Böhme, C. and Hirosawa, F.
Osaka J. Math.
**49** (2012), 297–323

# GENERALIZED ENERGY CONSERVATION FOR KLEIN–GORDON TYPE EQUATIONS

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(Received October 25, 2010)

**Abstract**

The aim of this paper is to derive energy estimates for solutions of the Cauchy problem for the Klein–Gordon type equation $u_{tt} - \triangle u + m(t)^2 u = 0$. The coefficient $m$ is given by $m(t)^2 = \lambda(t)^2 + p(t)$ with a decreasing, smooth shape function $\lambda$ and an oscillating, smooth and bounded perturbation function $p$. We study under which assumptions for $\lambda$ and $p$ one can expect results about a generalization of energy conservation. The main theorems of this note deal with $m$ belonging to $C^M$, $M \geq 2$, and $m$ belonging to the Gevrey class $\gamma^{(s)}$, $s \geq 1$.

## 1. Introduction

This note deals with the following Cauchy problem of Klein–Gordon type equation:

$$\begin{cases} (\partial_t^2 - \Delta + m(t)^2) u(t, x) = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ (u(0, x), (\partial_t u)(0, x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $m(t)$ is a positive function and $\Delta = \sum_{j=1}^n \partial_{x_j}^2$. Furthermore, we assume that $u_0 \in H^1$ and $u_1 \in L^2$. By partial Fourier transformation with respect to the space variable $x$ problem (1.1) is rewritten as

$$\begin{cases} (\partial_t^2 + \langle \xi \rangle_{m(t)}^2) v(t, \xi) = 0, & (t, \xi) \in (0, \infty) \times \mathbb{R}^n, \\ (v(0, \xi), (\partial_t v)(0, \xi)) = (v_0(\xi), v_1(\xi)), & \xi \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $\langle \xi \rangle_{m(t)} := \sqrt{|\xi|^2 + m(t)^2}$, $v(t, \xi) = \hat{u}(t, \xi)$ and $\hat{f}(\xi)$ denotes the partial Fourier transformation of $f(x)$ with respect to $x$.

Let us introduce an energy functional to the solution of (1.2) by

$$\mathcal{E}_0(v)(t, \xi) := \frac{1}{2}(|\partial_t v(t, \xi)|^2 + \langle \xi \rangle_{m(t)}^2 |v(t, \xi)|^2). \tag{1.3}$$

The aim of this paper is to derive some conditions for $m(t)$ in order to get some uniform estimates of $\mathcal{E}_0(v)(t, \xi)$ with respect to $(t, \xi)$ by the initial energy $\mathcal{E}_0(v)(0, \xi)$.

2010 Mathematics Subject Classification. 35L15, 35B40, 35B20.

If $m(t)$ is a constant, then we immediately have the property of energy conservation of microlocal version, that is

$$\mathcal{E}_0(v)(t,\xi) \equiv \mathcal{E}_0(v)(0,\xi) \tag{1.4}$$

for all time $t$. Moreover, we define an energy $E_0(u)$ to the solutions of (1.1) by

$$E_0(u)(t) := \frac{1}{2}(\|\partial_t u(t,\ \cdot)\|^2 + \|\nabla u(t,\ \cdot)\|^2 + m(t)^2\|u(t,\ \cdot)\|^2),$$

where $\nabla = (\partial_{x_1},\dots,\partial_{x_n})$ and $\|\cdot\|$ denotes the usual $L^2$ norm in $\mathbb{R}^n$. Then by Plancherel's equality the estimate (1.4) implies immediately the energy conservation in the usual meaning:

$$E_0(u)(t) \equiv E_0(u)(0)$$

for all time $t$.

However, such an identity cannot be expected in general for variable $m(t)$. Thus we introduce in a natural way the following partial estimates to (1.4), which is called *generalized energy conservation* (= GEC):

$$\mathcal{E}(v)(t,\xi) \simeq \mathcal{E}(v)(0,\xi) \tag{1.5}$$

for a suitable energy functional $\mathcal{E}(v)$ to the solution of (1.2).

Throughout the paper $f \simeq g$ denotes that the two positive functions $f$ and $g$ are uniformly equivalent. More precisely, we introduce the notations $f \lesssim g$ and $f \gtrsim g$ if there exist two positive constants $C_1$, $C_2$ such that two positive functions $f$ and $g$ uniformly satisfy $f \le C_1 g$ and $f \ge C_2 g$, respectively. Then $f \simeq g$ denotes that $f$ and $g$ satisfy both of the estimates $f \lesssim g$ and $f \gtrsim g$.

One of the trivial conditions of $m(t)$ for (1.5) is that $m'/m \in L^1$. Then we easily verify that (1.5) for $\mathcal{E}(v) = \mathcal{E}_0(v)$ holds true. On the other hand, (1.5) does not hold in general if $m'/m \notin L^1$; we will consider such a counter-example in Example 2.3.

However, one can expect a result about (1.5) even if $m'/m \notin L^1$ under some additional assumptions to $m$. Let us consider a function $\lambda \in C^2$ satisfying the following properties for any large time $t$:

$$\lambda(t) > 0, \quad \lambda'(t) \le 0, \quad |\lambda^{(k)}(t)| \le C_k \lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \tag{1.6}$$

for $k = 1, 2$, where $\Lambda(t) := \int_0^t \lambda(\tau)\,d\tau$. Moreover, let the functional $\mathcal{E}(v)$ be given by

$$\mathcal{E}(v)(t,\xi) = \frac{1}{\langle\xi\rangle_{\lambda(t)}}\mathcal{E}_0(v)(t,\xi). \tag{1.7}$$

Then it is proved in [1] the following result:

**Theorem 1.1** ([1]). *Let* $m(t) = \lambda(t)$, *where* $\lambda$ *satisfies* (1.6). *If*

$$\frac{\lambda(t)}{\Lambda(t)} = o(\lambda(t)) \tag{1.8}$$

*as* $t \to \infty$, *then* $\mathcal{E}(v)$ *satisfies* (1.5).

REMARK 1.1. We see from Theorem 1.1 that $\mathcal{E}(v)$ defined by (1.7) can be an appropriate energy of (1.2) for the generalized energy conservation (1.5). On the other hand, it is also proved in [1] that (1.5) does not hold in general if $\mathcal{E}(v) = \mathcal{E}_0(v)$.

Let us define $E(u)(t)$ by

$$E(u)(t) := \int_{\mathbb{R}^n} \mathcal{E}(v)(t, \xi)\, d\xi.$$

Then by Theorem 1.1 we immediately have the following corollary:

**Corollary 1.1.** *Let* $m(t) = \lambda(t)$. *Under the same assumptions to* $\lambda(t)$ *as in* Theorem 1.1 *we have*

$$E(u)(t) \simeq E(u)(0) \tag{1.9}$$

*and*

$$\lambda(t)E_0(u)(0) \lesssim E_0(u)(t) \lesssim E_0(u)(0).$$

EXAMPLE 1.1. Let $m(t) = \lambda(t) = (1+t)^{-\alpha}$ with $0 \le \alpha < 1$. Then the conditions to Theorem 1.1 are satisfied.

In contrast of (1.1), the weaker estimates $\lambda(t)^2 E_0(u)(0) \lesssim E_0(u)(t) \lesssim E_0(u)(0)$ is trivial by $E_0'(u)(t) = \lambda'(t)\lambda(t)\|u(\ ,\cdot)\|^2 \ge (\log\lambda(t)^2)' E_0(u)(t)$ and Gronwall's inequality. Generally, we can expect only the weaker estimate for $m(t) = \lambda(t) = (1+t)^{-\alpha}$ with $\alpha > 1$; for this reason it is called that $m(t) = (1+t)^{-\alpha}$ is an effective and non-effective mass for $0 \le \alpha < 1$ and $\alpha > 1$, respectively. Indeed, $\alpha = 1$ is the critical case and the situation is more complicate as follows.

We restrict ourselves to the scale-invariant Klein–Gordon equation (1.1) with $m(t) = m_0(1+t)^{-1}$, where $m_0$ is a positive constant. The decreasing function $m$ does not satisfy (1.8). Nevertheless, in [2] there is a result about energy estimates with respect to a non-standard energy $E_{m_0}(u)$ depending essentially on the constant $m_0$. Let $E_{m_0}(u)$ be given by

$$E_{m_0}(u)(t) := \frac{1}{2}(\|\partial_t u(t, \cdot)\|^2 + \|\nabla u(t, \cdot)\|^2 + \mu(t)^2\|u(t, \cdot)\|^2)$$

with

$$\mu(t)^2 := \begin{cases} (1+t)^{-1}, & m_0^2 > \frac{1}{4}, \\ (1+t)^{-1}(1+\ln(1+t))^{-2}, & m_0^2 = \frac{1}{4}, \\ (1+t)^{-1-\sqrt{1-4m_0^2}}, & m_0^2 \in \left(0, \frac{1}{4}\right). \end{cases}$$

The scale-invariance of the Klein–Gordon equation (1.1) with $m(t) = m_0(1+t)^{-1}$ allows us to carry out some transformations to a confluent hypergeometric equation which depends on the parameter $m_0$ as introduced in [4]. Exact solution representations imply the following estimates similar to those stated in Corollary 1.1.

**Theorem 1.2** ([2]). *Let $m(t) = m_0(1+t)^{-1}$. Then the solutions to* (1.1) *satisfy*

$$\mu(t)E_{m_0}(u)(0) \lesssim E_{m_0}(u)(t) \lesssim E_{m_0}(u)(0).$$

Briefly, our problem to investigate in this paper is a perturbation of Theorem 1.1 for non monotone decreasing mass $m(t)$. That is, we will represent $m(t)$ as

$$m(t) = \sqrt{\lambda(t)^2 + p(t)}, \tag{1.10}$$

where $\lambda(t)$ is a positive monotone decreasing shape function and $p(t)$ is the perturbation. Thus we suppose that

$$p(t) = o(\lambda(t)) \tag{1.11}$$

as $t \to \infty$, which provides that

$$m(t) \simeq \lambda(t). \tag{1.12}$$

Then we are interested in the interplay between the oscillations of $p(t)$ and the condition to $\lambda(t)$ corresponding to (1.6). Here we shall consider only the effective mass in order to avoid a delicate situation as in Theorem 1.2.

It is not the same equation of (1.1) but a different kind of Klein–Gordon type equations with time dependent oscillating coefficients which was considered in [7, 11]. Indeed, they studied some $L^p$-$L^q$ type estimates for the Klein–Gordon type equations

$$\partial_t^2 u - \lambda(t)^2 a(t)^2(\Delta - m^2(t))u = 0, \tag{1.13}$$

where $\lambda$ is a smooth, monotone increasing shape function, $a$ is a smooth function with oscillations and $m$ is a positive, decreasing function. We observe from these results that $L^p$-$L^q$ type estimates including $L^2$-$L^2$ type energy estimates are established under a suitable interplay between the properties of $\lambda$, $a$ and $m$. In the case that $\lambda(t) \equiv 1$ and $m \equiv 0$, that is, (1.13) is a wave equation with oscillating propagation speed, it was considered in [9] a sufficient condition to the oscillating speed of $a(t)$ for $L^p$-$L^q$ type estimates. Moreover, in [5, 6] the contributions of the smoothness properties of the coefficient $a \in C^M$ with $M \geq 2$ and $a \in \gamma^{(s)}$, where $\gamma^{(s)}$ is the Gevrey class defined in the next section, were studied for the $L^2$-$L^2$ type estimate $E_0(u)(t) \simeq E(u)(0)$ with $m \equiv 0$. This is called the generalized energy conservation.

