Title	A construction of surface bundles over surfaces with non-zero signature
Author(s)	Endo, Hisaaki
Citation	Osaka Journal of Mathematics. 1998, 35(4), p. 915-930
Version Type	VoR
URL	https://doi.org/10.18910/4596
rights	
Note	

Endo, H.
Osaka J. Math.
35 (1998), 915–930

A CONSTRUCTION OF SURFACE BUNDLES OVER SURFACES WITH NON-ZERO SIGNATURE

HISAAKI ENDO[†]

(Received June 11, 1997)

1. Introduction

Let Σ_g (respectively Σ_h) be a closed oriented surface of genus g (respectively h), where g (respectively h) is a non-negative integer. Let $\mathrm{Diff}_+\Sigma_h$ be the group of all orientation-preserving diffeomorphisms of Σ_h with C^∞-topology. A Σ_h-*bundle* over Σ_g (also called a *surface bundle* over a surface) is fiber bundle $\xi = (E, \Sigma_g, p, \Sigma_h, \mathrm{Diff}_+\Sigma_h)$ over Σ_g with total space E, fiber Σ_h, projection $p : E \longrightarrow \Sigma_g$ and structure group $\mathrm{Diff}_+\Sigma_h$. Our main concern is the signature $\tau(E)$ of the total s-pace E of ξ.

It is easily seen that if ξ is a trivial bundle then $\tau(E) = \tau(\Sigma_g)\tau(\Sigma_h) = 0$. Chern-Hirzebruch-Serre [5] proved that if the fundamental group $\pi(\Sigma_g)$ of Σ_g acts trivially on the cohomology ring $H^*(\Sigma_h; \mathbb{R})$ of Σ_h then $\tau(E) = 0$.

Kodaira [12] and Atiyah [1] gave examples of surface bundles over surfaces with non-zero signature. For each pair (m, t) of integers $m, t \in \mathbb{Z}\,(m \geq 2, t \geq 3)$, Kodaira constructed a surface bundle $\xi = \xi(m, t)$ with

$$\begin{aligned} g &= m^{2t}(t-1) + 1, \\ h &= mt, \\ \tau(E) &= \frac{4}{3}m^{2t-1}(t-1)(m^2-1). \end{aligned}$$

By setting $m = 2$ and $t = 3$, we obtain a surface bundle $\xi = \xi(2, 3)$ with $g = 129$, $h = 6$ and $\tau(E) = 256$. The total space E of the bundle $\xi = \xi(m, t)$ is an m-fold branched covering of $\Sigma_g \times \Sigma_t$ and its signature $\tau(E)$ can be calculated by using G-signature theorem(see [9] and [11]).

Meyer [16], [17] gave a signature formula for surface bundles over surfaces in terms of the *signature cocycle* τ_h, which is a 2-cocycle of the Siegel modular group $Sp(2h, \mathbb{Z})$ of degree h. Using the signature cocycle and Birman-Hilden's relations [3] of mapping class groups of surfaces, he showed that if $h = 1, 2$ or $g = 1$ then

[†]The author is partially supported by JSPS Research Fellowships for Young Scientists.

$\tau(E) = 0$. But he also showed that for every $h \geq 3$ and every $n \in \mathbb{Z}$ there exist an integer $g \geq 0$ and a Σ_h-bundle ξ over Σ_g such that $\tau(E) = 4n$.

We consider the following problem:

Problem 1.1. For each $h \geq 3$ and each $n \in \mathbb{Z}$, let $g(h, n)$ be the minimum value of the genus g such that there exists a Σ_h-bundle ξ over Σ_g with $\tau(E) = 4n$. Determine the value $g(h, n)$.

In this paper, we estimate the value $g(h, n)$ by using Wajnryb's presentation[19] of the mapping class group $\mathcal{M}_h$ of Σ_h.

Our main result is:

Theorem 1.2. *For each $h \geq 3$ and each $n \in \mathbb{Z}(n \neq 0)$, the following inequality holds*:

$$\frac{|n|}{h-1} + 1 \leq g(h, n) \leq 111|n|.$$

We construct a Σ_h-bundle ξ over Σ_g with $g = 111$, $h = 3$ and $\tau(E) = -4$ to prove Theorem 1.2. The genus of the base space of this bundle and that of a fiber of it are smaller than those of any example constructed by Kodaira [12] and Atiyah [1].

In Section 2, we review Meyer's work [16], [17] on signature of surface bundles over surfaces. And in Section 3, we calculate the values of Meyer's signature cocycle for the relators of Wajnryb's presentation [19] of the mapping class group $\mathcal{M}_h$ and characterize the 2-cycles of $\mathcal{M}_h$ as words in the generators of the presentation of $\mathcal{M}_h$. We prove our main theorem in Section 4 by using this characterization and a simple technique of the commutator collection process [7]. In Section 2, we review Meyer's work [16], [17] on signature of surface bundles over surfaces. And in Section 3, we calculate the values of Meyer's signature cocycle for the relators of Wajnryb's p-resentation [19] of the mapping class group $\mathcal{M}_h$ and characterize the 2-cycles of $\mathcal{M}_h$ as words in the generators of the presentation of $\mathcal{M}_h$. We prove our main theorem in Section 4 by using this characterization and a simple technique of the commutator collection process [7].

The author wishes to express his heartfelt gratitude to his adviser, Prof. Katsuo Kawakubo, for helpful comments and useful suggestions, and Kazunori Kikuchi and Toshiyuki Akita for helpful discussions.

2. Meyer's signature formula

In this section we review Meyer's signature cocycle and Meyer's signature formula [16], [17] for surface bundles over surfaces.

