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A CONSTRUCTION
OF SURFACE BUNDLES OVER SURFACES
WITH NON-ZERO SIGNATURE

Hisaoaki Endo†

(Received June 11, 1997)

1. Introduction

Let \( \Sigma_g \) (respectively \( \Sigma_h \)) be a closed oriented surface of genus \( g \) (respectively \( h \)), where \( g \) (respectively \( h \)) is a non-negative integer. Let \( \text{Diff}_+\Sigma_h \) be the group of all orientation-preserving diffeomorphisms of \( \Sigma_h \) with \( C^\infty \)-topology. A \( \Sigma_h \)-bundle over \( \Sigma_g \) (also called a surface bundle over a surface) is fiber bundle \( \xi = (E, \Sigma_g, p, \Sigma_h, \text{Diff}_+\Sigma_h) \) over \( \Sigma_g \) with total space \( E \), fiber \( \Sigma_h \), projection \( p : E \to \Sigma_g \) and structure group \( \text{Diff}_+\Sigma_h \). Our main concern is the signature \( \tau(E) \) of the total space \( E \) of \( \xi \).

It is easily seen that if \( \xi \) is a trivial bundle then \( \tau(E) = \tau(\Sigma_g)\tau(\Sigma_h) = 0 \). Chern-Hirzebruch-Serre [5] proved that if the fundamental group \( \pi(\Sigma_g) \) of \( \Sigma_g \) acts trivially on the cohomology ring \( H^*(\Sigma_h; \mathbb{R}) \) of \( \Sigma_h \) then \( \tau(E) = 0 \).

Kodaira [12] and Atiyah [1] gave examples of surface bundles over surfaces with non-zero signature. For each pair \( (m, t) \) of integers \( m, t \in \mathbb{Z} (m \geq 2, t \geq 3) \), Kodaira constructed a surface bundle \( \xi = \xi(m, t) \) with

\[
\begin{align*}
g &= m^2 t (t - 1) + 1, \\
h &= mt, \\
\tau(E) &= \frac{4}{3} m^{2t-1} (t - 1)(m^2 - 1).
\end{align*}
\]

By setting \( m = 2 \) and \( t = 3 \), we obtain a surface bundle \( \xi = \xi(2, 3) \) with \( g = 129, h = 6 \) and \( \tau(E) = 256 \). The total space \( E \) of the bundle \( \xi = \xi(m, t) \) is an \( m \)-fold branched covering of \( \Sigma_g \times \Sigma_t \) and its signature \( \tau(E) \) can be calculated by using \( G \)-signature theorem (see [9] and [11]).

Meyer [16], [17] gave a signature formula for surface bundles over surfaces in terms of the signature cocycle \( \tau_h \), which is a 2-cocycle of the Siegel modular group \( \text{Sp}(2h, \mathbb{Z}) \) of degree \( h \). Using the signature cocycle and Birman-Hilden’s relations [3] of mapping class groups of surfaces, he showed that if \( h = 1, 2 \) or \( g = 1 \) then

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\[ \tau(E) = 0. \] But he also showed that for every \( h \geq 3 \) and every \( n \in \mathbb{Z} \) there exist an integer \( g \geq 0 \) and a \( \Sigma_h \)-bundle \( \xi \) over \( \Sigma_g \) such that \( \tau(E) = 4n \).

We consider the following problem:

**Problem 1.1.** For each \( h \geq 3 \) and each \( n \in \mathbb{Z} \), let \( g(h, n) \) be the minimum value of the genus \( g \) such that there exists a \( \Sigma_h \)-bundle \( \xi \) over \( \Sigma_g \) with \( \tau(E) = 4n \). Determine the value \( g(h, n) \).

In this paper, we estimate the value \( g(h, n) \) by using Wajnryb's presentation\[19\] of the mapping class group \( \mathcal{M}_h \) of \( \Sigma_h \).

Our main result is:

**Theorem 1.2.** For each \( h \geq 3 \) and each \( n \in \mathbb{Z} (n \neq 0) \), the following inequality holds:

\[ \frac{|n|}{h - 1} + 1 \leq g(h, n) \leq 111|n|. \]

We construct a \( \Sigma_h \)-bundle \( \xi \) over \( \Sigma_g \) with \( g = 111 \), \( h = 3 \) and \( \tau(E) = -4 \) to prove Theorem 1.2. The genus of the base space of this bundle and that of a fiber of it are smaller than those of any example constructed by Kodaira \[12\] and Atiyah \[1\].

In Section 2, we review Meyer's work \[16\], \[17\] on signature of surface bundles over surfaces. And in Section 3, we calculate the values of Meyer's signature cocycle for the relators of Wajnryb's presentation \[19\] of the mapping class group \( \mathcal{M}_h \) and characterize the 2-cycles of \( \mathcal{M}_h \) as words in the generators of the presentation of \( \mathcal{M}_h \). We prove our main theorem in Section 4 by using this characterization and a simple technique of the commutator collection process \[7\]. In Section 2, we review Meyer's work \[16\], \[17\] on signature of surface bundles over surfaces. And in Section 3, we calculate the values of Meyer's signature cocycle for the relators of Wajnryb's presentation \[19\] of the mapping class group \( \mathcal{M}_h \) and characterize the 2-cycles of \( \mathcal{M}_h \) as words in the generators of the presentation of \( \mathcal{M}_h \). We prove our main theorem in Section 4 by using this characterization and a simple technique of the commutator collection process \[7\].

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### 2. Meyer's signature formula

In this section we review Meyer's signature cocycle and Meyer's signature formula \[16\], \[17\] for surface bundles over surfaces.