There are also relations between Klein–Gordon equations (1.1) and damped wave equations with time dependent dissipation

$$\partial_t^2 u - \Delta u + b(t)u_t = 0. \tag{1.14}$$

In [12] there were studied some precise energy estimates for scale-invariant problem (1.14) with $b(t) = b_0(1+t)^{-1}$ taking into account a dependence on the constant $b_0$ by the application of properties of Bessel functions. Moreover, in [8, 13] more general functions $b(t)$ including some oscillations as perturbation problems of [12] were investigated.

## 2. Main theorems

Let $M \geq 2$ be a positive integer, where $M = \infty$ is admissible. We consider the mass $m(t)$ in the Cauchy problem of Klein–Gordon equations (1.1) and (1.2) represented by (1.10) for some functions $\lambda(t)$, $p(t) \in C^M([0, \infty))$ satisfying (1.11) and

$$\lambda(t) > 0, \quad \lambda'(t) \leq 0, \quad |\lambda^{(k)}(t)| \leq C_k \lambda(t)\rho(t)^k \tag{2.1}$$

and

$$|p^{(k)}(t)| \leq C_k \lambda(t)^2 \rho(t)^k \tag{2.2}$$

for a positive, continuous and monotone decreasing function $\rho(t)$ and some positive constants $C_k$, $k = 1, 2, \ldots, M$ and all time $t$. Then one of our main theorems is represented as follows:

**Theorem 2.1.** *If* (2.1) *and* (2.2) *are valid for any* $k = 1, \ldots, M$ *with* $\rho(t)$ *satisfying*

$$\rho(t) = o(\lambda(t)) \quad (t \to \infty) \tag{2.3}$$

*and*

$$\rho\left(\frac{\rho}{\lambda}\right)^{M-1} \in L^1, \tag{2.4}$$

*then* $\mathcal{E}(v)$ *satisfies the estimate of generalized energy conservation* (1.5).

We observe from Theorem 2.1 that the condition (2.4) is weaker as $M$ becomes larger. It follows that smoother $m(t)$ can allow faster oscillations of the perturbation $p(t)$ for the estimate (1.5).

Let $b \in C^M([0,\infty))$ with $M \geq 2$ be a positive and periodic function. Then the following examples of $m(t)$ are applicable for Theorem 2.1 to prove (1.5).

EXAMPLE 2.1. Let $\alpha$ and $\beta$ be non-negative real numbers satisfying $0 \leq \alpha < \beta < 1$. We define $\lambda(t)$ and $p(t)$ by

$$\lambda(t) = (1+t)^{-\alpha} \quad \text{and} \quad p(t) = \lambda(t)^2 b((1+t)^{1-\beta}).$$

Then (2.1), (2.2) and (2.3) are fulfilled for $\rho(t) = (1+t)^{-\beta}$ for any $k = 1, \ldots, M$. According to Theorem 2.1 the inequality $\beta > \alpha + (1-\alpha)/M$ implies the estimate (1.5). In particular, if $M = \infty$, then (1.5) holds since $\beta > \alpha$.

EXAMPLE 2.2. Let $\nu$ and $\mu$ be real numbers satisfying $\mu < \nu \leq 0$. We define $\lambda(t)$ and $p(t)$ by

$$\lambda(t) = (1+t)^{-1}(\log(e+t))^{-\mu} \quad \text{and} \quad p(t) = \lambda(t)^2 b((\log(e+t))^{1-\nu}).$$

Then (2.1), (2.2) and (2.3) are fulfilled for $\rho(t) = (1+t)^{-1}(\log(e+t))^{-\nu}$ for any $k = 1, \ldots, M$. According to Theorem 2.1 the inequality $\nu > \mu + (1-\mu)/M$ implies the estimate (1.5). In particular, if $M = \infty$, then (1.5) holds since $\nu > \mu$.

In Example 2.1 we see that the oscillations of $p(t)$ are determined by the parameter $\beta$, where smaller $\beta$ means faster oscillations. That is, if the oscillations are too fast with respect to the decreasing function $\lambda(t)$ (described by the parameter $\alpha$), in general we cannot apply Theorem 2.1. Actually, we can state the following counter-example.

EXAMPLE 2.3. For $0 \leq \alpha < 1$ we consider the following $m(t)$:

$$m(t) = (1+t)^{-\alpha}\sqrt{2 + \sin((1+t)^{1-\alpha})}, \tag{2.5}$$

that is,

$$\lambda^2(t) = 2(1+t)^{-2\alpha} \quad \text{and} \quad p(t) = (1+t)^{-2\alpha}\sin((1+t)^{1-\alpha}).$$

Then setting $\rho(t) = (1+t)^{-\alpha}$ the conditions (2.3) and (2.4) are not valid for $M = \infty$; moreover, we can prove that (1.9) does not hold.

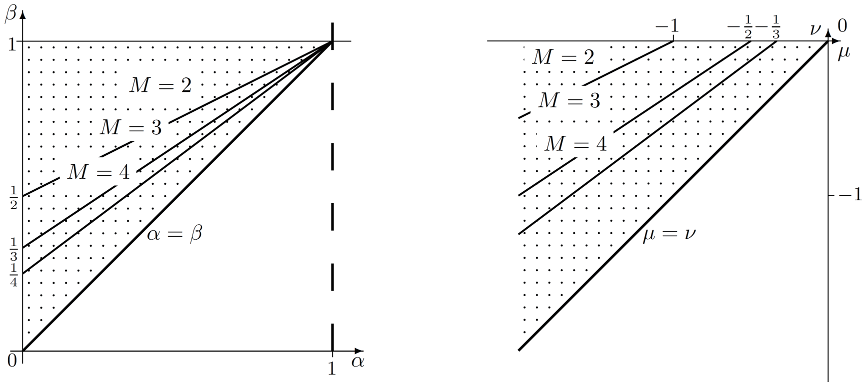


Fig. 1. Examples 2.1 and 2.3 and Example 2.2.

Example 2.3 is a consequence of the following theorem in [3], which is proved by applying Floquet's theory as in [10].

**Theorem 2.2** ([3]). *Let* $m(t)$ *be given by* (2.5). *There do not exist positive constants* $C$ *and* $\varepsilon$ *such that for every initial time* $t_0 \in [0, \infty)$ *and for every initial data* $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ *the estimate*

$$E_0(u)(t) \le C \exp(C(1+t)^{1-\alpha-\varepsilon}) E_0(u)(t_0)$$

*is fulfilled for all* $t \in [t_0, \infty)$.

Theorem 2.1 is applicable essentially for a finite $M$, thus we have no conclusion in the limit cases of slowly decaying $\rho(t)/\lambda(t)$, for instance, in the cases $\beta \to \alpha$ with $\alpha < 1$ in Example 2.1.

Let us consider in the limit case as $M \to \infty$ the Gevrey class $\gamma^{(s)}$ with $s \ge 1$:

$$\gamma^{(s)} := \{ f \in C^\infty((0,\infty));\ |f^{(k)}(t)| \le Ck!^s \rho^k,\ \exists \rho > 0,\ k = 0, 1, \dots \}.$$

REMARK 2.1. $\gamma^{(1)}$ is not usually called the Gevrey class but real-analytic class. However, we do not particularly distinguish both cases in this paper.

Let $m \in \gamma^{(s)}$ with $s \ge 1$ represented by (1.10). Then we suppose that $\lambda(t)$ and $p(t)$ satisfy (2.1) and (2.2) with $C_k = Ck!^s$ for any $k \in \mathbb{N}$. Then our second main theorem is represented as follows:

**Theorem 2.3.** *Let* $m(t) \in \gamma^{(s)}$ *represented by* (1.10) *satisfy* (2.1) *and* (2.2) *with* $C_k = \kappa k!^s$ *for a positive constant* $\kappa$ *and for any* $k \in \{0, 1, \dots\}$. *If* (2.3),

$$\frac{\rho'(t)}{\rho(t)} \le \frac{\lambda'(t)}{\lambda(t)} \tag{2.6}$$

*and*

$$\frac{\rho(t)}{\lambda(t)} \lesssim (\log t)^{-s} \tag{2.7}$$

*are valid as* $t \to \infty$, *then* $\mathcal{E}(v)$ *satisfies the estimate of generalized energy conservation* (1.5).

By Theorem 2.3 we can conclude the estimate (1.5) to the following example:

EXAMPLE 2.4. Let $0 \le \alpha < 1$, $s \ge 1$ and $b \in \gamma^{(s)}$ be a positive and periodic function. We define $\lambda(t)$ and $p(t)$ by

$$\lambda(t) = (1+t)^{-\alpha} \quad \text{and} \quad p(t) = \lambda(t)^2 b((1+t)^{1-\alpha}(\log(e+t))^{-\nu}).$$

Then (2.1), (2.2) and (2.3) are fulfilled for $C_k = \kappa k!^s$, $\rho(t) = (1+t)^{-\alpha}(\log(e+t))^{-\nu}$ and for any $k \in \mathbb{N}$. According to Theorem 2.3 the inequality $\nu \ge s$ implies the estimate (1.5).

## 3. Proof of Theorem 2.1

**3.1. Estimate on a finite time interval.** Let $T > 0$ be an arbitrarily fixed time. Then the derivative of the energy (1.3) can be estimated by

$$\partial_t \mathcal{E}_0(v)(t,\xi) = m'(t)m(t)|v(t,\xi)|^2 \lesseqgtr \pm \frac{2|m'(t)|}{m(t)} \mathcal{E}_0(v)(t,\xi).$$

The application of Gronwall's inequality yields

$$\begin{aligned} \mathcal{E}_0(v)(t,\xi) &\lesseqgtr \exp\left(\pm \int_0^t \frac{2|m'(\tau)|}{m(\tau)}\, d\tau\right) \mathcal{E}_0(0,\xi) \\ &\lesseqgtr \exp\left(\pm T \max_{\tau \in [0,T]} \left\{\frac{2|m'(\tau)|}{m(\tau)}\right\}\right) \mathcal{E}_0(v)(0,\xi) \end{aligned}$$

for any $(t,\xi) \in [0,T] \times \mathbb{R}^n$. Moreover, noting $\langle \xi \rangle_{\lambda(t)} / \langle \xi \rangle_{\lambda(0)} \simeq 1$ for $(t,\xi) \in [0,T] \times \mathbb{R}^n$, we have the estimate (1.5) in $[0,T] \times \mathbb{R}^n$. Therefore, we only have to discuss for large $t$.

