that $\operatorname{supp} h = [-1, 1]$ and $\int_{|s| \geq l} h(s)\, ds = p$. Such a function can be taken so that $|h'| \leq 2/l^2$. This bound is independent of p. Then (2.8) holds for sufficiently small p and l, since the right hand side of (2.8) is dominated from below by a quantity independent of p.

We next discuss the proof. To prove the Wegner type estimate, Hislop and Klopp [12] introduced Birman-Schwinger type operators by

$$\Gamma_{L,2}^{\lambda\omega} = (H_L^0 - E)^{-1/2} \lambda^2 |A^\omega|^2 (H_L^0 - E)^{-1/2}, \tag{2.13}$$

$$\Gamma_{L,1}^{\lambda\omega} = (H_L^0 - E)^{-1/2} \lambda \sum_{j=1}^{d} \{(i\partial_j + A_j) A_j^\omega + A_j^\omega (i\partial_j + A_j)\} (H_L^0 - E)^{-1/2} \tag{2.14}$$

and

$$\Gamma_L^{\lambda\omega} = \Gamma_{L,2}^{\lambda\omega} + \Gamma_{L,1}^{\lambda\omega}$$

for $0 \leq E < B_0$. Then we have the following reduction:

Lemma 2.1. *It holds that*

$$N([E-\eta, E+\eta] : H_L^{\lambda\omega}) \leq N([1-\kappa, 1+\kappa] : -\Gamma_L^{\lambda\omega}) \tag{2.15}$$

for any $0 \leq E < B_0$ *and* $0 < \eta < B_0 - E$, *where* $\kappa = \eta/(B_0 - E)$.

Proof. Let $\delta_1, \ldots, \delta_N$ be the all eigenvalues of $H_L^{\lambda\omega} - E$ in the interval $[-\eta, \eta]$ including the multiplicity and $v_1, \ldots, v_N$ be the orthonormal system consisting their eigenfunctions. These vectors satisfy

$$\delta_j (H_L^0 - E)^{-1/2} v_j = (I + \Gamma_L^{\lambda\omega}) w_j,$$

where $w_j = (H_L^0 - E)^{1/2} v_j$. The vectors $w_1, \ldots, w_N$ are linearly independent and

$$\left\| (I + \Gamma_L^{\lambda\omega}) \sum_j c_j w_j \right\|^2 \le \frac{1}{B_0 - E} \left\| \sum_j c_j \delta_j w_j \right\|^2 \le \frac{\eta^2}{B_0 - E} \left\| \sum_j c_j v_j \right\|^2$$
$$\le \kappa^2 \left\| \sum_j c_j w_j \right\|^2$$

for any $c_1, c_2, \ldots, c_N \in \mathbb{C}$. Then, by the min-max principle, we obtain (2.15). □

To treat the operator $\Gamma_L^{\lambda\omega}$, we prepare the following:

Lemma 2.2. *Let* $0 \le E < B_0$.
(i) *For any* $\mathcal{A} \in L^q(\Lambda_L \to \mathbb{R})$ *with some* $q \ge \hat{d}$, *the operator* $(H_L^0 - E)^{-1/2}\mathcal{A}$ *belongs to the class* $\mathcal{I}_q$ *and satisfies*

$$\|\|(H_L^0 - E)^{-1/2}\mathcal{A}\|\|_q \le c \frac{B_0^{d/(2q)}}{\sqrt{B_0 - E}} \|\mathcal{A}\|_q, \tag{2.16}$$

where c *is a finite constant depending only on* d *and* q, *and* $\|\|\cdot\|\|_q$ *is the trace norm defined by* $\|\|A\|\|_q := \mathrm{Tr}[|A|^q]^{1/q}$ *and* $|A| = \sqrt{A^* A}$ (*cf.* [24]).
(ii) *For any* $\mathcal{A} \in L^q(\Lambda_L \to \mathbb{R})$ *with some* $q \ge \hat{d}/2$, *we define an operator by*

$$\Gamma_L[\mathcal{A}] = (H_L^0 - E)^{-1/2}\mathcal{A}(H_L^0 - E)^{-1/2}.$$

Then this operator belongs to the class $\mathcal{I}_q$ *and satisfies*

$$\|\|\Gamma_L[\mathcal{A}]\|\|_q^q \le c \frac{B_0^{d/2}}{(B_0 - E)^q} \|\mathcal{A}\|_q^q, \tag{2.17}$$

where c *is a finite constant depending only on* d *and* q.
(iii) *For any* $\mathcal{B} \in L^q(\Lambda_L \to \mathbb{R}^d)$ *with some* $q \ge \hat{d}$, *we define an operator by*

$$\Gamma_L\langle\mathcal{B}\rangle = (H_L^0 - E)^{-1/2} \sum_{j=1}^{d} \{(i\partial_j + A_j)\mathcal{B}_j + \mathcal{B}_j(i\partial_j + A_j)\}(H_L^0 - E)^{-1/2}.$$

Then this operator belongs to the class $\mathcal{I}_q$ *and satisfies*

$$\|\|\Gamma_L\langle\mathcal{B}\rangle\|\|_q^q \le c \frac{B_0^{(d+q)/2}}{(B_0 - E)^q} \|\mathcal{B}\|_q^q, \tag{2.18}$$

where c *is a finite constant depending only on* d *and* q.

Proof. (i) We first assume that $q \in 2\mathbb{N}$. Then we have

$$|||(H_L^0 - E)^{-1/2}\mathcal{A}|||_q = |||\mathcal{A}(H_L^0 - E)^{-1}\mathcal{A}|||_{q/2}^{1/2}.$$

For any $R > 0$, by the resolvent equation

$$(H_L^0 - E)^{-1} = (H_L^0 + R)^{-1/2}\{1 + (R + E)(H_L^0 - E)^{-1}\}(H_L^0 + R)^{-1/2},$$

we have

$$|||(H_L^0 - E)^{-1/2}\mathcal{A}|||_q \leq |||(H_L^0 + R)^{-1/2}\mathcal{A}|||_q \sqrt{\frac{B_0 + R}{B_0 - E}}.$$

As in Lemma 2.1 of [30], we use the diamagnetic inequality to obtain

$$|(H_L^0 + R)^{-1/2}\varphi| \leq (-\Delta_L + R)^{-1/2}|\varphi| \tag{2.19}$$

for any $\varphi \in C_0^\infty(\Lambda_L)$, where Δ_L is the Laplacian with the Dirichlet boundary condition on $L^2(\mathbb{R}^d)$ (see (4.9) in [19] and (A.23) in [13]). By using (2.19) successively, we have

$$\left||(H_L^0 + R)^{-1/2}\mathcal{A}|^q\varphi\right| \leq \left|(-\Delta_L + R)^{-1/2}|\mathcal{A}|\right|^q|\varphi|.$$

From this, Lemma 15.11 in [27] and Theorem 4.1 in [26], we have

$$\begin{aligned}|||(H_L^0 + R)^{-1/2}\mathcal{A}|||_q &\leq |||(-\Delta_L + R)^{-1/2}|\mathcal{A}|\,|||_q \leq |||(-\Delta + R)^{-1/2}|\mathcal{A}|\chi_L|||_q \\ &\leq c\|\mathcal{A}\chi_L\|_q / R^{(q-d)/(2q)}\end{aligned}$$

if $q > d$, where c is a finite constant depending only on q and d, and χ_L is the characteristic function of Λ_L. Since $\min_{R>0}(B_0 + R)/R^{(q-d)/q} = cB_0^{d/q}$, we have (2.16). For $q \notin 2\mathbb{N}$, we use the Stein interpolation theorem (cf. [25] Theorem IX.21).

(ii) Since

$$|||\Gamma_L[\mathcal{A}]|||_q \leq |||(H_L^0 - E)^{-1/2}\sqrt{|\mathcal{A}|}\,|||_{2q}^2.$$

(i) implies (ii).

(iii) Since

$$|||(H_L^0 - E)^{-1/2}(i\partial_j + A_j)||| \leq \sqrt{\frac{B_0}{B_0 - E}}$$

for any j, (i) implies (iii). □

As in [4] and [5], the right hand side of (2.15) is dominated by

$$\int_{-3\kappa/2}^{3\kappa/2} dt\ \mathrm{Tr}[(\rho_\kappa)'(-\Gamma_L^{\lambda\omega} - 1 + t)],$$

where $\rho_\kappa(s) = \rho(s/\kappa)$ and ρ is a smooth function on $\mathbb{R}$ such that $0 \le \rho \le 1$ on $\mathbb{R}$, $\rho = 0$ on $(-\infty, -1/2]$ and $\rho = 1$ on $[1/2, \infty)$. Then Hislop and Klopp [12] used a vector field $\mathbb{A}$ on the probability space defined by

$$\mathbb{A} = \sum_{a \in \mathbb{Z}^d} \omega(a) \frac{\partial}{\partial \omega(a)}. \tag{2.20}$$

This vector field acts as

$$\mathbb{A}\,\mathrm{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)] = \mathrm{Tr}[(\rho_\kappa)'(-\Gamma_L^{\lambda\omega} - 1 + t)(-\Gamma_{L,1}^{\lambda\omega} - 2\Gamma_{L,2}^{\lambda\omega})].$$

Since $\Gamma_{L,2}^{\lambda\omega} \le (\lambda \bar{A})^2/(B_0 - E)$ and $-\Gamma_L^{\lambda\omega} \ge 1 - 2\kappa$ in the space of $(\rho_\kappa)'(-\Gamma_L^{\lambda\omega} - 1 + t) \ne 0$, we have

$$\left(1 - 2\kappa - \frac{(\lambda \bar{A})^2}{B_0 - E}\right) \mathrm{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)] \le \mathbb{A}\,\mathrm{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)].$$

Therefore, if $(B_0 - E)(1 - \delta) \ge (\lambda \bar{A})^2$ and $\kappa \le \delta/4$, then we have

$$\mathrm{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)] \le \frac{2}{\delta} \mathbb{A}\,\mathrm{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)]$$

and

$$\mathbb{E}[N([1 - \kappa, 1 + \kappa] : -\Gamma_L^{\lambda\omega})] \le \frac{2}{\delta} \int_{-3\kappa/2}^{3\kappa/2} dt \sum_{a \in \mathbb{Z}^2 \cap \Lambda_{L+6}} I(L, t; a),$$

where

$$I(L, t; a) = \mathbb{E}\left[\int_{-1}^{1} d\omega(a) h(\omega(a)) \omega(a) \frac{\partial}{\partial \omega(a)} \mathrm{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)]\right].$$

To prove Theorem 1 (i), the following simple estimate is enough:

Lemma 2.3. *There exists a finite constant c depending only on d and* $\mathrm{diam}\,\mathrm{supp}\,u$ *such that*

$$\sup_{-3\kappa/2 \le t \le 3\kappa/2} |I(L, t; a)| \le c \frac{\lambda^{\hat{d}} B^{(d+\hat{d})/2} \mathcal{H}'}{(B_0 - E)^{\hat{d}}} L^d \tag{2.21}$$

for any $a \in \mathbb{Z}^d$, $L \ge 1$ and $0 < \kappa \le 1/3$.

Proof. By the integration by parts, we have

$$I(L, t; a) = \mathbb{E}\left[-\int_{-1}^{1} ds (h(\omega(a)) + \omega(a) h'(\omega(a))) \operatorname{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - E + t)] \right]. \tag{2.22}$$

For any ω, we have

$$\begin{aligned} \operatorname{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega} - E + t)] &\leq \operatorname{Tr}[E([1 - 2\kappa, \infty) : -\Gamma_L^{\lambda\omega})] \\ &\leq \operatorname{Tr}[E([1 - 2\kappa, \infty) : -\Gamma_{L,1}^{\lambda\omega})] \\ &\leq (3 |\!|\!| \Gamma_{L,1}^{\lambda\omega} |\!|\!|_{\hat{d}})^{\hat{d}}. \end{aligned} \tag{2.23}$$

By Lemma 2.2 (ii), we can complete the proof. □

To prove Theorem 1 (ii), we use the following estimate:

Lemma 2.4. *Under the condition of* Theorem 1 (ii), *for any $q \geq \hat{d}$ and $q < l \in \mathbb{N}$, there exists a finite constant c depending only on l, q, α and d such that*

$$\sup_{-3\kappa/2 \leq t \leq 3\kappa/2} |I(L, t; a)| \leq \frac{c \lambda^q B_0^{d/2} (\lambda + \sqrt{B_0})^q \mathcal{H}' L^d}{\kappa^{q/l} (B_0 - E)^q (1 + \operatorname{dist}(a, \Lambda_L))^{(\alpha-1)q/l}} \tag{2.24}$$

for any $a \in \mathbb{Z}^d$, $\eta > 0$ and $L \geq 1$.

Proof. As in [4], [5] and [12], we rewrite (2.22) as follows and apply the theory of the spectral shift functions:

$$\begin{aligned} I(L, t; a) = \mathbb{E}\Bigg[\int_{-1}^{1} & d\omega(a) (h(\omega(a)) + \omega(a) h'(\omega(a))) \\ & \times \operatorname{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega[a,0]} - 1 + t) - \rho_\kappa(-\Gamma_L^{\lambda\omega} - 1 + t)] \Bigg], \end{aligned}$$

where $\omega[a, 0]$ is an element of the probability space defined by replacing $\omega(a)$ by 0. By Lemma 2.5 below and Theorem 2.1 in [4], [5] and [12], we have the spectral shift function $\xi(s : (\Gamma_L^{\lambda\omega[a,0]})^l, (\Gamma_L^{\lambda\omega})^l)$ for the pair $(\Gamma_L^{\lambda\omega[a,0]})^l, (\Gamma_L^{\lambda\omega})^l$ such that

$$\|\xi(s : (\Gamma_L^{\lambda\omega[a,0]})^l, (\Gamma_L^{\lambda\omega})^l)\|_{l/q} \leq |\!|\!| (\Gamma_L^{\omega[a,0]})^l - (\Gamma_L^{\lambda\omega})^l |\!|\!|_{q/l}^{q/l}$$

(cf. [4], [5], [12], [26]). As in [4], [5] and [12], we apply the Birman-Krein identity

[2], [33] as follows:

$$\begin{aligned}&\operatorname{Tr}[\rho_\kappa(-\Gamma_L^{\lambda\omega[a,0]}-1+t)-\rho_\kappa(-\Gamma_L^{\lambda\omega}-1+t)]\\&=\int_0^\infty\left\{\frac{\partial}{\partial s}\rho_\kappa(-s^{1/l}-1+t)\right\}\xi(s:(\Gamma_L^{\lambda\omega[a,0]})^l,(\Gamma_L^{\lambda\omega})^l)\,ds.\end{aligned}$$

Since

$$\left(\int_0^\infty\left|\frac{\partial}{\partial s}\rho_\kappa(-s^{1/l}-1+t)\right|^{l/(l-q)}dr\right)^{(l-q)/l}\le c_1\left(\int_{\mathbb{R}}|(\rho_\kappa)'(r)|^{l/(l-q)}\,dr\right)^{(l-q)/l}\le\frac{c_2}{\kappa^{q/l}},$$

we obtain (2.24). □

The following is the estimate of the difference of the operators used in the proof of the preceding lemma:

Lemma 2.5. *For any $q\ge\hat{d}$ and $l\in\mathbb{N}$, the operator $(\Gamma_L^{\lambda\omega[a,0]})^l-(\Gamma_L^{\lambda\omega})^l$ belongs to the super trace class $\mathcal{I}_{q/l}$ and satisfies*

$$|||(\Gamma_L^{\lambda\omega[a,0]})^l-(\Gamma_L^{\lambda\omega})^l|||_{q/l}^{q/l}\le\frac{c\lambda^qB_0^{d/2}(\lambda+\sqrt{B_0})^qL^d|\omega(a)|^{q/l}}{(B_0-E)^q(1+\operatorname{dist}(a,\Lambda_L))^{(\alpha-1)q/l}},\tag{2.25}$$

where c is a finite constant depending only on q, l, α and d.

