Kutami, M. and Oshiro, K.
Osaka J. Math.
24 (1987), 465–473

ON PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATISFYING THE COMPARABILITY AXIOM II

Dedicated to Professor Hisao Tominaga on his 60th birthday

MAMORU KUTAMI AND KIYOICHI OSHIRO

(Received April 7, 1986)

In [4], by observing the directly finiteness of projective modules, the first author classified directly finite (d.f. for short) regular rings satisfying the comparability axiom (c. axiom for short) into three types: Type A, Type B and Type C.

In the present paper, we give a more explicite criterion of the directly finiteness of projective modules over each types and show the following for a d.f. regular ring R satisfying the c. axiom: (a) R is Type A if and only if $Soc(R)=0$ and the intersection $I_0(R)$ of all nonzero ideals of R is nonzero. (b) R is Type B if and only if $Soc(R)=0$, $I_0(R)=0$ and the family $L(R)$ of all ideals of R has a cofinal subfamily. (c) R is Type C if and only if $Soc(R)\neq 0$, or $I_0(R)=0$ and $L(R)$ does not have any cofinal subfamilies. As an application we show the following for a projective module P over a d.f. regular ring satisfying the c. axiom: P is directly infinite (d.inf. for short) if and only if P contains a direct summand which is isomorphic to $\aleph_0 X$ for a suitable nonzero module X.

Throughout this paper we assume that R is a d.f. regular ring satisfying the c. axiom, and all R-modules considered are unital right R-modules.

1. Notations and definitions

For two R-modules X and Y, we use $X\lesssim Y$ (resp. $X\lesssim\oplus Y$) to mean that X is isomorphic to a submodule of Y (resp. a direct summand of Y). $X\underset{\neq}{\lesssim} Y$ means that $X\lesssim Y$ and $X\not\cong Y$. For a submodule X of an R-module Y, $X<\oplus Y$ means that X is a direct summand of Y. For a cardinal number α and an R-module X, αX denotes a direct sum of α-copies of X. For a set I, we denote by $|I|$ the cardinal number of I. We denote by $L(R)$ the family of all ideals of R. Since R satisfies the c. axiom, $L(R)$ is a linearly ordered set under inclusion ([1, Proposition 8.5]). We put $I_0(R)=\cap\{I\mid 0\neq I\in L(R)\}$. We denote by $Soc(R)$ the socle of R. We note that if $Soc(R)\neq 0$ then it is homogeneous and coincides with $I_0(R)$.

The reader is reffered to K.R. Goodearl [1] for the following elementary properties on R;

i) Every finitely generated projective R-module is d.f..

ii) For any finitely generated projective R-modules P and Q, either $P\lesssim Q$ or $Q\lesssim P$ holds.

iii) For any finitely generated projective R-module P and any R-modules X and Y, $P\oplus X\cong P\oplus Y$ implies $X\cong Y$.

iv) For any projective R-module X and any finitely generated projective R-modules $Y_1, Y_2, \cdots$ such that $Y_1\oplus\cdots\oplus Y_n\lesssim X$ for all n, we have that $\oplus Y_n\lesssim X$. In particular, for finitely generated projective R-modules P and Q, if $P\not\lesssim nQ$ for all n, then $\aleph_0 Q\lesssim P$.

v) R has a unique *dimension function* D, namely, D is a function from the family of all cyclic right ideals of R to [0, 1] such that

a) $D(R)=1$,

b) if $J\lesssim K$, then $D(J)\leqq D(K)$ and

c) if $J\oplus K$ is a cyclic right ideal of R, then $D(J\oplus K)=D(J)+D(K)$. D is said to be *strictly positive* if $D(J)>0$ for all nonzero cyclic right ideals J of R. We note that D is strictly positive dimension function if and only if R is a simple ring, and that R is not simple if and only if there exists a nonzero r in R such that $\aleph_0(rR)\lesssim R$. Furthermore we note that $\{r\in R\,|\,D(rR)=0\}$ is the unique maximal ideal of R.