**3.2. Refined diagonalization procedure for large times.** Let us consider the equivalent reduced system for the equation of (1.2)

$$\partial_t V_0 = \mathcal{Q}_0 V_0,$$

where

$$V_0 = V_0(t,\xi) := \begin{pmatrix} \langle\xi\rangle_{m(t)} v(t,\xi) \\ -i\,\partial_t v(t,\xi) \end{pmatrix} \quad \text{and} \quad \mathcal{Q}_0 = \mathcal{Q}_0(t,\xi) := \begin{pmatrix} \dfrac{\partial_t\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(t)}} & i\langle\xi\rangle_{m(t)} \\ i\langle\xi\rangle_{m(t)} & 0 \end{pmatrix}.$$

Let us carry out the first step of diagonalization procedure with the diagonalizer $\mathcal{N}_0$ for diag $\mathcal{Q}_0$, where

$$\mathcal{N}_0 := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Denoting

$$\mathcal{Q}_1 := \partial_t - \mathcal{N}_0^{-1}(\partial_t - \mathcal{Q}_0)\mathcal{N}_0, \quad \Phi_1 := \operatorname{diag} \mathcal{Q}_1 \quad \text{and} \quad \mathcal{R}_1 := \mathcal{Q}_1 - \Phi_1$$

we have

$$\Phi_1(t,\xi) := \begin{pmatrix} \overline{\phi_1} & 0 \\ 0 & \phi_1 \end{pmatrix} \quad \text{and} \quad \mathcal{R}_1(t,\xi) := \begin{pmatrix} 0 & r_1 \\ r_1 & 0 \end{pmatrix},$$

where

$$\phi_1 = \phi_1(t,\xi) := \frac{\partial_t\langle\xi\rangle_{m(t)}}{2\langle\xi\rangle_{m(t)}} + i\langle\xi\rangle_{m(t)} \quad \text{and} \quad r_1 = r_1(t,\xi) := \frac{\partial_t\langle\xi\rangle_{m(t)}}{2\langle\xi\rangle_{m(t)}}. \tag{3.1}$$

Moreover, we denote from now on that $\phi_{j,\Re} := \Re\{\phi_j\}$ and $\phi_{j,\Im} := \Im\{\phi_j\}$ for the functions $\phi_1, \phi_2, \ldots$.

Indeed, we can derive the hyperbolicity of the equation for the energy estimate by the diagonalization procedure due to $\mathcal{N}_0$. However, this step of the diagonalization procedure is not sufficient to prove (1.5) for $\rho \notin L^1$. Thus we introduce further steps of diagonalization procedure by use of the smoothness of $m(t)$, which is called *refined diagonalization procedure* introduced in [5].

The refined diagonalization procedure is inductively carried out by the following diagonalizers $\mathcal{N}_k$:

$$\mathcal{N}_k = \mathcal{N}_k(t,\xi) := I + \frac{1}{(\Phi_k)_{11} - (\Phi_k)_{22}} \begin{pmatrix} 0 & -(\mathcal{R}_k)_{12} \\ (\mathcal{R}_k)_{21} & 0 \end{pmatrix}$$

for $k = 1, \ldots, M-1$, where

$$\mathcal{Q}_{k+1} = \mathcal{Q}_{k+1}(t,\xi) := \partial_t - \mathcal{N}_k^{-1}(\partial_t - \mathcal{Q}_k)\mathcal{N}_k,$$
$$\Phi_{k+1} = \Phi_{k+1}(t,\xi) := \operatorname{diag} \mathcal{Q}_{k+1} \quad \text{and} \quad \mathcal{R}_{k+1} = \mathcal{R}_{k+1}(t,\xi) := \mathcal{Q}_{k+1} - \Phi_{k+1}.$$

Here we note that $\mathcal{Q}_{k+1} \in C^{M-k-1}$ with respect to $t$. Hence, such diagonalization procedure is valid since $\mathcal{N}_1, \dots, \mathcal{N}_k$ are invertible. Then the benefit of the refined diagonalization procedure is represented in the following lemma:

**Lemma 3.1.** *Assume that $\mathcal{N}_1, \dots, \mathcal{N}_k$ are invertible. Then we have*

$$(\Phi_{k+1})_{22} = \overline{(\Phi_{k+1})_{11}} \quad \text{and} \quad (\mathcal{R}_{k+1})_{21} = \overline{(\mathcal{R}_{k+1})_{12}}. \tag{3.2}$$

*Moreover, denoting $\phi_{k+1} = \phi_{k+1}(t,\xi) := (\Phi_{k+1})_{22}$, $r_{k+1} = r_{k+1}(t,\xi) := (\mathcal{R}_{k+1})_{21}$, and*

$$\mu_k = \mu_k(t,\xi) := 1 - \det \mathcal{N}_k = \frac{|r_k|^2}{4\phi_{k,\Im}^2}$$

*the functions $\phi_{k+1,\Re}$, $\phi_{k+1,\Im}$ and $r_{k+1}$ are inductively given by*

$$\phi_{k+1,\Re} = \frac{\partial_t \langle \xi \rangle_{m(t)}}{2\langle \xi \rangle_{m(t)}} + \sum_{j=1}^{k} \frac{\partial_t \mu_j}{2(1-\mu_j)}, \tag{3.3}$$

$$\phi_{k+1,\Im} = \phi_{k,\Im} + \frac{1}{1-\mu_k}\left( -2\mu_k \phi_{k,\Im} + \Im\left\{ \frac{r_k}{2\phi_{k,\Im}} \partial_t \frac{\overline{r_k}}{2\phi_{k,\Im}} \right\} \right) \tag{3.4}$$

*and*

$$r_{k+1} = \frac{1}{1-\mu_k}\left( \mu_k r_k - \partial_t \frac{i r_k}{2\phi_{k,\Im}} \right) \tag{3.5}$$

*for $k = 1, \dots, M-1$.*

*Proof.* Let us prove this lemma by induction. By the definition of $\Phi_1$ and $\mathcal{R}_1$ in (3.1) we see that $(\Phi_1)_{11} - (\Phi_1)_{22} = -2i\phi_{1,\Im}$ and $(\mathcal{R}_1)_{12} = r_1 = \overline{r_1} = \overline{(\mathcal{R}_1)_{21}}$. Therefore, we have

$$\mathcal{N}_1 = I + \frac{1}{2\phi_{1,\Im}} \begin{pmatrix} 0 & \overline{ir_1} \\ ir_1 & 0 \end{pmatrix}.$$

Then straightforward calculations give

$$\begin{aligned} &\partial_t - \mathcal{N}_1^{-1} \partial_t \mathcal{N}_1 \\ &= \frac{1}{1-\mu_1} \begin{pmatrix} \frac{\overline{r_1}}{2\phi_{1,\Im}} \partial_t \frac{r_1}{2\phi_{1,\Im}} & 0 \\ 0 & \frac{r_1}{2\phi_{1,\Im}} \partial_t \frac{\overline{r_1}}{2\phi_{1,\Im}} \end{pmatrix} + \frac{1}{1-\mu_1} \begin{pmatrix} 0 & -\partial_t \frac{\overline{ir_1}}{2\phi_{1,\Im}} \\ -\partial_t \frac{ir_1}{2\phi_{1,\Im}} & 0 \end{pmatrix} \end{aligned}$$

and

$$\mathcal{N}_1^{-1} \mathcal{Q}_1 \mathcal{N}_1 = \Phi_1 + \frac{1}{1-\mu_1} \begin{pmatrix} 2i\mu_1 \phi_{1,\Im} & 0 \\ 0 & -2i\mu_1 \phi_{1,\Im} \end{pmatrix} + \frac{\mu_1}{1-\mu_1} \mathcal{R}_1.$$

They imply that (3.2) is valid for $k = 1$. Moreover, we have

$$\phi_{2,\Re} = \phi_{1,\Re} + \frac{1}{1-\mu_1}\Re\left\{\frac{r_1}{2\phi_{1,\Im}}\partial_t\frac{\overline{r_1}}{2\phi_{1,\Im}}\right\} = \phi_{1,\Re} + \frac{\partial_t\mu_1}{2(1-\mu_1)},$$
$$\phi_{2,\Im} = \phi_{1,\Im} + \frac{1}{1-\mu_1}\left(-2\mu_1\phi_{1,\Im} + \Im\left\{\frac{r_1}{2\phi_{1,\Im}}\partial_t\frac{\overline{r_1}}{2\phi_{1,\Im}}\right\}\right)$$

and

$$r_2 = \frac{1}{1-\mu_1}\left(\mu_1 r_1 - \partial_t\frac{ir_1}{2\phi_{1,\Im}}\right),$$

which are (3.3), (3.4) and (3.5) with $k = 1$. Thus the lemma is valid for $k = 1$.

Assume that $(\Phi_j)_{11} - (\Phi_j)_{22} = -2i\phi_{j,\Im}$ and $(\mathcal{R}_j)_{21} = r_j = \overline{(\mathcal{R}_j)_{12}}$ for $j = 2, \dots, k$. Then we immediately achieve the representations

$$\phi_{k+1,\Re} = \phi_{k,\Re} + \frac{\partial_t\mu_k}{2(1-\mu_k)}, \tag{3.6}$$

(3.4) and (3.5) by the same way as in the case of $k = 1$. Moreover, assuming the representation

$$\phi_{k,\Re} = \frac{\partial_t\langle\xi\rangle_{m(t)}}{2\langle\xi\rangle_{m(t)}} + \sum_{j=1}^{k-1}\frac{\partial_t\mu_j}{2(1-\mu_j)}$$

we conclude (3.3) by (3.6). □

If the refined diagonalization procedure by $\mathcal{N}_k$ for $k = 1, \dots, M-1$ holds true as in Lemma 3.1, then we can state the following lemma for the energy estimates:

**Lemma 3.2.** *Assume that $\mathcal{N}_1, \dots, \mathcal{N}_{M-1}$ are invertible for $t \in [\tau_0, \tau_1]$. Then the following energy estimates are valid*:

$$\mathcal{E}_0(v)(t,\xi) \lesseqgtr \frac{\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(\tau_0)}}\zeta^{\pm}_{M-1}(t,\tau_0,\xi)\exp\left(\pm 4\int_{\tau_0}^{t}|r_M(\tau,\xi)|\,d\tau\right)\mathcal{E}_0(v)(\tau_0,\xi)$$

*for $t \in [\tau_0, \tau_1]$, where $\zeta^{\pm}_0 = 1$ and*

$$\zeta^{\pm}_j = \zeta^{\pm}_j(t,\tau,\xi) = \prod_{k=1}^{j}\frac{(1 \pm \sqrt{\mu_k(t,\xi)})(1 \pm \sqrt{\mu_k(\tau,\xi)})}{(1 \mp \sqrt{\mu_k(t,\xi)})(1 \mp \sqrt{\mu_k(\tau,\xi)})}$$

*for $j \geq 1$.*

Proof. Let us define $V_M = V_M(t,\xi)$ by $V_M := \mathcal{N}_{M-1}^{-1} \cdots \mathcal{N}_0^{-1} V_0$. Then $V_M$ is a solution of

$$\partial_t V_M = (\Phi_M + \mathcal{R}_M) V_M.$$

Therefore, we deduce

$$\begin{aligned}
\partial_t |V_M|^2 &= 2\Re(\partial_t V_M, V_M)_{\mathbb{C}^2} = 2\Re(\Phi_M V_M, V_M)_{\mathbb{C}^2} + 2\Re(\mathcal{R}_M V_M, V_M)_{\mathbb{C}^2} \\
&= 2\phi_{M,\Re} |V_M|^2 + 4\Re(r_M V_{M,1}, V_{M,2})_{\mathbb{C}^2} \\
&\lesseqgtr 2(\phi_{M,\Re} \pm 2|r_M|)|V_M|^2.
\end{aligned}$$