For a pair \( (\alpha, \beta) \) of symplectic matrices \( \alpha, \beta \in Sp(2h, \mathbb{Z}) \), the vector space \( V_{\alpha,\beta} \)
is defined by:

\[ V_{\alpha, \beta} := \{(x, y) \in \mathbb{R}^{2h} \times \mathbb{R}^{2h} \mid (\alpha^{-1} - I)x + (\beta - I)y = 0\}, \]

where \( I \) is the identity matrix. Consider the (possibly degenerate) symmetric bilinear form

\[ (\ , \ )_{\alpha, \beta} : V_{\alpha, \beta} \times V_{\alpha, \beta} \to \mathbb{R} \]
on \( V_{\alpha, \beta} \) defined by:

\[ \langle (x_1, y_1), (x_2, y_2) \rangle_{\alpha, \beta} := \langle x_1 + y_1, (I - \beta)x_2 \rangle, \]

\[ (x_i, y_i) \in V_{\alpha, \beta} \quad (i = 1, 2), \]

where \( \langle \ , \ \rangle \) is the standard symplectic form on \( \mathbb{R}^{2h} \) given by:

\[ \langle x, y \rangle = ^t x J y \quad (x, y \in \mathbb{R}^{2h}), \]

\[ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \text{M}_{2h}(\mathbb{R}). \]

**Meyer's signature cocycle** [16], [17]

\[ \tau_h : \text{Sp}(2h, \mathbb{Z}) \times \text{Sp}(2h, \mathbb{Z}) \to \mathbb{Z} \]
is defined by:

\[ \tau_h(\alpha, \beta) := \text{sign}(V_{\alpha, \beta}, \langle \ , \ \rangle_{\alpha, \beta}) \]

\[ (\alpha, \beta \in \text{Sp}(2h, \mathbb{Z})). \]

From the Novikov additivity, \( \tau_h \) is a 2-cocycle of \( \text{Sp}(2h, \mathbb{Z}) \) and represents a cohomology class \([\tau_h]\) \( \in H^2(\text{Sp}(2h, \mathbb{Z}), \mathbb{Z}). \)

Let \( \mathcal{M}_h \) be the mapping class group of a surface \( \Sigma_h \) of genus \( h \), namely it is the group of all isotopy classes of orientation-preserving diffeomorphisms of \( \Sigma_h \). By choosing a symplectic basis on \( H^1(\Sigma_h; \mathbb{Z}) \cong \mathbb{Z}^{2h} \), the natural action of \( \mathcal{M}_h \) on \( H^1(\Sigma_h; \mathbb{Z}) \) induces a representation \( \sigma : \mathcal{M}_h \to \text{Sp}(2h, \mathbb{Z}). \)

Next, we define a homomorphism \( k : H_2(\mathcal{M}_h; \mathbb{Z}) \to \mathbb{Z} \) by using \( \tau_h \) and \( \sigma \). It is known that the group \( \mathcal{M}_h \) is finitely presentable, so there exists an exact sequence:

\[ 1 \to R \to F \xrightarrow{\pi} \mathcal{M}_h \to 1, \]

where \( F \) is a free group of finite rank generated by a free basis \( E = \{e_\lambda\}_{\lambda \in \Lambda} \). By well known Hopf's theorem (cf. [4]) the following isomorphism holds:

\[ H_2(\mathcal{M}_h; \mathbb{Z}) \cong R \cap [F, F]/[R, F]. \]
The map \( c : F \to \mathbb{Z} \) is defined by:

\[
c(x) := \sum_{j=1}^{m} \tau_h(\sigma(\pi(\bar{x}_{j-1})), \sigma(\pi(x_j)))
\]

\[
\left( x = \prod_{j=1}^{m} x_i, x_i \in E \cup E^{-1}, \bar{x}_j = \prod_{i=1}^{j} x_i \right).
\]

It can be checked that the restriction \( c|_R : R \to \mathbb{Z} \) is actually a homomorphism and that \( c([R, F]) = 0 \). Hence \( c|_R \) naturally induces a homomorphism \( k : H_2(M_h; \mathbb{Z}) \cong R \cap [F, F]/[R, F] \to \mathbb{Z} \).

Now, we describe Meyer's signature formula for surface bundles over surfaces.

Let \( \xi = (E, \Sigma_g, p, \Sigma_h, D_{\Sigma}^+ \Sigma_h) \) be a \( \Sigma_h \)-bundle over \( \Sigma_g \) and \( f : \Sigma_g \to BD\Sigma^+ \Sigma_h \) its classifying map. The map \( f \) induces a homomorphism \( \chi \) between fundamental groups:

\[
\chi := f_\# : \pi_1(\Sigma_g) \to \pi_1(BD_{\Sigma}^+ \Sigma_h) \cong \pi_0(D_{\Sigma}^+ \Sigma_h) \cong \mathcal{M}_h,
\]

which is called the holonomy homomorphism of \( \xi \) (cf. [18]). By a theorem of Earle and Eells [6], which states that the connected component \( D_{\Sigma}^0 \Sigma_h \) of the identity of \( D_{\Sigma}^+ \Sigma_h \) is contractible if \( h \geq 2 \), the isomorphism class of \( \xi \) is completely determined by its holonomy homomorphism \( \chi \) when \( h \geq 2 \) (see [16], [17] and [18]). From now on, we suppose that \( h \geq 2 \) and \( g \geq 1 \).

