



Title	On projective modules over directly finite regular rings satisfying the comparability axiom. II
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Citation	Osaka Journal of Mathematics. 1987, 24(3), p. 465-473
Version Type	VoR
URL	https://doi.org/10.18910/4578
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Osaka J. Math.
24 (1987), 465–473

ON PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATISFYING THE COMPARABILITY AXIOM II

Dedicated to Professor Hisao Tominaga on his 60th birthday

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(Received April 7, 1986)

In [4], by observing the directly finiteness of projective modules, the first author classified directly finite (d.f. for short) regular rings satisfying the comparability axiom (c. axiom for short) into three types: Type A, Type B and Type C.

In the present paper, we give a more explicite criterion of the directly finiteness of projective modules over each types and show the following for a d.f. regular ring R satisfying the c. axiom: (a) R is Type A if and only if $\text{Soc}(R)=0$ and the intersection $I_0(R)$ of all nonzero ideals of R is nonzero. (b) R is Type B if and only if $\text{Soc}(R)=0$, $I_0(R)=0$ and the family $L(R)$ of all ideals of R has a cofinal subfamily. (c) R is Type C if and only if $\text{Soc}(R)\neq 0$, or $I_0(R)=0$ and $L(R)$ does not have any cofinal subfamilies. As an application we show the following for a projective module P over a d.f. regular ring satisfying the c. axiom: P is directly infinite (d.inf. for short) if and only if P contains a direct summand which is isomorphic to $\aleph_0 X$ for a suitable nonzero module X .

Throughout this paper we assume that R is a d.f. regular ring satisfying the c. axiom, and all R -modules considered are unital right R -modules.

1. Notations and definitions

For two R -modules X and Y , we use $X \leq Y$ (resp. $X \leq \bigoplus Y$) to mean that X is isomorphic to a submodule of Y (resp. a direct summand of Y). $X \lessdot Y$ means that $X \leq Y$ and $X \not\cong Y$. For a submodule X of an R -module Y , $X < \bigoplus Y$ means that X is a direct summand of Y . For a cardinal number α and an R -module X , αX denotes a direct sum of α -copies of X . For a set I , we denote by $|I|$ the cardinal number of I . We denote by $L(R)$ the family of all ideals of R . Since R satisfies the c. axiom, $L(R)$ is a linearly ordered set under inclusion ([1, Proposition 8.5]). We put $I_0(R) = \bigcap \{I \mid 0 \neq I \in L(R)\}$. We denote by $\text{Soc}(R)$ the socle of R . We note that if $\text{Soc}(R) \neq 0$ then it is homogeneous and coincides with $I_0(R)$.

The reader is referred to K.R. Goodearl [1] for the following elementary properties on R :

- i) Every finitely generated projective R -module is d.f..
- ii) For any finitely generated projective R -modules P and Q , either $P \leq Q$ or $Q \leq P$ holds.
- iii) For any finitely generated projective R -module P and any R -modules X and Y , $P \oplus X \cong P \oplus Y$ implies $X \cong Y$.
- iv) For any projective R -module X and any finitely generated projective R -modules Y_1, Y_2, \dots such that $Y_1 \oplus \dots \oplus Y_n \leq X$ for all n , we have that $\bigoplus Y_n \leq X$. In particular, for finitely generated projective R -modules P and Q , if $P \not\leq nQ$ for all n , then $\aleph_0 Q \leq P$.
- v) R has a unique *dimension function* D , namely, D is a function from the family of all cyclic right ideals of R to $[0, 1]$ such that

- a) $D(R) = 1$,
- b) if $J \leq K$, then $D(J) \leq D(K)$ and

c) if $J \oplus K$ is a cyclic right ideal of R , then $D(J \oplus K) = D(J) + D(K)$. D is said to be *strictly positive* if $D(J) > 0$ for all nonzero cyclic right ideals J of R . We note that D is strictly positive dimension function if and only if R is a simple ring, and that R is not simple if and only if there exists a nonzero r in R such that $\aleph_0(rR) \leq R$. Furthermore we note that $\{r \in R \mid D(rR) = 0\}$ is the unique maximal ideal of R .