Proof. We apply Lemma 2.2 to the each term of

$$\begin{aligned}&(\Gamma_L^{\lambda\omega[a,0]})^l-(\Gamma_L^{\lambda\omega})^l\\&=-\sum_{h=1}^{l}(\Gamma_L^{\lambda\omega[a,0]})^{h-1}\big(\Gamma_L[\lambda^2\omega(a)u(x-a)\cdot(A^{\omega[a,0]}+A^\omega)]\\&\qquad+\Gamma_L\langle\lambda\omega(a)u(x-a)\rangle\big)(\Gamma_L^{\lambda\omega})^{l-h}.\end{aligned}\tag{2.26}$$

To estimate the norm $\|u\|_q$, we use

$$|u(x)|\le(\alpha-1)^{-1}(1+|x|)^{-(\alpha-1)}.\tag{2.27}$$

□

To prove Theorem 1 (iii), we use the following instead of Lemma 2.5:

Lemma 2.6. *If* $\operatorname{supp}u$ *is compact, then* (2.25) *is replaced by*

$$\begin{aligned}&|||(\Gamma_L^{\lambda\omega[a,0]})^l-(\Gamma_L^{\lambda\omega})^l|||_{q/l}^{q/l}\\&\le c\left(\frac{\sqrt{B_0}}{B_0-E}\vee1\right)^{q(1-1/l)}\left(\frac{\lambda(\lambda+\sqrt{B_0})}{B_0-E}\right)^qB_0^{d/2}|\omega(a)|^{q/l},\end{aligned}\tag{2.28}$$

where c is a finite constant depending only on $\operatorname{diam}\operatorname{supp} u$, q, l *and* d.

Proof. As in Nakamura [22], we take smooth functions ϕ_k, $k = 1, 2, \dots,$ such that $0 \le \phi_k \le 1$, $\phi_k = 1$ on $\operatorname{supp}\phi_{k-1}$ and $\operatorname{supp}\phi_k \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x, \operatorname{supp} u) < 1\}$ for any $k = 1, 2, \dots,$ where $\phi_0 = |u|^2$, and consider the commutator of these functions with the resolvents to rewrite the expression (2.26) as follows:

$$
\begin{aligned}
&(\Gamma_L^{\lambda\omega[a,0]})^l - (\Gamma_L^{\lambda\omega})^l \\
&= \sum_{h=1}^{l} \sum_{m=1}^{2^{l-1}} \sigma(h, m)(\Gamma_L^{\lambda\omega[a,0]})^{N(h,m)} \\
&\qquad \times \left(\Gamma_L\bigl[\widetilde{\mathcal{A}_{h,m,l}}\bigr] + \Gamma_L\bigl\langle\widehat{\mathcal{A}_{h,m,l}}\bigr\rangle\right) \cdots \left(\Gamma_L\bigl[\widetilde{\mathcal{A}_{h,m,h+1}}\bigr] + \Gamma_L\bigl\langle\widehat{\mathcal{A}_{h,m,h+1}}\bigr\rangle\right) \\
&\qquad \times \left(\Gamma_L\bigl[\tilde{\mathcal{A}}\bigr] + \Gamma_L\bigl\langle\hat{\mathcal{A}}\bigr\rangle\right)\left(\Gamma_L\bigl[\widetilde{\mathcal{A}_{h,m,h-1}}\bigr] + \Gamma_L\bigl\langle\widehat{\mathcal{A}_{h,m,h-1}}\bigr\rangle\right) \\
&\qquad \times \cdots \times \\
&\qquad \times \left(\Gamma_L\bigl[\widetilde{\mathcal{A}_{h,m,1}}\bigr] + \Gamma_L\bigl\langle\widehat{\mathcal{A}_{h,m,1}}\bigr\rangle\right)(\Gamma_L^{\lambda\omega[a,0]})^{M(h,m)},
\end{aligned}
$$

where $\sigma(h, m) \in \{+, -\}$, $\hat{\mathcal{A}} = \lambda\omega(a)u(x - a)$, $\tilde{\mathcal{A}} = \lambda^2\omega(a)u(x - a) \cdot (A^{\omega[a,0]} + A^{\omega})$, each of $\left(\widehat{\mathcal{A}_{h,m,j}}, \widetilde{\mathcal{A}_{h,m,j}}\right)$ is one of $(\phi_k(x-a)\lambda A^{\omega[a,0]}(x), \phi_k(x-a)|\lambda A^{\omega[a,0]}|^2)$, $(\phi_k(x - a)\lambda A^{\omega}(x), \phi_k(x-a)|\lambda A^{\omega}|^2)$ or $(i\nabla\phi_k(x-a), 0)$ for some $k \in \mathbb{N}$, $N(h, m)$ is the number of j in $\{h + 1, \dots, l\}$ such that $\left(\widehat{\mathcal{A}_{h,m,j}}, \widetilde{\mathcal{A}_{h,m,j}}\right) = (i\nabla\phi_k(x - a), 0)$ for some $k \in \mathbb{N}$, and $M(h, m)$ is that of j in $\{1, \dots, h - 1\}$. Then we obtain a bound independent of L since the volumes of the supports of $\widehat{\mathcal{A}_{h,m,j}}$ and $\widetilde{\mathcal{A}_{h,m,j}}$ are dominated by a constant independent of L. □

For the proof of Theorem 2, we use the following extension of Theorem 3.4 in Germinet and Klein [11], which is an improved theory of the multiscale analysis founded by Fröhlich and Spencer [10]:

Proposition 2.2. *Let I_0 and $\widetilde{I}_0$ be compact and open intervals such that $I_0 \subset \widetilde{I}_0$. For each $L \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, let $\Lambda_L(x) = x + \Lambda_L$ and let $\{H^{\omega}_{L,x}\}_{\omega\in\Omega}$ be a family of random operators on $L^2(\Lambda_L(x) \to \mathbb{C}^{\nu})$ for some $\nu \in \mathbb{N}$, satisfying the following*:

(STA) (*Stationarity*) *The probability of each event determined by a condition on an operator $U_x H^{\omega}_{L,x} U_x^{-1}$ is independent of x, where U_x is the unitary operator from $L^2(\Lambda_L(x) \to \mathbb{C}^{\nu})$ to $L^2(\Lambda_L \to \mathbb{C}^{\nu})$ defined by $(U_x\varphi)(y) = \varphi(y + x)$ for any $\varphi \in L^2(\Lambda_L(x) \to \mathbb{C}^{\nu})$*;

(IAD) (*Independence at distance*) *There exists a finite constant ρ such that for any finite number of boxes $\{\Lambda_{L_j}(x_j)\}_{j=1}^n$ with $\operatorname{dist}(\Lambda_{L_i}(x_i), \Lambda_{L_j}(x_j)) \ge \rho$ for $i \ne j$, the events determined by conditions on operator $U_{x_j} H^{\omega}_{L_j,x_j} U_{x_j}^{-1}$ are mutually independent in $j = 1, 2, \dots, n$*;

(SLI) (*Simon-Lieb inequality*) *There exists a finite constant* γ *such that*

$$\begin{aligned} &|||\Gamma_{l_3,x_3}(H^{\omega}_{l_3,x_3} - E)^{-1}\chi_{l_1,x_1}||| \\ &\le \gamma\, |||\Gamma_{l_2,x_2}(H^{\omega}_{l_2,x_2} - E)^{-1}\chi_{l_1,x_1}||| \times |||\Gamma_{l_3,x_3}(H^{\omega}_{l_3,x_3} - E)^{-1}\Gamma_{l_2,x_2}||| \end{aligned} \tag{2.29}$$

for any $E \in I_0 - \sigma(H^{\omega}_{l_2,x_2}) - \sigma(H^{\omega}_{l_3,x_3})$, $l_1, l_2, l_3 \in 2\mathbb{N}$ *and* $x_1, x_2, x_3 \in \mathbb{Z}^d$ *such that* $\Lambda_{l_1}(x_1) \subset \Lambda_{l_2-3}(x_2) \subset \Lambda_{l_2}(x_2) \subset \Lambda_{l_3-2}(x_3)$, *where, for any* $L \in \mathbb{N}$ *and* $x \in \mathbb{Z}^d$, $\Gamma_{L,x}$ *and* $\chi_{L,x}$ *are the operators of multiplication of the characteristic functions of* $\overline{\Lambda_{L-1}(x)} - \Lambda_{L-3}(x)$ *and* $\Lambda_L(x)$, *respectively*;

(W) (*Wegner type estimate*) *There exist constants* $C_W, l_0, \eta_0 \in (0, \infty)$, $h \in (0, 1]$ *and* $b \in [1, \infty)$ *such that*

$$\mathbb{P}(d(E, \sigma(H^{\omega}_L)) \le \eta) \le C_W \eta^h L^{bd} \tag{2.30}$$

for any $E \in \widetilde{I}_0$, $0 < \eta < \eta_0$ *and* $l_0 \le L \in 2\mathbb{N}$;

(NE) (*Number of eigenvalues*) *There exist constants* $C_{NE} \in (0,\infty)$ *and* $v \in [1,\infty)$ *such that*

$$\mathbb{E}[N(\widetilde{I}_0 : H^{\omega}_L)] \le C_{NE} L^{vd} \tag{2.31}$$

for any $l_0 \le L \in 2\mathbb{N}$.

For any $\theta > 0$, $E \in \mathbb{R}$, $6 \le L \in \mathbb{N}$ *and* $x \in \mathbb{Z}^d$, *we say a box* $\Lambda_L(x)$ *is* (θ, E)-*suitable for* ω *if* $E \notin \sigma(H^{\omega}_{L,x})$ *and*

$$|||\Gamma_{L,x}(H^{\omega}_{L,x} - E)^{-1}\chi_{L/3,x}||| \le L^{-\theta}.$$

For any $m > 0$, $E \in \mathbb{R}$, $6 \le L \in \mathbb{N}$ *and* $x \in \mathbb{Z}^d$, *we say a box* $\Lambda_L(x)$ *is* (m, E)-*regular for* ω *if* $E \notin \sigma(H^{\omega,g}_{L,x_0})$ *and*

$$|||\Gamma_{L,x}(H^{\omega}_{L,x} - E)^{-1}\chi_{L/3,x}||| \le \exp\left(-m\frac{L}{2}\right).$$

For $m > 0$, $6 \le L \in \mathbb{N}$, *an interval* I *in* $\mathbb{R}$ *and* $x, y \in \mathbb{Z}^d$, *we set*

$$R(m, L, I, x, y) := \{\omega\colon \text{for every } E \in I, \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (m, E)\text{-regular}\}.$$

Then for any $\theta > 4(b+v)d/(3h)$, *there exists a positive finite constant* $\bar{\mathcal{L}}$ *depending only on* d, ρ, C_W, b, h, l_0, η_0, γ *and* θ, *and satisfying the following*: *if there exist* $E_0 \in I_0$ *and* $\bar{\mathcal{L}} \le \mathcal{L} \in 6\mathbb{N}$ *such that*

$$\mathbb{P}(\Lambda_{\mathcal{L}} \text{ is } (\theta, E_0)\text{-suitable}) > 1 - 841^{-d}, \tag{2.32}$$

then there exists $\delta_0 > 0$ *depending only on* d, ρ, C_W, b, h, η_0, C_{NE}, v, θ *and* $\mathcal{L}$ *such that, for any* $0 < \zeta < 1$ *and* $1 < \alpha < 1/\zeta$, *there exist* $L_0 \in 6\mathbb{N}$ *depending only on* d,

ρ, C_W, b, h, C_{NE}, υ, θ, $\mathcal{L}$ dist$(\widetilde{I_0}^c, I_0)$, ζ and α, and $m > 0$ depending only on ζ and L_0, satisfying

$$\mathbb{P}(R(m, L_k, I_0(E_0, \delta_0), x, y)) > 1 - \exp(-L_k^\zeta)$$

for any $k \in \mathbb{Z}_+$, x, $y \in \mathbb{Z}^d$ with $|x - y|_\infty > L_k + \rho$, where $L_{k+1} = \max\{(6\mathbb{N}) \cap [0, L_k^\alpha]\}$ and $I_0(E_0, \delta_0) := [E_0 - \delta_0, E_0 + \delta_0] \cap I_0$.

The constant $\bar{\mathcal{L}}$ can be taken as

$$\bar{\mathcal{L}} \leq (11^{-1}\eta_0^{-4h/(3h\theta+bd)}) \vee (c_1\gamma^{8h/(h\theta-bd)}) \vee (c_2C_W^{4/(h\theta-bd)}) \vee c_3, \tag{2.33}$$

where c_1, c_2 and c_3 are finite constants depending only on d, ρ, b, h, l_0 and θ.

This proposition can be proved as in §3.3 of [30] and is used to prove Theorems 2, 5, 7 and 9 below.

By this proposition and its application to the proof of the Anderson localization in [11], we have only to show that the assumptions in this proposition are satisfied with the compact interval $I_0 = [0, B - (\lambda\bar{A})^2/(1 - \delta)]$ and the open interval $\widetilde{I_0} = [0, B - (\lambda\bar{A})^2/(1 - \delta/2))$ for arbitrarily fixed $\delta \in (0, 1)$. Then (STA) obviously holds. (IAD) holds by taking ρ as diam supp u. (SLI) holds with $\gamma = c_1(1 + \sqrt{B_0})$, where c_1 is a finite constant depending only on d. Theorem 1 (i) implies (W) with $b = 2$, $h = 1$, $\eta_0 = (\lambda\bar{A})^2\delta/(4(2 - \delta))$ and $C_W = c_2(2 - \delta)^{\hat{d}+1}B_0^{(d+\hat{d})/2}\mathcal{H}'/(\lambda^{\hat{d}+2}\delta)$, where c_2 is a finite constant depending only on d and diam supp u. (NE) holds with $\upsilon = 1$ by the following lemma:

Lemma 2.7. *For any $q \geq \hat{d}$, there exists a finite constant c depending only on d and q such that*

$$N([0, B_0 - \zeta] : H_L^{\lambda\omega}) \leq cB_0^{(d+q)/2}\|\lambda A^\omega\chi_L\|_q^q\zeta^{-q} \tag{2.34}$$

for any $\zeta > 0$, ω and $L \geq 1$.