The fundamental group \( \pi_1(\Sigma_g) \) of \( \Sigma_g \) is finitely presented, so we have an exact sequence:

\[
1 \to \tilde{R} \to \tilde{F} \xrightarrow{	ilde{\phi}} \pi_1(\Sigma_g) \to 1,
\]

where

\[
\pi_1(\Sigma_g) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \left( = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right) = 1 \right\>,
\]

\[
\tilde{F} = \langle \tilde{a}_1, \ldots, \tilde{a}_g, \tilde{b}_1, \ldots, \tilde{b}_g \rangle,
\]

\[
\tilde{\pi} : \tilde{a}_i \mapsto a_i, \tilde{b}_i \mapsto b_i
\]

and \( \tilde{R} \) is the normal closure of \( \tilde{\phi} := \prod_{i=1}^{g} [\tilde{a}_i, \tilde{b}_i] = \prod_{i=1}^{g} \tilde{a}_i \tilde{b}_i \tilde{a}_i^{-1} \tilde{b}_i^{-1} \) in \( \tilde{F} \). Hopf's theorem allows us to identify \( H_2(\pi_1(\Sigma_g); \mathbb{Z}) \) with \( R \cap [F, F]/[R, F] \). For the holonomy homomorphism \( \chi \), we can choose a homomorphism \( \tilde{\chi} : \tilde{F} \to F \) so that \( \pi \circ \tilde{\chi} = \chi \circ \tilde{\pi} \). Then the induced homomorphism \( \chi_* : H_2(\pi_1(\Sigma_g); \mathbb{Z}) \to H_2(M_h; \mathbb{Z}) \) is defined by:

\[
\chi_*(\tilde{x}[\tilde{R}, \tilde{F}]) := \tilde{\chi}(\tilde{x})[R, F] \quad (\tilde{x} \in \tilde{R} \cap [\tilde{F}, \tilde{F}])
\]
A CONSTRUCTION OF SURFACE BUNDLES

and is not depend on a choice of \( \tilde{\chi} \).

Meyer proved the following theorem by using the Leray-Serre spectral sequence for \( \xi \) and the cohomology group \( H^1(\Sigma_g; H_1(\Sigma_h; \mathbb{R})) \) of \( \Sigma_g \) with local coefficients.

**Theorem 2.1** (Meyer [16, 17]). Let \( \xi = (E, \Sigma_g, p, \Sigma_h, \text{Diff}_+ \Sigma_h) \) be a \( \Sigma_h \)-bundle over \( \Sigma_g \) \((h \geq 2, g \geq 1)\) and \( \chi : \pi_1(\Sigma_g) \to \mathcal{M}_h \) its holonomy homomorphism. Then the following equality holds:

\[
\tau(E) = -k(\chi_*(\tilde{\tau}[\tilde{R}, \tilde{F}])) = -c(\chi(\tilde{r})).
\]

3. **Explicit description of 2-cycles of \( \mathcal{M}_h \)**

In this section, we calculate values of the map \( c : F \to \mathbb{Z} \) for the relators of the finite presentation of \( \mathcal{M}_h \) due to Wajnryb and give an explicit description of the homomorphism \( k \) defined in the preceding section in order to characterize the elements of \( R \cap [F, F] \) as words of \( F \).

Let \( \mathcal{M}_h \) be the mapping class group of a surface \( \Sigma_h \) of genus \( h \). A finite presentation of \( \mathcal{M}_2 \) was obtained by Birman-Hilden [3] and that of \( \mathcal{M}_h \) \((h \geq 3)\) by Hatcher-Thurston [8].

Wajnryb [19] simplified their presentation of \( \mathcal{M}_h \) \((h \geq 2)\) as foll ws. (We denote the commutator \( xyx^{-1}y^{-1} \) of \( x, y \in F \) by \([x, y]\).)

The generators, which are called the Lickorish-Humphries generators, of the presentation are:

\[
y_1, y_2, u_1, u_2, \ldots, u_h, z_1, z_2, \ldots, z_{h-1}
\]

and the relators of it are:

\[
A^1 := [y_1, y_2], \\
A_{i,j}^2 := [y_i, u_j] \quad (i = 1, 2, 1 \leq j \leq h, i \neq j), \\
A_{i,j}^3 := [y_i, z_j] \quad (i = 1, 2, 1 \leq j \leq h - 1), \\
A_{i,j}^4 := [u_i, u_j] \quad (1 \leq i < j \leq h), \\
A_{i,j}^5 := [u_i, z_j] \quad (1 \leq i \leq h, 1 \leq j \leq h - 1, j \neq i, i + 1), \\
A_{i,j}^6 := [z_i, z_j] \quad (1 \leq i < j \leq h - 1), \\
B_i^1 := y_iz_iy_iy_i^{-1}u_i^{-1}u_i^{-1} \quad (i = 1, 2), \\
B_i^2 := u_iz_iu_i^{-1}u_i^{-1}z_i^{-1} \quad (1 \leq i \leq h - 1), \\
B_i^3 := z_iu_{i+1}z_iu_{i+1}^{-1}z_i^{-1}u_{i+1}^{-1} \quad (1 \leq i \leq h - 1), \\
C^1 := (y_1u_1z_1)^{-1}y_2(u_2z_1u_1y_2^{-1}u_1z_1u_2)^{-1}y_2(u_2z_1u_1y_2^{-1}u_1z_1u_2), \\
D^1 := y_1z_1z_2u_1t_2(y_1y_2t_2^{-1}t_1t_2y_2^{-1}u_1u_2u_3)^{-1}u_1u_2u_3, \\
E^1 := [d_u, u_hz_h^{-1}u_h^{-1} \cdots z_1u_1y_1^2u_1z_1 \cdots u_{h-1}z_{h-1}u_h],
\]
where