Noting the equality

$$\int_{\tau_0}^t \phi_{M,\Re}(\tau,\xi)\,d\tau = \frac{1}{2}\log\left( \frac{\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(\tau_0)}} \prod_{k=1}^{M-1} \frac{1-\mu_k(\tau_0,\xi)}{1-\mu_k(t,\xi)} \right),$$

Gronwall's lemma yields the following estimates:

$$|V_M(t,\xi)|^2 \lesseqgtr \frac{\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(\tau_0)}} \left( \prod_{k=1}^{M-1} \frac{1-\mu_k(\tau_0,\xi)}{1-\mu_k(t,\xi)} \right) \exp\left( \pm 4 \int_{\tau_0}^t |r_M(\tau,\xi)|\,d\tau \right) |V_M(\tau_0,\xi)|^2. \tag{3.7}$$

Denoting $V_{k+1} = V_{k+1}(t,\xi) := \mathcal{N}_k^{-1} V_k$ for $k = 0, \dots, M-1$, we achieve

$$\begin{aligned}
V_{k+1} &= \frac{1}{1-\mu_k}\left( I + \frac{1}{2\phi_{k,\Im}} \begin{pmatrix} 0 & -\overline{ir_k} \\ -ir_k & 0 \end{pmatrix} \right) \begin{pmatrix} V_{k,1} \\ V_{k,2} \end{pmatrix} \\
&= \frac{1}{1-\mu_k} \begin{pmatrix} V_{k,1} - \dfrac{\overline{ir_k}}{2\phi_{k,\Im}} V_{k,2} \\ V_{k,2} - \dfrac{ir_k}{2\phi_{k,\Im}} V_{k,1} \end{pmatrix}.
\end{aligned}$$

Noting the estimates

$$\begin{aligned}
|V_{k+1}|^2 &= \frac{1}{(1-\mu_k)^2} \left( \left| V_{k,1} - \frac{\overline{ir_k}}{2\phi_{k,\Im}} V_{k,2} \right|^2 + \left| V_{k,2} - \frac{ir_k}{2\phi_{k,\Im}} V_{k,1} \right|^2 \right) \\
&= \frac{1}{(1-\mu_k)^2} \left( (1+\mu_k)|V_k|^2 - 2\Re\left( \frac{ir_k}{2\phi_{k,\Im}} V_{k,1}, V_{k,2} \right)_{\mathbb{C}^2} \right) \\
&\lesseqgtr \frac{(1 \pm \sqrt{\mu_k})^2}{(1-\mu_k)^2} |V_k|^2 = (1 \mp \sqrt{\mu_k})^{-2} |V_k|^2
\end{aligned} \tag{3.8}$$

for $k = 1, \dots, M-1$ and the equalities

$$|V_1(t,\xi)|^2 = \frac{1}{2}|V_0(t,\xi)|^2 = \mathcal{E}_0(v)(t,\xi) \tag{3.9}$$

we have

$$|V_M(t,\xi)|^2 \lesseqgtr \left(\prod_{k=1}^{M-1}(1 \mp \sqrt{\mu_k(t,\xi)})^{-2}\right)\mathcal{E}_0(v)(t,\xi).$$

Therefore, by (3.7) we obtain

$$\begin{aligned}
\mathcal{E}_0(v)(t,\xi) &\le \left(\prod_{k=1}^{M-1}\left(1+\sqrt{\mu_k(t,\xi)}\right)^2\right)|V_M(t,\xi)|^2 \\
&\le \frac{\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(\tau_0)}}\left(\prod_{k=1}^{M-1}\frac{(1-\mu_k(\tau_0,\xi))(1+\sqrt{\mu_k(t,\xi)})}{1-\sqrt{\mu_k(t,\xi)}}\right) \\
&\quad \times \exp\left(4\int_{\tau_0}^t |r_M(\tau,\xi)|\,d\tau\right)|V_M(\tau_0,\xi)|^2 \\
&\le \frac{\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(\tau_0)}}\zeta_{M-1}^{+}(t,\tau_0,\xi)\exp\left(4\int_{\tau_0}^t |r_M(\tau,\xi)|\,d\tau\right)\mathcal{E}_0(v)(\tau_0,\xi).
\end{aligned}$$

Analogously, we have

$$\mathcal{E}_0(v)(t,\xi) \ge \frac{\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(\tau_0)}}\zeta_{M-1}^{-}(t,\tau_0,\xi)\exp\left(-4\int_{\tau_0}^t |r_M(\tau,\xi)|\,d\tau\right)\mathcal{E}_0(v)(\tau_0,\xi).$$

Thus the proof of the lemma is concluded. □

**3.3. Symbol calculus.** Let us discuss the invertibility of $\mathcal{N}_k$ for $k = 1,\dots,M-1$ and a benefit of the refined diagonalization procedure to introduce some symbol classes. More precisely, we shall prove that the invertibility of $\mathcal{N}_k$, the property $r_M(t,\xi) \in L^1((T,\infty); L^\infty(\mathbb{R}^n_\xi))$ for a large $T$ and Lemma 3.2 conclude the proof of Theorem 2.1.

Let $l$, $p$ and $q$ be integers satisfying $l \ge 0$, $p \le 0$ and $q \ge 0$. A function $f = f(t,\xi) \in C^l([0,\infty)\times\mathbb{R}^n)$ belongs to the symbol class $\mathcal{S}^l\{p,q\}$ if there exist positive constants $C_1, \dots, C_k$ such that the estimates

$$|\partial_t^k f(t,\xi)| \le C_k\langle\xi\rangle_{\lambda(t)}^p\rho(t)^{q+k} \tag{3.10}$$

are valid for $(t,\xi) \in [T,\infty)\times\mathbb{R}^n$ and $k = 0,1,\dots,l$. By the definition of the symbol classes and straightforward estimates we can immediately state the following lemma:

**Lemma 3.3.** (i) *If* $f \in \mathcal{S}^l\{p, q\}$, *then* $\partial_t^k f \in \mathcal{S}^{l-k}\{p, q+k\}$ *for* $k \le l$.
(ii) *If* $f_1 \in \mathcal{S}^{l_1}\{p, q\}$ *and* $f_2 \in \mathcal{S}^{l_2}\{p, q\}$, *then* $f_1 + f_2 \in \mathcal{S}^{\min\{l_1,l_2\}}\{p, q\}$.
(iii) *If* $f_1 \in \mathcal{S}^{l_1}\{p_1, q_1\}$ *and* $f_2 \in \mathcal{S}^{l_2}\{p_2, q_2\}$, *then* $f_1 f_2 \in \mathcal{S}^{\min\{l_1,l_2\}}\{p_1 + p_2, q_1 + q_2\}$.
(iv) *If* $f \in \mathcal{S}^l\{p,q\}$, *then* $f \in \mathcal{S}^l\{p+\sigma, q-\sigma\}$ *for all non-negative integers* $\sigma$ *satisfying* $p + \sigma \le 0$ *and* $q - \sigma \ge 0$.

Proof. The properties (i)–(iii) are valid by the definitions of the symbol classes. Property (iv) follows from (2.3) and the natural property $\lambda(t) \le \langle\xi\rangle_{\lambda(t)}$. □

Here we note that the following property holds:

**Lemma 3.4.** *Under the assumptions* (1.12), (2.1) *and* (2.2) *we have* $\langle\xi\rangle_{m(t)}^{-1} \in \mathcal{S}^{M-1}\{-1, 0\}$.

Proof. Let $k$ be an integer satisfying $1 \le k \le M-2$. We will prove that the estimate

$$|\partial_t^k \langle\xi\rangle_{m(t)}^{-1}| \le C_k \langle\xi\rangle_{\lambda(t)}^{-1}\rho(t)^k \tag{3.11}$$

is valid for $k = 1, \dots, M-1$ by induction. The estimate (3.11) is evident for $k = 1$. Assume that the estimates (3.11) for $j = 1, \dots, k$ are valid. Leibniz rule gives

$$0 = \partial_t^{k+1}(\langle\xi\rangle_{m(t)}\langle\xi\rangle_{m(t)}^{-1}) = \langle\xi\rangle_{m(t)}\partial_t^{k+1}\langle\xi\rangle_{m(t)}^{-1} + \sum_{j=1}^{k+1}\binom{k+1}{j}(\partial_t^j\langle\xi\rangle_{m(t)})(\partial_t^{k-j+1}\langle\xi\rangle_{m(t)}^{-1}).$$

It follows that

$$\begin{aligned}\partial_t^{k+1}\langle\xi\rangle_{m(t)}^{-1} &= -\langle\xi\rangle_{m(t)}^{-1}\sum_{j=1}^{k+1}\binom{k+1}{j}(\partial_t^j\langle\xi\rangle_{m(t)})(\partial_t^{k-j+1}\langle\xi\rangle_{m(t)}^{-1})\\ &= -\langle\xi\rangle_{m(t)}^{-1}\sum_{j=1}^{k+1}\binom{k+1}{j}\left(\partial_t^{j-1}\frac{(m(t)^2)'}{2\langle\xi\rangle_{m(t)}}\right)(\partial_t^{k-j+1}\langle\xi\rangle_{m(t)}^{-1})\\ &= -\frac{1}{2}\langle\xi\rangle_{m(t)}^{-1}\sum_{j=1}^{k+1}\sum_{l=0}^{j-1}\binom{k+1}{j}\binom{j-1}{l}(m(t)^2)^{(l+1)}(\partial_t^{j-l-1}\langle\xi\rangle_{m(t)}^{-1})(\partial_t^{k-j+1}\langle\xi\rangle_{m(t)}^{-1}).\end{aligned} \tag{3.12}$$

Therefore, noting the estimate

$$|(m(t)^2)^{(l+1)}| \le \sum_{j=0}^{l+1}\binom{l+1}{j}|\lambda^{(j)}(t)\lambda^{(l-j+1)}(t)| + |p^{(l+1)}(t)| \le C_{l+1}\lambda(t)^2\rho(t)^{l+1}, \tag{3.13}$$

which follows from (2.1) and (2.2), we have

$$|\partial_t^{k+1}\langle\xi\rangle_{m(t)}^{-1}| \le C_{k+1}\langle\xi\rangle_{m(t)}^{-1}\langle\xi\rangle_{\lambda(t)}^{-2}\lambda(t)^2\rho(t)^{k+1} \le C_{k+1}\langle\xi\rangle_{\lambda(t)}^{-1}\rho(t)^{k+1}.$$

Thus the estimate (3.11) is valid for $k+1$, which concludes the proof of the lemma. □

According to the definition of the symbol classes we can prove the following statements which imply the invertibility of $\mathcal{N}_k$ for $k = 1, \dots, M-1$ and $r_M(t,\xi) \in L^1((T,\infty); L^\infty(\mathbb{R}^n_\xi))$.