For a pair (α, β) of symplectic matricies $\alpha, \beta \in Sp(2h, \mathbb{Z})$, the vector space $V_{\alpha,\beta}$

is defined by:

$$V_{\alpha,\beta} := \{(x,y) \in \mathbb{R}^{2h} \times \mathbb{R}^{2h} \mid (\alpha^{-1} - I)x + (\beta - I)y = 0\},$$

where I is the identity matrix. Consider the (possibly degenerate) symmetric bilinear form

$$\langle \ \ , \ \ \rangle_{\alpha,\beta} : V_{\alpha,\beta} \times V_{\alpha,\beta} \longrightarrow \mathbb{R}$$

on $V_{\alpha,\beta}$ defined by:

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{\alpha,\beta} := \langle x_1 + y_1, (I - \beta) y_2 \rangle,$$
$$(x_i, y_i) \in V_{\alpha,\beta} \quad (i = 1, 2),$$

where $\langle \ \ , \ \ \rangle$ is the standard symplectic form on $\mathbb{R}^{2h}$ given by:

$$\langle x, y \rangle = {}^t x J y \quad (x, y \in \mathbb{R}^{2h}),$$
$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in M_{2h}(\mathbb{R}).$$

Meyer's signature cocycle [16], [17]

$$\tau_h : Sp(2h, \mathbb{Z}) \times Sp(2h, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is defined by:

$$\tau_h(\alpha, \beta) := \operatorname{sign}(V_{\alpha,\beta}, \langle \ \ , \ \ \rangle_{\alpha,\beta})$$
$$(\alpha, \beta \in Sp(2h, \mathbb{Z})).$$

From the Novikov additivity, τ_h is a 2-cocycle of $Sp(2h, \mathbb{Z})$ and represents a cohomology class $[\tau_h] \in H^2(Sp(2h, \mathbb{Z}), \mathbb{Z})$.

Let $\mathcal{M}_h$ be the mapping class group of a surface Σ_h of genus h, namely it is the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_h. By choosing a symplectic basis on $H^1(\Sigma_h; \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2h}$, the natural action of $\mathcal{M}_h$ on $H^1(\Sigma_h; \mathbb{Z})$ induces a representation $\sigma : \mathcal{M}_h \longrightarrow Sp(2h, \mathbb{Z})$.

Next, we define a homomorphism $k : H_2(\mathcal{M}_h; \mathbb{Z}) \longrightarrow \mathbb{Z}$ by using τ_h and σ. It is known that the group $\mathcal{M}_h$ is finitely presentable, so there exists an exact sequence:

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} \mathcal{M}_h \longrightarrow 1,$$

where F is a free group of finite rank generated by a free basis $E = \{e_\lambda\}_{\lambda \in \Lambda}$. By well known Hopf's theorem (cf. [4]) the following isomorphism holds:

$$H_2(\mathcal{M}_h; \mathbb{Z}) \cong R \cap [F, F]/[R, F].$$

The map $c : F \longrightarrow \mathbb{Z}$ is defined by:

$$c(x) := \sum_{j=1}^{m} \tau_h(\sigma(\pi(\widetilde{x}_{j-1})), \sigma(\pi(x_j)))$$
$$\left(x = \prod_{j=1}^{m} x_i,\ x_i \in E \cup E^{-1},\ \widetilde{x}_j = \prod_{i=1}^{j} x_i \right).$$

It can be checked that the restriction $c|_R : R \longrightarrow \mathbb{Z}$ is actually a homomorphism and that $c([R,F]) = 0$. Hence $c|_R$ naturally induces a homomorphism $k : H_2(\mathcal{M}_h;\mathbb{Z}) \cong R \cap [F,F]/[R,F] \longrightarrow \mathbb{Z}$.

Now, we describe Meyer's signature formula for surface bundles over surfaces.

Let $\xi = (E, \Sigma_g, p, \Sigma_h, \mathrm{Diff}_+\Sigma_h)$ be a Σ_h-bundle over Σ_g and $f : \Sigma_g \longrightarrow B\mathrm{Diff}_+\Sigma_h$ its classifying map. The map f induces a homomorphism χ between fundamental groups:

$$\chi := f_\sharp : \pi_1(\Sigma_g) \longrightarrow \pi_1(B\mathrm{Diff}_+\Sigma_h) \cong \pi_0(\mathrm{Diff}_+\Sigma_h) \cong \mathcal{M}_h,$$

which is called the *holonomy homomorphism* of ξ (cf. [18]). By a theorem of Earle and Eells [6], which states that the connected component $\mathrm{Diff}_0\Sigma_h$ of the identity of $\mathrm{Diff}_+\Sigma_h$ is contractible if $h \geq 2$, the isomorphism class of ξ is completely determined by its holonomy homomorphism χ when $h \geq 2$ (see [16], [17] and [18]). From now on, we suppose that $h \geq 2$ and $g \geq 1$.