Let $\{P_i\}_{i=1}^\infty$ be a subfamily of the family of all cyclic projective R -modules. We say that $\{P_i\}_{i=1}^\infty$ is a *cofinal* subfamily if all P_i are nonzero, $P_1 \geq P_2 \geq \dots$ and for any nonzero cyclic projective R -module X there exists a positive integer n satisfying $X \geq P_n$. Similarly a subfamily $\{I_i\}_{i=1}^\infty$ of $L(R)$ is said to be a *cofinal* subfamily of $L(R)$ if all I_i are nonzero, $I_1 \geq I_2 \geq \dots$ and for any nonzero X in $L(R)$ there exists a positive integer n satisfying $X \geq I_n$.

2. Directly finiteness and directly infiniteness

We start the following

Theorem 2.1. (a) For countably generated projective R -modules P and Q , either $P \leq Q$ or $Q \leq P$ holds.

(b) If P and Q are countably generated projective R -modules such that $P \leq Q$ and $Q \leq P$, then $P \cong Q$.

Proof. We show the theorem by modifying the proof of [3, Lemma 2.5]. Let $P = \bigoplus_{i=1}^\infty P_i$ and $Q = \bigoplus_{i=1}^\infty Q_i$ be cyclic decompositions of P and Q . (a) Assume $P \not\leq Q$. Then there exists a positive integer n such that $P_1 \oplus \dots \oplus P_n \not\leq Q$ and so $Q_1 \oplus \dots \oplus Q_m \not\leq P_1 \oplus \dots \oplus P_n$ for all m . Therefore $Q \not\leq P_1 \oplus \dots \oplus P_n < \bigoplus P$. (b) It is sufficient to assume that P and Q are non-finitely generated projective and that $Q \not\leq P_1 \oplus \dots \oplus P_n$ for all n from the assumption. Since $Q \not\leq P$, there exists a

positive integer n_1 such that $Q_1 \leq P_1 \oplus \cdots \oplus P_{n_1}$ and $Q_1 \not\leq P_1 \oplus \cdots \oplus P_{n_1-1}$, and there exists a positive integer m_1 such that $Q_1 \oplus \cdots \oplus Q_{m_1} \leq P_1 \oplus \cdots \oplus P_{n_1}$ and $Q_1 \oplus \cdots \oplus Q_{m_1+1} \not\leq P_1 \oplus \cdots \oplus P_{n_1}$. Then we have a decomposition $Q_{m_1+1} = X_{m_1+1} \oplus Y_{m_1+1}$ such that $Y_{m_1+1} \neq 0$ and

$$P_1 \oplus \cdots \oplus P_{n_1} \cong Q_1 \oplus \cdots \oplus Q_{m_1} \oplus X_{m_1+1}.$$

Since $Q \leq P$, $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \leq P_{n_1+1} \oplus P_{n_1+2} \oplus \cdots$. Then there exists a positive integer $n_2 (> n_1)$ such that $Y_{m_1+1} \leq P_{n_1+1} \oplus \cdots \oplus P_{n_2}$ and $Y_{m_1+1} \not\leq P_{n_1+1} \oplus \cdots \oplus P_{n_2-1}$. So, there exists a positive integer $m_2 (> m_1)$ such that $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2} \leq P_{n_1+1} \oplus \cdots \oplus P_{n_2}$ and $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2+1} \not\leq P_{n_1+1} \oplus \cdots \oplus P_{n_2}$. Then we have a decomposition $Q_{m_2+1} = X_{m_2+1} \oplus Y_{m_2+1}$ such that $Y_{m_2+1} \neq 0$ and

$$P_{n_1+1} \oplus \cdots \oplus P_{n_2} \cong Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2} \oplus X_{m_2+1}.$$

Continuing this procedure, we have that $P \cong Q$.