Let $\{P_i\}_{i=1}^{\infty}$ be a subfamily of the family of all cyclic projective R-modules. We say that $\{P_i\}_{i=1}^{\infty}$ is a *cofinal* subfamily if all P_i are nonzero, $P_1\gtrsim P_2\gtrsim\cdots$ and for any nonzero cyclic projective R-module X there exists a positive integer n satisfying $X\gtrsim P_n$. Similarly a subfamily $\{I_i\}_{i=1}^{\infty}$ of $L(R)$ is said to be a *cofinal* subfamily of $L(R)$ if all I_i are nonzero, $I_1\geqq I_2\geqq\cdots$ and for any nonzero X in $L(R)$ there exists a positive integer n satisfying $X\gtrsim I_n$.

2. Directly finiteness and directly infiniteness

We start the following

Theorem 2.1. (*a*) *For countably generated projective R-modules P and Q, either $P\lesssim Q$ or $Q\lesssim P$ holds.*

(*b*) *If P and Q are countably generated projective R-modules such that $P\lesssim Q$ and $Q\lesssim P$, then $P\cong Q$.*

Proof. We show the theorem by modifying the proof of [3, Lemma 2.5]. Let $P=\oplus_{i=1}^{\infty}P_i$ and $Q=\oplus_{i=1}^{\infty}Q_i$ be cyclic decompositions of P and Q. (a) Assume $P\not\lesssim Q$. Then there exists a positive integer n such that $P_1\oplus\cdots\oplus P_n\not\lesssim Q$ and so $Q_1\oplus\cdots\oplus Q_m\lesssim P_1\oplus\cdots\oplus P_n$ for all m. Therefore $Q\lesssim P_1\oplus\cdots\oplus P_n<\oplus P$. (b) It is sufficient to assume that P and Q are non-finitely generated projective and that $Q\not\lesssim P_1\oplus\cdots\oplus P_n$ for all n from the assumption. Since $Q\lesssim P$, there exists a positive integer n_1 such that $Q_1 \lesssim P_1 \oplus \cdots \oplus P_{n_1}$ and $Q_1 \not\lesssim P_1 \oplus \cdots \oplus P_{n_1-1}$, and there exists a positive integer m_1 such that $Q_1 \oplus \cdots \oplus Q_{m_1} \lesssim P_1 \oplus \cdots \oplus P_{n_1}$ and $Q_1 \oplus \cdots \oplus Q_{m_1+1} \not\lesssim P_1 \oplus \cdots \oplus P_{n_1}$. Then we have a decomposition $Q_{m_1+1} = X_{m_1+1} \oplus Y_{m_1+1}$ such that $Y_{m_1+1} \neq 0$ and

$$P_1 \oplus \cdots \oplus P_{n_1} \cong Q_1 \oplus \cdots \oplus Q_{m_1} \oplus X_{m_1+1}\,.$$

Since $Q \lesssim P$, $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \lesssim P_{n_1+1} \oplus P_{n_1+2} \oplus \cdots$. Then there exists a positive integer $n_2\,(>n_1)$ such that $Y_{m_1+1} \lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2}$ and $Y_{m_1+1} \not\lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2-1}$. So, there exists a positive integer $m_2(>m_1)$ such that $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2} \lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2}$ and $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2+1} \not\lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2}$. Then we have a decomposition $Q_{m_2+1} = X_{m_2+1} \oplus Y_{m_2+1}$ such that $Y_{m_2+1} \neq 0$ and

$$P_{n_1+1} \oplus \cdots \oplus P_{n_2} \cong Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2} \oplus X_{m_2+1}\,.$$

Continuing this procedure, we have that $P \cong Q$.

REMARK. When R is simple, we can drop the assumption 'countably generated' from above Theorem 2.1 (see [3, Theorems 2.4 and 2.6]). But the assumption can not be removed in general. For, if R is a non-simple d.f. regular ring satisfying the c. axiom, then there exists a nonzero r in R such that $\aleph_0(rR) \lesssim R$. So, $R \not\lesssim \alpha(rR)$ and $\alpha(rR) \not\lesssim R$, where $|R| < \alpha$. Next, if we take R as in [1, Example 5.15], then there exists a simple right ideal S of R such that $\aleph S \lesssim R$. Then $\aleph_0 R \lesssim \aleph S \oplus \aleph_0 R$ and $\aleph S \oplus \aleph_0 R \lesssim \aleph_0 R$, but $\aleph_0 R \not\cong \aleph S \oplus \aleph_0 R$.