The fundamental group $\pi_1(\Sigma_g)$ of Σ_g is finitely presented, so we have an exact sequence:

$$1 \longrightarrow \widetilde{R} \longrightarrow \widetilde{F} \xrightarrow{\widetilde{\pi}} \pi_1(\Sigma_g) \longrightarrow 1,$$

where

$$\pi_1(\Sigma_g) = \left\langle a_1, \cdots, a_g, b_1, \cdots, b_g \,\middle|\, \prod_{i=1}^{g}[a_i, b_i] \left(= \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right) = 1 \right\rangle,$$
$$\widetilde{F} = \langle \widetilde{a}_1, \cdots, \widetilde{a}_g, \widetilde{b}_1, \cdots, \widetilde{b}_g \rangle,$$
$$\widetilde{\pi} : \widetilde{a}_i \longmapsto a_i, \widetilde{b}_i \longmapsto b_i$$

and $\widetilde{R}$ is the normal closure of $\widetilde{r} := \prod_{i=1}^{g}[\widetilde{a}_i, \widetilde{b}_i] (= \prod_{i=1}^{g} \widetilde{a}_i \widetilde{b}_i \widetilde{a}_i^{-1} \widetilde{b}_i^{-1})$ in $\widetilde{F}$. Hopf's theorem allows us to identify $H_2(\pi_1(\Sigma_g);\mathbb{Z})$ with $\widetilde{R} \cap [\widetilde{F}, \widetilde{F}]/[\widetilde{R}, \widetilde{F}]$. For the holonomy homomorphism χ, we can choose a homomorphism $\widetilde{\chi} : \widetilde{F} \longrightarrow F$ so that $\pi \circ \widetilde{\chi} = \chi \circ \widetilde{\pi}$. Then the induced homomorphism $\chi_* : H_2(\pi_1(\Sigma_g);\mathbb{Z}) \longrightarrow H_2(\mathcal{M}_h;\mathbb{Z})$ is defined by:

$$\chi_*(\widetilde{x}[\widetilde{R}, \widetilde{F}]) := \widetilde{\chi}(\widetilde{x})[R, F] \quad (\widetilde{x} \in \widetilde{R} \cap [\widetilde{F}, \widetilde{F}])$$

and is not depend on a choice of $\widetilde{\chi}$.

Meyer proved the following theorem by using the Leray-Serre spectral sequence for ξ and the cohomology group $H^1(\Sigma_g; H_1(\Sigma_h; \mathbb{R}))$ of Σ_g with local coefficients.

Theorem 2.1 (Meyer [16], [17]). *Let $\xi = (E, \Sigma_g, p, \Sigma_h, \mathrm{Diff}_+\Sigma_h)$ be a Σ_h-bundle over Σ_g $(h \geq 2, g \geq 1)$ and $\chi : \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_h$ its holonomy homomorphism. Then the following equality holds*:

$$\tau(E) = -k(\chi_*(\widetilde{r}[\widetilde{R}, \widetilde{F}])) \, (= -c(\widetilde{\chi}(\widetilde{r}))).$$

3. Explicit description of 2-cycles of $\mathcal{M}_h$

In this section, we calculate values of the map $c : F \longrightarrow \mathbb{Z}$ for the relators of the finite presentation of $\mathcal{M}_h$ due to Wajnryb and give an explicit description of the homomorphism k defined in the preceding section in order to characterize the elements of $R \cap [F, F]$ as words of F.

Let $\mathcal{M}_h$ be the mapping class group of a surface Σ_h of genus h. A finite presentation of $\mathcal{M}_2$ was obtained by Birman-Hilden [3] and that of $\mathcal{M}_h$ $(h \geq 3)$ by Hatcher-Thurston [8].

Wajnryb [19] simplified their presentation of $\mathcal{M}_h$ $(h \geq 2)$ as foll ws. (We denote the commutator $xyx^{-1}y^{-1}$ of $x, y \in F$ by $[x, y]$.)

The generators, which are called the Lickorish-Humphries generators, of the presentation are:

$$y_1, y_2, u_1, u_2, \cdots, u_h, z_1, z_2, \cdots, z_{h-1}$$

and the relators of it are:

$$\begin{aligned}
A^1 &:= [y_1, y_2], \\
A^2_{i,j} &:= [y_i, u_j] \quad (i = 1, 2, 1 \leq j \leq h, i \neq j), \\
A^3_{i,j} &:= [y_i, z_j] \quad (i = 1, 2, 1 \leq j \leq h-1), \\
A^4_{i,j} &:= [u_i, u_j] \quad (1 \leq i < j \leq h), \\
A^5_{i,j} &:= [u_i, z_j] \quad (1 \leq i \leq h, 1 \leq j \leq h-1, j \neq i, i+1), \\
A^6_{i,j} &:= [z_i, z_j] \quad (1 \leq i < j \leq h-1), \\
B^1_i &:= y_i u_i y_i u_i^{-1} y_i^{-1} u_i^{-1} \quad (i = 1, 2), \\
B^2_i &:= u_i z_i u_i z_i^{-1} u_i^{-1} z_i^{-1} \quad (1 \leq i \leq h-1), \\
B^3_i &:= z_i u_{i+1} z_i u_{i+1}^{-1} z_i^{-1} u_{i+1}^{-1} \quad (1 \leq i \leq h-1), \\
C^1 &:= (y_1 u_1 z_1)^{-4} y_2 (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2)^{-1} y_2 (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2), \\
D^1 &:= y_1 z_1 z_2 t_1 t_2 (y_2 t_2 y_2 t_2^{-1} t_1 t_2 y_2)^{-1} (w u_1 z_1 u_2 z_2 u_3)^{-1} v w u_1 z_1 u_2 z_2 u_3, \\
E^1 &:= [d, u_h z_{h-1} u_{h-1} \cdots z_1 u_1 y_1^2 u_1 z_1 \cdots u_{h-1} z_{h-1} u_h],
\end{aligned}$$

where

$$\begin{aligned} t_1 &:= u_1y_1z_1u_1, \\ t_i &:= u_iz_{i-1}z_iu_i \quad (2 \le i \le h-1), \\ v &:= y_1u_1z_1u_2y_2(y_1u_1z_1u_2)^{-1}, \\ w &:= z_2u_3t_2y_2(z_2u_3t_2)^{-1}, \\ v_1 &:= (u_2z_1u_1y_1^2u_1z_1u_2)^{-1}y_2(u_2z_1u_1y_1^2u_1z_1u_2), \\ v_i &:= t_{i-1}t_iv_{i-1}(t_{i-1}t_i)^{-1} \quad (2 \le i \le h-1), \\ w_1 &:= u_1z_1u_2v_1(u_1z_1u_2)^{-1}, \\ w_i &:= u_iz_iu_{i+1}v_i(u_iz_iu_{i+1})^{-1} \quad (2 \le i \le h-1), \\ d &:= (w_1w_2\cdots w_{h-1})^{-1}y_1w_1w_2\cdots w_{h-1}. \end{aligned}$$