REMARK. When R is simple, we can drop the assumption ‘countably generated’ from above Theorem 2.1 (see [3, Theorems 2.4 and 2.6]). But the assumption can not be removed in general. For, if R is a non-simple d.f. regular ring satisfying the c. axiom, then there exists a nonzero r in R such that $\aleph_0(rR) \leq R$. So, $R \not\leq \alpha(rR)$ and $\alpha(rR) \not\leq R$, where $|R| < \alpha$. Next, if we take R as in [1, Example 5.15], then there exists a simple right ideal S of R such that $\aleph S \leq R$. Then $\aleph_0 R \leq \aleph S \oplus \aleph_0 R$ and $\aleph S \oplus \aleph_0 R \leq \aleph_0 R$, but $\aleph_0 R \not\cong \aleph S \oplus \aleph_0 R$.

Corollary 2.2. *Let P and Q be countably generated projective R -modules and let n be a positive integer.*

- (a) *If $nP \cong nQ$, then $P \cong Q$.*
- (b) *If $nP \leq nQ$, then $P \leq Q$.*

Proof. (a) We prove the statement by induction on n . So, assume that this holds for n and $(n+1)P \cong (n+1)Q$. Then we have decompositions $nP = X_1 \oplus X_2$ and $P = Y_1 \oplus Y_2$ such that $X_1 \oplus Y_1 \cong nQ$ and $X_2 \oplus Y_2 \cong Q$. By Theorem 2.1 (a), either $Y_1 \leq X_2$ or $X_2 \leq Y_1$ holds. If $Y_1 \leq X_2$, then $nQ \cong X_1 \oplus Y_1 \leq X_1 \oplus X_2 = nP$ and $P = Y_1 \oplus Y_2 \leq X_2 \oplus Y_2 \cong Q$; so $nQ \leq nP$ and $nP \leq nQ$. Hence the induction hypothesis says that $P \cong Q$. If $X_2 \leq Y_1$, similarly, we have that $P \cong Q$. (b) Assume that $nP \leq nQ$ and $P \not\leq Q$. Then $Q \not\leq P$, and so $nQ \not\leq nP$. Therefore $nP \cong nQ$ by Theorem 2.1 (b); whence $P \cong Q$ by (a), a contradiction.

Now, for our purpose we define a relation “ \sim ” on the family of all cyclic projective R -modules $CP(R)$ by the rule: For any P and Q in $CP(R)$, $P \sim Q$ if and only if $P \leq mQ$ and $Q \leq nP$ for some positive integers m and n . For P in $CP(R)$ we put $[P] = \{Q \in CP(QR) \mid Q \sim P\}$. Then the relation “ \sim ” is a congruence relation.

Proposition 2.3. *Let P be a non-finitely generated, countably generated projective R -module with a cyclic decomposition $P = \bigoplus_{i=1}^{\infty} P_i$ such that $P_1 \supseteq P_2 \supseteq \dots$ and $[P_1] = [P_2] = \dots$. If $\{P_i\}_{i=1}^{\infty}$ is cofinal, then P is d.inf. if and only if $P \cong \aleph_0 P_i$ for all i .*

Proof. “If” part is clear. “Only if” part. Let P be a d.inf. projective R -module. From [4, Theorem 6], $tP_i \not\supseteq P_{i+1} \oplus P_{i+2} \oplus \dots$ for all t and hence $\aleph_0 P_i \leq P_{i+1} \oplus P_{i+2} \oplus \dots \leq P$. Since $P \leq \aleph_0 P_i$, it follows from Theorem 2.1 that $P \cong \aleph_0 P_i$ for all i .

REMARK. Let P be a non-finitely generated, countably generated projective R -module with a cyclic decomposition $P = \bigoplus_{i \in I} P_i$ such that $|I| = \aleph_0$ and $[P_i] = [P_j]$ for any $i, j \in I$. If there exists $i \in I$ such that $|\{j \in I \mid P_i \leq P_j\}| = \aleph_0$, then $P \cong \aleph_0 P_i$ for all $i \in I$ by Theorem 2.1.

Proposition 2.4. *Let P be a non-countably generated (d.inf.) projective R -module with a cyclic decomposition $P = \bigoplus_{\alpha \in I} P_{\alpha}$ such that $[P_{\alpha}] = [P_{\beta}]$ for any $\alpha, \beta \in I$. Then there exists an infinite cardinal $\tau (> \aleph_0)$ such that $P \cong \tau P_{\gamma}$ for any $\gamma \in I$.*

Proof (cf. the proof of [3, Theorem 2.6]). Let B be the set of all countably infinite subsets of I , and let $\gamma \in I$. We consider the family consisting of all subsets F of B satisfying the following properties:

- (1) each member of F is pairwise disjoint, and
- (2) for each member I' of F , $P_{I'} = \bigoplus_{\alpha \in I'} P_{\alpha} \cong \aleph_0 P_{\gamma}$.