\[ t_1 := u_1 y_1 z_1 u_1, \]
\[ t_i := u_i z_{i-1} z_i u_i \quad (2 \leq i \leq h - 1), \]
\[ v := y_1 u_1 z_1 u_2 y_2 (y_1 u_1 z_1 u_2)^{-1}, \]
\[ w := z_2 u_3 t_2 y_2 (z_2 u_3 t_2)^{-1}, \]
\[ v_1 := (u_2 z_1 u_1 y_2^2 u_1 z_1 u_2)^{-1} y_2 (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2), \]
\[ v_i := t_{i-1} t_i v_{i-1} (t_{i-1} t_i)^{-1} \quad (2 \leq i \leq h - 1), \]
\[ w_1 := u_1 z_1 u_2 v_1 (u_1 z_1 u_2)^{-1}, \]
\[ w_i := u_i z_i u_{i+1} v_i (u_i z_i u_{i+1})^{-1} \quad (2 \leq i \leq h - 1), \]
\[ d := (w_1 u_2 \cdots w_{h-1})^{-1} y_1 w_1 u_2 \cdots w_{h-1}. \]

Elements \( y_i, u_i, z_i \) can be interpreted as Dehn twists with respect to curves \( Y_i, U_i, Z_i \) in Fig.1 of [3] (see also [13] and [10]). For \( h = 2 \) we can omit the relator \( D^1 \).

By choosing a symplectic basis of \( H^1(\Sigma_h; \mathbb{Z}) \) as in [17], we fix an explicit representation \( \sigma : \mathcal{M}_h \to Sp(2h, \mathbb{Z}) \) by:

\[
\sigma : y_i \mapsto \begin{pmatrix} I & 0 \\ -E_{ii} & I \end{pmatrix} \quad (i = 1, 2), \]
\[
\sigma : u_i \mapsto \begin{pmatrix} I & E_{ii} \\ 0 & I \end{pmatrix} \quad (1 \leq i \leq h), \]
\[
\sigma : z_i \mapsto \begin{pmatrix} I \\ -E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} & 0 \end{pmatrix} \quad (1 \leq i \leq h - 1),
\]

where \( E_{ij} \in M_h(\mathbb{Z}) \) is the \((i,j)\)-matrix unit.

We also fix an exact sequence:

\[ 1 \to R \to F \xrightarrow{\pi} \mathcal{M}_h \to 1, \]

where

\[ F := \langle y_1, y_2, u_1, \ldots, u_h, z_1, \ldots, z_{h-1} \rangle \]

and \( R \) is the normal closure of the set of all relators \( A^1_{i,j}, B^1_i, C^1, D^1, E^1 \) in \( F \). Let \( c : F \to \mathbb{Z} \) be the map defined as in Section 2 by using explicit homomorphisms \( \sigma \) and \( \pi \) fixed above.

Now we calculate values of the map \( c : F \to \mathbb{Z} \) for relators \( A^1_{i,j}, B^1_i, C^1, D^1, E^1 \) of the presentation and describe the homomorphism \( c|_R : R \to \mathbb{Z} \).

To compute values of \( c \), Meyer showed the following lemma:
Lemma 3.1 (Meyer[16], [17]). The map $c : F \to \mathbb{Z}$ satisfies the following properties:

1. $c(xy) = c(x) + c(y) + \tau_h(\sigma(\pi(x)), \sigma(\pi(y))) \ (x, y \in F)$;
2. $c(x^{-1}) = -c(x) \ (x \in F)$;
3. $c(xy^{-1}) = c(y) \ (x, y \in F)$;
4. $c(xzyz^{-1}) = c(x) + c(y)$ if $\pi(xzyz^{-1}) = 1 \in \mathcal{M}_h \ (x, y, z \in F)$.

Values of $c$ for relators are computed by using Lemma 3.1

Lemma 3.2. The values of $c$ for the relators of Wajnryb's presentation of $\mathcal{M}_h (h \geq 3)$ are calculated as follows:

1. $c(A_{l,j}^l) = 0 \ (\text{for every } l, i, j)$;
2. $c(B_{l}^l) = 0 \ (\text{for every } l, i)$;
3. $c(C^1) = -6$;
4. $c(D^1) = 1$;
5. $c(E^1) = 0$.

Proof. We denote $\tau_h(\sigma(\pi(x)), \sigma(\pi(y)))$ by $\tilde{\tau}_h(x, y)$ for $x, y \in F$. By virtue of Lemma 3.1, it follows immediately that $c(A_{l,j}^l) = c(B_{l}^l) = c(E^1) = 0$. For example,

$$c(B_{l}^l) = c(y_1 \cdot u_1 \cdot y_1^{-1} u_1^{-1})$$
$$= c(y_1) + c(y_1^{-1} u_1^{-1})$$
$$= c(y_1) + c(u_1^{-1}) = c(y_1) - c(u_1)$$
$$= 0.$$

Using Lemma 3.1 and calculating signature of symmetric bilinear forms concretely, we obtain values $c(C^1)$ and $c(D^1)$.