Elements y_i, u_i, z_i can be interpreted as Dehn twists with respect to curves Y_i, U_i, Z_i in Fig.1 of [3] (see also [13] and [10]). For $h = 2$ we can omit the relator D^1.

By choosing a symplectic basis of $H^1(\Sigma_h;\mathbb{Z})$ as in [17], we fix an explicit representation $\sigma : \mathcal{M}_h \longrightarrow Sp(2h,\mathbb{Z})$ by:

$$\begin{aligned} \sigma : y_i &\longmapsto \begin{pmatrix} I & 0 \\ -E_{ii} & I \end{pmatrix} \quad (i = 1,2), \\ \sigma : u_i &\longmapsto \begin{pmatrix} I & E_{ii} \\ 0 & I \end{pmatrix} \quad (1 \le i \le h), \\ \sigma : z_i &\longmapsto \begin{pmatrix} I & 0 \\ -E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} & I \end{pmatrix} \quad (1 \le i \le h-1), \end{aligned}$$

where $E_{ij} \in M_h(\mathbb{Z})$ is the (i,j)-matrix unit.

We also fix an exact sequence:

$$1 \longrightarrow R \longrightarrow F \stackrel{\pi}{\longrightarrow} \mathcal{M}_h \longrightarrow 1,$$

where

$$F := \langle y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1} \rangle$$

and R is the normal closure of the set of all relators $A^l_{i,j}, B^l_i, C^1, D^1, E^1$ in F. Let $c : F \longrightarrow \mathbb{Z}$ be the map defined as in Section 2 by using explicit homomorphisms σ and π fixed above.

Now we calculate values of the map $c : F \longrightarrow \mathbb{Z}$ for relators $A^l_{i,j}, B^l_i, C^1, D^1, E^1$ of the presentation and describe the homomorphism $c|_R : R \longrightarrow \mathbb{Z}$.

To compute values of c, Meyer showed the following lemma:

Lemma 3.1 (Meyer[16], [17]). *The map* $c : F \longrightarrow \mathbb{Z}$ *satisfies the following properties*:
(1) $c(xy) = c(x) + c(y) + \tau_h(\sigma(\pi(x)), \sigma(\pi(y))) \quad (x, y \in F)$;
(2) $c(x^{-1}) = -c(x) \quad (x \in F)$;
(3) $c(xyx^{-1}) = c(y) \quad (x, y \in F)$;
(4) $c(xzyz^{-1}) = c(x) + c(y)$ *if* $\pi(xzyz^{-1}) = 1 \in \mathcal{M}_h \quad (x, y, z \in F)$.

Values of c for relators are computed by using Lemma 3.1

Lemma 3.2. *The values of* c *for the relators of Wajnryb's presentation of* $\mathcal{M}_h (h \geq 3)$ *are calculated as follows*:
(1) $c(A^l_{i,j}) = 0$ (*for every* l, i, j);
(2) $c(B^l_i) = 0$ (*for every* l, i);
(3) $c(C^1) = -6$;
(4) $c(D^1) = 1$;
(5) $c(E^1) = 0$.

Proof. We denote $\tau_h(\sigma(\pi(x)), \sigma(\pi(y)))$ by $\widetilde{\tau}_h(x, y)$ for $x, y \in F$. By virtue of Lemma 3.1, it follows immediately that $c(A^l_{i,j}) = c(B^l_i) = c(E^1) = 0$. For example,

$$\begin{aligned} c(B^1_1) &= c(y_1 \cdot u_1 \cdot y_1 u_1^{-1} y_1^{-1} \cdot u_1^{-1}) \\ &= c(y_1) + c(y_1 u_1^{-1} y_1^{-1}) \\ &= c(y_1) + c(u_1^{-1}) = c(y_1) - c(u_1) \\ &= 0. \end{aligned}$$

Using Lemma 3.1 and calculating signature of symmetric bilinear forms concretely, we obtain values $c(C^1)$ and $c(D^1)$.

$$\begin{aligned} c(C^1) &= c((y_1 u_1 z_1)^{-4} y_2 (u_2 z_1 u_1 y_2^2 u_1 z_1 u_2)^{-1} y_2 (u_2 z_1 u_1 y_2^2 u_1 z_1 u_2)) \\ &= c((y_1 u_1 z_1)^{-4} y_2) \qquad (c(y_2) = 0) \\ &= c((y_1 u_1 z_1)^{-4}) + \widetilde{\tau}_h((y_1 u_1 z_1)^{-4}, y_2) \qquad (c(y_2) = 0) \\ &= 2c((y_1 u_1 z_1)^{-2}) + \widetilde{\tau}_h((y_1 u_1 z_1)^{-2}, (y_1 u_1 z_1)^{-2}) + \widetilde{\tau}_h((y_1 u_1 z_1)^{-4}, y_2) \\ &= 4(\widetilde{\tau}_h(1, z_1^{-1}) + \widetilde{\tau}_h(z_1^{-1}, u_1^{-1}) + \widetilde{\tau}_h(z_1^{-1} u_1^{-1}, y_1^{-1})) \\ &\quad + 2\widetilde{\tau}_h((y_1 u_1 z_1)^{-1}, (y_1 u_1 z_1)^{-1}) \\ &\quad + \widetilde{\tau}_h((y_1 u_1 z_1)^{-2}, (y_1 u_1 z_1)^{-2}) + \widetilde{\tau}_h((y_1 u_1 z_1)^{-4}, y_2) \\ &= 4(0 + 0 + 0) + 2 \cdot (-3) + (-1) + 1 \\ &= -6. \end{aligned}$$