Then this family is non-empty set from the proof of [4, Theorem 6] and Remark of Proposition 2.3. Since this family is inductively ordered set under inclusion, there exists a maximal member F by Zorn’s Lemma. Put $I^* = \bigcup_{K \in F} K$. If $I^* = I$, then our proof is complete. Next, consider the case that $I^* \neq I$. Let I^{**} be a complement of I^* in I . From the proof of [4, Theorem 6], Remark of Proposition 2.3 and the maximality of F , I^{**} is a countable set. Choose one member K' of F and put $F' = F - \{K'\}$ and $K'' = K' \cup I^{**}$. Then K'' is a countably infinite set and $P_{K''} \cong \aleph_0 P_{\gamma}$ since $[P_{\alpha}] = [P_{\beta}]$ for any $\alpha, \beta \in I$. Therefore $P = (\bigoplus_{K \in F'} (\bigoplus_{\alpha \in K} P_{\alpha})) \oplus (\bigoplus_{\alpha \in K''} P_{\alpha}) \cong \tau P_{\gamma}$ for some infinite cardinal $\tau > \aleph_0$.

Corollary 2.5 ([3, Theorem 2.6]). *Assume that R is simple. Then every d.inf. projective R -module is a free R -module.*

Proof. Let P be a d.inf. projective R -module with a cyclic decomposition $P = \bigoplus_{\alpha \in I} P_{\alpha}$ and we consider $R \oplus P = R \oplus (\bigoplus_{\alpha \in I} P_{\alpha})$. Noting that R is simple, we see that $[X] = [R]$ for a nonzero cyclic projective R -module X ; whence $[P_{\alpha}] = [R]$ for all $\alpha \in I$. By Proposition 2.3, its Remark and Proposition 2.4, $R \oplus P \cong \tau R$ for some infinite cardinal τ . Therefore $P \cong \tau R$ by the cancellation property of R .

Theorem 2.6. *Let P and Q be d.inf. projective R -modules with cyclic decom-*

positions $P = \bigoplus_{\alpha \in I} P_\alpha$ and $Q = \bigoplus_{\beta \in J} Q_\beta$ such that $[P_\alpha] = [P_{\alpha'}]$ and $[Q_\beta] = [Q_{\beta'}]$ for any $\alpha, \alpha' \in I$ and $\beta, \beta' \in J$. If $P \leq Q$ and $Q \leq P$, then $P \cong Q$.

Proof. Since $P \leq Q$ and $Q \leq P$, we note that $[P_\alpha] = [Q_\beta]$ for any P_α and Q_β . If $|I| \leq \aleph_0$ and $|J| \leq \aleph_0$, then $P \cong Q$ from Theorem 2.1. Therefore we may consider the following cases:

- 1) $|I| \leq \aleph_0$ and $|J| > \aleph_0$.
- 2) $|I| > \aleph_0$ and $|J| > \aleph_0$.

In order to prove for these cases, we show the following for any nonzero cyclic projective R -module T and cardinal numbers σ and ρ :

(#) If $\rho T \leq \sigma T$, then $\rho \leq \sigma$.