**Lemma 3.5.** *There exists a large time $T$ such that*

$$\mu_k \in \mathcal{S}^{M-k}\{-2k, 2k\} \quad \textit{with} \quad |\mu_k| \le \frac{1}{2} \tag{3.14}$$

*for $k = 1, \dots, M-1$ and*

$$r_k \in \mathcal{S}^{M-k}\{-k+1, k\} \quad \textit{and} \quad \phi_{k,\Im}^{-1} \in \mathcal{S}^{M-k}\{-1, 0\} \tag{3.15}$$

*for $k = 1, \dots, M$.*

Proof. By (1.12), (2.1), (2.2) and Lemma 3.4 we see that

$$\frac{\lambda^{(k)}(t)}{\lambda(t)}, \frac{p^{(k)}(t)}{\lambda(t)^2} \in \mathcal{S}^{M-k}\{0, k\} \quad \text{and} \quad \frac{\lambda(t)}{\langle\xi\rangle_{m(t)}} \in \mathcal{S}^M\{0, 0\}.$$

Therefore, by Lemma 3.3 and the representation

$$r_1(t,\xi) = \frac{\lambda'(t)\lambda(t)}{2\langle\xi\rangle_{m(t)}^2} + \frac{p'(t)}{4\langle\xi\rangle_{m(t)}^2}$$

we have (3.15) for $k = 1$. Moreover, by the definition of $\phi_{1,\Im}$ we conclude $\phi_{1,\Im}^{-1} \in \mathcal{S}^{M-1}\{-1, 0\}$ and $\mu_1 = |r_1|^2/(4\phi_{1,\Im}^2) \in \mathcal{S}^{M-1}\{-2, 2\}$. Therefore, by (2.3) we estimate

$$|\mu_1| \lesssim \langle\xi\rangle_{\lambda(t)}^{-2}\rho(t)^2 \lesssim \frac{\rho(t)^2}{\lambda(t)^2} < \frac{1}{2}$$

for any $t \ge T_1$ with a large $T_1$. Thus the invertibility of $\mathcal{N}_1$ is valid for $t \ge T_1$.

Let us suppose that (3.14) and (3.15) are satisfied for $j = 1, \dots, k$ with $k < M$. Thus there exists a large time $T_k$ such that $\mathcal{N}_j$ is invertible for any $j = 1, \dots, k$. We see for $j = 1, \dots, k$ that $(1-\mu_j)^{-1} \in \mathcal{S}^{M-j}\{0, 0\}$. This follows from the representation

$$\partial_t^l(1-\mu_j)^{-1} = \sum_{1h_1+\cdots+lh_l=l} C_{l,h}(1-\mu_j)^{-(1+|h|)}(\mu_j^{(1)})^{h_1}\cdots(\mu_j^{(l)})^{h_l},$$

where $h = (h_1, \dots, h_l) \in \{0, 1, \dots, l\}^l$, $l = 0, \dots, M - j$, $|h| = h_1 + \cdots + h_l$, and $C_{l,h}$ are constants. Indeed, by $\mu_j \in \mathcal{S}^{M-j}\{-2j, 2j\}$ and $|\mu_j| \le 1/2$ we deduce

$$\begin{aligned} |\partial_t^l (1 - \mu_j)^{-1}| &\lesssim \langle \xi \rangle_{\lambda(t)}^{-2jh_1 - 2jh_2 - \cdots - 2jh_l} \rho(t)^{(2j+1)h_1 + (2j+2)h_2 + \cdots + (2j+l)h_l} \\ &= \langle \xi \rangle_{\lambda(t)}^{-2j|h|} \rho(t)^{2j|h|+l} \lesssim \rho(t)^l. \end{aligned}$$

Therefore, by Lemma 3.3 and the representation (3.5) we have $r_{k+1} \in \mathcal{S}^{M-k-1}\{-k, k+1\}$. Moreover, by the representation (3.4) we conclude

$$\phi_{k+1,\Im}^{-1} = \phi_{k,\Im}^{-1}(1 + \delta_k)^{-1},$$

where

$$\delta_{k+1} = \delta_{k+1}(t, \xi) = \frac{1}{1 - \mu_k}\left(-2\mu_k + \frac{1}{\phi_{k,\Im}} \Im\left\{\frac{r_k}{2\phi_{k,\Im}} \partial_t \frac{\overline{r_k}}{2\phi_{k,\Im}}\right\}\right) \in \mathcal{S}^{M-k-1}\{-2k, 2k\}.$$

By the same argument as to show $\mu_j \in \mathcal{S}^{M-j}\{-2j, 2j\}$ we get $(1+\delta_k)^{-1} \in \mathcal{S}^{M-k-1}\{0, 0\}$. Thus we obtain $\phi_{k+1,\Im}^{-1} \in \mathcal{S}^{M-k-1}\{-1, 0\}$. Consequently, we have $\mu_{k+1} \in \mathcal{S}^{M-k-1}\{-2(k+1), 2(k+1)\}$. It follows that there exists a large time $T_{k+1}$ such that $|\mu_{k+1}| \le 1/2$ for any $t \ge T_{k+1}$.

Therefore, the proof of the lemma is concluded by induction. □

**3.4. Conclusion of the proof of Theorem 2.1.** Lemma 3.5 ensures that there exists a positive constant $T_M$ such that $\mathcal{N}_1, \dots, \mathcal{N}_{M-1}$ are uniformly invertible for $(t, \xi) \in [T_M, \infty) \times \mathbb{R}^n$; thus the estimates of Lemma 3.2 is valid. By Lemma 3.5 we conclude

$$|r_M(t, \xi)| \lesssim \langle \xi \rangle_{\lambda(t)}^{-M+1} \rho(t)^M \le \rho(t)\left(\frac{\lambda(t)}{\rho(t)}\right)^{M-1}.$$

Therefore, the assumption (2.4) gives $\int_{T_M}^{\infty} |r_M(\tau, \xi)|\, d\tau < \infty$. Moreover, recalling $|\mu_j(t, \xi)| \le 1/2$ we achieve $|\zeta_{M-1}(t, T_M, \xi)| \simeq 1$ for $t \ge T_M$. Consequently, since (1.5) is valid for $t \le T_M$ we finish the proof of Theorem 2.1. □

## 4. Proof of Theorem 2.3

**4.1. Division of the zones.** Theorem 2.1 was proved by dividing the phase space into two zones; for $t < T_M$ and $t \ge T_M$. On the other hand Theorem 2.3 will be proved by estimating the energy in different ways in infinitely many zones of the divided phase space. The basic idea of the proof is introduced in [6].

For a large constant $N$ we define $t_0 = t_0(\xi)$ implicitly by

$$\langle \xi \rangle_{\lambda(t_0)} = N\rho(t_0).$$

Then we define $Z_\Psi = Z_\Psi(N)$, which is called the pseudo-differential zone, by

$$Z_\Psi = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n;\ t \le t_0(\xi)\}.$$

Here we note that $Z_\Psi$ is bounded.

Let us define $Z_{\mathrm{H}} = Z_{\mathrm{H}}(N)$, which is called the hyperbolic zone, by

$$Z_{\mathrm{H}} = \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n;\ (t,\xi) \notin Z_\Psi\}.$$

Moreover, we divide $Z_{\mathrm{H}}$ by an infinite number of zones $Z_{\mathrm{H},1}, Z_{\mathrm{H},2}, \ldots$ defined by

$$Z_{\mathrm{H},k} = \{(t,\xi) \in Z_{\mathrm{H}};\ Nk^s\rho(t) < \langle\xi\rangle_{\lambda(t)} \le N(k+1)^s\rho(t)\},$$

where $t_k = t_k(\xi)$ is implicitly defined by

$$\langle\xi\rangle_{\lambda(t_k)} = N(k+1)^s\rho(t_k)$$

for $k = 1, 2, \ldots$. Here we note that $\{t_k\}$ is uniquely determined by (2.6) for any fixed $\xi$. Then we immediately have the following lemma:

**Lemma 4.1.** *If* (2.7) *holds, then there exists a positive constant* $c_0$ *such that* $t_k \le e^{c_0 k}$ *for any* $k \in \mathbb{N}$.

Proof. Noting (2.7), we have the estimates

$$N(k+1)^s\rho(t_k) = \langle\xi\rangle_{\lambda(t_k)} \ge \lambda(t_k) \ge \frac{1}{c_0}\rho(t_k)(\log(e+t_k))^s.$$

They imply the conclusion of the lemma. □

**4.2. Gevrey symbol class in $Z_{\mathrm{H},K}$.** Firstly, we note the following fact: *if there exists a positive constant* $C$ *such that* $f \in \gamma^{(s)}$ satisfies

$$|f^{(k)}(t)| \le Ck!^s\rho(t)^k \quad (\forall k \in \mathbb{N}),$$

*then the following estimates also hold*:

$$|f^{(k)}(t)| \le C\frac{k!^s(4\rho(t))^k}{(k+1)^2} \quad (\forall k \in \mathbb{N}).$$

Therefore, we can replace the definition of the constants $C_k$ in Theorem 2.3 by $C_k = \kappa_0 k!^s/(k+1)^2$ with a positive constant $\kappa_0$ without loss of generality.

We have introduced the symbol class $\mathcal{S}^l\{p,q\}$ for the proof of Theorem 2.1, but this is not sufficient for the proof of Theorem 2.3. In the proof of Theorem 2.3 we have to derive a benefit of a property of the Gevrey functions, which is represented by order of the constants $C_k$ in (3.10) as $k \to \infty$. Therefore, the new symbol class for the Gevrey functions has to be precise as follows.

We fix a positive integer $K$ from now on. Let $N$, $\kappa$ and $\rho_0$ be positive constants and $\rho(t)$ a positive monotone decreasing function. A function $f = f(t,\xi) \in C^\infty([0,\infty)\times\mathbb{R}^n)$ belongs to the symbol class $\mathcal{S}_K\{p,q;\kappa,\rho_0,N\}$ with integers $p$ and $q$ satisfying $p \le 0$ and $q \ge 0$ if the estimates

$$|\partial_t^k f(t,\xi)| \le \kappa \frac{(q+k)!^s}{(q+k+1)^2}\langle\xi\rangle_{\lambda(t)}^p(\rho_0\rho(t))^{q+k}$$

hold true for any $(t,\xi)\in Z_{\mathrm{H},K}$ and all $k\in\{0,1,\dots\}$. We shall denote $\mathcal{S}_K\{p,q\} = \mathcal{S}_K\{p,q;\kappa\} = \mathcal{S}_K\{p,q;\kappa,\rho_0\} = \mathcal{S}_K\{p,q;\kappa,\rho_0,N\}$ according to need without any confusion. Moreover, for two functions $f\in\mathcal{S}_K\{p_1,q_1\}$ and $g\in\mathcal{S}_K\{p_2,q_2\}$ we denote the symbol class of the product $fg$ by $fg\in\mathcal{S}_K\{p_1,q_1\}\mathcal{S}_K\{p_2,q_2\}$ as a matter of convenience. By the definition of zone $Z_{\mathrm{H},K}$ and the symbol class we have the following properties:

**Lemma 4.2.** (i) *If $f\in\mathcal{S}_K\{p,q\}$, then $\partial_t^k f\in\mathcal{S}_K\{p,q+k\}$.*
(ii) *If $f_1\in\mathcal{S}_K\{p,q;\kappa_1\}$, $f_2\in\mathcal{S}_K\{p,q;\kappa_2\}$, then $f_1+f_2\in\mathcal{S}_K\{p,q;\kappa_1+\kappa_2\}$ and $f_1, f_2\in\mathcal{S}_K\{p,q;\max\{\kappa_1,\kappa_2\}\}$.*
(iii) *If $f_1\in\mathcal{S}_K\{p_1,q_1;\kappa_1\}$ and $f_2\in\mathcal{S}_K\{p_2,q_2;\kappa_2\}$, then $f_1f_2\in\mathcal{S}_K\{p_1+p_2,q_1+q_2;4\pi^2\kappa_1\kappa_2/3\}$.*
(iv) *If $f\in\mathcal{S}_K\{p,q;\kappa\}$ for $q\le 2K$, then $f\in\mathcal{S}_K\{p+\sigma,q-\sigma;\kappa(3^s\rho_0N^{-1})^\sigma\}$ for all positive integers $\sigma$ satisfying $p+\sigma\le 0$ and $q-\sigma\ge 0$.*

Proof. The items (i) and (ii) are trivial from the definitions of the symbol classes.