$$
\begin{aligned}
c(D^1) &= c(y_1z_1z_2t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}(wu_1z_1u_2z_2u_3)^{-1}vwu_1z_1u_2z_2u_3)\\
&= c(y_1z_1z_2t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1})\\
&\qquad (c(v) = c(y_1u_1z_1u_2y_2(y_1u_1z_1u_2)^{-1}) = c(y_1) = 0)\\
&= c(y_1z_1z_2) + c(t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1})\\
&\quad +\widetilde{\tau}_h(y_1z_1z_2, t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1})\\
&= \widetilde{\tau}_h(y_1,z_1) + \widetilde{\tau}_h(y_1z_1,z_2) + c(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}) + c(t_2y_2^{-1}t_2^{-1}y_2^{-1})\\
&\quad +\widetilde{\tau}_h(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}, t_2y_2^{-1}t_2^{-1}y_2^{-1}) + \widetilde{\tau}_h(y_1z_1z_2, t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1})\\
&= \widetilde{\tau}_h(y_1,z_1) + \widetilde{\tau}_h(y_1z_1,z_2) + \widetilde{\tau}_h(t_2y_2^{-1}t_2^{-1}, y_2^{-1})\\
&\quad +\widetilde{\tau}_h(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}, t_2y_2^{-1}t_2^{-1}y_2^{-1}) + \widetilde{\tau}_h(y_1z_1z_2, t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1})\\
&\qquad (c(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}) = c(y_2^{-1}) = -c(y_2) = 0,\\
&\qquad\ c(t_2y_2^{-1}t_2^{-1}y_2^{-1}) = c(t_2y_2^{-1}t_2^{-1}) + c(y_2^{-1}) + \widetilde{\tau}_h(t_2y_2^{-1}t_2^{-1}, y_2^{-1})\\
&\qquad\qquad = \widetilde{\tau}_h(t_2y_2^{-1}t_2^{-1}, y_2^{-1}))\\
&= 0+0+0+0+1\\
&= 1. \qquad \square
\end{aligned}
$$

REMARK 3.3. All values of Meyer's signature cocycle τ_h calculated in Lemma 3.2 are independent of the genus h (≥ 3) because all generators which appear in C^1 and D^1 are $y_1, y_2, u_1, u_2, u_3, z_1$ and z_2. We can easily check by using a computer that the values are correct in the case $h = 3$. (We used *Mathematica*).

DEFINITION 3.4. Let F_n be a free group of rank n. *Algebraic m copies* of an element $x \in F_n$ are m_+ copies of x and m_- copies of x^{-1}, where $m_+, m_- \geq 0$ and $m_+ - m_- = m$. The integer m is called the *algebraic number* of these algebraic copies.

For each generator $e = y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}$, the homomorphism $e^* : F \longrightarrow \mathbb{Z}$ is defined by:

$$
e^*(x) := \begin{cases} +1 & (x = e), \\ 0 & (x : \text{other generators}). \end{cases}
$$

An element $x \in F$ belongs to $[F, F]$ if and only if $e^*(x) = 0$ for every generator e. Combining this with Lemma 3.2, we characterize the elements of $R \cap [F, F]$ as words in y_i, u_i, z_i and calculate the value of c for each element $x \in R \cap [F, F]$.

Proposition 3.5. *Suppose that $h \geq 3$. For an element $x \in F$, the following two conditions are equivalent:*

(1) *$x \in R \cap [F, F]$ and $c(x) = 4n$ $(n \in \mathbb{Z})$;*

(2) *x is equal to a product of conjugates of algebraic copies of relators and the*

algebraic number $m(R^1)$ of algebraic copies of a relator R^1 included in x is determined as follows:

R^1	$A^l_{i,j}$	B^1_1	B^1_2	B^2_1	B^2_2	$B^2_i(i \geq 3)$
$m(R^1)$	$\forall$	$-6n$	$18n$	$-2n$	$10n$	0

B^3_1	$B^3_i(i \geq 2)$	C^1	D^1	E^1
$-8n$	0	n	$10n$	$\forall$

where $\forall$ stands for arbitrary number of algebraic copies of R^1.

Proof. (1) $\Longrightarrow$ (2): Since R is the normal closure of the set $\{A^l_{i,j}, B^l_i, C^1, D^1, E^1\}$ of all relators, any $x \in R$ is a product of conjugates of algebraic copies of relators. For $x \in R \cap [F, F]$, let $a^l_{i,j}$ (respectively b^l_i, c^1, d^1, e^1) be the algebraic number of algebraic copies of $A^l_{i,j}$ (respectively B^l_i, C^1, D^1, E^1) included in x. These numbers must satisfy the following system of equations because x belongs to $[F, F]$.

$$\sum_{i=1}^{2} b^1_i e^*(B^1_i) + \sum_{i=1}^{h-1} b^2_i e^*(B^2_i) + \sum_{i=1}^{h-1} b^3_i e^*(B^3_i) + c^1 e^*(C^1) + d^1 e^*(D^1) = 0$$
$$(e = y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}).$$

($e^*(A^l_{i,j}) = e^*(E^1) = 0$ for every generator e because $A^l_{i,j}$ and E^1 belong to $[F, F]$. Values of e^* and c for other relators are exhibited in Table 3.6 below). Solving this, we get

$$b^1_1 = -6n,\ b^1_2 = 18n,\ b^2_1 = -2n,\ b^2_2 = 10n,\ b^2_i = 0\,(3 \leq i \leq h-1),$$
$$b^3_1 = -8n,\ b^3_i = 0\,(2 \leq i \leq h-1),\ c^1 = n,\ d^1 = 10n,$$

where n is an integer, while $a^l_{i,j}$ and e^1 are arbitrary integers.