Let τ be a cardinal number. We regard τ an initial ordinal; so $|\{\text{ordinal } \alpha \mid \alpha < \tau\}| = \tau$. Put $\Lambda(\tau) = \{\text{ordinal } \alpha \mid \alpha < \tau\}$. We shall prove (#) by the transfinite induction on σ . First assume that $\sigma = \aleph_0$ and let f be a monomorphism from ρT to $\sigma T = \aleph_0 T$. Putting $\Gamma_m = \{\alpha \in \Lambda(\rho) \mid f(T_\alpha) \leq \bigoplus_{i=1}^m T_i\}$, we see that $f(\bigoplus_{\alpha \in \Gamma_m} T_\alpha) \leq \bigoplus_{i=1}^m T_i$, $|\Gamma_m| \leq m$ and $\bigcup_{\alpha \in \Gamma_m} \Gamma_m = \Lambda(\rho)$. Therefore $\rho = |\Lambda(\rho)| = |\bigcup_{\alpha \in \Gamma_m} \Gamma_m| \leq \aleph_0 = \sigma$. Next assume that (#) holds for any cardinal number $\sigma' < \sigma$, and let f be a monomorphism from ρT to σT . For any x in $\Lambda(\sigma)$, put $\Gamma_x = \{\alpha \in \Lambda(\rho) \mid f(T_\alpha) \leq \bigoplus_{\beta \leq x} T_\beta\}$. Then $f(\bigoplus_{\alpha \in \Gamma_x} T_\alpha) \leq \bigoplus_{\beta \leq x} T_\beta$, $\bigcup_{\alpha \in \Gamma_x} \Gamma_x = \Lambda(\rho)$ and $|\{\beta \mid \beta \leq x\}| < \sigma$ because σ is an initial ordinal. From the induction hypothesis, $|\Gamma_x| < \sigma$. Therefore we see that $\rho = |\Lambda(\rho)| = |\bigcup_{x \in \Lambda(\sigma)} \Gamma_x| \leq \sigma^2 = \sigma$ as desired.

Case 1) Let $P_\beta \in \{P_\alpha\}_{\alpha \in I}$. Since $|J| > \aleph_0$ and $[P_\alpha] = [Q_\beta]$ for all Q_β , $Q \cong \tau P_\alpha$ for a suitable cardinal number τ . Since $P \leq Q \cong \tau P_\alpha$ and $Q \leq P$, we see that $\aleph_0 P_\alpha \leq P$ and $P \leq \aleph_0 P_\alpha$, whence $\aleph_0 P_\alpha \cong P$ by Theorem 2.1. As $\tau P_\alpha \cong Q \leq \aleph_0 P_\alpha$, $\tau \leq \aleph_0$ by (#), a contradiction.

Case 2) By Proposition 2.4 and (#), we immediately have that $P \cong Q$.

Corollary 2.7 ([3, Proposition 2.7]). *Assume that R is simple. If P and Q are d.inf. projective R -modules such that $P \leq Q$ and $Q \leq P$, then $P \cong Q$.*

3. Types A, B and C

In [4] we showed the following result, which already used in Proposition 2.3: A non-finitely generated projective R -module P is d.f. if and only if P is countably generated with a cyclic decomposition $P = \bigoplus_{i=1}^\infty P_i$ satisfying the conditions (*) and (A), or (*) and (B) below:

(*) $P_i \geq P_{i+1}$ for all i , and there exists no nonzero R -module X such that $X \leq P_i$ for all i .

(A) There exists a positive integer m such that

- (1) For each $i \geq m$, $P_i \leq t_i P_{i+1}$ for some positive integer t_i , and
- (2) $\bigoplus_{i=m}^\infty P_i \leq t P_m$ for some positive integer t .

(B) There exists an increasing sequence $1 = i_1 < i_2 < \dots$, of positive integers such that $P_{i_n} \geq \aleph_0 P_{i_{n+1}}$ for $n = 1, 2, \dots$.

And, from this result, we classified d.f. regular rings R satisfying the c. axiom into three types:

Type A: There exists a non-finitely generated d.f. projective R -module P with a cyclic decomposition $P = \bigoplus_{i=1}^{\infty} P_i$ satisfying (*) and (A).

Type B: There exists a non-finitely generated d.f. projective R -module P with a cyclic decomposition $P = \bigoplus_{i=1}^{\infty} P_i$ satisfying (*) and (B).

Type C: All d.f. projective R -modules are finitely generated.

REMARK. If a ring R is Type A (resp. Type B), then all non-finitely generated d.f. projective R -module P have a cyclic decomposition $P = \bigoplus_{i=1}^{\infty} P_i$ satisfying (*) and (A) (resp. (*) and (B)) by [4, Theorem 6 and Remark 2]. We note that (*) holds then $\text{Soc}(R) = 0$. So, if $\text{Soc}(R) \neq 0$ then R is type C.