(iii): Let $f_1\in\mathcal{S}_K\{p_1,q_1\}$ and $f_2\in\mathcal{S}_K\{p_2,q_2\}$. We assume that $q_1\le q_2$ without loss of generality. By Leibniz rule for any $k\in\{0,1,\dots\}$ we calculate

$$\begin{aligned}
&|\partial_t^k(f_1f_2)|\\
&\le \sum_{l=0}^k\binom{k}{l}|\partial_t^l f_1||\partial_t^{k-l}f_2|\\
&\le \langle\xi\rangle_{\lambda(t)}^{p_1+p_2}\rho(t)^{q_1+q_2+k}\sum_{l=0}^k\binom{k}{l}\frac{(q_1+l)!^s}{(q_1+l+1)^2}\frac{(q_2+k-l)!^s}{(q_2+k-l+1)^2}\\
&= \frac{(q_1+q_2+k)!^s}{(q_1+q_2+k+1)^2}\langle\xi\rangle_{\lambda(t)}^{p_1+p_2}\rho(t)^{q_1+q_2+k}\\
&\quad\times\sum_{l=0}^k\frac{k!}{l!\,(k-l)!}\left(\frac{(q_1+l)!\,(q_2+k-l)!}{(q_1+q_2+k)!}\right)^s\left(\frac{q_1+q_2+k+1}{(q_1+l+1)(q_2+k-l+1)}\right)^2.
\end{aligned}$$

Then, noting

$$\frac{k!}{l!\,(k-l)!}\left(\frac{(q_1+l)!\,(q_2+k-l)!}{(q_1+q_2+k)!}\right)^s \le \left(\frac{k!}{l!\,(k-l)!}\right)^{1-s} \le 1 \tag{4.1}$$

and

$$\begin{aligned}&\sum_{l=0}^{k}\left(\frac{(q_1+q_2+k+1)}{(q_1+l+1)(q_2+k-l+1)}\right)^2\\ &\le 2\left(\frac{2q_2+k+1}{q_2+[(k+1)/2]+1}\right)^2\sum_{l=0}^{[k/2]}\frac{1}{(l+1)^2} \le 8\sum_{l=1}^{\infty}\frac{1}{l^2} \le \frac{4}{3}\pi^2,\end{aligned} \tag{4.2}$$

where $[\,\cdot\,]$ denotes Gaussian symbol, we deduce (iii) for $\kappa = 4\pi^2/3$.

(iv): Let $f \in \mathcal{S}_K\{p, q;\kappa\}$ with $\sigma \ge 1$, $p+\sigma \le 0$ and $q-\sigma \ge 0$. Noting $NK^s\rho(t) \le \langle\xi\rangle_{\lambda(t)}$ we have

$$\begin{aligned}|\partial_t^k f| &\le \kappa\frac{(q+k)!^s}{(q+k+1)^2}\langle\xi\rangle_{\lambda(t)}^{p}\rho(t)^{q+k}\\ &\le \kappa\left(\frac{\rho_0}{N}\left(\frac{q+k}{K}\right)^s\right)^\sigma\frac{(q-\sigma+k)!^s}{(q-\sigma+k+1)^2}\langle\xi\rangle_{\lambda(t)}^{p+\sigma}\rho(t)^{q-\sigma+k}\\ &\le \kappa(3^s\rho_0N^{-1})^\sigma\frac{(q-\sigma+k)!^s}{(q-\sigma+k+1)^2}\langle\xi\rangle_{\lambda(t)}^{p+\sigma}\rho(t)^{q-\sigma+k}.\end{aligned}$$

Thus the proof of the lemma is concluded. □

Let us denote

$$\nu := \max\left\{\frac{4\pi^2}{3}, 3^s\rho_0\right\}.$$

Then the properties Lemma 4.2 (iii) and (iv) are represented by

$$\mathcal{S}_K\{p_1, q_1;\kappa_1\}\mathcal{S}_K\{p_2, q_2;\kappa_2\} \subset \mathcal{S}_K\{p_1+p_2, q_1+q_2;\nu\kappa_1\kappa_2\} \tag{4.3}$$

and

$$\mathcal{S}_K\{p, q;\kappa, \rho_0\} \subset \mathcal{S}_K\{p+\sigma, q-\sigma;\kappa(\nu N^{-1})^\sigma, \rho_0\} \tag{4.4}$$

for $\sigma \le \min\{-p, q\}$ and $q \le 2K$.

By using Lemma 4.2 we have the following lemma, corresponding to Lemma 3.4 in the proof of Theorem 2.1:

**Lemma 4.3.** *Suppose that* (1.12), (2.1) *and* (2.2) *are valid with* $C_k = \kappa_0 k!^s/(k+1)^2$. *Then we have* $\langle\xi\rangle_{m(t)}^{-1} \in \mathcal{S}_K\{-1, 0;\ \kappa_1, \rho_1\}$, *where* $\kappa_1 = 1/\inf_{(t,\xi)}\{\langle\xi\rangle_{m(t)}/\langle\xi\rangle_{\lambda(t)}\}$ *and* $\rho_1 = \nu^2\kappa_1\tilde{\kappa}_0/2$ *with* $\tilde{\kappa}_0 = \nu\kappa_0^2 + \kappa_0$.

Proof. Let us prove that there exist positive constants $\kappa_1$ and $\rho_1$ such that the estimates

$$|\partial_t^k \langle\xi\rangle_{m(t)}^{-1}| \le \kappa_1 \frac{k!^s}{(k+1)^2}\langle\xi\rangle_{\lambda(t)}^{-1}(\rho_1\rho(t))^k \tag{4.5}$$

are valid for any $k \in \{0, 1, \dots\}$ by induction. Clearly, (4.5) holds for $k = 0$. Assume that (4.5) is valid for any $j \le k$. Then we can represent $\partial_t^{k+1}\langle\xi\rangle_{m(t)}^{-1}$ by (3.12) for the proof of Lemma 3.4 and therefore the estimate (3.13) can be precise as follows:

$$\begin{aligned}
|(m(t)^2)^{(l+1)}| &\le \sum_{j=0}^{l+1}\binom{l+1}{j}|\lambda^{(j)}(t)\lambda^{(l-j+1)}(t)| + |p^{(l+1)}(t)| \\
&\le \lambda(t)^2\rho(t)^{l+1}\left(\kappa_0^2\sum_{j=0}^{l+1}\frac{(l+1)!}{j!(l-j+1)!}\frac{j!^s}{(j+1)^2}\frac{(l-j+1)!^s}{(l-j+2)^2} + \kappa_0\frac{(l+1)!^s}{(l+2)^2}\right) \\
&\le (\nu\kappa_0^2 + \kappa_0)\frac{(l+1)!^s}{(l+2)^2}\lambda(t)^2\rho(t)^{l+1}.
\end{aligned}$$

Noting the representation of $\partial_t^{k+1}\langle\xi\rangle_{m(t)}^{-1}$ in the proof of Lemma 3.4 we obtain

$$\begin{aligned}
&|\partial_t^{k+1}\langle\xi\rangle_{m(t)}^{-1}| \\
&\le \frac{\kappa_1^2\tilde{\kappa}_0}{2}\langle\xi\rangle_{m(t)}^{-3}\lambda(t)^2 \\
&\quad\times\sum_{j=0}^{k}\sum_{l=0}^{k-j}\binom{k+1}{j}\binom{k-j}{l}\frac{(l+1)!^s(k-j-l)!^s j!^s}{(l+2)^2(k-j-l+1)^2(j+1)^2}\rho(t)^{l+1}(\rho_1\rho(t))^{k-l} \\
&\le \frac{\kappa_1^2\tilde{\kappa}_0}{2\rho_1}\frac{(k+1)!^2}{(k+2)^2}\langle\xi\rangle_{m(t)}^{-1}(\rho_1\rho(t))^{k+1} \\
&\quad\times\sum_{j=0}^{k}\sum_{l=0}^{k-j}\frac{(l+1)^s}{(k-j+1)^s}\frac{(k+2)^2}{(l+2)^2(k-j-l+1)^2(j+1)^2}\rho_1^{-l} \\
&\le \kappa_1\frac{(k+1)!^2}{(k+2)^2}\langle\xi\rangle_{m(t)}^{-1}(\rho_1\rho(t))^{k+1},
\end{aligned}$$

where we used (4.2) with $(q_1, q_2, k, j) = (0, 0, j, k-l)$ and $(q_1, q_2, k, j) = (1, 0, l, k)$

in order to estimate

$$\sum_{j=0}^{k}\sum_{l=0}^{k-j}\frac{(k+2)^2}{(l+2)^2(k-j-l+1)^2(j+1)^2}$$
$$=\sum_{l=0}^{k}\frac{(k+2)^2}{(l+2)^2(k-l+1)^2}\sum_{j=0}^{k-l}\frac{(k-l+1)^2}{(k-j-l+1)^2(j+1)^2}\le \nu^2.$$

Thus the lemma is inductively proved. □

Finally, we introduce a lemma corresponding to Lemma 3.5, which is crucial for the proof of Theorem 2.3.

**Lemma 4.4.** *There exist positive constants $\gamma_1$ and $N$ independent of $K$ such that*

$$\mu_j \le 2^{-j} \tag{4.6}$$

*for $j=1,\dots,K-1$ and*

$$r_j \in \mathcal{S}_K\{-j+1,\, j;\, \gamma_1^j,\, \rho_1,\, N\}$$

*for $j=1,\dots,K$ in $Z_{\mathrm{H},K}$.*

Proof. The key difference from the proof of Lemma 3.5 is to derive the restriction in $Z_{\mathrm{H},K}$. For a philosophy of the proof we refer to [6].