(2) $\Longrightarrow$ (1): Such an element x belongs to $R \cap [F, F]$ because $e^*(x) = 0$ for every generator e. The value $c(x)$ can be calculated by using Lemma 3.2:

$$\begin{aligned} c(x) &= n\,c(C^1) + 10n\,c(D^1) \\ &= -6n + 10n \\ &= 4n. \end{aligned}$$

This completes the proof of Proposition 3.5. □

REMARK 3.7. Proposion 3.5 implies that the 'signature' $c(x)$ of a '2-cycle' $x \in R \cap [F, F]$ of $\mathcal{M}_h$ is concentrated on relators $B^1_1, B^1_2, B^2_1, B^2_2, B^3_1, C^1, D^1$ of Wajnryb's

	y_1^*	y_2^*	u_1^*	u_2^*	$\cdots$	u_{h-2}^*	u_{h-1}^*	u_h^*	z_1^*	z_2^*	$\cdots$	z_{h-2}^*	z_{h-1}^*	c
B_1^1	1		-1											0
B_2^1		1		-1										0
B_1^2			1						-1					0
B_2^2				1						-1				0
$\vdots$					$\ddots$						$\ddots$			$\vdots$
B_{h-2}^2						1						-1		0
B_{h-1}^2							1						-1	0
B_1^3				-1					1					0
B_2^3					$\ddots$					1				0
$\vdots$						$\ddots$					$\ddots$			$\vdots$
B_{h-2}^3							-1					1		0
B_{h-1}^3								-1					1	0
C^1	-4	2	-4	0	$\cdots$	0	0	0	-4	0	$\cdots$	0	0	-6
D^1	1	-2	0	0	$\cdots$	0	0	0	1	1	$\cdots$	0	0	1

(The blanks in the table above mean that the corresponding value is equal to zero.)

Table 3.6.

presentation and the algebraic number $m(R^1)$ of a relator R^1 is independent of the genus $h(\geq 3)$.

4. A construction of holonomy homomorphisms

We now construct the holonomy homomorphism $\chi : \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_h$ of a surface bundle ξ over a surface Σ_g with non-zero signature. We use a simple technique of the commutator collection process (see [7], [15]) to construct χ.

DEFINITION 4.1. Let F_n be the free group on the n free generators $e_1, \cdots, e_n$ and let a, b, u, v and w be words in $e_1, \cdots, e_n$. Two words u and v are called *freely equal* (denoted $u \approx v$) if they determine the same element of F_n.

The α-*skip* is the following sequence of free equalities:

$$\begin{aligned} uava^{-1}w &\approx u(ava^{-1}v^{-1})vw \\ &= u[a, v]vw \end{aligned}$$

and the *β-skip* is the following sequence of free equalities:

$$\begin{aligned} uavba^{-1}b^{-1}w &\approx u(avba^{-1}b^{-1}v^{-1})vw \\ &= u[a, vb]vw, \end{aligned}$$

where $[a, b] := aba^{-1}b^{-1}$. (We used the commutator relation $ba \approx [b, a]ab$.)

We apply α- and β-skips to elements of the free group F on the generators $y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}$ defined in the preceding section and prove the following lemma.

Lemma 4.2. *Suppose that $h \geq 3$. There exists a word W in $y_1, y_2, u_1, \cdots, u_h$, $z_1, \cdots, z_{h-1}$ with the following properties:*
(1) *W is a product of 111 commutators;*
(2) *W belongs to $R \cap [F, F]$ as an element of F;*
(3) $c(W) = 4$.

Proof. We set

$$\begin{aligned} \widetilde{W}_1 &:= (B_1^2)^{-1}(B_1^1)^{-3}B_2^1B_2^2D^1, \\ \widetilde{W}_2 &:= B_2^1(B_1^3)^{-1}B_2^1B_2^2D^1, \\ \widetilde{W} &:= C^1\widetilde{W}_8^2\widetilde{W}_2^8. \end{aligned}$$

Since the word $\widetilde{W}$ satisfies the condition (2) of Proposition 3.5 in case $n = 1$, $\widetilde{W}$ has the properties (2) and (3) above. We decompose $\widetilde{W}$ to a product W of 111 commutators by using α- and β-skips repeatedly.

We rewrite some of Wajnryb's relators as follows:

$$\begin{aligned} B_1^1 &= y_1R_1u_1^{-1} \quad (R_1 = [u_1, y_1]), \\ B_2^1 &= y_2R_2u_2^{-1} \quad (R_2 = [u_2, y_2]), \\ B_1^2 &= u_1R_3z_1^{-1} \quad (R_3 = [z_1, u_1]), \\ B_2^2 &= u_2R_4z_2^{-1} \quad (R_4 = [z_2, u_2]), \\ B_1^3 &= z_1R_5u_2^{-1} \quad (R_5 = [u_2, z_1]), \\ C^1 &= (y_1u_1z_1)^{-4}y_2^2R_6 \quad (R_6 = [y_2^{-1}, (u_2z_1u_1y_1^2u_1z_1u_2)^{-1}]), \\ D^1 &= y_1z_1z_2t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}y_2^{-1}t_2^{-1}R_7R_8 \\ &\qquad (R_7 = [y_2^{-1}, y_1u_1z_1u_2], \quad R_8 = [v^{-1}, (wu_1z_1u_2z_2u_3)^{-1}]), \end{aligned}$$

where $R_1, \cdots R_8$ are commutators.