$$c(C^1) = c((y_1 u_1 z_1)^{-4} y_2 (u_2 z_1 u_1 y_2^2 u_1 z_1 u_2)^{-1} y_2 (u_2 z_1 u_1 y_2^2 u_1 z_1 u_2)$$
$$= c((y_1 u_1 z_1)^{-4} y_2) \ (c(y_2) = 0)$$
$$= c(y_1 u_1 z_1)^{-4} y_2 + \tilde{\tau}_h((y_1 u_1 z_1)^{-4}, y_2) \ (c(y_2) = 0)$$
$$= 2 \tilde{\tau}_h((y_1 u_1 z_1)^{-2}, (y_1 u_1 z_1)^{-2}) + \tilde{\tau}_h((y_1 u_1 z_1)^{-4}, y_2)$$
$$= 4(\tilde{\tau}_h(1, z_1^{-1}) + \tilde{\tau}_h(z_1^{-1}, u_1^{-1}) + \tilde{\tau}_h(z_1^{-1} u_1^{-1}, y_1^{-1}))$$
$$+ 2 \tilde{\tau}_h((y_1 u_1 z_1)^{-1}, (y_1 u_1 z_1)^{-1})$$
$$+ \tilde{\tau}_h((y_1 u_1 z_1)^{-2}, (y_1 u_1 z_1)^{-2}) + \tilde{\tau}_h((y_1 u_1 z_1)^{-4}, y_2)$$
$$= 4(0 + 0 + 0) + 2 \cdot (-3) + (-1) + 1$$
$$= -6.$$

Remark 3.3. All values of Meyer's signature cocycle \( \tau_h \) calculated in Lemma 3.2 are independent of the genus \( h (\geq 3) \) because all generators which appear in \( C^1 \) and \( D^1 \) are \( y_1, y_2, u_1, u_2, u_3, z_1, z_2 \). We can easily check by using a computer that the values are correct in the case \( h = 3 \). (We used Mathematica).

Definition 3.4. Let \( F_n \) be a free group of rank \( n \). Algebraic \( m \) copies of an element \( x \in F_n \) are \( m_+ \) copies of \( x \) and \( m_- \) copies of \( x^{-1} \), where \( m_+, m_- \geq 0 \) and \( m_+ - m_- = m \). The integer \( m \) is called the algebraic number of these algebraic copies.

For each generator \( e = y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1} \), the homomorphism \( e^* : F \to \mathbb{Z} \) is defined by:

\[
e^*(x) := \begin{cases} 
+1 & (x = e), \\
0 & (x : \text{other generators}).
\end{cases}
\]

An element \( x \in F \) belongs to \( [F, F] \) if and only if \( e^*(x) = 0 \) for every generator \( e \). Combining this with Lemma 3.2, we characterize the elements of \( R \cap [F, F] \) as words in \( y_i, u_i, z_i \) and calculate the value of \( c \) for each element \( x \in R \cap [F, F] \).

Proposition 3.5. Suppose that \( h \geq 3 \). For an element \( x \in F \), the following two conditions are equivalent:

1. \( x \in R \cap [F, F] \) and \( c(x) = 4n (n \in \mathbb{Z}) \);
2. \( x \) is equal to a product of conjugates of algebraic copies of relators and the
algebraic number \( m(R^1) \) of algebraic copies of a relator \( R^1 \) included in \( x \) is determined as follows:

\[
\begin{array}{cccccc}
R^1 & A^i_{i,j} & B^1_i & B^2_i & B^2_i & B^2_i (i \geq 3) \\
m(R^1) & \forall & -6n & 18n & -2n & 10n & 0 \\
B^3_i & B^3_i (i \geq 2) & C^1 & D^1 & E^1 \\
-8n & 0 & n & 10n & \forall
\end{array}
\]

where \( \forall \) stands for arbitrary number of algebraic copies of \( R^1 \).

Proof. (1) \( \implies \) (2): Since \( R \) is the normal closure of the set \( \{ A^i_{i,j}, B^1_i, C^1, D^1, E^1 \} \) of all relators, any \( x \in R \) is a product of conjugates of algebraic copies of relators. For \( x \in R \cap [F,F] \), let \( a^i_{i,j} \) (respectively \( b^1_i, c^1, d^1, e^1 \)) be the algebraic number of algebraic copies of \( A^i_{i,j} \) (respectively \( B^1_i, C^1, D^1, E^1 \)) included in \( x \). These numbers must satisfy the following system of equations because \( x \) belongs to \([F,F]\).

\[
\sum_{i=1}^{2} b^1_i e^*(B^1_i) + \sum_{i=1}^{h-1} b^2_i e^*(B^2_i) + \sum_{i=1}^{h-1} b^3_i e^*(B^3_i) + c^1 e^*(C^1) + d^1 e^*(D^1) = 0
\]

\((e = y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1})\).

\((e^*(A^i_{i,j}) = e^*(E^1) = 0 \) for every generator \( e \) because \( A^i_{i,j} \) and \( E^1 \) belong to \([F,F]\). Values of \( e^* \) and \( c \) for other relators are exhibited in Table 3.6 below). Solving this, we get

\[
\begin{align*}
& b^1_1 = -6n, \quad b^1_2 = 18n, \quad b^1_i = -2n, \quad b^2_i = 10n, \quad b^3_i = 0 \text{ (for } 3 \leq i \leq h - 1), \\
& b^3_i = -8n, \quad b^3_i = 0 \text{ (for } 2 \leq i \leq h - 1), \quad c^1 = n, \quad d^1 = 10n,
\end{align*}
\]

where \( n \) is an integer, while \( a^i_{i,j} \) and \( e^1 \) are arbitrary integers.

(2) \( \implies \) (1): Such an element \( x \) belongs to \( R \cap [F,F] \) because \( e^*(x) = 0 \) for every generator \( e \). The value \( c(x) \) can be calculated by using Lemma 3.2:

\[
c(x) = n c(C^1) + 10n c(D^1)
\]

\[
= -6n + 10n
\]

\[
= 4n.
\]

This completes the proof of Proposition 3.5. \( \square \)

Remark 3.7. Proposition 3.5 implies that the ‘signature’ \( c(x) \) of a ‘2-cycle’ \( x \in R \cap [F,F] \) of \( M_h \) is concentrated on relators \( B^1_1, B^1_2, B^2_1, B^2_2, B^3_1, C^1, D^1 \) of Wajnryb’s
<table>
<thead>
<tr>
<th></th>
<th>$y_1^*$</th>
<th>$y_2^*$</th>
<th>$u_1^*$</th>
<th>$u_2^*$</th>
<th>...</th>
<th>$u_{h-2}^*$</th>
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<tr>
<td>$B_1^1$</td>
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<tr>
<td>$D^1$</td>
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<td>...</td>
<td>0</td>
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</tr>
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</table>

(The blanks in the table above mean that the corresponding value is equal to zero.)