In this section, as is mentioned in the introduction, we shall give ideal theoretic characterizations of each types.

Lemma 3.1 ([1, Corollary 2.23]). *Let H and J be right ideals of R , and assume that H is finitely generated. Then $H \leq R J$ if and only if $H \leq nJ$ for some positive integer n .*

For an element a of a ring R , we put

$$\Sigma_a = \Sigma\{xR \mid x \in R \text{ and } xR \leq aR\}.$$

Lemma 3.2. (a) *For each $a \in R$, Σ_a is the smallest ideal of R containing a , and hence $\Sigma_a = RaR$.*

(b) *For each $a, b \in R$, $\Sigma_a \leq \Sigma_b$ if and only if $aR \leq n(bR)$ for some positive integer n .*

(c) *For $a, b \in R$, $\Sigma_a \leq \Sigma_b$ if and only if $\text{Soc}(aR) \leq bR$.*

Proof. (a) Let $r \in R$ and $\sum_{i=1}^n x_i r_i \in \Sigma_a$ such that $r_i \in R$ and $x_i R \leq aR$ for each i . Then $(rx_i r_i)R \leq \bigoplus (x_i r_i)R \leq x_i R \leq aR$ and $rx_i r_i \in \Sigma_a$ for each i , and so $r(\sum_{i=1}^n x_i r_i) \in \Sigma_a$. Thus Σ_a is an ideal of R containing a . Let I be an ideal of R containing a . If $xR \leq aR$ and $x \in R$, then $xR \leq RaR \leq I$ from Lemma 3.1. Therefore $\Sigma_a \leq I$ and hence $\Sigma_a = RaR$. (b) is clear from (a) and Lemma 3.1, and (c) follows from (b).

Theorem 3.3. *The following are equivalent:*

(a) *R is Type A.*

(b) *$\text{Soc}(R) = 0$ and $I_0(R) \neq 0$.*

(c) *There exists a non-finitely generated d.f. projective R -module P with a cyclic decomposition $P = \bigoplus_{i=1}^{\infty} P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal, $[P_1] = [P_2] = \dots$ and $tP_1 \geq P_2 \oplus P_3 \oplus \dots$ for some positive integer t .*

Proof. (a) \rightarrow (b). Assume that Type A. Then of course $\text{Soc}(R) = 0$. Now

assume that $I_0(R)=0$. Since R is Type A, we have a non-finitely generated d.f. projective R -module P with a cyclic decomposition $P=\bigoplus_{i=1}^{\infty} P_i$ which satisfies (*) and $[P_m]=[P_{m+1}]=\dots$ for some positive integer m . Let $P_i \cong x_i R$ for some $x_i \in R$. Noting that $I_0(R)=0$, we have a nonzero ideal X of R such that $X \leq \Sigma_{x_m} = \Sigma_{x_{m+1}} = \dots$, which contradicts that $P=\bigoplus_{i=1}^{\infty} P_i$ satisfying (*). Therefore we see that $I_0(R) \neq 0$. (b) \rightarrow (c). Take a nonzero element x_1 in $I_0(R)$; then $\Sigma_{x_1} = I_0(R)$. Since $Soc(R)=0$, there exist nonzero cyclic right ideals $x_{i+1}R$ and $y_{i+1}R$ of R such that $x_iR = x_{i+1}R \oplus y_{i+1}R$ and $x_{i+1}R \leq y_{i+1}R$ for each i ; so $2(x_{i+1}R) \leq x_iR$. Put $P=\bigoplus_{i=1}^{\infty} x_iR$. If there exists a nonzero element y of R such that $yR \leq x_iR$ for all i , then $\Sigma_{x_1} = \Sigma_{x_i} = \Sigma_y$ for all i by the smallness of Σ_{x_1} . Hence there exist positive integers t and m such that $2t(x_mR) \leq x_iR \leq t(yR) \leq t(x_mR)$ from Lemma 3.2; whence $2t(x_mR) \leq t(x_mR)$ which contradicts the directly finiteness of $t(x_mR)$. Therefore $\{x_iR\}_{i=1}^{\infty}$ is cofini cofinal. By the smallness of Σ_{x_1} , we see that $[x_1R]=[x_iR]$ for all i and $\bigoplus_{i=1}^{\infty} x_iR \leq 2(x_1R)$, and hence $\bigoplus_{i=1}^{\infty} x_iR$ is d.f.. (c) \rightarrow (a) is clear.