$\widetilde{W}_i (i = 1, 2)$ is transformed into another word $W_i (i = 1, 2)$ by using α- and β-skips in the following way:

$$
\begin{aligned}
\widetilde{W}_1 &= (B_1^2)^{-1}(B_1^1)^{-3}B_2^1B_2^2D^1 \\
&\approx z_1R_3^{-1}R_1^{-1}y_1^{-1}(u_1R_1^{-1}y_1^{-1})^2y_2R_2R_4z_2^{-1}y_1z_1z_2t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\underset{(\beta)}{\widetilde{\approx}} z_1R_3^{-1}R_1^{-1}y_1^{-1}(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}y_1z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (S_1 := [y_2, R_2R_4z_2^{-1}y_1z_1z_2t_1t_2]) \\
&\underset{(\alpha)}{\widetilde{\approx}} z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (S_2 := [y_1^{-1}, (u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}]) \\
&=: W_1;
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{W}_2 &= B_2^1(B_1^3)^{-1}B_2^1B_2^2D^1 \\
&\approx y_2R_2R_5^{-1}z_1^{-1}y_2R_2R_4z_2^{-1}y_1z_1z_2t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\underset{(\beta)}{\widetilde{\approx}} y_2R_2R_5^{-1}z_1^{-1}S_3R_2R_4z_2^{-1}y_1z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (S_3 := [y_2, R_2R_4z_2^{-1}y_1z_1z_2t_1t_2]) \\
&\underset{(\beta)}{\widetilde{\approx}} S_4R_2R_5^{-1}z_1^{-1}S_3R_2R_4z_2^{-1}y_1z_1z_2R_7R_8 \\
&\qquad (S_4 := [y_2, R_2R_5^{-1}z_1^{-1}S_3R_2R_4z_2^{-1}y_1z_2t_2]) \\
&\underset{(\alpha)}{\widetilde{\approx}} S_4R_2R_5^{-1}S_5S_3R_2R_4z_2^{-1}y_1z_2R_7R_8 \\
&\qquad (S_5 := [z_1^{-1}, S_3R_2R_4z_2^{-1}y_1]) \\
&=: W_2.
\end{aligned}
$$

The word W_1 obtained above naturally includes 10 commutators and the word W_2 9 ones. Hence the word $C^1W_1^2W_2^8$ naturally includes 93 commutators.

Furthermore we perform 6 α-skips and 4 β-skips to $C^1W_1^2$ and get a word $\widehat{W}$ in the following way:

$$
\begin{aligned}
C^1W_1^2 &= (y_1u_1z_1)^{-4}y_2y_2R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2 \\
&\qquad \cdot S_1R_2R_4z_2^{-1}z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8W_1 \\
&\underset{(\beta)}{\widetilde{\approx}} (y_1u_1z_1)^{-3}z_1^{-1}u_1^{-1}y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2 \\
&\qquad \cdot S_1R_2R_4z_2^{-1}z_1z_2R_7R_8W_1 \\
&\qquad\quad (S_6 := [y_2, R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2t_2]) \\
&\underset{(\beta)}{\widetilde{\approx}} (y_1u_1z_1)^{-3}S_7u_1^{-1}y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2 \\
&\qquad \cdot S_1R_2R_4R_7R_8W_1 \\
&\qquad (S_7 := [z_1^{-1}, u_1^{-1}y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}]) \\
&\underset{(\alpha)}{\widetilde{\approx}} (y_1u_1z_1)^{-2}z_1^{-1}u_1^{-1}y_1^{-1}S_7S_8y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1} \\
&\qquad \cdot u_1R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8W_1
\end{aligned}
$$

$$
\begin{aligned}
&\quad (S_8 := [u_1^{-1}, y_1^{-1} y_2 S_6 R_6 z_1 R_3^{-1} R_1^{-1} S_2]) \\
&\underset{(\alpha)}{\approx} (y_1 u_1 z_1)^{-2} S_9 u_1^{-1} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot u_1 R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 W_1 \\
&\quad (S_9 := [z_1^{-1}, u_1^{-1} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6]) \\
&\underset{(\alpha)}{\approx} (y_1 u_1 z_1)^{-2} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 \\
&\quad \cdot z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1} z_1 z_2 t_2 y_2^{-1} t_2^{-1} R_7 R_8 \\
&\quad (S_{10} := [u_1^{-1}, y_1^{-1} S_7 S_8 y_1^{-1} y_2 S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}]) \\
&\underset{(\beta)}{\approx} (z_1^{-1} u_1^{-1} y_1^{-1})^2 S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1} z_1 z_2 R_7 R_8 \\
&\quad (S_{11} := [y_2, S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 \\
&\qquad \cdot z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1} z_1 z_2 t_2]) \\
&\underset{(\beta)}{\approx} z_1^{-1} u_1^{-1} y_1^{-1} S_{12} u_1^{-1} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 R_7 R_8 \\
&\quad (S_{12} := [z_1^{-1}, u_1^{-1} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\qquad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 (u_1 R_1^{-1} y_1^{-1})^2 S_1 R_2 R_4 z_2^{-1}]) \\
&\underset{(\alpha)}{\approx} z_1^{-1} u_1^{-1} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} u_1 R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 \\
&\quad (S_{13} := [u_1^{-1}, y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\qquad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2]) \\
&\underset{(\alpha)}{\approx} z_1^{-1} S_{14} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 \\
&\quad (S_{14} := [u_1^{-1}, y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} \\
&\qquad \cdot R_1^{-1} S_2 R_1^{-1} y_1^{-1} R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 z_1 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1}]) \\
&\underset{(\alpha)}{\approx} S_{15} S_{14} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} \\
&\quad \cdot R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 R_3^{-1} R_1^{-1} S_2 R_1^{-1} y_1^{-1} R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8 \\
&\quad (S_{15} := [z_1^{-1}, S_{14} y_1^{-1} S_{12} S_{13} y_1^{-1} S_9 S_{10} y_1^{-1} S_7 S_8 y_1^{-1} S_{11} S_6 R_6 R_3^{-1} \\
&\qquad \cdot R_1^{-1} S_2 R_1^{-1} y_1^{-1} R_1^{-1} y_1^{-1} S_1 R_2 R_4 R_7 R_8]) \\
&=: \widehat{W}
\end{aligned}
$$