Proof. We use Lemma 2.1 with $E = \eta = (B_0 - \zeta)/2$. Then by using also the positivity of $\Gamma_{L,2}^{\lambda\omega}$, we have

$$N([0, B_0 - \zeta] : H_L^{\lambda\omega}) \leq N\left(\left[\frac{2\zeta}{B_0 + \zeta}, \infty\right) : -\Gamma_{L,1}^{\lambda\omega}\right) \leq \left(\frac{B_0 + \zeta}{2\zeta}\right)^q \mathrm{Tr}[(\Gamma_{L,1}^{\lambda\omega})^q].$$

By Lemma 2.2, we obtain (2.34). $\square$

Now, by Proposition 2.2, we have only to show the corresponding initial estimate (2.32) for any $E_0 \in I_0$, some $\theta > 4d$ and $\mathcal{L} \in 6\mathbb{N}$ satisfying

$$\mathcal{L} \geq \frac{c_3}{(\lambda^2\delta)^{4/(3\theta+2d)}} \vee (c_4(1+B_0)^{4/(\theta-2d)}) \vee \left\{ c_5 \left(\frac{B^{(d+\hat{d})/2}\mathcal{H}'}{\lambda^{\hat{d}+2}\delta} \right)^{4/(\theta-2d)} \right\} \vee c_6, \tag{2.35}$$

where c_3, c_4, c_5 and c_6 are finite constants depending only on θ, d and diam supp u. For this we use the following Combes-Thomas type estimate (cf. [6]):

Lemma 2.8. *We take $\nu > 0$, $E_0 \in [0, B_0)$, $L \geq 1$, $A, B \subset \Lambda_L$ so that*

$$\operatorname{dist}(E_0, \sigma(H_L^{\lambda\omega})) \geq \nu \quad \text{and} \quad D(A, B) > 0,$$

where $D(A, B) = \sup_{v\in\mathbb{R}^d:\,|v|=1} \inf_{x\in A, y\in B} v\cdot(x-y)$. Then there exist universal finite positive constants c_1 and c_2 such that

$$|||\chi_A(H_L^{\lambda\omega} - E_0)^{-1}\chi_B||| \leq \frac{c_1}{\nu}\exp\left(-\frac{c_2\sqrt{\nu}}{1+\sqrt{E_0/\nu}}D(A, B)\right), \tag{2.36}$$

where χ_A and χ_B are the operators of multiplying the characteristic functions of A and B, respectively.

Proof. For any $w \in \mathbb{R}^d$ such that $|w| < \nu$, we have

$$e^{-w\cdot x}H_L^{\lambda\omega}e^{w\cdot x} = H_L^{\lambda\omega} + 2iw\cdot(i\nabla + A^\omega) - |w|^2.$$

Since

$$\|w\cdot(i\nabla + A^\omega)\varphi\| \leq \frac{1}{2R}\|(H_L^{\lambda\omega} - E_0)\varphi\| + \left(\frac{R}{2}|w|^2 + \sqrt{E_0}|w|\right)\|\varphi\|$$

for any $R > 0$ and $\varphi \in C_0^\infty$, we have

$$|||e^{-w\cdot x}(H_L^{\lambda\omega} - E_0)^{-1}e^{w\cdot x}||| \leq \left\{\nu\left(1-\frac{1}{R}\right) - (1+R)|w|^2 - 2\sqrt{E_0}|w|\right\}_+^{-1}$$

and

$$\begin{aligned} &|||\chi_A(H_L^{\lambda\omega} - E_0)^{-1}\chi_B||| \\ &\leq \sup_{x\in A, y\in B}\exp(w\cdot(x-y))\left\{\nu\left(1-\frac{1}{R}\right) - (1+R)|w|^2 - 2\sqrt{E_0}|w|\right\}_+^{-1}. \end{aligned}$$

By taking $w = -kv$ with $k > 0$ and $v \in \mathbb{R}^d$ such that $|v| = 1$ and taking the infimum with respect to v, we have

$$|||\chi_A(H_L^{\lambda\omega} - E_0)^{-1}\chi_B||| \le \exp(-kD(A, B))\left\{\nu\left(1 - \frac{1}{R}\right) - (1+R)k^2 - 2\sqrt{E_0}\,k\right\}_+^{-1}.$$

As an simple bound we take k so that

$$\nu\left(1 - \frac{1}{R}\right) - (1+R)k^2 - 2\sqrt{E_0}\,k = \frac{\nu(1 - 1/R)}{2}.$$

Then we obtain (2.36). □

By this lemma, for $\Lambda_{\mathcal{L}}$ to be (θ, E_0)-suitable, it is enough that

$$\inf \sigma(H_{\mathcal{L}}^{\lambda\omega}) \ge E_0 + f(\theta, \mathcal{L})\sqrt{E_0} + f(\theta, \mathcal{L})^2 \tag{2.37}$$

where $f(\theta, \mathcal{L}) = c_7(\theta \log \mathcal{L} + \log(8d))/\mathcal{L}$ and c_7 is a finite constant. Since

$$\begin{aligned} H_{\mathcal{L}}^{\lambda\omega} &\ge \left(1 - \frac{1}{t}\right)H_{\mathcal{L}}^0 - (t-1)\sup_{\Lambda_{\mathcal{L}}}|\lambda A^\omega|^2 \\ &\ge \left(1 - \frac{1}{t}\right)B_0 - (t-1)(\lambda\bar{A})^2 \sup_{a \in \Lambda_{\mathcal{L}+R}\cap\mathbb{Z}^d}|\omega(a)| \end{aligned}$$

for any $t > 1$, (2.37) is replaced by

$$\begin{aligned} &\sup_{t>1}\{(1 - 1/t)B_0 - (t-1)(\lambda\bar{A})^2 \sup_{a\in\Lambda_{\mathcal{L}+R}\cap\mathbb{Z}^d}|\omega(a)|^2 - E_0\} \\ &\ge f(\theta, \mathcal{L})\sqrt{E_0} + f(\theta, \mathcal{L})^2, \end{aligned} \tag{2.38}$$

where R is an integer such that $\operatorname{supp} u \subset \Lambda_R$. Therefore a sufficient condition for the corresponding (2.32) is

$$\mathbb{P}\left(\lambda\bar{A} \sup_{a\in\Lambda_{\mathcal{L}+R}\cap\mathbb{Z}^d}|\omega(a)| > \sqrt{B_0} - \sqrt{E_0} - f(\theta, \mathcal{L})\right) \le 841^{-d}. \tag{2.39}$$

Since ω is identically distributed, this condition is replaced by

$$\mathbb{P}\left(|\omega(0)| \ge \frac{\sqrt{B_0} - \sqrt{E_0} - f(\theta, \mathcal{L})}{\lambda\bar{A}}\right) \le c_8\mathcal{L}^{-d},$$

where c_8 is a finite positive constant depending only on d and $\operatorname{diam}\operatorname{supp} u$. We now let $E_0 = \sup I_0$, substitute the right hand side of (2.35) to $\mathcal{L}$, and take θ and $\mathcal{L}$ sufficiently largely. Then we obtain the condition in (2.8).

To prove Theorem 3 following Kirsch, Stollmann and Stolz [15], we use the following to estimate the probability of events on $H^{\lambda\omega}_{L,x}$ by taking the supremum with respect to $\{\omega(a)\colon a \in \mathbb{Z}^d - \Lambda_{2L}(x)\}$ so that events on $H^{\lambda\omega}_{L,x}$ and $H^{\lambda\omega}_{L',x'}$ can be treated as if they are independent when $\Lambda_{2L}(x) \cap \Lambda_{2L'}(x') = \emptyset$, where $H^{\lambda\omega}_{L,x}$ is the restriction of $H^{\lambda\omega}$ to $L^2(\Lambda_L(x))$ with the Dirichlet boundary condition:

Lemma 2.9. *We assume* (2.3) *for some* $\alpha > d+1$, $\lim_{|x|\to\infty} u(x) = 0$, $L \geq 1$ *and* $\omega = \omega'$ *on* $\Lambda_{2L} \cap \mathbb{Z}^d$. *Then we have*

$$\left| \sqrt{\mu_j(H^{\lambda\omega}_L)} - \sqrt{\mu_j(H^{\lambda\omega'}_L)} \right| \leq \frac{c\lambda}{L^{\alpha-1-d}} \tag{2.40}$$

for any $j \in \mathbb{N}$, *where* $\mu_j(H^{\lambda\omega}_L)$ *is the* j*-th eigenvalue of* $H^{\lambda\omega}_L$ *including the multiplicity, and* c *is a finite constant depending only on* d *and* α.

This lemma is proved by using the min-max principle and $\sup_{x\in\Lambda_L} |(A^\omega - A^{\omega'})(x)| \leq c/L^{\alpha-1-d}$.

In the theory of [15], the main part is to dominate the following probability of the resonance to proceed well the multiscale analysis:

$$\mathbb{P}\big(\widetilde{\operatorname{dist}}(\sigma_I(H^{\lambda\omega}_{L,x}), \sigma_I(H^{\lambda\omega}_{L',x'})) \leq \Xi\big), \tag{2.41}$$

where x, x', L, L' are taken so that $\Lambda_{2L}(x) \cap \Lambda_{2L'}(x') = \emptyset$, $\sigma_I(H^{\lambda\omega}_{L,x}) = \sigma(H^{\lambda\omega}_{L,x}) \cap [0, B_0 - (\lambda\overline{A})^2/(1-\delta) + \Xi/2]$,

$$\begin{aligned}
&\widetilde{\operatorname{dist}}(\sigma_I(H^{\lambda\omega}_{L,x}), \sigma_I(H^{\lambda\omega}_{L',x'}))\\
&= \inf\big\{\operatorname{dist}(\sigma_I(H^{\lambda\overline{\omega}}_{L,x}), \sigma_I(H^{\lambda\overline{\omega}}_{L',x'}))\colon\\
&\qquad \overline{\omega}, \overline{\omega'} \in \Omega \text{ s.t. } \overline{\omega} = \omega \text{ on } \Lambda_{2L}(x) \cap \mathbb{Z}^d, \overline{\omega'} = \omega \text{ on } \Lambda_{2L'}(x') \cap \mathbb{Z}^d\big\}
\end{aligned}$$

and Ξ is a number in the interval $(0, B_0)$ we should specify. We here note that

$$|\mu_j(H^{\lambda\omega}_L) - \mu_j(H^{\lambda\omega'}_L)| \leq \frac{c_1(\sqrt{B_0} + \lambda)\lambda}{L^{\alpha-1-d}}$$

holds in the situation of Lemma 2.9, if $\mu_j(H^{\lambda\omega}_L) \wedge \mu_j(H^{\lambda\omega'}_L) \leq B_0$. Thus we can cover $\sigma_I(H^{\lambda\overline{\omega'}}_{L',x'})$ by

$$\begin{aligned}
N_{\omega,x,x'} = \inf\Bigg\{ & N\Bigg(\Bigg[0,\ B_0 - \frac{(\lambda\overline{A})^2}{1-\delta} + \frac{c_1(\sqrt{B_0}+\lambda)\lambda}{L^{\alpha-1-d}}\Bigg] : H^{\lambda\overline{\omega'}}_{L',x'}\Bigg)\colon\\
& \overline{\omega'} \in \Omega \text{ s.t. } \overline{\omega'} = \omega \text{ on } \Lambda_{2L'}(x') \cap \mathbb{Z}^d \Bigg\}
\end{aligned}$$

number of intervals $\{I_{j,\omega,x,x'} : j \leq N_{\omega,x,x,'}\}$ depending only on $\{\omega(a) : a \in \mathbb{Z}^d \cap \Lambda_{2L'}(x')\}$ with the length less than $c_2(\sqrt{B_0} + \lambda)\lambda/L^{\alpha-1-d}$. Then the probability in (2.41) is dominated by

$$\mathbb{E}_{2L',x'}\left[\sum_{j \leq N_{\omega,x,x'}} \mathbb{P}_{2L,x}\Big(\inf\big\{\mathrm{dist}(\sigma_I(H_{L,x}^{\lambda\omega_x}),\ I_{j,\omega,x,x'}) : \overline{\varpi} = \omega \text{ on } \Lambda_{2L}(x) \cap \mathbb{Z}^d\big\} \leq \Xi\Big)\right],$$

where $\mathbb{P}_{2L,x}$ is the probability with respect to $\{\omega(a) : a \in \Lambda_{2L}(x) \cap \mathbb{Z}^d\}$ and $\mathbb{E}_{2L,x}$ is its expectation. The probability in these expectation and summation is dominated by

$$\mathbb{P}_{2L,x}\left(\mathrm{dist}(\sigma_I(H_{L,x}^{\lambda\omega}),\ E_{j,\omega,x,x'}) \leq \Xi + c_2\frac{(\sqrt{B_0} + \lambda)\lambda}{L^{\alpha-1-d}}\right),$$

which is estimated by Theorem 1 (ii), and $\mathbb{E}_{2L',x'}[N_{\omega,x,x'}]$ is estimated by Lemma 2.7, where $E_{j,\omega,x,x'}$ is the middle point of $I_{j,\omega,x,x'}$. Now we see that $\Xi = (\sqrt{B_0} + \lambda)\lambda/(L \wedge L')^{\alpha-1-d}$ is an appropriate choice in (2.41). The rest of the proof is same as in [15].

Finally we prove the upper bound of the infimum of the spectrum in the simple 2-dimensional case:

Proof of Proposition 2.1. By the definition, we have

$$A_2^{\omega_0}(x) = \begin{cases} 2 & \text{if} \quad \dfrac{-3}{2} + 4n \leq x_1 \leq \dfrac{-1}{2} + 4n \quad \text{for some} \quad n \in \mathbb{Z}, \\ 4(4n - x_1) & \text{if} \quad \dfrac{-1}{2} + 4n \leq x_1 \leq \dfrac{1}{2} + 4n \quad \text{for some} \quad n \in \mathbb{Z}, \\ -2 & \text{if} \quad \dfrac{1}{2} + 4n \leq x_1 \leq \dfrac{3}{2} + 4n \quad \text{for some} \quad n \in \mathbb{Z}, \\ 4(x_1 - 2 - 4n) & \text{if} \quad \dfrac{3}{2} + 4n \leq x_1 \leq \dfrac{5}{2} + 4n \quad \text{for some} \quad n \in \mathbb{Z}. \end{cases}$$

Thus we have

$$B^{\omega_0}(x) = \begin{cases} -4 & \text{if} \quad \dfrac{-1}{2} + 4n \leq x_1 \leq \dfrac{1}{2} + 4n \quad \text{for some} \quad n \in \mathbb{Z}, \\ 4 & \text{if} \quad \dfrac{3}{2} + 4n \leq x_1 \leq \dfrac{5}{2} + 4n \quad \text{for some} \quad n \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

We set

$$F^{\omega_0}(x) = \int_0^{x_1} A_2^{\omega_0}(s, x_2)\, ds$$

and

$$\phi^{\lambda\omega_0}(x) = \exp\left(-\frac{B}{4}|x|^2 - \lambda F^{\omega_0}(x)\right).$$

Then we have

$$
\begin{aligned}
&(\phi^{\lambda\omega_0}, H^{\lambda\omega_0}\phi^{\lambda\omega_0}) \\
&= B\|\phi^{\lambda\omega_0}\|^2 - 4\lambda \sum_{a\in 4\mathbb{Z}} \int_{a-1/2}^{a+1/2} dx_1 \int dx_2 |\phi^{\lambda\omega_0}(x)|^2 \\
&\quad + 4\lambda \sum_{a\in 4\mathbb{Z}} \int_{a+3/2}^{a+5/2} dx_1 \int dx_2 |\phi^{\lambda\omega_0}(x)|^2.
\end{aligned}
$$

Since $0 \geq F^{\omega_0}(x) \geq -3$, we have $2\pi/B \leq \|\phi^{\lambda\omega_0}\|^2 \leq e^{6\lambda}2\pi/B$,

$$
\int_{-1/2}^{1/2} dx_1 \int dx_2 |\phi^{\lambda\omega_0}(x)|^2 \geq \frac{2\pi}{B}(1 - e^{-B/8})
$$

and

$$
\int_{a-1/2}^{a+1/2} dx_1 \int dx_2 |\lambda\phi^{\omega_0}(x)|^2 \begin{cases} \geq \sqrt{\dfrac{2\pi}{B}} \exp\left(-\dfrac{((a+1/2)^2 \vee (a-1/2)^2)B}{2} \right), \\ \leq \sqrt{\dfrac{2\pi}{B}} \exp\left(6\lambda - \dfrac{((a+1/2)^2 \wedge (a-1/2)^2)B}{2} \right) \end{cases}
$$

for $a \neq 0$. Therefore we see that $(\phi^{\lambda\omega_0}, H^{\lambda\omega_0}\phi^{\lambda\omega_0})/\|\phi^{\lambda\omega_0}\|^2$ is dominated by the right hand side of (2.11). By the min-max principle, we can complete the proof. □