Table 3.6.

presentation and the algebraic number $m(R^1)$ of a relator $R^1$ is independent of the genus $h(\geq 3)$.

4. A construction of holonomy homomorphisms

We now construct the holonomy homomorphism $\chi : \pi_1(\Sigma_g) \rightarrow M_h$ of a surface bundle $\xi$ over a surface $\Sigma_g$ with non-zero signature. We use a simple technique of the commutator collection process (see [7], [15]) to construct $\chi$.

**Definition 4.1.** Let $F_n$ be the free group on the $n$ free generators $e_1, \cdots, e_n$ and let $a, b, u, v$ and $w$ be words in $e_1, \cdots, e_n$. Two words $u$ and $v$ are called freely equal (denoted $u \approx v$) if they determine the same element of $F_n$.

The $\alpha$-skip is the following sequence of free equalities:

\[ uava^{-1}w \approx u(ava^{-1}v^{-1})vw \]

\[ = u[a, v]vw \]
and the \(\beta\)-skip is the following sequence of free equalities:

\[
u avb^{-1} b^{-1} w \approx u (avb^{-1} b^{-1} v^{-1}) v w \\
= u [a, v b] v w,
\]

where \([a, b] := aba^{-1} b^{-1}\). (We used the commutator relation \(ba \approx [b, a] a b\).)

We apply \(\alpha\)- and \(\beta\)-skips to elements of the free group \(F\) on the generators \(y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}\) defined in the preceding section and prove the following lemma.

**Lemma 4.2.** Suppose that \(h \geq 3\). There exists a word \(W\) in \(y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}\) with the following properties:

1. \(W\) is a product of 111 commutators;
2. \(W\) belongs to \(R \cap [F, F]\) as an element of \(F\);
3. \(c(W) = 4\).

**Proof.** We set

\[
\widetilde{W}_1 := (B_1^2)^{-1} (B_1^3)^{-3} B_1^3 B_2 D^1, \\
\widetilde{W}_2 := B_2^2 (B_1^3)^{-1} B_1^2 B_2 D^1, \\
\widetilde{W} := C^1 \widetilde{W}_8 \widetilde{W}_8^8.
\]

Since the word \(\widetilde{W}\) satisfies the condition (2) of Proposition 3.5 in case \(n = 1\), \(\widetilde{W}\) has the properties (2) and (3) above. We decompose \(\widetilde{W}\) to a product \(W\) of 111 commutators by using \(\alpha\)- and \(\beta\)-skips repeatedly.

We rewrite some of Wajnryb's relators as follows:

\[
B_1^1 = y_1 R_1 u_1^{-1} (R_1 = [u_1, y_1]), \\
B_2^1 = y_2 R_2 u_2^{-1} (R_2 = [u_2, y_2]), \\
B_1^2 = u_1 R_3 z_1^{-1} (R_3 = [z_1, u_1]), \\
B_2^2 = u_2 R_4 z_2^{-1} (R_4 = [z_2, u_2]), \\
B_1^3 = z_1 R_5 u_1^{-1} (R_5 = [u_2, z_1]), \\
C^1 = (y_1 u_1 z_1)^{-4} y_2 R_6 (R_6 = [u_2 z_1 u_1 y_1^2 u_1 z_1 u_2]^{-1}), \\
D^1 = y_1 z_2 t_2 y_2^{-1} t_2^{-1} (y_2 t_2^{-1} y_2 t_2^{-1} R_7 R_8 (R_7 = [y_2^{-1}, y_1 u_1 z_1 u_2], R_8 = [v^{-1}, (w u_1 z_1 u_2 z_2 u_3)]),
\]

where \(R_1, \cdots, R_8\) are commutators.

\(\widetilde{W}_i(i = 1, 2)\) is transformed into another word \(W_i(i = 1, 2)\) by using \(\alpha\)- and \(\beta\)-skips in the following way:
\[ \begin{align*}
\tilde{W}_1 &= (B_1^2)^{-1}(B_1^3)^{-3}B_1^2 B_2^3 D^1 \\
&\approx z_1 R_3^{-1} R_1^{-1} y_1^{-1} (u_1 R_1^{-1} y_1^{-1})^2 y_2 R_2 R_4 z_2^{-1} y_1 z_1 z_2 t_1 t_2 y_2^{-1} t_2^{-1} t_1^{-1} t_2 y_2^{-1} t_2^{-1} R_7 R_8 \\
&\overset{(\beta)}{=} z_1 R_3^{-1} R_1^{-1} y_1^{-1} (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1} y_1 z_1 z_2 t_2 y_2^{-1} t_2^{-1} R_7 R_8 \\
&\quad (S_1 := [y_2, R_2 R_4 z_2^{-1} y_1 z_1 z_2 t_1 t_2]) \\
&\overset{(\alpha)}{=} z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1} z_1 z_2 t_2 y_2^{-1} t_2^{-1} R_7 R_8 \\
&\quad (S_2 := [y_1^{-1}, (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1}]) \\
=: &\ W_1;
\end{align*} \]