Corollary 2.2. *Let P and Q be countably generated projective R-modules and let n be a positive integer.*

(*a*) *If $nP \cong nQ$, then $P \cong Q$.*

(*b*) *If $nP \lesssim nQ$, then $P \lesssim Q$.*

Proof. (a) We prove the statement by induction on n. So, assume that this holds for n and $(n+1)P \cong (n+1)Q$. Then we have decompositions $nP = X_1 \oplus X_2$ and $P = Y_1 \oplus Y_2$ such that $X_1 \oplus Y_1 \cong nQ$ and $X_2 \oplus Y_2 \cong Q$. By Theorem 2.1 (a), either $Y_1 \lesssim X_2$ or $X_2 \lesssim Y_1$ holds. If $Y_1 \lesssim X_2$, then $nQ \cong X_1 \oplus Y_1 \lesssim X_1 \oplus X_2 = nP$ and $P = Y_1 \oplus Y_2 \lesssim X_2 \oplus Y_2 \cong Q$; so $nQ \lesssim nP$ and $nP \lesssim nQ$. Hence the induction hypothesis says that $P \cong Q$. If $X_2 \lesssim Y_1$, similarly, we have that $P \cong Q$. (b) Assume that $nP \lesssim nQ$ and $P \not\lesssim Q$. Then $Q \lesssim P$, and so $nQ \lesssim nP$. Therefore $nP \cong nQ$ by Theorem 2.1 (b); whence $P \cong Q$ by (a), a contradiction.

Now, for our purpose we define a relation "$\sim$" on the family of all cyclic projective R-modules $CP(R)$ by the rule: For any P and Q in $CP(R)$, $P \sim Q$ if and only if $P \lesssim mQ$ and $Q \lesssim nP$ for some positive integers m and n. For P in $CP(R)$ we put $[P] = \{Q \in CP(QR) \mid Q \sim P\}$. Then the relation "$\sim$" is a congruence relation.

Proposition 2.3. *Let P be a non-finitely generated, countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P_1 \gtrsim P_2 \gtrsim \cdots$ and $[P_1]=[P_2]=\cdots$. If $\{P_i\}_{i=1}^{\infty}$ is cofinal, then P is d.inf. if and only if $P \cong \aleph_0 P_i$ for all i.*

Proof. "If" part is clear. "Only if" part. Let P be a d.inf. projective R-module. From [4, Theorem 6], $tP_i \not\gtrsim P_{i+1} \oplus P_{i+2} \oplus \cdots$ for all t and hence $\aleph_0 P_i \lesssim P_{i+1} \oplus P_{i+2} \oplus \cdots \leqq P$. Since $P \lesssim \aleph_0 P_i$, it follows from Theorem 2.1 that $P \cong \aleph_0 P_i$ for all i.

REMARK. Let P be a non-finitely generated, countably generated projective R-module with a cyclic decomposition $P=\oplus_{i \in I}P_i$ such that $|I|=\aleph_0$ and $[P_i]=[P_j]$ for any $i, j \in I$. If there exists $i \in I$ such that $|\{j \in I \mid P_i \lesssim P_j\}|=\aleph_0$, then $P \cong \aleph_0 P_i$ for all $i \in I$ by Theorem 2.1.