3. Random Pauli Hamiltonians with an anomalous magnetic moment

In this section we consider a random family of Pauli Hamiltonians defined as follows: let d be an integer greater than 1, and u and $\omega = (\omega(a))_{a\in\mathbb{Z}^d}$ be an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ and random variables satisfying the same conditions as in the last section. We consider the random vector potential $A^\omega(x)$ defined as in (2.2) and the corresponding magnetic field $B^\omega(x) = (B^\omega_{jk}(x))_{1\leq j<k\leq d}$ represented as

$$
B^\omega_{jk}(x) = \sum_{a\in\mathbb{Z}^d} \omega(a)(du)_{jk}(x-a),
$$

where $(du)_{jk} = \partial_j u_k - \partial_k u_j$. We set $\overline{A}$ as in Theorem 1 (i) and $\overline{B} = \sup\{|||B^\omega(x)|||_1 \colon x \in \mathbb{R}^d,\ \omega \in [-1,1]^{\mathbb{Z}^d}\}$, where $|||\cdot|||_1$ is the trace norm. Let $\gamma_1, \gamma_2, \ldots, \gamma_d$ be Hermitian matrices satisfying the commutation relation

$$
\gamma_j\gamma_k + \gamma_k\gamma_j = \begin{cases} 2I & \text{if} \quad j = k, \\ O & \text{if} \quad j \neq k, \end{cases}
$$

where I and O are the identity and the zero matrices, respectively. We can construct such matrices acting on $\mathbb{C}^D$, where $D = 2^{[d/2]}$ and $[d/2]$ is the largest integer less than or equal to $d/2$ (see e.g. [7] §12.2). Then our object is the operator

$$H^{\lambda\omega,g} = I\sum_{j=1}^{d}(i\partial_j + \lambda A_j^{\omega}(x))^2 + \frac{g}{2}\sum_{j<k} i\gamma_j\gamma_k\lambda B_{jk}^{\omega}(x) \tag{3.1}$$

acting on $\mathbb{C}^D$-valued functions, where g is a constant greater than 2 and $\lambda > 0$. This operator with the domain $C_0^{\infty}(\mathbb{R}^d \to \mathbb{C}^D)$ is known to be essentially self-adjoint on $L^2(\mathbb{R}^d \to \mathbb{C}^D)$ (cf. [19]). Accordingly we take the unique self-adjoint extension, and denote it by the same symbol. This operator is called the Pauli Hamiltonian with the magnetic moment g and the magnetic field λB^{ω}. If the magnetic moment g is 2, then this operator is the square of a Dirac operator and this is the most studied case. The case of $g > 2$ is called anomalous (cf. [1]). In this case, the operator $H^{\lambda\omega,g}$ has spectrum in the negative half line. For the negative spectrum, we can discuss the Wegner type estimate and the Anderson localization.

To give the Wegner type estimates, we consider the restriction $H_L^{\lambda\omega,g}$ to $L^2(\Lambda_L \to \mathbb{C}^D)$ with the Dirichlet boundary condition for each $L > 0$: $H_L^{\lambda\omega,g}$ is the self-adjoint operator corresponding to the closure of the quadratic form

$$\begin{aligned} q^{\lambda\omega,g}(\Phi, \Psi) := \sum_{j=1}^{d}\langle(i\partial_j + \lambda A_j^{\omega}(x))\Phi(x), (i\partial_j + \lambda A_j^{\omega}(x))\Psi(x)\rangle \\ + \left\langle \Phi(x), \frac{g}{2}\sum_{j<k} i\gamma_j\gamma_k\lambda B_{jk}^{\omega}(x)\Psi(x)\right\rangle \end{aligned} \tag{3.2}$$

with the domain $C_0^{\infty}(\Lambda_L \to \mathbb{C}^D)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Hermitian inner product of the space $L^2(\Lambda_L \to \mathbb{C}^D)$. The spectra of $H_L^{\lambda\omega,g}$ are purely discrete. Then our Wegner type estimates are stated as follows:

Theorem 4 (Wegner type estimate). (i) *We assume* $\operatorname{supp} u$ *is compact. Then there exists a finite constant c depending only on d and* $\operatorname{diam}\operatorname{supp} u$, *and satisfying the following*:

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega,g})] \le cW_d(\delta, g\lambda, \mathcal{H}')\eta L^{2d} \tag{3.3}$$

for any $\delta > 0$, $L \ge 1$, $E < 0$ and $\eta > 0$ such that $E + 2\eta \le -(\lambda\bar{A})^2 - \delta$, where

$$W_d(\delta, g\lambda, \mathcal{H}') = \frac{\mathcal{H}'(g\lambda)^{d/2}}{\delta}.$$

(ii) *For any $q \ge \hat{d}$ and $l \in \mathbb{N}$ satisfying $(q-1)/2 < l < (q(\alpha-1)/d - 1)/2$, there exists a finite constant c depending only on l, q, α and d, and satisfying the following*:

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega,g})] \le c\mathcal{W}_{q,l}(\delta, g\lambda, \mathcal{H}')\eta^{1-q/(2l+1)}L^{2d} \tag{3.4}$$

for any $\delta > 0$, $L \geq 1$, $E < 0$ *and* $0 < \eta < 1$ *such that* $E + 2\eta \leq -(\lambda \bar{A})^2 - \delta$, *where*

$$\mathcal{W}_{q,l}(\delta, g\lambda, \mathcal{H}') = \frac{\mathcal{H}'(g\lambda + 1)^{q(l+2)/(2l+1)}}{\delta}.$$

(iii) *We assume* $\operatorname{supp} u$ *is compact. Then for any* $q \geq \hat{d}$ *and* $(q-1)/2 < l \in \mathbb{N}$, *there exists a finite constant c depending only on* l, q, d *and* $\operatorname{diam} \operatorname{supp} u$, *and satisfying the following*:

$$\mathbb{E}[N([E - \eta, E + \eta] : H_L^{\lambda\omega, g})] \leq c W_{q,l}(\delta, g\lambda, \mathcal{H}')\eta^{1-q/(2l+1)} L^d \tag{3.5}$$

for any $\delta > 0$, $L \geq 1$, $E < 0$ *and* $0 < \eta < 1$ *such that* $E + 2\eta \leq -(\lambda \bar{A})^2 - \delta$, *where*

$$W_{q,l}(\delta, g\lambda, \mathcal{H}') = \frac{\mathcal{H}'(g\lambda + 1)^{q(l+2)/(2l+1)}}{\delta}.$$

REMARK 3.1. By the same method, we can also treat the operator

$$H^{\lambda\omega, A, g} = I \sum_{j=1}^{d} (i\partial_j + A_j(x) + \lambda A_j^{\omega}(x))^2 + \frac{g}{2} \sum_{j<k} i\gamma_j\gamma_k (B_{jk}(x) + \lambda B_{jk}^{\omega}(x)),$$

where $A(x)$ is another deterministic vector potential and $B(x)$ is its magnetic field. If we replace $H_L^{\lambda\omega, g}$ by $H_L^{\lambda\omega, A, g}$ in the right hand side of (3.3), then $W_d(\delta, g\lambda, \mathcal{H}')$ is replaced by

$$g^{d/2} \left(\sup_x \||B(x)\||_1 + \lambda \bar{B} \right)^{d/2} \frac{\mathcal{H}'}{\delta}$$

and the condition $E + 2\eta \leq -(\lambda\bar{A})^2 - \delta$ is replaced by $E + 2\eta \leq \inf \sigma(H^{0,A,g}) - (\lambda\bar{A})^2 - \delta$. (3.4) and (3.5) are also modified similarly.

In (iii), the bound is linear in the volume of the domain Λ_L. Therefore, as in Theorem 1 (iii), we can obtain a Hölder estimate of the density of states $N(B)$, $B \in \mathcal{B}(\mathbb{R})$, of the random Pauli Hamiltonian $H^{\lambda\omega, g}$ defined as a deterministic Borel measure such that the Borel measures $L^{-d} N(\,\cdot\, : H_L^{\lambda\omega, g})$ on $\mathbb{R}$ converges vaguely to $N(\,\cdot\,)$ as $L \to \infty$ for almost all ω. For this definition and the fundamental properties of the density of states, see [29]. The Hölder estimate is the following:

Corollary. *Under the situation of* Theorem 4 (iii), *the density of states* $N(\,\cdot\,)$ *of the random Pauli Hamiltonian* $H^{\lambda\omega, g}$ *satisfies*

$$N([E - \eta, E + \eta]) \leq c W_{q,l}(\delta, g\lambda, \mathcal{H}')\eta^{1-q/(2l+1)} \tag{3.6}$$

for any $\delta > 0$, $E < 0$ *and* $0 < \eta < 1$ *such that* $E + 2\eta \leq -(\lambda\bar{A})^2 - \delta$.

From (i) and (ii) of Theorem 4, we obtain the following results on the Anderson localization by referring Germinet and Klein [11] and Kirsch, Stollmann and Stolz [15]:

Theorem 5. (i) *We assume* $\operatorname{supp} u$ *is compact. Then, for any positive number* ϵ*, there exist finite positive constants* c_1*,* c_2 *and* c_3 *depending only on* d*,* ϵ *and* $\operatorname{diam}\operatorname{supp} u$ *such that* $E_0 < -(\lambda\bar{A})^2$ *and*

$$\begin{aligned}(3.7)\qquad &\mathbb{P}\left(|\omega(0)| \geq 4\frac{-E_0-\epsilon}{g\bar{B}}\right)\\ &< \{c_1(-E_0-\lambda^2\bar{A}^2)^\epsilon\} \wedge \frac{c_2}{(1+g\lambda)^\epsilon} \wedge \left\{c_3\left(\frac{-E_0-\lambda^2\bar{A}^2}{(g\lambda)^{d/2}\mathcal{H}'}\right)^\epsilon\right\}\end{aligned}$$

imply the same results in Theorem 2 *for the operator* $H^{\lambda\omega,g}$ *on the interval* $I=(-\infty,E_0]$.
(ii) *We assume* $\alpha > (5+\sqrt{21})d/2+1$*. Then, for any positive number* ϵ*, there exist finite positive constants* c_1*,* c_2*,* c_3 *and* c_4 *depending only on* d*,* α *and* ϵ *such that* $E_0 < -(\lambda\bar{A})^2$ *and*

$$\begin{aligned}(3.8)\qquad &\mathbb{P}\left(|\omega(0)| \geq 4\frac{-E_0-\epsilon}{g\lambda\bar{B}}\right)\\ &< \left\{c_1\left(\frac{(-E_0-(\lambda\bar{A})^2)\wedge 1}{\lambda(g+\lambda)}\right)^{(d+\epsilon)/(\alpha-1-d)}\right\} \wedge \frac{c_2}{(\log\lambda(g+\lambda))_-^{d+\epsilon}}\\ &\quad\wedge\left\{\frac{c_3}{(g\lambda+1)^{c_4}}\left(\frac{-E_0-(\lambda\bar{A})^2}{\mathcal{H}'(g\lambda(g+\lambda))^{1-d/(\alpha-1)}}\right)^{d(\alpha-1)/((\alpha-1-d)^2-3d(\alpha-1))+\epsilon}\right\}\end{aligned}$$

imply the exponential localization on the interval $I=(-\infty E_0]$*: for a.e.* ω*, the spectrum of* $H^{\lambda\omega,g}$ *is pure point on the interval* I *with exponentially decaying eigenfunctions.*

In order to show the intervals in this theorem intersect with the spectrum, we prove the following:

Proposition 3.1. *We assume* $0\in\operatorname{supp} h$*. Then, for any* ε *and* $R>0$*, there exist* $k_{\varepsilon,R}\in(0,\infty)$ *depending only on* d*,* u*,* ε *and* R *such that*

$$(3.9)\qquad \left[\inf_{\mu\in\lambda\operatorname{supp} h}\left(k_{\varepsilon,R}+(1+R)(\mu\bar{A})^2-g|\mu|\frac{\bar{B}-\varepsilon}{4}\right)\wedge 0,\ \infty\right)\subset\sigma(H^{\lambda\omega,g})$$

for a.e. ω.

Since the lower bound of the interval in (3.9) is less than $-(\lambda\bar{A})^2$ if g is sufficiently large, we can construct examples such that the Anderson localization occurs as in Example 2.1.

We next proceed to the proof. To prove the Wegner type estimates, we dominate the number of eigenvalues $N([E-\eta, E+\eta] : H_L^{\lambda\omega,g})$ directly by

$$\int_{-3\eta/2}^{3\eta/2} dt\ \mathrm{Tr}[(-\rho_\eta)'(H_L^{\lambda\omega,g} - E + t)].$$

Since $H_L^{\lambda\omega,g}$ is bounded below, we here take ρ_η so that its support is bounded above as follows: $\rho_\eta(s) = \rho(s/\eta)$ and ρ is a smooth function on $\mathbb{R}$ such that $0 \le \rho \le 1$ on $\mathbb{R}$, $\rho = 1$ on $(-\infty, -1/2]$ and $\rho = 0$ on $[1/2, \infty)$. The vector field $\mathbb{A}$ defined in (2.20) acts as

$$\begin{aligned}
&\mathbb{A}\ \mathrm{Tr}[(-\rho_\eta)(H_L^{\lambda\omega,g} - E + t)] \\
&= \mathrm{Tr}[(-\rho_\eta)'(H_L^{\lambda\omega,g} - E + t)(H_L^{\lambda\omega,g} + \Delta + |\lambda A^\omega|^2)] \\
&\le \mathrm{Tr}[(-\rho_\eta)'(H_L^{\lambda\omega,g} - E + t)](E + 2\eta + \lambda^2 \overline{A}^2).
\end{aligned}$$

Therefore, under the condition that $E + 2\eta + \lambda^2\overline{A}^2 \le -\delta$, we have the bound

$$\mathrm{Tr}[(-\rho_\eta)'(H_L^{\lambda\omega,g} - E + t)] \le \frac{1}{\delta}\mathbb{A}\ \mathrm{Tr}[\rho_\eta(H_L^{\lambda\omega,g} - E + t)]$$

and

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega,g})] \le \frac{1}{\delta}\int_{-3\eta/2}^{3\eta/2} dt \sum_{a\in\mathbb{Z}^d} I(L, E, t; a), \tag{3.10}$$

where

$$I(L, E, t; a) = \mathbb{E}\left[\int_{-1}^{1} d\omega(a) h(\omega(a))\omega(a) \frac{\partial}{\partial\omega(a)}\ \mathrm{Tr}[\rho_\eta(H_L^{\lambda\omega,g} - E + t)]\right].$$

To prove Theorem 4 (i), the following simple estimate is enough:

Lemma 3.1. *There exists a finite constant c depending only on d and* diam supp u *such that*

$$\sup_{-3\eta/2 \le t \le 3\eta/2} |I(L, E, t; a)| \le cL^d (g\lambda)^{d/2}\mathcal{H}' \tag{3.11}$$

for any $a \in \mathbb{Z}^d$, $L \ge 1$, $E < 0$ and $\eta > 0$ such that $E + 2\eta \le 0$.