\[ \begin{align*}
\tilde{W}_2 &= B_2^1 (B_3^3)^{-1} B_1^3 B_2^3 D^1 \\
&\approx y_2 R_2 R_5^{-1} z_1^{-1} y_2 R_2 R_4 z_2^{-1} y_1 z_1 z_2 t_1 t_2 y_2^{-1} t_2^{-1} t_1^{-1} t_2 y_2^{-1} t_2^{-1} R_7 R_8 \\
&\overset{(\beta)}{=} y_2 R_2 R_5^{-1} z_1^{-1} S_3 R_2 R_4 z_2^{-1} y_1 z_1 z_2 t_2 y_2^{-1} t_2^{-1} R_7 R_8 \\
&\quad (S_3 := [y_2, R_2 R_4 z_2^{-1} y_1 z_1 z_2 t_1 t_2]) \\
&\overset{(\beta)}{=} S_4 R_2 R_5^{-1} z_1^{-1} S_3 R_2 R_4 z_2^{-1} y_1 z_1 z_2 R_7 R_8 \\
&\quad (S_4 := [y_2, R_2 R_5^{-1} z_1^{-1} S_3 R_2 R_4 z_2^{-1} y_1 z_2 t_2]) \\
&\overset{(\alpha)}{=} S_4 R_2 R_5^{-1} S_6 S_3 R_2 R_4 z_2^{-1} y_1 z_2 R_7 R_8 \\
&\quad (S_5 := [z_1^{-1}, S_3 R_2 R_4 z_2^{-1} y_1]) \\
=: &\ W_2.
\end{align*} \]

The word \( W_1 \) obtained above naturally includes 10 commutators and the word \( W_2 \) 9 ones. Hence the word \( C^1 W_1^2 W_2^8 \) naturally includes 93 commutators.

Furthermore we perform 6 \( \alpha \)-skips and 4 \( \beta \)-skips to \( C^1 W_1^2 \) and get a word \( \tilde{W} \) in the following way:

\[ \begin{align*}
C^1 W_1^2 &= (y_1 u_1 z_1)^{-4} y_2 y_2 R_6 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 \\
&\quad \cdot S_1 R_2 R_4 z_2^{-1} z_1 z_2 t_2 y_2^{-1} t_2^{-1} R_7 R_8 W_1 \\
&\overset{(\beta)}{=} (y_1 u_1 z_1)^{-3} z_1^{-1} u_1^{-1} y_1^{-1} y_2 S_6 R_6 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 \\
&\quad \cdot S_1 R_3 R_4 z_2^{-1} z_1 z_2 R_7 R_8 W_1 \\
&\quad (S_6 := [y_2, R_6 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1} z_1 z_2 t_2]) \\
&\overset{(\beta)}{=} (y_1 u_1 z_1)^{-3} S_7 R_4 z_1^{-1} y_1^{-1} y_2 S_6 R_6 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 \\
&\quad \cdot S_1 R_3 R_4 R_7 R_8 W_1 \\
&\quad (S_7 := [z_1^{-1}, u_1^{-1} y_1^{-1} y_2 S_6 R_6 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1}]) \\
&\overset{(\alpha)}{=} (y_1 u_1 z_1)^{-2} z_1^{-1} u_1^{-1} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 z_1 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot u_1 R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 W_1
\end{align*} \]
\[(S_8 := [u_1^{-1}, y_1^{-1} y_2 S_6 R_6 z_1 R_3^{-1} R_7^{-1} S_2])\]
\[(\approx) (y_1 u_1 z_1)^{-2} S_9 u_1^{-1} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 R_3^{-1} R_7^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot u_1 R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 W_1\]
\[(S_9 := [z_1^{-1}, u_1^{-1} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6])\]
\[(\approx) (y_1 u_1 z_1)^{-2} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 R_3^{-1} R_7^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8\]
\[\cdot z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_1^{-1} z_2 t z_2 y_1^{-1} R_7 R_8\]
\[(S_{10} := [u_1^{-1}, y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 R_3^{-1} R_7^{-1} S_2 R_1^{-1} y_1^{-1}])\]
\[(\approx) (z_1^{-1} u_1^{-1} y_1^{-1})^2 S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_1^{-1} z_2 R_7 R_8\]
\[(S_{11} := [y_2, S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} S_1 R_3 R_4 R_7 R_8\]
\[\cdot z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_1^{-1} z_2 t z_2])\]
\[(\approx) z_1^{-1} u_1^{-1} y_1^{-1} S_{12} u_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 R_7 R_8\]
\[(S_{12} := [z_1^{-1}, u_1^{-1} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1}])\]
\[(\approx) z_1^{-1} u_1^{-1} y_1^{-1} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 R_7 R_8\]
\[(S_{13} := [u_1^{-1}, y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2])\]
\[(\approx) z_1^{-1} S_{14} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 R_7 R_8\]
\[(S_{14} := [u_1^{-1}, y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1}\]
\[\cdot R_1^{-1} S_2 R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}])\]
\[(\approx) S_{15} S_{14} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}\]
\[\cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 R_7 R_8\]
\[(S_{15} := [z_1^{-1}, S_{14} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1}\]
\[\cdot R_1^{-1} S_2 R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8])\]

\[= \widehat{W}\]

The word \(\widehat{W}\) is a product of 31 commutators and 8 copies of \(y_1^{-1}\). The word \(W^g\) is a product of 72 commutators and 8 copies of \(z_1^{-1} y_1^{-1} z_1\).
We perform 8 $\beta$-skips to the word $\mathcal{W}_W^8$ repeatedly by setting $a = y_1^{-1}$ and $b = z_1^{-1}$ in Definition 4.1. Then we obtain a word $W$ which is a product of 111 ($= 31 + 72 + 8$) commutators and is freely equal to $\mathcal{W}$. This completes the proof of Lemma 4.2.