Proposition 2.4. *Let P be a non-countably generated (d.inf.) projective R-module with a cyclic decomposition $P=\oplus_{\alpha \in I}P_\alpha$ such that $[P_\alpha]=[P_\beta]$ for any α, $\beta \in I$. Then there exists an infinite cardihal τ $(>\aleph_0)$ such that $P \cong \tau P_\gamma$ for any $\gamma \in I$.*

Proof (cf. the proof of [3, Theorem 2.6]). Let B be the set of all countably infinite subsets of I, and let $\gamma \in I$. We consider the family consisting of all subsets F of B satisfying the following properties:
(1) each member of F is pairwise disjoint, and
(2) for each member I' of F, $P_{I'}=\oplus_{\alpha \in I'}P_\alpha \cong \aleph_0 P_\gamma$.
Then this family is non-empty set from the proof of [4, Theorem 6] and Remark of Proposition 2.3. Since this family is inductively ordered set under inclusion, there exists a maximal member F by Zorn's Lemma. Put $I^*=\cup_{K \in F}K$. If $I^*=I$, then our proof is complete. Next, consider the case that $I^* \neq I$. Let I^{**} be a complement of I^* in I. From the proof of [4, Theorem 6], Remark of Proposition 2.3 and the maximality of F, I^{**} is a countable set. Choose one member K' of F and put $F'=F-\{K'\}$ and $K''=K' \cup I^{**}$. Then K'' is a countably infinite set and $P_{K''} \cong \aleph_0 P_\gamma$ since $[P_\alpha]=[P_\beta]$ for any $\alpha, \beta \in I$. Therefore $P=(\oplus_{K \in F'}(\oplus_{\alpha \in K}P_\alpha)) \oplus (\oplus_{\alpha \in K''}P_\alpha) \cong \tau P_\gamma$ for some infinite cardinal $\tau > \aleph_0$.

Corollary 2.5 ([3, Theorem 2.6]). *Assume that R is simple. Then every d.inf. projective R-module is a free R-module.*

Proof. Let P be a d.inf. projective R-module with a cyclic decomposition $P=\oplus_{\alpha \in I}P_\alpha$ and we consider $R \oplus P = R \oplus (\oplus_{\alpha \in I}P_\alpha)$. Noting that R is simple, we see that $[X]=[R]$ for a nonzero cyclic projective R-module X; whence $[P_\alpha]=[R]$ for all $\alpha \in I$. By Proposition 2.3, its Remark and Proposition 2.4, $R \oplus P \cong \tau R$ for some infinite cardinal τ. Therefore $P \cong \tau R$ by the cancellation property of R.

Theorem 2.6. *Let P and Q be d.inf. projective R-modules with cyclic decom-*

positions $P=\oplus_{\alpha\in I}P_\alpha$ *and* $Q=\oplus_{\beta\in J}Q_\beta$ *such that* $[P_\alpha]=[P_{\alpha'}]$ *and* $[Q_\beta]=[Q_{\beta'}]$ *for any* α, $\alpha'\in I$ *and* β, $\beta'\in J$. *If* $P\lesssim Q$ *and* $Q\lesssim P$, *then* $P\cong Q$.

Proof. Since $P\lesssim Q$ and $Q\lesssim P$, we note that $[P_\alpha]=[Q_\beta]$ for any P_α and Q_β. If $|I|\leqq\aleph_0$ and $|J|\leqq\aleph_0$, then $P\cong Q$ from Theorem 2.1. Therefore we may consider the following cases:

1) $|I|\leqq\aleph_0$ and $|J|>\aleph_0$.

2) $|I|>\aleph_0$ and $|J|>\aleph_0$.

In order to prove for these cases, we show the following for any nonzero cyclic projective R-module T and cardinal numbers σ and ρ:

(♯) If $\rho T\lesssim\sigma T$, then $\rho\leqq\sigma$.