Proof. As in the proof of Theorem 2.3, the proof is reduced to dominating $\mathrm{Tr}[\rho_\eta(H_L^{\omega,g} - E + t)]$. We dominate this as

$$\begin{aligned}
\mathrm{Tr}[\rho_\eta(H_L^{\omega,g} - E + t)] &\le \mathrm{Tr}[E((-\infty, E + 2\eta] : H_L^{\omega,g})] \\
&\le \mathrm{Tr}[\exp(-tH_L^{\omega,g})]\exp(t(E + 2\eta)).
\end{aligned} \tag{3.12}$$

Since $E+2\eta \leq 0$, we dominate the second term by 1. Using the representation of the heat semigroup by the Feynman-Kac-Itô formula (cf. [14], [29]), we dominate the first term by

$$\frac{2DL^d}{(4\pi t)^{d/2}} e^{tg\lambda \bar{B}/4}.$$

By taking t so that the second term of the right hand side is e, we have

$$\mathrm{Tr}[\exp(-tH_L^{\omega,g})] \leq cL^d (g\lambda)^{d/2}$$

and the same bound of $\mathrm{Tr}[\rho_\eta(H_L^{\omega,g}-E+t)]$. □

To prove Theorem 4 (ii), we use the following:

Lemma 3.2. *Under the condition of* Theorem 4 (ii), *for any $q \geq \hat{d}$ and $(q-1)/2 < l \in \mathbb{N}$, there exists a finite constant c depending only on l, q,* diam supp u *and d such that*

$$\sup_{-3\eta/2 \leq t \leq 3\eta/2} |I(L,E,t;a)| \leq c\mathcal{H}' \left(\frac{g\lambda (g\lambda+1)^{l+1}}{\eta(\mathrm{dist}(a,\Lambda_L)+1)^{\alpha-1}} \right)^{q/(2l+1)} L^d \tag{3.13}$$

for any $a \in \mathbb{Z}^d$, $0<\eta<1$ and $L \geq 1$.

To prove this lemma, we use the following as in the proof of Lemma 2.4:

Lemma 3.3. *Under the condition of* Theorem 4 (ii), *for any $q \geq \hat{d}$ and $(q-1)/2 < l \in \mathbb{N}$, the operator $(H_L^{\lambda\omega[a,0],g}+\mathcal{M})^{-l} - (H_L^{\lambda\omega,g}+\mathcal{M})^{-l}$ belongs to the super trace class $\mathcal{I}_{q/(2l-1)}$ and satisfies*

$$\begin{aligned} &|||(H_L^{\lambda\omega[a,0],g}+\mathcal{M})^{-l} - (H_L^{\lambda\omega,g}+\mathcal{M})^{-l}|||_{q/(2l+1)}^{q/(2l+1)} \\ &\leq c\left(\frac{g\lambda|\omega(a)|}{(1+\mathrm{dist}(a,\Lambda_L))^{\alpha-1}} \right)^{q/(2l+1)} L^d, \end{aligned} \tag{3.14}$$

where $\mathcal{M} = 1+g\lambda\overline{B}/4$ and c is a finite constant depending only on q, l and d.

To prove this lemma, we use the following, which is proved as in Lemma 3.4 (i):

Lemma 3.4. *For any $\mathcal{A} \in L^q(\Lambda_L \to \mathbb{R})$ with some $q \geq \hat{d}$, the operator $(H_L^{\lambda\omega,g}+\mathcal{M})^{-1/2}\mathcal{A}$ belongs to the class $\mathcal{I}_q$ and satisfies*

$$|||(H_L^{\lambda\omega,g}+\mathcal{M})^{-1/2}\mathcal{A}|||_q \leq c\|\mathcal{A}\|_q^q, \tag{3.15}$$

where c is a finite constant depending only on d and q.

To prove Theorem 4 (iii), we use the following instead of Lemma 3.3:

Lemma 3.5. *If* $\operatorname{supp} u$ *is compact, then* (3.14) *is replaced by*

$$\|\|(H_L^{\lambda\omega[a,0],g} + \mathcal{M})^{-l} - (H_L^{\lambda\omega,g} + \mathcal{M})^{-l}\|\|_{q/(2l+1)}^{q/(2l+1)} \le c(g\lambda|\omega(a)|)^{q/(2l+1)}, \tag{3.16}$$

where c is a finite constant depending only on $\operatorname{diam}\operatorname{supp} u$, q, l *and* d.

We next proceed to the proof of the localization.

Proof of Theorem 5 (i). We take $I_0 = (-\infty, E_0]$, $\widetilde{I_0} = (-\infty, (E_0 - \lambda^2 \overline{A}^2)/2)$ and $H_{L,x}^{\omega} = H_{L,x}^{\omega,g}$ in Proposition 2.2, where $H_{L,x}^{\omega,g}$ is the restriction of $H^{\lambda\omega,g}$ to $L^2(\Lambda_L(x) \to \mathbb{C}^D)$ with the Dirichlet boundary condition. Then the assumptions in that proposition hold with $\rho = \operatorname{diam}\operatorname{supp} u$, $l_0 = 1$, $\eta_0 = (-E_0 - \lambda^2 \overline{A}^2)/8$, $h = 1$, $b = 2$, $\upsilon = 1$, $\gamma = c_1\sqrt{1 + g\lambda}$ and $C_W = c_2(g\lambda)^{d/2}\mathcal{H}'/(-E_0 - \lambda^2 \overline{A}^2)$, where c_1 and c_2 are finite constants depending only on d and $\operatorname{diam}\operatorname{supp} u$. Then we have only to show the corresponding (2.32) for any $E \in I_0$, some $\theta > 4d$ and $\mathcal{L} \in 6\mathbb{N}$ greater than the right hand side of the corresponding (2.33). We now use a Combes-Thomas type estimate obtained by estimating the heat semigroup in Lemma A.1 of Fisher, Leschke and Müller [9]: under the condition that

$$0 < -E_0 - g\lambda\overline{B} \sup_{a\in\Lambda_{L+2\rho}\cap\mathbb{Z}^d} \frac{|\omega(a)|}{4} := \mathcal{B},$$

we have

$$\|\|\Gamma_{\mathcal{L}}(H_{\mathcal{L}}^{\lambda\omega,g} - E_0)^{-1}\chi_{\mathcal{L}/3}\|\| \le \frac{c_3}{\mathcal{B}} \exp(-c_4\sqrt{\mathcal{B}}\mathcal{L}),$$

where c_3 and c_4 are positive finite constants depending only on d. Thus, for $\Lambda_{\mathcal{L}}$ to be (θ, E_0)-suitable, it is enough that

$$\sup_{a\in\Lambda_{\mathcal{L}+2\rho}\cap\mathbb{Z}^d} |\omega(a)| \le 4\frac{-E_0 - ((\theta + 2)\log\mathcal{L} + \log c_4)^2/(c_5\mathcal{L})^2}{g\overline{B}} =: \mathcal{C}, \tag{3.17}$$

where c_5 is a positive finite constant depending only on d. Therefore a sufficient condition for the corresponding (2.32) at the energy E_0 is

$$\mathbb{P}(|\omega(0)| \ge \mathcal{C}) \le c_6(\mathcal{L} + 2\rho)^{-d}, \tag{3.18}$$

where c_6 is a positive constant depending only on d. By taking L largely so that $\mathcal{C} \ge 4(-E_0 - \varepsilon)/(g\overline{B})$ and Proposition 2.2 can be applied, we have the condition (3.7). □

To prove Theorem 5 (ii) following Kirsch, Stollmann and Stolz [15], we use the following as in the proof of Theorem 3:

Lemma 3.6. *We assume* (2.3) *for some* $\alpha > d+1$, $\lim_{|x|\to\infty} u(x) = 0$, $L \geq 1$ *and* $\omega = \omega'$ *on* $\Lambda_{2L} \cap \mathbb{Z}^d$. *Then we have*

$$\begin{aligned} &\mu_j(H_L^{\lambda\omega,g}) \\ &\leq \mu_j(H_L^{\lambda\omega',g}) + \frac{c_1\lambda}{L^{\alpha-1-d}} \left\{ \left(\mu_j(H_L^{\lambda\omega',g}) + \frac{g\lambda\bar{B}}{4} \right)^{1/2} + \frac{\lambda}{L^{\alpha-1-d}} \right\} + \frac{c_2 g\lambda}{L^{\alpha-d}} \end{aligned} \tag{3.19}$$

for any $j \in \mathbb{N}$, *where* $\mu_j(H_L^{\lambda\omega,g})$ *is the* j*-th eigenvalue of* $H_L^{\lambda\omega,g}$ *including the multiplicity, and* c_1 *and* c_2 *are finite constants depending only on* d *and* α. *Moreover, if* $\alpha > d+2$ *and* $L \geq c_3\lambda^{1/(\alpha-2-d)}$, *then we have*

$$\mu_j(H_L^{\lambda\omega,g}) \geq \mu_j(H_L^{\lambda\omega',g}) - \frac{c_1\lambda}{L^{\alpha-1-d}} \left(\mu_j(H_L^{\lambda\omega',g}) + \frac{g\lambda\bar{B}}{4} \right)^{1/2} - \frac{c_2 g\lambda}{L^{\alpha-d}}, \tag{3.20}$$

where c_3 *is a finite constant depending only on* d *and* α.

Proof. For any $\varepsilon > 0$ and $\Phi \in C_0^\infty(\Lambda_L \to \mathbb{C}^D)$ such that $\|\Phi\| = 1$, we have

$$\begin{aligned} &q^{\lambda\omega,g}(\Phi, \Phi) \\ &\leq (1+\varepsilon) q^{\lambda\omega',g}(\Phi,\Phi) + \left(1 + \frac{1}{\varepsilon}\right)\lambda^2 \sup_{x\in\Lambda_L} |A^\omega(x) - A^{\omega'}(x)|^2 \\ &\quad + \frac{g}{4}\lambda \sup_{x\in\Lambda_L} |||B^\omega(x) - B^{\omega'}(x)|||_1 + \varepsilon\frac{g\lambda\bar{B}}{4}. \end{aligned}$$

Since $|u(x)| \leq c_1'(1+|x|)^{-\alpha+1}$, the other terms are estimated by

$$\sup_{x\in\Lambda_L} |A^\omega(x) - A^{\omega'}(x)| \leq \frac{c_2'}{L^{\alpha-1-d}}$$

and

$$\sup_{x\in\Lambda_L} |||B^\omega(x) - B^{\omega'}(x)|||_1 \leq \frac{c_3'}{L^{\alpha-d}}.$$

Thus, by the min-max principle, we have

$$\mu_j(H_L^{\lambda\omega,g}) \leq (1+\varepsilon)\mu_j(H_L^{\lambda\omega',g}) + \left(1+\frac{1}{\varepsilon}\right)\left(\frac{c_2'\lambda}{L^{\alpha-1-d}}\right)^2 + \frac{g}{2}\frac{c_3'\lambda}{L^{\alpha-d}} + \varepsilon\frac{g\lambda\bar{B}}{4}.$$

The right hand side attains its minimum at

$$\varepsilon = \frac{c_2'\lambda}{L^{\alpha-1-d}}\left(\frac{g\lambda\bar{B}}{4} + \mu_j(H_L^{\lambda\omega',g})\right)^{-1/2}$$

and this gives (3.19). If

$$\|(i\nabla + \lambda A^{\omega'})\Phi\| \geq \lambda \sup_{x\in\Lambda_L} |A^{\omega}(x) - A^{\omega'}(x)|, \tag{3.21}$$

we also obtain

$$\begin{aligned} &q^{\lambda\omega,g}(\Phi,\Phi) \\ &\geq (1-\varepsilon)q^{\lambda\omega',g}(\Phi,\Phi) - \frac{\lambda^2}{\varepsilon}\sup_{x\in\Lambda_L}|A^{\omega}(x) - A^{\omega'}(x)|^2 \\ &\quad - \frac{g\lambda}{4}\sup_{x\in\Lambda_L}|\!|\!| B^{\omega}(x) - B^{\omega'}(x)|\!|\!|_1 - \varepsilon\frac{g\lambda\bar{B}}{4} \end{aligned}$$

and (3.20) by the same method. By using the diamagnetic inequality (cf. [3]) and recalling the well known first eigenvalue of the Dirichlet Laplacian, we obtain $\alpha > d+2$ and $L \geq c_3\lambda^{1/(\alpha-2-d)}$ as a simple sufficient condition for (3.21). $\square$

By this lemma, we see that

$$|\mu_j(H_L^{\lambda\omega,g}) - \mu_j(H_L^{\lambda\omega',g})| \leq \frac{c_1(g+\lambda)\lambda}{L^{\alpha-1-d}}$$

holds in the situation of Lemma 3.6, if $\mu_j(H_L^{\lambda\omega,g})\wedge\mu_j(H_L^{\lambda\omega',g}) \leq 0$. Therefore we dominate the probability (2.41) where $H_{L,x}^{\lambda\omega}$, $H_{L',x'}^{\lambda\omega}$, $\sigma_I(H_{L,x}^{\lambda\omega})$ and Ξ are replaced by $H_{L,x}^{\lambda\omega,g}$, $H_{L',x'}^{\lambda\omega,g}$, $\sigma_I(H_{L,x}^{\lambda\omega,g}) = \sigma(H_{L,x}^{\lambda\omega,g}) \cap (-\infty, E_0 + (g+\lambda)\lambda/(2(L \wedge L')^{\alpha-1-d})]$ and $(g+\lambda)\lambda/(L\wedge L')^{\alpha-1-d}$, respectively. The rest of the proof is same as in [15] and the last section.

Finally we prove the results on the spectral set:

Proof of Proposition 3.1. We take $x_0 \in \mathbb{R}^d$ and $\omega \in [-1,1]^{\mathbb{Z}^d}$ so that $|\!|\!| B^{\omega}(x_0)|\!|\!|_1 \geq \bar{B} - \varepsilon/3$. There exists $N \in \mathbb{N}$ such that $|\!|\!| B^{\omega_N}(x_0)|\!|\!|_1 \geq |\!|\!| B^{\omega}(x_0)|\!|\!|_1 - \varepsilon/3$, where ω_N is the periodic extension to $\mathbb{Z}^d$ of the restriction of ω to $\Lambda_N \cap \mathbb{Z}^d$. For any $0 \neq \mu \in \lambda\,\mathrm{supp}\, h$ and $R > 0$, we have

$$\begin{aligned} &\inf\sigma(H^{\mu\omega_N,g}) \\ &\leq \inf\Bigg\{\left(1+\frac{1}{R}\right)\|\nabla\Phi\|^2 + (1+R)\|\mu A^{\omega_N}\Phi\|^2 + \left(\Phi, \frac{g}{2}\sum_{j<k} i\gamma_j\gamma_k\mu B_{jk}^{\omega_N}\Phi\right) : \\ &\qquad \Phi \in C_0^{\infty}(\mathbb{R}^d \to \mathbb{C}^D), \|\Phi\| = 1\Bigg\} \end{aligned}$$

by the min-max principle. We use a coordinate so that $\mu B_{jk}^{\omega_N}(x_0) = 0$ for $(j,k) \notin \{(1,2),(3,4),\ldots,(d-1,d),(2,1),(4,3),\ldots,(d,d-1)\}$ and $\mu B_{jk}^{\omega_N}(x_0) \geq 0$ for $(j,k) \in$