Let us apply (4.1), (4.2), Lemma 4.2 (iii) and Lemma 4.3, the estimate

$$|(m(t)^2)^{(l+1)}| \le \tilde{\kappa}_0\frac{(l+1)!^s}{(l+2)^2}\lambda(t)^2\rho(t)^{l+1}$$

in the proof of Lemma 4.3 and the equality

$$\frac{\partial_t\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(t)}}=\frac{(m(t)^2)'}{2\langle\xi\rangle_{m(t)}^2}$$

in order to estimate

$$\left|\partial_t^l\left(\frac{\partial_t\langle\xi\rangle_{m(t)}}{\langle\xi\rangle_{m(t)}}\right)\right| \le \frac{1}{2}\sum_{j=0}^{l}\binom{l}{j}|(m(t)^2)^{(j+1)}||\partial_t^{l-j}\langle\xi\rangle_{m(t)}^{-2}|$$
$$\le \frac{\nu\tilde{\kappa}_0\kappa_1^2}{2}\langle\xi\rangle_{\lambda(t)}^{-2}\lambda(t)^2(\rho_1\rho(t))^{l+1}\sum_{j=0}^{l}\binom{l}{j}\frac{(l-j)!^s(j+1)!^s}{(l-j+1)^2(j+2)^2}$$

$$\le \frac{\nu\tilde{\kappa}_0\kappa_1^2}{2}\frac{(l+1)!^s}{(l+2)^2}(\rho_1\rho(t))^{l+1}\sum_{j=0}^{l}\frac{(j+1)^s}{(l+1)^s}\frac{(l+2)^2}{(l-j+1)^2(j+2)^2}$$
$$\le \iota_1\frac{(l+1)!^s}{(l+2)^2}(\rho_1\rho(t))^{l+1},$$

where $\iota_1 := \nu^2\tilde{\kappa}_0\kappa_1^2/2$. It follows that

$$r_1 \in \mathcal{S}_K\{0, 1;\ \iota_1, \rho_1\}$$

for $\gamma_1 \ge \iota_1$. Therefore, by (4.3), (4.4), Lemma 4.2 (iv) and Lemma 4.3 we have

$$\mu_1 = \frac{|r_1|^2}{4\phi_{1,\Im}^2} = \frac{1}{4}|r_1|^2\langle\xi\rangle_{m(t)}^{-2} \in \mathcal{S}_K\{0, 2;\ \nu\iota_1^2, \rho_1\}\mathcal{S}_K\left\{-2, 0;\ \frac{\nu\kappa_1^2}{4}, \rho_1\right\}$$
$$\subset \mathcal{S}_K\left\{-2, 2;\ \frac{\nu^3\kappa_1^2\iota_1^2}{4}, \rho_1\right\} \subset \mathcal{S}_K\{0, 0;\ (\iota_2 N^{-1})^2, \rho_1\},$$

where $\iota_2 := \nu^3\kappa_1\iota_1/2$. Hence, this implies (4.6) for $j = 1$ with $N \ge \sqrt{2}\iota_2$. Moreover, we introduce

$$d_j = d_j(t, \xi) := \phi_{j,\Im} - \langle\xi\rangle_{m(t)}.$$

By Lemma 3.1 and noting

$$-2\mu_1\phi_{1,\Im} = -\frac{1}{2}|r_1|^2\langle\xi\rangle_{m(t)}^{-1} \in \mathcal{S}_K\left\{-1, 2;\ \frac{\nu^2\iota_1^2\kappa_1}{2}, \rho_1\right\} \subset \mathcal{S}_K\left\{0, 1;\ \frac{\nu^3\iota_1^2\kappa_1 N^{-1}}{2}, \rho_1\right\},$$
$$\frac{r_1}{2\phi_{1,\Im}}\partial_t\frac{\overline{r_1}}{2\phi_{1,\Im}} \in \mathcal{S}_K\left\{-2, 3;\ \frac{\nu^3\iota_1^2\kappa_1^2}{4}, \rho_1\right\} \subset \mathcal{S}_K\left\{0, 1;\ \frac{\nu^5\iota_1^2\kappa_1^2 N^{-2}}{16}, \rho_1\right\},$$
$$(1-\mu_1)^{-1} = \sum_{l=0}^{\infty}\mu_1^l \in \mathcal{S}_K\{0, 0; 2, \rho_1\},$$

we get

$$d_2 = \frac{1}{1-\mu_1}\left(-2\mu_1\phi_{1,\Im} + \Im\left\{\frac{r_1}{2\phi_{1,\Im}}\partial_t\frac{\overline{r_1}}{2\phi_{1,\Im}}\right\}\right) \in \mathcal{S}_K\{0, 1; 1-2^{-2}, \rho_1\}$$

for any sufficiently large $N$. Let us assume that

$$r_j \in \mathcal{S}_K\{-j+1, j;\ \gamma_1^j, \rho_1\}$$

and

$$d_j \in \mathcal{S}_K\{0, 1;\ 1-2^{-j}, \rho_1\}$$

for $j \geq 2$. By (4.3) and (4.4) we have

$$d_j \langle \xi \rangle_{m(t)}^{-1} \in \mathcal{S}_K\{-1, 1;\ \nu\kappa_1, \rho_1\}.$$

It follows that

$$(-d_j \langle \xi \rangle_{m(t)}^{-1})^l \in \mathcal{S}_K\{-l, l;\ (\nu^2\kappa_1)^l, \rho_1\} \subset \mathcal{S}_K\{0, 0;\ (\nu^3\kappa_1 N^{-1})^l, \rho_1\} \subset \mathcal{S}_K\{0, 0;\ 2^{-l}, \rho_1\}$$

for $N \geq 2\nu^3\kappa_1$. Therefore, noting

$$\frac{1}{\phi_{k,\Im}} = \frac{\langle \xi \rangle_{m(t)}^{-1}}{1 + d_j \langle \xi \rangle_{m(t)}^{-1}} = \langle \xi \rangle_{m(t)}^{-1} \sum_{l=0}^{\infty} (-d_j \langle \xi \rangle_{m(t)}^{-1})^l \in \mathcal{S}_K\{-1, 0;\ 2\nu\kappa_1, \rho_1\}, \tag{4.7}$$

we obtain

$$\begin{aligned} \mu_j = \frac{|r_j|^2}{4\phi_{j,\Im}^2} &\in \mathcal{S}_K\{-2j+2, 2j;\ \nu\gamma_1^{2j}, \rho_1\}\mathcal{S}_K\{-2, 0;\ \nu^3\kappa_1^2, \rho_1\} \\ &\subset \mathcal{S}_K\{-2j, 2j;\ \nu^5\kappa_1^2\gamma_1^{2j}, \rho_1\}, \end{aligned} \tag{4.8}$$

moreover,

$$\mu_j \in \mathcal{S}_K\{0, 0;\ (\nu^3\kappa_1)^2(\gamma_1\nu N^{-1})^{2j}, \rho_1\} \subset \mathcal{S}_K\{0, 0;\ (\nu^4\kappa_1\gamma_1 N^{-1})^{2j}, \rho_1\}.$$

Thus

$$\mu_j^l \in \mathcal{S}_K\{0, 0;\ (\nu^5\kappa_1\gamma_1 N^{-1})^{2jl}, \rho_1\} \subset \mathcal{S}_K\{0, 0;\ (2^{-j})^l, \rho_1\}$$

for $N \geq \nu^5\kappa_1\gamma_1/\sqrt{2}$. This implies

$$\frac{1}{1-\mu_j} \in \mathcal{S}_K\{0, 0;\ (1-2^{-j})^{-1}, \rho_1\} \subset \mathcal{S}_K\{0, 0;\ 2, \rho_1\}.$$

Consequently, by (4.2), (4.7) and (4.8) we have

$$\begin{aligned} \mu_j r_j &\in \mathcal{S}_K\{-2j, 2j;\ \nu^5\kappa_1^2\gamma_1^{2j}, \rho_1\}\mathcal{S}_K\{-j+1, j;\ \gamma_1^j, \rho_1\} \\ &\subset \mathcal{S}_K\{-1, 1;\ \nu^5\kappa_1^2\gamma_1^{2j}(\nu N^{-1})^{2j-1}, \rho_1\}\mathcal{S}_K\{-j+1, j;\ \gamma_1^j, \rho_1\} \\ &\subset \mathcal{S}_K\{-j, j+1;\ \nu^6\kappa_1^2\gamma_1^{3j}(\nu N^{-1})^{2j-1}, \rho_1\} \end{aligned}$$

and

$$\frac{r_j}{2\phi_{j,\Im}} \in \mathcal{S}_K\{-j, j;\ \nu^2\kappa_1\gamma_1^j, \rho_1\}$$

for $N \geq \nu$. According to representation (3.5) we get

$$\begin{aligned} r_{j+1} &= \frac{1}{1-\mu_j}\left(\mu_j r_j + \partial_t \frac{i r_j}{2\phi_{j,\Im}}\right) \\ &\in \mathcal{S}_K\{-j, j+1;\ 2\nu(\nu^6\kappa_1^2\gamma_1^{3j}(\nu N^{-1})^{2j-1} + \nu^2\kappa_1\gamma_1^j), \rho_1\}. \end{aligned}$$

As well, we conclude

$$\frac{|r_j|^2}{2\phi_{j,\Im}} \in \mathcal{S}_K\{-2j+1, 2j;\ \nu^3\kappa_1\gamma_1^{2j}, \rho_1\} \subset \mathcal{S}_K\{0, 1;\ \nu^3\kappa_1\gamma_1(\gamma_1\nu N^{-1})^{2j-1}, \rho_1\}$$

and

$$\frac{\overline{r_j}}{2\phi_{j,\Im}}\partial_t \frac{r_j}{2\phi_{j,\Im}} \in \mathcal{S}_K\{-2j, 2j+1;\ \nu^5\kappa_1^2\gamma_1^{2j}, \rho_1\} \subset \mathcal{S}_K\{0, 1;\ \nu^5\kappa_1^2(\gamma_1\nu N^{-1})^{2j}, \rho_1\}.$$

This leads to

$$\begin{aligned}&\frac{1}{1-\mu_j}\left(-\frac{|r_j|^2}{2\phi_{j,\Im}} + \Im\left\{\frac{r_j}{2\phi_{j,\Im}}\partial_t \frac{\overline{r_j}}{2\phi_{j,\Im}}\right\}\right)\\ &\in \mathcal{S}_K\{0, 1; 2\nu(\nu^3\kappa_1\gamma_1(\gamma_1\nu N^{-1})^{2j-1} + \nu^5\kappa_1^2(\gamma_1\nu N^{-1})^{2j}), \rho_1\}.\end{aligned}$$

As consequence we have $r_{j+1} \in \mathcal{S}_K\{-j, j+1;\ \gamma_1^{j+1}, \rho_1\}$ and

$$d_{j+1} = d_j + \frac{1}{1-\mu_j}\left(-\frac{|r_j|^2}{2\phi_{j,\Im}} + \Im\left\{\frac{r_j}{2\phi_{j,\Im}}\partial_t \frac{\overline{r_j}}{2\phi_{j,\Im}}\right\}\right) \in \mathcal{S}_K\{0, 1;\ 1-2^{-j-1}, \rho_1\}$$

if the constants $N$ and $\gamma_1$ exist such that

$$\begin{cases} 2\nu^7\kappa_1^2\gamma_1^{3j}(\nu N^{-1})^{2j-1} + 2\nu^3\kappa_1\gamma_1^j \le \gamma_1^{j+1}, \\ 2\nu^4\kappa_1\gamma_1(\gamma_1\nu N^{-1})^{2j-1} + 2\nu^6\kappa_1^2(\gamma_1\nu N^{-1})^{2j} \le 2^{-j-1}. \end{cases} \tag{4.9}$$

Indeed, we easily observe that for $\gamma_1 := 4\nu^3\kappa_1$ there exists a positive constant $N_0$ such that for any $N \ge N_0$ the estimates (4.9) are valid uniformly with respect to $j \ge 2$. □

**4.3. Conclusion of the proof of Theorem 2.3.** We shall prove the estimate from above for $\mathcal{E}(v)(t, \xi)$ of (1.5); the other case is analogously proved.

Let us fix $(\tau, \eta) \in Z_{\mathrm{H}}$ arbitrarily. Then we can suppose that there exists a positive integer $K$ such that $(\tau, \eta) \in Z_{\mathrm{H},K+1}$. We restrict ourselves that the order $t_0(\eta) < t_1(\eta) < \cdots < t_K(\eta) < \tau \le t_{K+1}(\eta)$ is valid, because in the other cases we would have better estimates.