Let τ be a cardinal number. We regard τ an initial ordinal; so $|\{\text{ordinal }\alpha\,|\,\alpha<\tau\}|=\tau$. Put $\Lambda(\tau)=\{\text{ordinal }\alpha\,|\,\alpha<\tau\}$. We shall prove (♯) by the transfinite induction on σ. First assume that $\sigma=\aleph_0$ and let f be a monomorphism from ρT to $\sigma T=\aleph_0 T$. Putting $\Gamma_m=\{\alpha\in\Lambda(\rho)\,|\,f(T_\alpha)\leqq\oplus_{i=1}^m T_i\}$, we see that $f(\oplus_{\alpha\in\Gamma_m}T_\alpha)\leqq\oplus_{i=1}^m T_i$, $|\Gamma_m|\leqq m$ and $\cup_m\Gamma_m=\Lambda(\rho)$. Therefore $\rho=|\Lambda(\rho)|=|\cup_m\Gamma_m|\leqq\aleph_0=\sigma$. Next assume that (♯) holds for any cardinal number $\sigma'<\sigma$, and let f be a monomorphism from ρT to σT. For any x in $\Lambda(\sigma)$, put $\Gamma_x=\{\alpha\in\Lambda(\rho)\,|\,f(T_\alpha)\leqq\oplus_{\beta\leqq x}T_\beta\}$. Then $f(\oplus_{\alpha\in\Gamma_x}T_\alpha)\leqq\oplus_{\beta\leqq x}T_\beta$, $\cup_{x\in\Lambda(\sigma)}\Gamma_x=\Lambda(\rho)$ and $|\{\beta\,|\,\beta\leqq x\}|<\sigma$ because σ is an initial ordinal. From the induction hypothesis, $|\Gamma_x|<\sigma$. Therefore we see that $\rho=|\Lambda(\rho)|=|\cup_{x\in\Lambda(\sigma)}\Gamma_x|\leqq\sigma^2=\sigma$ as desired.

Case 1) Let $P_\beta\in\{P_\alpha\}_{\alpha\in I}$. Since $|J|>\aleph_0$ and $[P_\alpha]=[Q_\beta]$ for all Q_β, $Q\cong\tau P_\alpha$ for a suitable cardinal number τ. Since $P\lesssim Q\cong\tau P_\alpha$ and $Q\lesssim P$, we see that $\aleph_0 P_\alpha\lesssim P$ and $P\lesssim\aleph_0 P_\alpha$, whence $\aleph_0 P_\alpha\cong P$ by Theorem 2.1. As $\tau P_\alpha\cong Q\lesssim\aleph_0 P_\alpha$, $\tau\leqq\aleph_0$ by (♯), a contradiction.

Case 2) By Proposition 2.4 and (♯), we immediately have that $P\cong Q$.

Corollary 2.7 ([3, Proposition 2.7]). *Assume that* R *is simple. If* P *and* Q *are d.inf. projective* R*-modules such that* $P\lesssim Q$ *and* $Q\lesssim P$, *then* $P\cong Q$.

3. Types A, B and C

In [4] we showed the following result, which already used in Proposition 2.3: A non-finitely generated projective R-module P is d.f. if and only if P is countably generated with a cyclic decomposition $P=\oplus_{i=1}^\infty P_i$ satisfying the conditions (*) and (A), or (*) and (B) below:

(*) $P_i\gtrsim P_{i+1}$ for all i, and there exists no nonzero R-module X such that $X\lesssim P_i$ for all i.

(A) There exists a positive integer m such that

(1) For each $i\geqq m$, $P_i\lesssim t_iP_{i+1}$ for some positive integer t_i, and

(2) $\oplus_{i=m}^\infty P_i\lesssim tP_m$ for some positive integer t.

(B) There exists an increasing sequence $1=i_1<i_2<\cdots$, of positive integers such that $P_{i_n}\gtrsim\aleph_0 P_{i_{n+1}}$ for $n=1, 2, \cdots$.

And, from this result, we classified d.f. regular rings R satisfying the c. axiom into three types:

Type A: There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (A).

Type B: There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (B).

Type C: All d.f. projective R-modules are finitely generated.

Remark. If a ring R is Type A (resp. Type B), then all non-finitely generated d.f. projective R-module P have a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (A) (resp. (*) and (B)) by [4, Theorem 6 and Remark 2]. We note that (*) holds then $Soc(R)=0$. So, if $Soc(R)\neq 0$ then R is type C.

In this section, as is mentioned in the introduction, we shall give ideal theoritic characterizations of each types.