$\{(1, 2), (3, 4), \ldots, (d-1, d)\}$. Then we have

$$\frac{1}{2}\sum_{j<k} i\gamma_j\gamma_k\mu B_{jk}^{\omega_N}(x_0)Q = -\frac{|||\mu B^{\omega_N}(x_0)|||_1}{4}Q,$$

where Q is the projector to the intersection of the eigenspaces of the matrices $i\gamma_{2j-1}\gamma_{2j}$, $j = 1, 2, \ldots, d'$ of the eigenvalue -1. We can take a small ball $B_r(x_0) = \{x : |x - x_0| < r\}$ so that

$$\frac{1}{2}\sum_{j<k} i\gamma_j\gamma_k\mu B_{jk}^{\omega_N}(x)Q \le -|\mu|\frac{|||B^{\omega_N}(x_0)|||_1 - \varepsilon/3}{4}Q$$

on $B_r(x_0)$. By restricting $C_0^\infty(\mathbb{R}^d \to \mathbb{C}^D)$ to $C_0^\infty(B_r(x_0) \to Q\mathbb{C}^D)$, we have

$$\inf\sigma(H^{\mu\omega_N,g}) \le c\left(1 + \frac{1}{R}\right) + (1+R)\lambda^2\bar{A}^2 - g|\mu|\frac{\bar{B}-\varepsilon}{4},$$

where c is a constant depending only on d, u and ε. Since $\inf\sigma(H^{\mu\omega_N,g})$ is continuous in μ and $\sigma(H^{0,g}) = [0, \infty)$, we have

$$\begin{aligned}&\left[\inf_{\mu\in\lambda\,\mathrm{supp}\,h}\left(\left(1+\frac{1}{R}\right)c + (1+R)(\mu\bar{A})^2 - g|\mu|\frac{\bar{B}-\varepsilon}{4}\right)\wedge 0, \infty\right)\\ &\subset \{\sigma(H^{\mu\omega_N,g}) : \mu \in \lambda\,\mathrm{supp}\,h\}.\end{aligned}$$

As in (2.12), we obtain (3.9) by the representation of the spectral set of the ergodic operators by those of periodic operators (e.g. Theorem (5.33) in [23], §1.4 in [28]). □

4. Random magnetic Schrödinger operators with certain scalar potentials

In this section we consider random Schrödinger operators defined by

$$H^{\omega,\mu\vartheta} = \sum_{j=1}^{d}(i\partial_j + A_j(x) + A_j^\omega(x))^2 + \mu v^\vartheta(x)\sum_{j=1}^{d}(A_j^\omega(x))^2 \tag{4.1}$$

on $L^2(\mathbb{R}^d)$, where $d = 2d'$, $A(x)$ and $A^\omega(x)$ are same as in Section 2 except for that we do not assume the compactness of the support of the probability density function h of $\omega(a)$. $\mu > 0$ and $\vartheta = (\vartheta(a))_{a\in\mathbb{Z}^d}$ is another family of independently and identically distributed real random variables independent of ω. We assume that their distribution has a C^1 density $k(s)$ such that $\mathrm{supp}\,k \subset [0, 1]$. $v^\vartheta(x)$ is the alloy type random potential defined by using ϑ and a partition of unity as follows:

$$v^\vartheta(x) = \sum_{a\in\mathbb{Z}^d}\vartheta(a)v(x-a),$$

where $v \in L^\infty(\mathbb{R}^d \to [0,1])$ with a compact support such that $\partial_j v \in L^\infty$ for any $j = 1, 2, \dots, d$, and $\sum_{a\in\mathbb{Z}^d} v(x-a) \equiv 1$. The operator in (4.1) with the domain $C_0^\infty(\mathbb{R}^d)$ are known to be essentially self-adjoint on $L^2(\mathbb{R}^d)$ (cf. [19]). Accordingly we take the unique self-adjoint extension, and denote it by the same symbol.

To give Wegner type estimates for these operators, we consider the restriction $H_L^{\omega,\mu\vartheta}$ to $L^2(\Lambda_L)$ with the Dirichlet boundary condition for each $L > 0$. We set

$$\mathcal{K}' = \int |k'(s)|\,ds, \quad \mathcal{H}_p = \int |s|^p h(s)\,ds \quad \text{and} \quad \mathcal{H}'_p = \int |s|^p |sh'(s)|\,ds$$

for any $p \geq 0$. Then our Wegner type estimate is stated as follows:

Theorem 6 (Wegner type estimate). (i) *We assume* $\operatorname{supp} u$ *is compact and* $\mathcal{H}_{\hat{d}} < \infty$. *Then there exists a finite constant c depending only on d,* $\operatorname{diam}\operatorname{supp} u$ *and* $\operatorname{diam}\operatorname{supp} v$ *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\omega,\mu\vartheta})] \leq c W_d(E, \delta, B_0, \mu, h, k)\eta L^{2d} \tag{4.2}$$

for any $L \geq 1$, $0 \leq E < B_0$ and $0 < \eta < (B_0 - E)/2$, where

$$W_d(E, \delta, B_0, \mu, h, k) = \frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}+1}} \left\{ \left(1 + \frac{\mathcal{K}'}{\mu \wedge 1}\right)\mathcal{H}_{\hat{d}} + \mathcal{H}'_{\hat{d}} \right\}.$$

(ii) *We assume $\mathcal{H}_{2q} < \infty$ for some $q \geq \hat{d}$. Then, for any $l \in \mathbb{N} \cap (q, (\alpha-1)q/d)$, there exists a finite constant c depending only on d, q, l, α and* $\operatorname{diam}\operatorname{supp} v$ *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\omega,\mu\vartheta})] \leq c \mathcal{W}_{d,q,l}(E, \delta, B_0, \mu, h, k)\eta^{1-q/l} L^{2d} \tag{4.3}$$

for any $L \geq 1$, $0 \leq E < B_0$ and $0 < \eta < (B_0 - E)/3$, where

$$\mathcal{W}_{d,q,l}(E, \delta, B_0, \mu, h, k) = \frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}+1}} \left\{ \left(1 + \frac{\mathcal{K}'}{\mu \wedge 1}\right)\mathcal{H}_{\hat{d}} + \mathcal{H}'_{\hat{d}} \right\}.$$

(iii) *We assume* $\operatorname{supp} u$ *is compact and* $\operatorname{supp} h \subset [-1, 1]$. *Then, for any $q \geq \hat{d}$ and $q < l \in \mathbb{N}$, there exists a finite constant c depending only on d, q, l,* $\operatorname{diam}\operatorname{supp} u$ *and* $\operatorname{diam}\operatorname{supp} v$ *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\lambda\omega,\mu\vartheta})] \leq c W_{d,q,l}(E, \delta, B_0, \mu, h, k)\eta^{1-q/l} L^d \tag{4.4}$$

for any $L \geq 1$, $0 \leq E < B_0$ and $0 < \eta < (B_0 - E)/3$, where

$$\begin{aligned} &W_{d,q,l}(E, \delta, B_0, \mu, h, k) \\ &= \frac{B_0^{d/2}\lambda^q(\sqrt{B_0} + (\mu \vee 1)\lambda)^q}{(B_0 - E)^{q+1-q/l}} \left(\frac{\sqrt{B_0}}{B_0 - E} \vee 1\right)^{q(1-1/l)} \left\{ \mathcal{H}' + \frac{1}{\mu^{q/l} \wedge 1}\left(1 + \frac{\mathcal{K}'}{\mu \wedge 1}\right) \right\}. \end{aligned}$$

REMARK 4.1. If $\operatorname{supp} h \subset [-1, 1]$, $\mathcal{W}_{d,q,l}(E, \delta, B_0, \mu, h, k)$ is replaced by

$$\frac{B_0^{d/2}}{(B_0 - E)^{1-q/l}} \left(\frac{\lambda((\mu \vee 1)\lambda + \sqrt{B_0})}{B_0 - E} \right)^q \left(\mathcal{H}' + \frac{\mathcal{K}'}{\mu \wedge 1} \right).$$

We use this estimate in the application to the localization (see Theorem 7 (ii) below).

As in previous sections, we can obtain a Hölder estimate of the density of states $N(B)$, $B \in \mathcal{B}(\mathbb{R})$, of the random Schrödinger operators $H^{\lambda\omega,\mu\vartheta}$ defined as a deterministic Borel measure such that the Borel measures $L^{-d} N(\,\cdot\, : H_L^{\lambda\omega,\mu\vartheta})$ on $\mathbb{R}$ converges vaguely to $N(\,\cdot\,)$ as $L \to \infty$ for almost all ω.

Corollary. *Under the situation of* Theorem 6 (iii), *the density of states $N(\,\cdot\,)$ of the random Schrödinger operators $H^{\lambda\omega,\mu\vartheta}$ satisfies*

$$N([E - \eta, E + \eta]) \leq c W_{d,q,l}(E, \delta, B_0, \mu, h, k) \eta^{1-q/l} \tag{4.5}$$

for any $0 \leq E < B_0$ and $0 < \eta < (B_0 - E)/3$.

As in the last sections, we obtain also the following results on the Anderson localization:

Theorem 7. (i) *In the situation of* Theorem 6 (i), *for any $\epsilon > 0$, there exist finite positive constants c_1, c_2 and c_3 depending only on d, ϵ,* $\operatorname{diam} \operatorname{supp} u$ *and* $\operatorname{diam} \operatorname{supp} v$ *such that $0 < \zeta < B_0$ and*

$$\begin{aligned} &\mathbb{P}\left(|\omega(0)| \geq \frac{1}{A}(\sqrt{B_0} - \sqrt{B_0 - \zeta} - \epsilon) \right) \\ &< (c_1 \zeta^\epsilon) \wedge \frac{c_2}{(1 + B_0)^\epsilon} \wedge \left\{ c_3 \left(\frac{\zeta^{\hat{d}+1}}{B_0^{(d+\hat{d})/2} \big((1 + \mathcal{K}'/(\mu \wedge 1)) \mathcal{H}_{\hat{d}} + \mathcal{H}'_{\hat{d}} \big)} \right)^\epsilon \right\} \end{aligned} \tag{4.6}$$

imply the results in Theorem 2 *on the interval $I = [0, B_0 - \zeta]$ for the operator $H^{\omega,\mu\vartheta}$.*

In particular, for any $\epsilon > 0$, there exist a finite positive constant c depending only on d, ϵ, $\mathcal{H}_{\hat{d}}$, $\mathcal{K}$, $\operatorname{diam} \operatorname{supp} u$ *and* $\operatorname{diam} \operatorname{supp} v$ *such that $B_0 \geq c$ implies the results in* Theorem 2 *on the interval $I = [0, B_0 - B_0^{\epsilon+1/2}]$ for the operator $H^{\omega,\mu\vartheta}$.*

(ii) *In the situation of* Theorem 6 (ii), *we assume* $\operatorname{supp} h$ *is compact and $\alpha > (5 + \sqrt{21})d/2 + 1$. Moreover we assume $0 < \mu \leq 1 \leq \lambda$ for simplicity. Then, for any $\epsilon > 0$, there exist finite positive constants $c_1, c_2, \ldots, c_6$ depending only on d, ϵ, α and* diam supp v *such that* $0 < \zeta < B_0$ *and*

$$
\begin{aligned}
&\mathbb{P}\left(|\omega(0)| \ge \frac{1}{\lambda A}(\sqrt{B_0} - \sqrt{B_0 - \zeta} - \epsilon) \right) \\
&\quad < \frac{c_1}{(\log(1+B_0))^{d+\epsilon}} \wedge \frac{c_2}{\lambda^{(d+\epsilon)/(\alpha-2-d)}} \\
&\quad \wedge \left\{ c_3 \left(\frac{\zeta}{\lambda(\sqrt{B_0} + (\mu \vee 1)\lambda)} \right)^{(d+\epsilon)/(\alpha-1-d)} \right\} \\
&\qquad \wedge \frac{c_4}{(\log \lambda(\sqrt{B_0} + (\mu \vee 1)\lambda))_-^{d+\epsilon}} \\
&\qquad \wedge \left\{ c_5 \left(\frac{\zeta}{\lambda(\sqrt{B_0} + (\mu \vee 1)\lambda)} \right)^{c_6} \right. \\
&\qquad\qquad \left. \times \left(\frac{\zeta^{\hat{d}}}{B_0^{d+\hat{d}/2} \lambda^{\hat{d}} (\mathcal{H}' + \mathcal{K}'/(\mu \wedge 1))} \right)^{(d+\epsilon)/\{(\alpha-1-d)^2/(\alpha-1)-3d\}} \right\}
\end{aligned}
\tag{4.7}
$$

imply the results in Theorem 3 *on the interval* $I = [0, B_0 - \zeta]$ *for the operator* $H^{\lambda\omega,\mu\vartheta}$.

We can construct an example such that the spectrum $\sigma(H^{\omega,\mu\vartheta})$ intersects with the intervals in this theorem as follows:

EXAMPLE 4.1. As in Proposition 2.1, let $d = 2$, $A(x) = (-Bx_2/2, Bx_1/2)$ and $u(x) = (0, \zeta(x_1)\zeta(x_2))$, where $\zeta(t) = 0$ for $|t| \ge 3/2$, $\zeta(t) = 1$ for $|t| \le 1/2$, and $\zeta(t) = (3 - 2|t|)/2$ for $1/2 \le |t| \le 3/2$. Moreover we assume supp $h = \mathbb{R}$. Then we have $\sigma(H^{\omega,\vartheta}) \supset [B_0\sqrt{\vartheta_0/(\vartheta_0+1)}, \infty)$, where $\vartheta_0 = \inf \operatorname{supp} k$. The necessary condition on k for this is only supp $k \subset [0,\infty)$. In particular, we have $\sigma(H^{\omega,\vartheta}) = [0,\infty)$ if $\vartheta_0 = 0$. In fact, for any $r \in \mathbb{R}$, $\epsilon > 0$ and $L > 0$, the following holds with a positive probability: $|\omega(a) - ra_1| < \epsilon$ and $\vartheta_0 \le \vartheta(a) \le \vartheta_0 + \epsilon$ for any $a \in \mathbb{Z}^2 \cap \Lambda_{L+2}$. Then we have $|A_2^\omega(x) - 4rx_1| \le 9\epsilon$ on Λ_L by (2.10) and

$$
\sum_{a_1 \in \mathbb{Z}} a_1 \zeta(x_1 - a_1) = 2x_1. \tag{4.8}
$$

Therefore our operator is close to the operator

$$
\mathcal{H} = \left(i\partial_1 - \frac{B}{2}x_2 \right)^2 + \left(i\partial_2 + \frac{B}{2}x_1 + 4Rx_1 \right)^2 + \underline{\vartheta}(4rx_1)^2.
$$

The spectrum of this operator is

$$
\sigma(\mathcal{H}) = [((B+4r)^2 + \underline{\vartheta}(4r)^2)^{1/2}, \infty)
$$

(cf. [20], [21]). Then we can show that the infimum of the spectrum of $H_L^{\omega,\vartheta}$ is in a neighborhood of $\inf\sigma(\mathcal{H})$ by the min-max principle. Then by the same method as in Theorem (5.33) of [23], we have $\inf\sigma(\mathcal{H}) \in \sigma(H^{\omega,\vartheta})$. $\inf\sigma(\mathcal{H})$ varies over $[B_0\sqrt{\vartheta_0/(\vartheta_0+1)}, \infty)$ as r varies over $\mathbb{R}$.

We next discuss the proof. For $0 \le E < B_0$, we introduce Birman-Schwinger type operators by

$$\Gamma_{L,2}^{\omega,\mu\vartheta} = (H_L^{0,0} - E)^{-1/2}(\mu v^{\vartheta}(x) + 1)|A^{\omega}|^2(H_L^{0,0} - E)^{-1/2}, \tag{4.9}$$

$$\Gamma_{L,1}^{\omega} = (H_L^{0,0} - E)^{-1/2} \sum_{j=1}^{d}\{(i\partial_j + A_j)A_j^{\omega} + A_j^{\omega}(i\partial_j + A_j)\}(H_L^{0,0} - E)^{-1/2} \tag{4.10}$$

and

$$\Gamma_L^{\omega,\mu\vartheta} = \Gamma_{L,2}^{\omega,\mu\vartheta} + \Gamma_{L,1}^{\omega}.$$

By the same proof for Lemma 2.1, we have the following:

Lemma 4.1. *It holds that*

$$N([E-\eta, E+\eta] : H_L^{\omega,\mu\vartheta}) \le N([1-\kappa, 1+\kappa] : -\Gamma_L^{\omega,\mu\vartheta}) \tag{4.11}$$

for any $0 \le E < B_0$ *and* $0 < \eta < B_0 - E$, *where* $\kappa = \eta/(B_0 - E)$.