Let us denote $\Lambda_k := \int_{t_{k-1}}^{t_k} |r_k(\tau,\eta)|\,d\tau$, $\mu_{j,k} := \mu_j(t_k,\eta)$ and $\vartheta_k := \prod_{j=1}^{k}(1-\mu_{j,k})(1-\mu_{j,k+1})^{-1}$. By (3.7) and (3.8) we derive the estimates

$$\begin{aligned}|V_1(\tau, \eta)|^2 &\le \left(1 + \sqrt{\mu_1(\tau, \eta)}\right)^2 |V_2(\tau, \eta)|^2\\ &\cdots\\ &\le \left(\prod_{j=1}^{K}\left(1 + \sqrt{\mu_j(\tau, \eta)}\right)^2\right)|V_{K+1}(\tau, \eta)|^2,\end{aligned}$$

$$
\begin{aligned}
|V_{K+1}(\tau,\eta)|^2 &\le \frac{\langle\eta\rangle_{m(\tau)}}{\langle\eta\rangle_{m(t_K)}}\left(\prod_{j=1}^{K}\frac{1-\mu_j(t_K,\eta)}{1-\mu_j(\tau,\eta)}\right)\exp\left(4\int_{t_K}^{t}|r_{K+1}(\tau,\eta)|\,d\tau\right)|V_{K+1}(t_K,\eta)|^2\\
&\le \frac{\langle\eta\rangle_{m(\tau)}}{\langle\eta\rangle_{m(t_K)}}\left(\prod_{j=1}^{K}\frac{1-\mu_{j,K}}{1-\mu_j(\tau,\eta)}\right)\frac{\exp(4\Lambda_{K+1})}{(1-\sqrt{\mu_{K,K}})^2}|V_K(t_K,\eta)|^2
\end{aligned}
$$

and

$$
\begin{aligned}
|V_K(t_K,\eta)|^2 &\le \frac{\langle\eta\rangle_{m(t_K)}}{\langle\eta\rangle_{m(t_{K-1})}}\vartheta_{K-1}\exp(4\Lambda_K)|V_K(t_{K-1},\eta)|^2\\
&\le \frac{\langle\eta\rangle_{m(t_K)}}{\langle\eta\rangle_{m(t_{K-1})}}\frac{\vartheta_{K-1}\exp(4\Lambda_K)}{(1-\sqrt{\mu_{K-1,K-1}})^2}|V_{K-1}(t_{K-1},\eta)|^2\\
&\le \frac{\langle\eta\rangle_{m(t_K)}}{\langle\eta\rangle_{m(t_{K-2})}}\frac{\vartheta_{K-1}\vartheta_{K-2}\exp(4\Lambda_K+4\Lambda_{K-1})}{(1-\sqrt{\mu_{K-1,K-1}})^2(1-\sqrt{\mu_{K-2,K-2}})^2}|V_{K-2}(t_{K-2},\eta)|^2\\
&\dots\\
&\le \frac{\langle\eta\rangle_{m(t_K)}}{\langle\eta\rangle_{m(t_1)}}\frac{\left(\prod_{k=1}^{K-1}\vartheta_k\right)\exp\left(4\sum_{k=2}^{K}\Lambda_k\right)}{\prod_{j=1}^{K-1}(1-\sqrt{\mu_{j,j}})^2}|V_1(t_1,\eta)|^2.
\end{aligned}
$$

Here we note that

$$
\prod_{k=1}^{K-1}\vartheta_k=\prod_{k=1}^{K-1}\prod_{j=1}^{k}\frac{1-\mu_{j,k}}{1-\mu_{j,k+1}}=\prod_{j=1}^{K-1}\prod_{k=j}^{K-1}\frac{1-\mu_{j,k}}{1-\mu_{j,k+1}}=\prod_{j=1}^{K-1}\frac{1-\mu_{j,j}}{1-\mu_{j,K}}.
$$

Hence, we calculate

$$
\frac{\prod_{k=1}^{K-1}\vartheta_k}{\prod_{j=1}^{K-1}(1-\sqrt{\mu_{j,j}})^2}=\prod_{j=1}^{K-1}\frac{1-\mu_{j,j}}{(1-\mu_{j,K})(1-\sqrt{\mu_{j,j}})^2}=\prod_{j=1}^{K-1}\frac{1+\sqrt{\mu_{j,j}}}{(1-\mu_{j,K})(1-\sqrt{\mu_{j,j}})}.
$$

Moreover, we have

$$
\begin{aligned}
&\left(\prod_{j=1}^{K}(1+\sqrt{\mu_j(\tau,\eta)})^2\right)\left(\prod_{j=1}^{K}\frac{1-\mu_{j,K}}{1-\mu_j(\tau,\eta)}\right)\frac{1}{(1-\sqrt{\mu_{K,K}})^2}\prod_{j=1}^{K-1}\frac{1+\sqrt{\mu_{j,j}}}{(1-\mu_{j,K})(1-\sqrt{\mu_{j,j}})}\\
&=\left(\prod_{j=1}^{K}\frac{1+\sqrt{\mu_j(\tau,\eta)}}{1-\sqrt{\mu_j(\tau,\eta)}}\right)\prod_{j=1}^{K}\frac{1+\sqrt{\mu_{j,j}}}{1-\sqrt{\mu_{j,j}}}\\
&=:\Xi_K(\tau,\eta).
\end{aligned}
$$

By Lemma 4.4 we estimate

$$\begin{aligned}\Xi_K(\tau,\eta) &\le \left(\prod_{j=1}^{K}\frac{1+2^{-j/2}}{1-2^{-j/2}}\right)^2 = \exp\left(2\sum_{j=1}^{K}\log\left(1+\frac{2^{1-j/2}}{1-2^{-j/2}}\right)\right)\\ &\le \exp\left(2\sum_{j=1}^{K}\frac{2^{1-j/2}}{1-2^{-j/2}}\right) \le \exp\left(\frac{4}{1-2^{-1/2}}\sum_{j=1}^{K}2^{-j/2}\right) \le \exp\left(\frac{4\sqrt{2}}{(\sqrt{2}-1)^2}\right)\\ &\le e^{33}.\end{aligned}$$

Then, by Lemma 4.1, Lemma 4.4, definition of $Z_{\mathrm{H},j}$ and Stirling's formula $k! = \sigma_k\sqrt{k}e^{-k}k^k$, where $\sigma_k$ satisfies $\sigma_- \le \sigma_k \le \sigma_+$ by positive constants $\sigma_\pm$, we achieve

$$\begin{aligned}\Lambda_k &\le \gamma_1^k\frac{k!^s}{(k+1)^2}\int_{t_{k-1}}^{t_k}\langle\xi\rangle_{\lambda(t)}^{-k+1}(\rho_1\rho(t))^k\,dt\\ &\le \gamma_1^k\frac{k!^s}{(k+1)^2}\int_{t_{k-1}}^{t_k}(Nk^s\rho(t))^{-k+1}(\rho_1\rho(t))^k\,dt\\ &\le \rho(0)N\left(\frac{k!}{k^k}\right)^s\frac{k^s}{(k+1)^2}\left(\frac{\gamma_1\rho_1}{N}\right)^k t_k\\ &\le \rho(0)N\sigma_+^s k^{(3s/2)-2}e^{-sk}\left(\frac{\gamma_1\rho_1 e^{c_0}}{N}\right)^k\\ &\le CN2^{-k}\end{aligned}$$

for $N \ge 2\gamma_1\rho_1 e^{c_0}$, where $C = \rho(0)\sigma_+^s\sup_j\{j^{(3s/2)-2}e^{-sj}\}$. Summarizing the estimates above we obtain

$$\begin{aligned}|V_1(\tau,\eta)|^2 &\le \Xi(\tau,\eta)\exp\left(4\sum_{k=2}^{K}\Lambda_k\right)\frac{\langle\eta\rangle_{m(\tau)}}{\langle\eta\rangle_{m(t_1)}}|V_1(\tau,\eta)|^2\\ &\le e^{33+CN}\frac{\langle\eta\rangle_{m(\tau)}}{\langle\eta\rangle_{m(t_1)}}|V_1(\tau,\eta)|^2\\ &\simeq \frac{\langle\eta\rangle_{m(\tau)}}{\langle\eta\rangle_{m(t_1)}}|V_1(\tau,\eta)|^2.\end{aligned}$$

The function $t_1 = t_1(\xi)$ is uniformly bounded with respect to $\eta \in \mathbb{R}^n$, hence (1.5) with $t = t_1$ is trivial. Consequently, recalling (3.9) the proof of Theorem 2.3 is concluded. □

ACKNOWLEDGEMENTS. Part of the research was done during the stay of the first author at the Faculty of Science of Yamaguchi University in March–August 2009. The stay was supported by a JSPS grant. The first author thanks the Department of Mathematical Sciences of Yamaguchi University for hospitality and useful discussions.

## References


[1] C. Böhme: *Generalized energy conservation for wave models with time-dependent coefficients*, Master's thesis, TU Bergakademie Freiberg, Germany, 2008.

[2] C. Böhme and M. Reissig: *Energy estimates for wave models with time-dependent coefficients*; in Further Progress in Analysis, World Sci. Publ., Hackensack, NJ., 2009, 415–424.

[3] C. Böhme and M. Reissig: *Energy bounds for Klein–Gordon equations with time-dependent potential*, submitted to Annali Dell'Universita' di Ferrara.

[4] D. Del Santo, T. Kinoshita and M. Reissig: *Klein–Gordon type equations with a singular time-dependent potential*, Rend. Istit. Mat. Univ. Trieste **39** (2007), 1–35.

[5] F. Hirosawa: *On the asymptotic behavior of the energy for the wave equations with time depending coefficients*, Math. Ann. **339** (2007), 819–838.

[6] F. Hirosawa: *Energy estimates for wave equations with time dependent propagation speeds in the Gevrey class*, J. Differential Equations **248** (2010), 2972–2993.

[7] F. Hirosawa and M. Reissig: *From wave to Klein–Gordon type decay rates*; in Nonlinear Hyperbolic Equations, Spectral Theory, and Wavelet Transformations, Oper. Theory Adv. Appl. **145** Birkhäuser, Basel, 95–155, 2003.

[8] F. Hirosawa and J. Wirth: *$C^m$-theory of damped wave equations with stabilisation*, J. Math. Anal. Appl. **343** (2008), 1022–1035.

[9] M. Reissig and J. Smith: *$L^p$-$L^q$ estimate for wave equation with bounded time dependent coefficient*, Hokkaido Math. J. **34** (2005), 541–586.

[10] M. Reissig and K. Yagdjian: *One application of Floquet's theory to $L_p$–$L_q$ estimates for hyperbolic equations with very fast oscillations*, Math. Methods Appl. Sci. **22** (1999), 937–951.

[11] M. Reissig and K. Yagdjian: *Klein–Gordon type decay rates for wave equations with time-dependent coefficients*; in Evolution Equations: Existence, Regularity and Singularities (Warsaw, 1998), Banach Center Publ. **52**, Polish Acad. Sci., Warsaw, 189–212, 2000.

[12] J. Wirth: *Solution representations for a wave equation with weak dissipation*, Math. Methods Appl. Sci. **27** (2004), 101–124.

[13] J. Wirth: *Wave equations with time-dependent dissipation*, II: *Effective dissipation*, J. Differential Equations **232** (2007), 74–103.



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