Lemma 3.1 ([1, Corollary 2.23]). *Let H and J be right ideals of R, and assume that H is finitely generated. Then $H\leqq RJ$ if and only if $H\lesssim nJ$ for some positive integer n.*

For an element a of a ring R, we put

$$\Sigma_a = \Sigma\{xR\,|\,x\in R \text{ and } xR\lesssim aR\}\ .$$

Lemma 3.2. (*a*) *For each $a\in R$, Σ_a is the smallest ideal of R containing a, and hence $\Sigma_a=RaR$.*

(*b*) *For each $a, b\in R$, $\Sigma_a\leqq\Sigma_b$ if and only if $aR\lesssim n(bR)$ for some positive integer n.*

(*c*) *For $a, b\in R$, $\Sigma_a\lneqq\Sigma_b$ if and only if $\aleph_0(aR)\lesssim bR$.*

Proof. (a) Let $r\in R$ and $\sum_{i=1}^{n}x_ir_i\in\Sigma_a$ such that $r_i\in R$ and $x_iR\lesssim aR$ for each i. Then $(rx_ir_i)R\lesssim\oplus(x_ir_i)R\leqq x_iR\lesssim aR$ and $rx_ir_i\in\Sigma_a$ for each i, and so $r(\sum_{i=1}^{n}x_ir_i)\in\Sigma_a$. Thus Σ_a is an ideal of R containing a. Let I be an ideal of R containing a. If $xR\lesssim aR$ and $x\in R$, then $xR\leqq RaR\leqq I$ from Lemma 3.1. Therefore $\Sigma_a\leqq I$ and hence $\Sigma_a=RaR$. (b) is clear from (a) and Lemma 3.1, and (c) follows from (b).

Theorem 3.3. *The following are equivalent*:

(*a*) *R is Type A.*

(*b*) *$Soc(R)=0$ and $I_0(R)\neq 0$.*

(*c*) *There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal, $[P_1]=[P_2]=\cdots$ and $tP_1\gtrsim P_2\oplus P_3\oplus\cdots$ for some positive integer t.*

Proof. (a)$\rightarrow$(b). Assume that Type A. Then of course $Soc(R)=0$. Now assume that $I_0(R)=0$. Since R is Type A, we have a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ which satisfies (*) and $[P_m]=[P_{m+1}]=\cdots$ for some positive integer m. Let $P_i\cong x_iR$ for some $x_i\in R$. Noting that $I_0(R)=0$, we have a nonzero ideal X of R such that $X\lneqq\Sigma_{x_m}=\Sigma_{x_{m+1}}=\cdots$, which contradicts that $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*). Therefore we see that $I_0(R)\neq 0$. (b)$\to$(c). Take a nonzero element x_1 in $I_0(R)$; then $\Sigma_{x_1}=I_0(R)$. Since $Soc(R)=0$, there exist nonzero cyclic right ideals $x_{i+1}R$ and $y_{i+1}R$ of R such that $x_iR=x_{i+1}R\oplus y_{i+1}R$ and $x_{i+1}R\lesssim y_{i+1}R$ for each i; so $2(x_{i+1}R)\lesssim x_iR$. Put $P=\oplus_{i=1}^{\infty}x_iR$. If there exists a nonzero element y of R such that $yR\lesssim x_iR$ for all i, then $\Sigma_{x_1}=\Sigma_{x_i}=\Sigma_y$ for all i by the smallity of Σ_{x_1}. Hence there exist positive integers t and m such that $2t(x_mR)\lesssim x_1R\lesssim t(yR)\lesssim t(x_mR)$ from Lemma 3.2; whence $2t(x_mR)\lesssim t(x_mR)$ which contradicts the directly finiteness of $t(x_mR)$. Therefore $\{x_iR\}_{i=1}^{\infty}$ is cofini cofinal. By the smallity of Σ_{x_1}, we see that $[x_1R]=[x_iR]$ for all i and $\oplus_{i=1}^{\infty}x_iR\lesssim 2(x_1R)$, and hence $\oplus_{i=1}^{\infty}x_iR$ is d.f.. (c)$\to$(a) is clear.