Proposition 3.4. *Assume that $Soc(R)=0$ and $I_0(R)=0$. Then a non-finitely generated projective R -module P is d.f. (if and only) if P has a cyclic decomposition $P=\bigoplus_{i=1}^{\infty} P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal.*

Proof. Assume that P is a countably generated projective R -module with a cyclic decomposition $P=\bigoplus_{i=1}^{\infty} P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal. We express each P_i as $P_i \cong x_i R$, where $x_i \in R$. Then $\Sigma_{x_1} \geq \Sigma_{x_2} \geq \dots$. If there exists a positive integer j such that $\Sigma_{x_j} = \Sigma_{x_{j+1}} = \dots$, we have a nonzero ideal RxR such that $RxR \leq \Sigma_{x_j}$ since $I_0(R)=0$. By Lemma 3.2, we have $xR \leq x_iR$ for all i , which contradicts that $\{P_i\}_{i=1}^{\infty}$ is cofinal. Therefore we have an increasing sequence $i_1 < i_2 < \dots$, of positive integers such that $\Sigma_{x_{i_1}} \geq \Sigma_{x_{i_2}} \geq \dots$. Then $P_{i_n} \cong x_{i_n}R \geq \Sigma_0(x_{i_{n+1}}R) \cong P_{i_{n+1}}$ by Lemma 3.2. Thus above (B) holds and hence P is d.f..

Theorem 3.5. *The following are equivalent:*

- (a) *R is type B.*
- (b) *$Soc(R)=0$, $I_0(R)=0$ and $L(R)$ has a cofinal subfamily.*
- (c) *There exists a non-finitely generated d.f. projective R -module P with a cyclic decomposition $P=\bigoplus_{i=1}^{\infty} P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal and $[P_1] \neq [P_2] \neq \dots$*

Proof. (a) \rightarrow (b). Assume that R is Type B. Then it must hold that $Soc(R)=0$. We have a countably generated d.f. projective R -module P with a cyclic decomposition $P=\bigoplus_{i=1}^{\infty} P_i$ satisfying (*) and (B). Let $P_i \cong x_i R$ for $x_i \in R$. Then $\bigcap_{i=1}^{\infty} \Sigma_{x_i} = 0$ and $\{\Sigma_{x_i}\}_{i=1}^{\infty}$ is a cofinal subfamily of $L(R)$. (b) \rightarrow (c). From the assumption, we have a cofinal subfamily $\{I_i\}_{i=1}^{\infty}$ of $L(R)$ such that $I_1 \geq I_2 \geq \dots$. Take $x_i \in I_i - I_{i+1}$. Since $L(R)$ is a linearly ordered set under inclusion, we see that $I_i \geq \Sigma_{x_i} \geq I_{i+1}$; so $\Sigma_{x_1} \geq \Sigma_{x_2} \geq \dots$. Putting that $P=\bigoplus_{i=1}^{\infty} x_i R$, we see that $\{x_i R\}_{i=1}^{\infty}$ is cofinal and $[x_1 R] \neq [x_2 R] \neq \dots$; and hence P is d.f. from Proposition 3.4. (c) \rightarrow (a) is clear, since (B) follows from (c).

Theorem 3.6. *The following are equivalent:*

- (a) *R is Type C.*
- (b) *$Soc(R) \neq 0$, or $I_0(R) = 0$ and $L(R)$ does not have any cofinal subfamilies.*

Proof. This is immediate from Theorems 3.3 and 3.5.

REMARK. By theorems above, we see that Types A, B and C are right-left symmetric.