For Theorem 6 (i), as in [16], we write

$$\begin{aligned} &N([1-\kappa, 1+\kappa) : -\Gamma_L^{\omega,\mu\vartheta}) \\ &= \mathrm{Tr}[\chi_{[1-\kappa,\infty)}(-\Gamma_L^{\omega,\mu\vartheta})] - \mathrm{Tr}[\chi_{[1+\kappa,\infty)}(-\Gamma_L^{\omega,\mu\vartheta})] \\ &= \mathrm{Tr}[\chi_{[1+\kappa,\infty)}(-K\Gamma_L^{\omega,\mu\vartheta})] - \mathrm{Tr}[\chi_{[1+\kappa,\infty)}(-\Gamma_L^{\omega,\mu\vartheta})], \end{aligned} \tag{4.12}$$

where, for any interval I, χ_I is its characteristic function and $K = (1+\kappa)/(1-\kappa)$. Introducing an ordering $a(1), a(2), \ldots$ of $\mathbb{Z}^d$ so that the supremum norm $|a(j)|_\infty$ is an increasing function in j, we define transformations K_j, $j \in \mathbb{N}$, on the probability space $\mathbb{R}^{\mathbb{Z}^d}$ by

$$(K_j\omega)[k] = \begin{cases} K\omega[k] & \text{for} \quad k < j, \\ \omega[k] & \text{for} \quad k \ge j, \end{cases} \tag{4.13}$$

where $\omega[k] = \omega(a(k))$. Then we can write as

$$\mathbb{E}[N([1-\kappa, 1+\kappa) : -\Gamma_L^{\omega,\vartheta})] = \sum_{j=1}^{M(u,L)} I_j + I_0, \tag{4.14}$$

where $M(u, L)$ is the maximal integer j such that $\operatorname{supp} u(\cdot - a(j)) \cap \Lambda_L \neq \emptyset$.

$$I_j = \mathbb{E}\left[\operatorname{Tr}\left[\chi_{[1+\kappa,\infty)}\left(-\Gamma_{L,1}^{K_{j+1}\omega} - \frac{\Gamma_{L,2}^{K_{j+1}\omega,\mu\vartheta}}{K}\right) - \chi_{[1+\kappa,\infty)}\left(-\Gamma_{L,1}^{K_j\omega} - \frac{\Gamma_{L,2}^{K_j\omega,\mu\vartheta}}{K}\right)\right]\right]$$

for $j \geq 1$ and

$$I_0 = \mathbb{E}\left[\operatorname{Tr}\left[\chi_{[1+\kappa,\infty)}\left(-\Gamma_{L,1}^{\omega} - \frac{\Gamma_{L,2}^{\omega,\mu\vartheta}}{K}\right) - \chi_{[1+\kappa,\infty)}(-\Gamma_{L}^{\omega,\mu\vartheta})\right]\right].$$

For each $j \geq 1$, we have

$$I_j = \mathbb{E}\left[\int \left\{\frac{1}{K}h\left(\frac{\omega[j]}{K}\right) - h(\omega[j])\right\} \operatorname{Tr}\left[\chi_{[1+\kappa,\infty)}\left(-\Gamma_{L,1}^{K_j\omega} - \frac{\Gamma_{L,2}^{K_j\omega,\vartheta}}{K}\right)\right] d\omega[j]\right]$$

by changing the variable on the probability space. By the positivity of $\Gamma_{L,2}^{K_j\omega,\vartheta}$ and Lemma 2.2 (ii), we have

$$\begin{aligned} \operatorname{Tr}\left[\chi_{[1+\kappa,\infty)}\left(-\Gamma_{L,1}^{K_j\omega} - \frac{\Gamma_{L,2}^{K_j\omega,\vartheta}}{K}\right)\right] &\leq \operatorname{Tr}[\chi_{[1+\kappa,\infty)}(-\Gamma_{L,1}^{K_j\omega})] \\ &\leq \left\|\Gamma_{L,1}^{K_j\omega}\right\|_{\hat{d}}^{\hat{d}} \leq c_d \frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}}} \|A^\omega \chi_L\|_{\hat{d}}^{\hat{d}}. \end{aligned} \tag{4.15}$$

Since

$$\frac{1}{K}h\left(\frac{\omega[j]}{K}\right) - h(\omega[j]) = -\int_{1/K}^{1} \{h(t\omega[j]) + t\omega[j]h'(t\omega[j])\}\, dt,$$

we have

$$|I_j| \leq c_d \frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}}} \int_{1/K}^{1} \mathbb{E}\left[\int \{h(t\omega[j]) + |t\omega[j]h'(t\omega[j])|\}\|A^\omega \chi_L\|_{\hat{d}}^{\hat{d}}\, d\omega[j]\right] dt.$$

The expectation in the right hand side is estimated as

$$\begin{aligned} &\mathbb{E}\left[\int \{h(t\omega[j]) + |t\omega[j]h'(t\omega[j])|\}\|A^\omega \chi_L\|_{\hat{d}}^{\hat{d}}\, d\omega[j]\right] \\ &\leq (\mathcal{H}_{\hat{d}} + \mathcal{H}'_{\hat{d}}) \int_{\Lambda_L} \left(\sum_{a\in\mathbb{Z}^d} |u(x-a)|\right)^{\hat{d}} dx \leq (d\bar{A})^{\hat{d}}(\mathcal{H}_{\hat{d}} + \mathcal{H}'_{\hat{d}})L^d. \end{aligned} \tag{4.16}$$

Thus we have

$$|I_j| \leq c(\mathcal{H}_{\hat{d}} + \mathcal{H}'_{\hat{d}})\frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}}}(K-1)L^d, \tag{4.17}$$

where c is a finite constant depending only on d and $\operatorname{diam}\operatorname{supp} u$. On the other hand, by changing the variable on the probability space, we have

$$I_0 = \mathbb{E}\Bigg[\Bigg(\prod_{j=1}^{M(v,L)} \int d\vartheta[j] K k\bigg(K\vartheta[j] + \frac{K-1}{\mu}\bigg) - \prod_{j=1}^{M(v,L)} \int d\vartheta(a)[j] k(\vartheta[j])\Bigg) \operatorname{Tr}[\chi_{[1+\kappa,\infty)}(-\Gamma_L^{\omega,\mu\vartheta})]\Bigg],$$

where $\vartheta[j] = \vartheta(a(j))$ and $M(v, L)$ is the maximal integer j such that $\operatorname{supp} v(\,\cdot\, - a(j)) \cap \Lambda_L \neq \emptyset$. Since

$$\begin{aligned} &Kk\bigg(K\vartheta[j] + \frac{K-1}{\mu}\bigg) - k(\vartheta[j]) \\ &= \int_1^K \bigg\{k\bigg(t\vartheta[j] + \frac{t-1}{\mu}\bigg) + t\bigg(\vartheta[j] + \frac{1}{\mu}\bigg)k'\bigg(t\vartheta[j] + \frac{t-1}{\mu}\bigg)\bigg\}\, dt, \end{aligned}$$

we have

$$|I_0| \le c\mathcal{H}_{\hat{d}}\bigg(1 + \frac{\mathcal{K}'}{\mu \vee 1}\bigg)\frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}}}(K-1)L^{2d}, \tag{4.18}$$

where c is a finite constant depending only on d, $\operatorname{diam}\operatorname{supp} u$ and $\operatorname{diam}\operatorname{supp} v$. By all these, we can complete the proof of Theorem 6 (i).

For Theorem 6 (ii), we introduce a smooth function ρ on $\mathbb{R}$ such that $0 \le \rho \le 1$, $\rho = 0$ on $(-\infty, 1+\kappa]$ and $\rho = 1$ on $[\mathbb{K}(1-\kappa), \infty)$, where $\mathbb{K} = (1+\kappa)/(1-2\kappa)$. Then we obtain

$$N([1-\kappa, 1+\kappa] : -\Gamma_L^{\omega,\mu\vartheta}) \le \operatorname{Tr}[\rho(-\mathbb{K}\Gamma_L^{\omega,\mu\vartheta})] - \operatorname{Tr}[\rho(-\Gamma_L^{\omega,\mu\vartheta})]. \tag{4.19}$$

As in Lemma 2.4 of [30], we have

$$\mathbb{E}[\operatorname{Tr}[\rho(-\mathbb{K}\Gamma_L^{\omega,\mu\vartheta})]] = \lim_{j\to\infty} \mathbb{E}\big[\operatorname{Tr}\big[\rho\big(-\Gamma_{L,1}^{\mathbb{K}_j\omega} - \Gamma_{L,2}^{\mathbb{K}_j\omega,\mu\vartheta}/\mathbb{K}\big)\big]\big],$$

where $\mathbb{K}_j$ is the transformation on the probability space defined as in (4.13) using $\mathbb{K}$ instead of K. Therefore we obtain

$$\mathbb{E}[N([1-\kappa, 1+\kappa] : -\Gamma_L^{\omega,\mu\vartheta})] \le \sum_{j=1}^{\infty} \mathbb{I}_j + \mathbb{I}_0, \tag{4.20}$$

where

$$\mathbb{I}_j = \mathbb{E}\big[\operatorname{Tr}\big[\rho\big(-\Gamma_{L,1}^{\mathbb{K}_{j+1}\omega} - \Gamma_{L,2}^{\mathbb{K}_{j+1}\omega,\mu\vartheta}/\mathbb{K}\big)\big] - \operatorname{Tr}\big[\rho\big(-\Gamma_{L,1}^{\mathbb{K}_j\omega} - \Gamma_{L,2}^{\mathbb{K}_j\omega,\mu\vartheta}/\mathbb{K}\big)\big]\big]$$

for $j \geq 1$ and

$$\mathbb{I}_0 = \mathbb{E}[\mathrm{Tr}[\rho(-\Gamma^{\omega}_{L,1} - \Gamma^{\omega,\mu\vartheta}_{L,2}/\mathbb{K})] - \mathrm{Tr}[\rho(-\Gamma^{\omega,\mu\vartheta}_{L})]].$$

We here note that (4.16) holds even if $\hat{d}$ is replaced by any $q \geq 1$. Moreover we have the following more general estimate:

$$\mathbb{E}\left[\int_{\mathbb{R}} \| A^{k(\omega)} \chi_L \|_q^q \tilde{h}(\omega(a))\, d\omega(a)\right] \leq (d\bar{A})^q L^d \int_{\mathbb{R}} |k(s)|^q \tilde{h}(s)\, ds \tag{4.21}$$

for any $q \geq 1$ and $k \in C(\mathbb{R} \to \mathbb{R})$, where $\tilde{h}$ is an integrable function such that $\tilde{h} \geq h$, and

$$A^{k(\omega)}(x) = \sum_{a \in \mathbb{Z}^d} k(\omega(a)) u(x-a).$$

In fact we can show this estimate as in (4.16) if q is an integer. Then the general case is treated by the Stein interpolation theorem (cf. [25] Theorem IX.21).

By using (4.21) as in (4.18), we have

$$|\mathbb{I}_0| \leq c\mathcal{H}_q \left(1 + \frac{\mathcal{K}'}{\mu \vee 1}\right) \frac{B_0^{(d+q)/2}}{(B_0 - E)^q} (\mathbb{K} - 1) L^{2d} \tag{4.22}$$

for any $q \geq \hat{d}$, where c is a finite constant depending only on d, α, q and $\mathrm{supp}\, v$. On the other hand, as in Lemma 2.4, we have

$$\begin{aligned} \mathbb{I}_j = -\mathbb{E}\bigg[\int & \left\{\frac{1}{\mathbb{K}} h\left(\frac{\omega[j]}{\mathbb{K}}\right) - h(\omega[j])\right\} \\ & \times \mathrm{Tr}\big[\rho\big(-\Gamma^{\mathbb{K}_j\omega}_{L,1} - \Gamma^{\mathbb{K}_j\omega,\mu\vartheta}_{L,2} \big/ \mathbb{K}\big) \\ & \quad - \rho\big(-\Gamma^{\mathbb{K}_j^0\omega}_{L,1} - \Gamma^{\mathbb{K}_j^0\omega,\mu\vartheta}_{L,2}/\mathbb{K}\big)\big]\, d\omega[j]\bigg], \end{aligned} \tag{4.23}$$

where

$$(\mathbb{K}_j^0\omega)[k] = \begin{cases} (\mathbb{K}_j\omega)[k] & \text{for} \quad k \neq j, \\ 0 & \text{for} \quad k = j. \end{cases}$$

As in Lemma 2.4, we apply the theory of the spectral shift functions to obtain

$$|\mathbb{I}_j| \leq \frac{c(\mathbb{K}-1)B_0^{d/2} L^d \{(\mu+1)^q (\mathcal{H}_{2q} + \mathcal{H}'_{2q}) + B_0^{q/2}(\mathcal{H}_q + \mathcal{H}'_q)\}}{\kappa^{q/l}(B_0 - E)^q (1 + \mathrm{dist}(a(j), \Lambda_L))^{(\alpha-1)q/l}} \tag{4.24}$$

for any $j \in \mathbb{N}$, where c is a finite constant depending only on d, q, α and l. By (4.22) and (4.24), we obtain Theorem 6 (ii).

For Theorem 6 (iii), we use the bound by the summation (4.20) and the representation (4.23), and apply the theory of the spectral shift functions as above. In this case we use also the calculation of commutators in [12] as in the proof of Lemma 2.6. Then we obtain

$$|\mathbb{I}_j| \leq c\mathcal{H}' \frac{(\mathbb{K}-1)B_0^{d/2}(\lambda\bar{A})^q(\sqrt{B_0}+(\mu+1)\lambda\bar{A})^q}{\kappa^{q/l}(B_0-E)^q}\left(\frac{\sqrt{B_0}}{B_0-E}\vee 1\right)^{q(1-1/l)}, \tag{4.25}$$

where c is a finite constant depending only on d, q, l and $\operatorname{diam}\operatorname{supp} u$. Moreover we apply the same theories to $\mathbb{I}_0$:

$$\mathbb{I}_0 = \sum_{j=1}^{m(v,L)} \mathbb{J}_j,$$

where

$$\mathbb{J}_j = \mathbb{E}\left[\operatorname{Tr}\left[\rho\left(-\Gamma_L^{\mu\omega,(\mathbb{K})_{j+1}(\mu\vartheta)}\right)\right] - \operatorname{Tr}\left[\rho\left(-\Gamma_L^{\mu\omega,(\mathbb{K})_j(\mu\vartheta)}\right)\right]\right]$$

and $(\mathbb{K})_j$, $j \in \mathbb{N}$, are the transformations on $\mathbb{R}^{\mathbb{Z}^d}$ defined by

$$\{(\mathbb{K})_j(\mu\vartheta)\}[k] = \begin{cases} (\mu\vartheta[k]+1-\mathbb{K})/\mathbb{K} & \text{for} \quad k < j, \\ (\mu\vartheta)[k] & \text{for} \quad k \geq j. \end{cases}$$

By changing the variables, we rewrite as

$$\begin{aligned} \mathbb{J}_j = \mathbb{E}\Bigg[\int \left\{\mathbb{K}k\left(\mathbb{K}\vartheta[j]+\frac{\mathbb{K}-1}{\mu}\right) - k(\vartheta[j])\right\} \\ \times \operatorname{Tr}\left[\rho\left(-\Gamma_L^{\lambda\omega,(\mathbb{K})_j(\mu\vartheta)}\right) - \rho\left(-\Gamma_L^{\lambda\omega,(\mathbb{K})_j^0(\mu\vartheta)}\right)\right] d\vartheta[j]\Bigg], \end{aligned}$$

where

$$\{(\mathbb{K})_j^0(\mu\vartheta)\}[k] = \begin{cases} (\mathbb{K})_j^0(\mu\vartheta)\}[k] & \text{for} \quad k \neq j, \\ 0 & \text{for} \quad k = j. \end{cases}$$

By the same estimate for $\mathbb{I}_j$, we obtain

$$|\mathbb{J}_j| \leq c\frac{\mathbb{K}-1}{\kappa^{q/l}}\left(1+\frac{\mathcal{K}'}{\mu\vee 1}\right)\frac{B_0^{d/2}(\lambda\bar{A})^q(\sqrt{B_0}+(\mu+1)\lambda\bar{A})^q}{(\mu^{q/l}\vee 1)(B_0-E)^q}\left(\frac{\sqrt{B_0}}{B_0-E}\vee 1\right)^{q(1-1/l)}, \tag{4.26}$$

where c is a finite constant depending only on d, q, l and $\operatorname{diam}\operatorname{supp} v$. By (4.25) and (4.26), we obtain Theorem 6 (iii).