Proposition 3.4. *Assume that $Soc(R)=0$ and $I_0(R)=0$. Then a non-finitely generated projective R-module P is d.f. (if and only) if P has a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal.*

Proof. Assume that P is a countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal. We express each P_i as $P_i\cong x_iR$, where $x_i\in R$. Then $\Sigma_{x_1}\geqq\Sigma_{x_2}\geqq\cdots$. If there exists a positive integer j such that $\Sigma_{x_j}=\Sigma_{x_{j+1}}=\cdots$, we have a nonzero ideal RxR such that $RxR\lneqq\Sigma_{x_j}$ since $I_0(R)=0$. By Lemma 3.2, we have $xR\lesssim x_iR$ for all i, which contradicts that $\{P_i\}_{i=1}^{\infty}$ is cofinal. Therefore we have an increasing sequence $i_1<i_2<\cdots$, of positive integers such that $\Sigma_{x_{i_1}}\gneqq\Sigma_{x_{i_2}}\gneqq\cdots$. Then $P_{i_n}\cong x_{i_n}R\gtrsim\aleph_0(x_{i_{n+1}}R)\cong P_{i_{n+1}}$ by Lemma 3.2. Thus above (B) holds and hence P is d.f..

Theorem 3.5. *The following are equivalent:*

(*a*) *R is type B.*

(*b*) *$Soc(R)=0$, $I_0(R)=0$ and $L(R)$ has a cofinal subfamily.*

(*c*) *There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal and $[P_1]\neq[P_2]\neq\cdots$.*

Proof. (a)$\to$(b). Assume that R is Type B. Then it must hold that $Soc(R)=0$. We have a countably generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (B). Let $P_i\cong x_iR$ for $x_i\in R$. Then $\cap_{i=1}^{\infty}\Sigma_{x_i}=0$ and $\{\Sigma_{x_i}\}_{i=1}^{\infty}$ is a cofinal subfamily of $L(R)$. (b)$\to$(c). From the assumption, we have a cofinal subfamily $\{I_i\}_{i=1}^{\infty}$ of $L(R)$ such that $I_1\gneqq I_2\gneqq\cdots$. Take $x_i\in I_i-I_{i+1}$. Since $L(R)$ is a linearly ordered set under inclusion, we see that $I_i\geqq\Sigma_{x_i}\gneqq I_{i+1}$; so $\Sigma_{x_1}\gneqq\Sigma_{x_2}\gneqq\cdots$. Putting that $P=\oplus_{i=1}^{\infty}x_iR$, we see that $\{x_iR\}_{i=1}^{\infty}$ is cofinal and $[x_1R]\neq[x_2R]\neq\cdots$; and hence P is d.f. from Proposition 3.4. (c)$\to$(a) is clear, since (B) follows from (c).

Theorem 3.6. *The following are equivalent:*
(*a*) *R is Type C.*
(*b*) *$Soc(R) \neq 0$, or $I_0(R)=0$ and $L(R)$ does not have any cofinal subfamilies.*

Proof. This is immediate from Theorems 3.3 and 3.5.

REMARK. By theorems above, we see that Types A, B and C are right-left symmetric.

As an application we show the following

Theorem 3.7. *A projective R-module P is d.inf. if and only if there exists a nonzero R-module X such that $\aleph_0 X$ is isomorphic to a direct summand of P.*