As an application we show the following

Theorem 3.7. *A projective R -module P is d.inf. if and only if there exists a nonzero R -module X such that $\aleph_0 X$ is isomorphic to a direct summand of P .*

Proof. “If” part is clear. “Only if” part. Let $P = \bigoplus_{\alpha \in I} P_\alpha$ be a d.inf. projective R -module where each P_α is nonzero cyclic. If $Soc(R) \neq 0$, then for any nonzero simple right ideal $X \leq Soc(R)$, clearly $|I|X \leq \bigoplus P = \bigoplus_{\alpha \in I} P_\alpha$, whence $\aleph_0 X \leq \bigoplus P$. So, we may consider the case $Soc(R) = 0$. If $|I| > \aleph_0$, by the proof of [4, Theorem 6], there exists $P_\beta \in \{P_\alpha\}_{\alpha \in I}$ such that $|\{P_\alpha \in \{P_\alpha\}_{\alpha \in I} \mid P_\beta \leq P_\alpha\}| \geq \aleph_0$; so $\aleph_0 P_\beta \leq \bigoplus P$. Hence we may further assume that $|I| = \aleph_0$, so say $P = \bigoplus_{i=1}^{\infty} P_i$. If $\{P_i\}_{i=1}^{\infty}$ is not cofinal, then clearly there exists a desired X . Hence assume that $\{P_i\}_{i=1}^{\infty}$ is cofinal. Since P is d.inf., we see from Proposition 3.4 that $I_0(R) \neq 0$. Noting that P is d.inf., together with Theorem 3.3, we see that $[P_m] = [P_{m+1}] = \dots$ for positive integer m and $tP_m \not\leq P_{m+1} \oplus P_{m+2} \oplus \dots$ for all t . Then there exists an ascending chain $m = m_1 < m_2 < \dots$, of positive integers such that $P_m \leq P_{m_{i+1}} \oplus \dots \oplus P_{m_{i+1}}$ for $i = 1, 2, \dots$, and so $\aleph_0 P_m \leq \bigoplus P_{m_1+1} \oplus P_{m_1+2} \oplus \dots < \bigoplus P$ as desired.

Finally we give an example of Type A which has infinitely many ideals.

EXAMPLE (cf. [2, p. 486–p. 489]). Choose a field F and set $R_0 = F$. For each positive integer n , let R_n be the ring of all $\aleph_0 \times \aleph_0$ matrices over R_{n-1} of the form

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} & 0 \\ \vdots & & \vdots & \\ x_{n1} & \cdots & x_{nn} & a \\ 0 & & & a \\ & & & \ddots \end{pmatrix}$$

where $x_{ij} \in R_{n-1}$ and $a \in F$, and put $\alpha_n = \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \ddots \end{pmatrix} \in R_n$, where 1_{n-1} is the

identity element in R_{n-1} . We define a ring homomorphism $p_n: R_n \rightarrow F$ by the rule $p_n(x) = a$ for x above, and define a ring homomorphism $f_{n+1, n}: R_n \rightarrow R_{n+1}$ by

the rule

$$f_{n+1,n}(y) = \begin{pmatrix} y & & 0 \\ & p_n(y) & \\ 0 & & p_n(y) \\ & & \ddots \end{pmatrix}$$

for all $y \in R_n$. Then each R_n is a non-simple unit-regular ring satisfying the c. axiom. Put $R = \varinjlim R_n$ and let $\phi_n: R_n \rightarrow R$ be the canonical map. Then we see that R is a non-simple unit-regular ring satisfying the c. axiom with a nonzero socle of R . Now set $S_n = M_{2^n}(R)$ for $n = 1, 2, \dots$. Map each $R_n \rightarrow R_{n+1}$ along the diagonal, i.e., map $x \rightarrow \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, and set $S = \varinjlim S_n$, and let $\psi_n: S_n \rightarrow S$ be the canonical map. Then S is a non-simple d.f. regular ring satisfying the c. axiom which is Type A and has an ascending chain $S\psi_1(\phi_1(\alpha_1))S \leq S\psi_1(\phi_2(\alpha_2))S \leq \dots$ of ideals of S .

Unfortunately we do not have any examples of d.f. regular rings R satisfying the c. axiom such that $I_0(R) = 0$; so we do not have any examples of Type B and non-trivial Type C.

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