By virtue of Lemma 4.2, we can show the following theorem.

**Theorem 4.3.** There exists a $\Sigma_h$-bundle $\xi = (E, \Sigma_g, p, \Sigma_h, \text{Diff}_+ \Sigma_h)$ over $\Sigma_g$ with $g = 111$, $h = 3$ and $\tau(E) = -4$.

**Proof.** Set $g = 111$ and $h = 3$. We choose a word $W$ which satisfies conditions (1)-(3) of Lemma 4.2 and write

$$W = \prod_{i=1}^{g} [\alpha_i, \beta_i] \quad (\alpha_i, \beta_i \in F(i = 1, \ldots, g)).$$

Let $\tilde{\chi} : \tilde{F} \to F$ the homomorphism defined by:

$$\tilde{\chi}(\tilde{a}_i) = \alpha_i, \quad \tilde{\chi}(\tilde{b}_i) = \beta_i \quad (i = 1, \ldots, g),$$

where $\tilde{F} = \langle \tilde{a}_1, \ldots, \tilde{a}_g, \tilde{b}_1, \ldots, \tilde{b}_g \rangle$. Since $\tilde{\chi}(\tilde{r}) = W \in R \cap [F, F]$, $\tilde{\chi}$ induces the homomorphism $\chi : \pi_1(\Sigma_g) \to \mathcal{M}_h$ (i.e., $\pi \circ \tilde{\chi} = \chi \circ \pi$) as in Section 2. For the $\Sigma_h$-bundle $\xi$ over $\Sigma_g$ which has $\chi$ as its holonomy homomorphism, we calculate the signature of its total space $E$:

$$\tau(E) = -c(\chi(\tilde{r}))$$
$$= -c(W)$$
$$= -4.$$

We have thus proved the theorem. \qed

Finally, we prove our main theorem (Theorem 1.2) by using Lemma 4.2 and results of Lück [14] concerning about $L^2$-Betti numbers of groups.

**Proof of Theorem 1.2.** Let $W$ be the word constructed in the proof of Lemma 4.2. For every $h \geq 3$ and each $n \in \mathbb{Z}(n \neq 0)$, we can construct a $\Sigma_h$-bundle $\xi = \xi(h, n)$ with $g = 111|n|$ and $\tau(E) = 4n$ by using the word $W^{-n}$ as in the proof of Theorem 4.3 (see Remark 3.7). Therefore we have

$$g(h, n) \leq 111|n|.$$

On the other hand, for every $\Sigma_h$-bundle $\xi$ over $\Sigma_g$ with $g \geq 1, h \geq 3$ and $\tau(E) =
4n, the associated exact sequence:

$$1 \rightarrow \pi_1(\Sigma_h) \rightarrow \pi_1(E) \rightarrow \pi_1(\Sigma_g) \rightarrow 1$$

of fundamental groups satisfies the assumption of [14], Theorem 4.1. Then the first $L^2$-Betti number $b_1(\pi_1(E))$ of $\pi_1(E)$ is equal to zero and the Winkelnkemper-type inequality $\chi(E) \geq |\tau(E)|$ holds from [14], Theorem 5.1. By substituting $\chi(E) = \chi(\Sigma_h)\chi(\Sigma_g) = 4(h - 1)(g - 1)$, $\tau(E) = 4n$

for the inequality, we obtain

$$g(h, n) \geq \frac{|n|}{h - 1} + 1$$

and this completes the proof of our theorem.

REMARK 4.4. The $\Sigma_h$-bundle $\xi = \hat{\xi}(h, n)$ over $\Sigma_g$ constructed in the first half of the proof of Theorem 1.2 has $g = 111|n|$, $\tau(E) = 4n$, $b_1(E) = 2(111|n| + h - 3)$, $b_2(E) = 2(222|n|h - 5)$ and $\chi(E) = 4(111|n| - 1)(h - 1)$, where $h(\geq 3)$ and $n \in \mathbb{Z}(n \neq 0)$. If the total space $E$ admits a complex structure, $E$ is an algebraic surface of general type and satisfies the Noether condition, the Noether inequality and the Bogomolov-Miyaoka-Yau inequality (cf. [2]). But $E$ cannot be a geometric 4-manifold in the sense of Thurston [20], in particular, a compact Kähler surface covered by the unit ball in $\mathbb{C}^2$.

Let $\Gamma(h, n)$ be the fundamental group of the total space of $\xi = \hat{\xi}(h, n)(h \geq 3, n \geq 1)$ constructed in the first half of the proof of Theorem 1.2. Computing an invariant defined by Johnson [11], we obtain the following result.

**Corollary 4.5.** The family $\{\Gamma(h, n)\}_{h \geq 3, n \geq 1}$ contains infinitely many commensurability classes of discrete groups. In particular, $\{\Gamma(h, n)\}_{n \geq 1}$ is a family of infinitely many non-commensurable discrete groups for each $h (\geq 3)$.

Proof. The commensurability invariant $\gamma(\Gamma)$ [11] for $\Gamma = \Gamma(h, n)$ is

$$\gamma(\Gamma(h, n)) = \frac{n}{(111n - 1)(h - 1)} \quad (h \geq 3, n \geq 1),$$

which runs over infinitely many rational numbers.

REMARK 4.6. Although the author attempted to show that the value $g(h, n)$ does not depend on the genus $h \geq 3$ of fiber $\Sigma_h$ for each $n \in \mathbb{Z} (n \neq 0)$, it was not achieved because of some serious transformation problems on words in free generators.
References