Proof. "If" part is clear. "Only if" part. Let $P=\oplus_{\alpha\in I}P_\alpha$ be a d.inf. projective R-module where each P_α is nonzero cyclic. If $Soc(R)\neq 0$, then for any nonzero simple right ideal $X\leqq Soc(R)$, clearly $|I|X\lesssim\oplus P=\oplus_{\alpha\in I}P_\alpha$, whence $\aleph_0 X\lesssim\oplus P$. So, we may consider the case $Soc(R)=0$. If $|I|>\aleph_0$, by the proof of [4, Theorem 6], there exists $P_\beta\in\{P_\alpha\}_{\alpha\in I}$ such that $|\{P_\alpha\in\{P_\alpha\}_{\alpha\in I}|P_\beta\lesssim P_\alpha\}|$ $\geqq\aleph_0$; so $\aleph_0 P_\beta\lesssim\oplus P$. Hence we may further assume that $|I|=\aleph_0$, so say $P=\oplus_{i=1}^{\infty}P_i$. If $\{P_i\}_{i=i}^{\infty}$ is not cofinal, then clearly there exists a desired X. Hence assume that $\{P_i\}_{i=1}^{\infty}$ is cofinal. Since P is d.inf., we see from Proposition 3.4 that $I_0(R)\neq 0$. Noting that P is d.inf., together with Theorem 3.3, we see that $[P_m]=[P_{m+1}]=\cdots$ for positive integer m and $tP_m\not\gtrsim P_{m+1}\oplus P_{m+2}\oplus\cdots$ for all t. Then there exists an ascending chain $m=m_1<m_2<\cdots$, of positive integers such that $P_m\lesssim\oplus P_{m_i+1}\oplus\cdots\oplus P_{m_{i+1}}$ for $i=1, 2, \cdots$, and so $\aleph_0 P_m\lesssim\oplus P_{m_1+1}\oplus P_{m_1+2}\oplus\cdots<\oplus P$ as desired.

Finally we give an example of Type A which has infinitely many ideals.

EXAMPLE (cf. [2, p. 486–p. 489]). Choose a field F and set $R_0=F$. For each positive integer n, let R_n be the ring of all $\aleph_0\times\aleph_0$ matrices over R_{n-1} of the form

$$x=\begin{pmatrix} x_{11} & \cdots & x_{1n} & & & 0 \\ \vdots & & \vdots & & & \\ x_{n1} & \cdots & x_{nn} & & & \\ & & & a & & \\ & & & & a & \\ 0 & & & & & \ddots \end{pmatrix}$$

,where $x_{ij}\in R_{n-1}$ and $a\in F$, and put $\alpha_n=\begin{pmatrix} 1_{n-1} & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}\in R_n$, where 1_{n-1} is the identity element in R_{n-1}. We define a ring homomorphism $p_n\colon R_n\to F$ by the rule $p_n(x)=a$ for x above, and define a ring homomorphism $f_{n+1,n}\colon R_n\to R_{n+1}$ by the rule

$$f_{n+1,n}(y)=\begin{bmatrix} y & & & 0 \\ & p_n(y) & & \\ & & p_n(y) & \\ 0 & & & \ddots \end{bmatrix}$$

for all $y\in R_n$. Then each R_n is a non-simple unit-regular ring satisfying the c. axiom. Put $R=\varinjlim_{n\geqq 1} R_n$ and let ϕ_n: $R_n\to R$ be the canonical map. Then we see that R is a non-simple unit-regular ring satisfying the c. axiom with a nonzero socle of R. Now set $S_n=M_{2^n}(R)$ for $n=1, 2, \cdots$. Map each $R_n\to R_{n+1}$ along the diagonal, i.e., map $x\to\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, and set $S=\varinjlim S_n$, and let ψ_n: $S_n\to S$ be the canonical map. Then S is a non-simple d.f. regular ring satisfying the c. axiom which is Type A and has an ascending chain $S\psi_1(\phi_1(\alpha_1))S\subsetneqq S\psi_1(\phi_2(\alpha_2))S\subsetneqq\cdots$ of ideals of S.

Unfortunately we do not have any examples of d.f. regular rings R satisfying the c. axiom such that $I_0(R)=0$; so we do not have any examples of Type B and non-trivial Type C.

References

[1] K.R. Goodearl: Von Neumann regular rings, Pitman, London-San Francisco-Melbourne, 1979.

[2] K.R. Goodearl: *Artinian and noetherian modules over regular rings*, Comm. Algebra **8**(5) (1980), 477–504.

[3] J. Kado: *Projective modules over simple regular rings*, Osaka J. Math. **16** (1979), 405–412.

[4] M. Kutami: *On projective modules over directly finite regular rings satisfying the comparability axiom*, Osaka J. Math. **22** (1985), 671–677.

Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi 753
Japan