We next proceed to the proof of Theorem 7. To prove (i), we take I_0, $\widetilde{I}_0$ and $H_{L,x}^{\omega}$ as $[0, B_0-\zeta]$, $[0, B_0-\zeta/2)$ and $H_{L,x}^{\omega,\vartheta}$, respectively, in Proposition 2.2. Then the assumptions in that proposition are satisfied with $\rho = \operatorname{diam}\operatorname{supp} u \vee \operatorname{diam}\operatorname{supp} v$, $b = 2$, $h = 1$, $\eta_0 = \zeta/6$, $\upsilon = 1$, $\gamma = c_1(1 + \sqrt{B_0})$ and $C_W = c_2\{(1 + \mathcal{K}'/(\mu \wedge 1)\mathcal{H}_{\hat{d}} + \mathcal{H}'_{\tilde{d}}\} B_0^{(d+\tilde{d})/2} / \zeta^{\tilde{d}+1}$, where c_1 and c_2 are finite constants depending only on d, $\operatorname{diam}\operatorname{supp} u$ and $\operatorname{diam}\operatorname{supp} v$. Then we have only to show the corresponding (2.32) for $E_0 = B_0 - \zeta$, some $\theta > 4d$ and $\mathcal{L} \in 6\mathbb{N}$ greater than the right hand side of the corresponding (2.33). As in the proof of Theorem 2, we can show that (4.6) is a sufficient condition. The second statement of (i) is proved by applying $\sqrt{B_0} - \sqrt{B_0 - \zeta} \geq \zeta/(2\sqrt{B_0})$ and the Chebyshev inequality $\mathbb{P}(|\omega(0)| \geq \eta) \leq \mathbb{E}[|\omega(0)|^{\hat{d}}]/\eta^{\hat{d}}$ to the left hand side of (4.6).

We next consider the situation of Theorem 7 (ii). As in the proof of Theorem 3 and Theorem 5 (ii), we apply the theory of Kirsch, Stollmann and Stolz [15]. Lemma 3.6 is modified as follows:

Lemma 4.2. *We assume* (2.3) *for some* $\alpha > d + 1$. *If* $L \geq 2\operatorname{diam}\operatorname{supp} v$ *and* $(\omega, \vartheta) = (\omega', \vartheta')$ *on* Λ_{2L}, *then we have*

$$\mu_j(H_L^{\lambda\omega,\mu\vartheta}) \leq \mu_j(H_L^{\lambda\omega',\mu\vartheta'}) + \frac{c\lambda}{L^{\alpha-1-d}}\left(\sqrt{\mu_j(H_L^{\lambda\omega',\mu\vartheta'})} + \lambda\left(\mu + \frac{1}{L^{\alpha-1-d}}\right)\right) \tag{4.27}$$

for any $j \in \mathbb{N}$, *where* $\mu_j(H_L^{\lambda\omega,\vartheta})$ *is the* j*-th eigenvalue of* $H_L^{\omega,\vartheta}$ *including the multiplicity, and* c *is a finite constant depending only on* d *and* α. *Moreover, if* $\alpha > d + 2$ *and* $L \geq c'\lambda^{1/(\alpha-2-d)}$, *then we have*

$$\mu_j(H_L^{\lambda\omega,\mu\vartheta}) \geq \mu_j(H_L^{\lambda\omega',\mu\vartheta'}) - \frac{c\lambda}{L^{\alpha-1-d}}\left(\sqrt{\mu_j(H_L^{\lambda\omega',\mu\vartheta'})} + \lambda\mu\right), \tag{4.28}$$

where c' *is a finite constant depending only on* d *and* α.

By this lemma, we see that

$$|\mu_j(H_L^{\lambda\omega,\mu\vartheta}) - \mu_j(H_L^{\lambda\omega',\mu\vartheta'})| \leq \frac{c_1(\sqrt{B_0} + \lambda(\mu \vee 1))\lambda}{L^{\alpha-1-d}}$$

holds in the situation of Lemma 4.2, if $\mu_j(H_L^{\lambda\omega,\mu\vartheta}) \wedge \mu_j(H_L^{\lambda\omega',\mu\vartheta'}) \leq B_0$. Therefore we dominate the probability (2.41) where $H_{L,x}^{\lambda\omega}$, $H_{L',x'}^{\lambda\omega}$, $\sigma_I(H_{L,x}^{\lambda\omega})$ and Ξ are replaced by $H_{L,x}^{\lambda\omega,\mu\vartheta}$, $H_{L',x'}^{\lambda\omega,\mu\vartheta'}$, $\sigma_I(H_{L,x}^{\lambda\omega,\mu\vartheta}) = \sigma(H_{L,x}^{\lambda\omega,\mu\vartheta}) \cap [0, B_0 - \zeta + (\sqrt{B_0} + \lambda(\mu \vee 1))\lambda/(2L^{\alpha-1-d})]$ and $(\sqrt{B_0} + \lambda(\mu \vee 1))\lambda/(L \wedge L')^{\alpha-1-d}$, respectively. The rest of the proof is same as in [15] and the last sections.

5. Gaussian random vector potentials

In this section we consider the random magnetic Schrödinger operator (4.1) where the random vector potential $A^\omega(x)$ is an $\mathbb{R}^d$-valued stationary ergodic Gaussian random field with mean zero and the covariance $\beta_{jk}(x) = \mathbb{E}[A_j^\omega(x)A_k^\omega(0)]$ represented as

$$\beta_{jk}(x) = \sum_{\iota=1}^{r} \int \sigma_\iota^j(x+y)\overline{\sigma_\iota^k(y)}\,dy$$

for some complex valued C^s functions $\sigma_\iota^j(x)$, $0 \le j \le d$, $1 \le \iota \le r$, on $\mathbb{R}^d$ with compact support and $s \ge d+3$. Then our Wegner type estimate is the following:

Theorem 8 (Wegner type estimate). *There exists a finite constant c depending only on d, β and* diam supp v *such that*

$$\mathbb{E}[N([E-\eta, E+\eta] : H_L^{\omega,\mu\vartheta})] \le cW_{d,s}(E, B_0, \mu, \mathcal{K}')\eta^{1-d/s}L^{2d} \tag{5.1}$$

for any $L \ge 1$, $0 \le E < B_0$ and $0 < \eta < (B_0 - E)/3$, where

$$W_{d,s}(E, B_0, \mu, \mathcal{K}') = \frac{B_0^{(d+\hat{d})/2}}{(B_0 - E)^{\hat{d}+1-d/s}}\left(1 + \frac{\mathcal{K}'}{\mu \wedge 1}\right)\left(1 + \frac{\mu \vee 1}{\sqrt{B_0}}\right)^{(\hat{d}+1)d/s}.$$

By using this estimate and Germinet and Klein theory [11] as in the last sections, we have the following result on the Anderson localization:

Theorem 9. *For any $\epsilon > 0$, there exist a finite positive constant c depending only on d, ϵ, β, μ, $\mathcal{K}'$ and* diam supp v *such that $B_0 \ge c$ implies the results in* Theorem 2 *on the interval $I = [0, B_0 - B_0^{\epsilon+1/2}]$ for the operator $H^{\omega,\mu\vartheta}$.*

To prove Theorem 8, we use the following representation in Lemma 2.2 in [30]:

Lemma 5.1. *The Gaussian random field $A^\omega(x)$ is represented as*

$$A_j^\omega(x) = \sum_{\iota=1}^{2r} \sum_{a \in \mathbb{Z}^d} W_{\iota,a}^\omega e_{\iota,a,j}(x) \tag{5.2}$$

in $L^p(\Lambda_L \times \Omega)$ for any $1 \le p < \infty$, where $\{W_{\iota,a}^\omega\}_{1 \le \iota \le 2r, a \in \mathbb{Z}^d}$ is a family of independently and identically distributed random variables with the standard normal distribution and $e_{\iota,a,j}$, $1 \le \iota \le 2r$, $a \in \mathbb{Z}^d$, $1 \le j \le d$, are $\mathbb{R}^d$-valued C^s functions such that, for any $0 \le l \le s$,

$$\sup_{x \in \Lambda_L} |\nabla^l e_{\iota,a,j}(x)| \le cL^{s-l-d/2}|a|_\infty^{l-s}, \tag{5.3}$$

where c is some finite constant depending only on the covariance β.

By this lemma, we can treat the Gaussian random field similarly as the alloy type random potential whose single site potentials have noncompact supports. Therefore we can give a Wegner type estimate similarly as in the proof of Theorem 6 (ii). We estimate the each term of the bound by the summation

$$\mathbb{E}[N([1-\kappa, 1+\kappa] : -\Gamma_L^{\omega,\vartheta})] \le \sum_{j=1}^{\infty} \mathbb{I}_j + \mathbb{I}_0, \tag{5.4}$$

corresponding to (4.20), where $\omega[j]$ is identified with $W^{\omega}_{\iota(j),a(j)}$ and $(\iota(j), a(j))$, $j = 1, 2, \dots,$ is an ordering of $\{1, 2, \dots, 2r\} \times \mathbb{Z}^d$ such that $|a(j)|_\infty$ is an increasing function in j. We here use the estimate

$$\mathbb{E}[\|A^{\mathbb{K}_j\omega}\chi_L\|_q^q |W^{\omega}_{\iota,a}|^r] \le c|\beta(0)|^{(q+r)/2} L^d \tag{5.5}$$

for any $q \ge 1$, $r \ge 0$, $j \in \mathbb{N}$, $\iota \in \{1, 2, \dots, 2r\}$, $a \in \mathbb{Z}^d$ and $\mathbb{K} \le 4$, where $|\beta(0)|$ is a norm of the matrix $\beta(0)$ and c is a finite constant depending only on q, r and d. Similarly we have

$$\mathbb{E}[\|A^{\mathbb{K}_j^0\omega}\chi_L\|_q^q |W^{\omega}_{\iota,a}|^r] \le c|\beta(0)|^{(q+r)/2} L^d. \tag{5.6}$$

Then, as in (4.22) and (4.24), we have

$$|\mathbb{I}_0| \le c_1\left(1 + \frac{\mathcal{K}'}{\mu \wedge 1}\right)\frac{B_0^{(d+q)/2}}{(B_0 - E)^q}(\mathbb{K} - 1)L^{2d} \tag{5.7}$$

and

$$|\mathbb{I}_j| \le \frac{c_2 B_0^{d/2}((\mu+1)^q + B_0^{q/2})(\mathbb{K}-1)L^{d+(s-d/2)q/l}}{\kappa^{q/l}(B_0 - E)^q(|a(j)|_\infty \vee 1)^{qs/l}}, \tag{5.8}$$

where c_1 and c_2 are finite constants depending only on d, β, q, l and $\operatorname{diam}\operatorname{supp} v$. However the bound of (5.8) is unnecessarily big for small j. Thus we use this bound only for the terms $\mathbb{I}_j$ satisfying

$$|a(j)|_\infty \ge \frac{(\mu \vee 1 + \sqrt{B_0})^{l/s} L^{1-d/(2s)}}{B_0^{l/(2s)}\kappa^{1/s}}$$

and use the simpler bound

$$|\mathbb{I}_j| \le \frac{c_3 B_0^{(d+q)/2}(\mathbb{K}-1)L^d}{(B_0 - E)^q}, \tag{5.9}$$

for the other terms, where c_3 is a finite constant depending only on d, β and q. The bound (5.9) is obtained as in (4.17). In the obtained bound, the dependence on B_0 and

μ is better for smaller l and q. Accordingly we take $q = \hat{d}$ and $l = \hat{d} + 1$. Then we obtain Theorem 8.

To prove Theorem 9, we take I_0, $\widetilde{I}_0$ and $H^{\omega}_{L,x}$ in Proposition 2.2 as in the last section. Then the assumptions in that proposition are satisfied with $\rho = \operatorname{diam}\operatorname{supp}\beta \vee \operatorname{diam}\operatorname{supp} v$, $b = 2$, $h = 1 - d/s$, $\eta_0 = \zeta/6$, $v = 1$, $\gamma = c_1(1 + \sqrt{B_0})$ and

$$C_W = c_2\left(1 + \frac{\mathcal{K}'}{\mu \wedge 1}\right)\left(1 + \frac{\mu \vee 1}{\sqrt{B_0}}\right)^{d(\hat{d}+1)/s} \frac{B_0^{(d+\hat{d})/2}}{\zeta^{\hat{d}+1-d/s}},$$

where c_1 and c_2 are finite constants depending only on d, β and $\operatorname{diam}\operatorname{supp} v$. Then we have only to show the corresponding (2.32) for $E_0 = B_0 - \zeta$, some $\theta > 4d/(1 - d/s)$ and some $\mathcal{L}$ greater than the right hand side of the corresponding (2.33). The corresponding (2.32) is reduced to (2.39) as in Section 2. We now use the following lemma by Fischer, Leschke and Müller [9], which is based on Fernique's theorem [8]:

Lemma 5.2 (Lemma 5.3 in [9]). *There exists a positive finite constant* $L_0 = L_0(\beta_{jj}(0), \|\nabla\beta_{jj}\|_\infty)$ *such that*

$$\mathbb{P}\left(\sup_{x\in\Lambda_L} |A^{\omega}_j(x)| \geq \eta\right) \leq 2^{2(d+1)} \exp\left(-\frac{\eta^2}{200\beta_{jj}(0)\log L}\right)$$

for any $L \geq L_0$ *and* $\eta \geq 0$.

By this lemma, (2.39) is reduced to

$$\sqrt{B_0} - \sqrt{B_0 - \zeta} \geq c_3\sqrt{\log \mathcal{L}} + f(\theta, \mathcal{L}),$$

where c_3 is a finite positive constant depending only on d and β. Since $\sqrt{B_0} - \sqrt{B_0 - \zeta} \geq \zeta/(2\sqrt{B_0})$ and $f(\theta, \mathcal{L})$ decays as $\mathcal{L} \to \infty$, this condition is reduced to

$$\zeta \geq c_4\sqrt{B_0 \log \mathcal{L}}, \tag{5.10}$$

where c_4 is a finite positive constant appending only on d, β and θ. By substituting the bound of $\mathcal{L}$ given by the corresponding (2.33) to the right hand side of (5.10), we obtain the condition in terms of ζ and B_0. However since the bound appears in the order of the logarithm in (5.10), we easily see from (5.10) that $\zeta = B_0^{\epsilon+1/2}$ and $B_0 \geq c_5$ constitute a sufficient condition, where c_5 is a finite constant depending only on d, β, μ, $\mathcal{K}'$, $\operatorname{diam}\operatorname{supp} v$, θ and ϵ.

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