The word $\widehat{W}$ is a product of 31 commutators and 8 copies of y_1^{-1}.The word W_2^8 is a product of 72 commutators and 8 copies of $z_1^{-1} y_1 z_1$.

We perform 8 β-skips to the word $\widehat{W}W_2^8$ repeatedly by setting $a = y_1^{-1}$ and $b = z_1^{-1}$ in Definition 4.1. Then we obtain a word W which is a product of 111 $(= 31 + 72 + 8)$ commutators and is freely equal to $\widetilde{W}$. This completes the proof of Lemma 4.2. □

By virtue of Lemma 4.2, we can show the following theorem.

Theorem 4.3. *There exists a* Σ_h*-bundle* $\xi = (E, \Sigma_g, p, \Sigma_h, \mathrm{Diff}_+\Sigma_h)$ *over* Σ_g *with* $g = 111$*,* $h = 3$ *and* $\tau(E) = -4$.

Proof. Set $g = 111$ and $h = 3$. We choose a word W which satisfies conditions (1)-(3) of Lemma 4.2 and write

$$W = \prod_{i=1}^{g}[\alpha_i, \beta_i] \quad (\alpha_i, \beta_i \in F(i = 1, \cdots, g)).$$

Let $\widetilde{\chi} : \widetilde{F} \longrightarrow F$ the homomorphism defined by:

$$\widetilde{\chi}(\widetilde{a}_i) = \alpha_i, \quad \widetilde{\chi}(\widetilde{b}_i) = \beta_i \quad (i = 1, \cdots, g),$$

where $\widetilde{F} = \langle \widetilde{a}_1, \cdots, \widetilde{a}_g, \widetilde{b}_1, \cdots, \widetilde{b}_g \rangle$. Since $\widetilde{\chi}(\widetilde{r}) = W \in R \cap [F, F]$, $\widetilde{\chi}$ induces the homomorphism $\chi : \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_h$ (i.e., $\pi \circ \widetilde{\chi} = \chi \circ \widetilde{\pi}$) as in Section 2. For the Σ_h-bundle ξ over Σ_g which has χ as its holonomy homomorphism, we calculate the signature of its total space E:

$$\begin{aligned} \tau(E) &= -c(\widetilde{\chi}(\widetilde{r})) \\ &= -c(W) \\ &= -4. \end{aligned}$$

We have thus proved the theorem. □

Finally, we prove our main theorem (Theorem 1.2) by using Lemma 4.2 and results of Lück [14] concerning about L^2-Betti numbers of groups.

Proof of Theorem 1.2. Let W be the word constructed in the proof of Lemma 4.2. For every $h \geq 3$ and each $n \in \mathbb{Z}(n \neq 0)$, we can construct a Σ_h-bundle $\xi = \hat{\xi}(h, n)$ with $g = 111|n|$ and $\tau(E) = 4n$ by using the word W^{-n} as in the proof of Theorem 4.3 (see Remark 3.7). Therefore we have

$$g(h, n) \leq 111|n|.$$

On the other hand, for every Σ_h-bundle ξ over Σ_g with $g \geq 1, h \geq 3$ and $\tau(E) =$

$4n$, the associated exact sequence:

$$1 \longrightarrow \pi_1(\Sigma_h) \longrightarrow \pi_1(E) \xrightarrow{p_\sharp} \pi_1(\Sigma_g) \longrightarrow 1$$

of fundamental groups satisfies the assumption of [14], Theorem 4.1. Then the first L^2-Betti number $b_1(\pi_1(E))$ of $\pi_1(E)$ is equal to zero and the Winkelnkemper-type inequality $\chi(E) \geq |\tau(E)|$ holds from [14], Theorem 5.1. By substituting

$$\chi(E) = \chi(\Sigma_h)\chi(\Sigma_g) = 4(h-1)(g-1), \quad \tau(E) = 4n$$

for the inequality, we obtain

$$g(h,n) \geq \frac{|n|}{h-1} + 1$$

and this completes the proof of our theorem. □

REMARK 4.4. The Σ_h-bundle $\xi = \hat{\xi}(h,n)$ over Σ_g constructed in the first half of the proof of Theorem 1.2 has $g = 111|n|$, $\tau(E) = 4n$, $b_1(E) = 2(111|n| + h - 3)$, $b_2(E) = 2(222|n|h - 5)$ and $\chi(E) = 4(111|n| - 1)(h-1)$, where $h(\geq 3)$ and $n \in \mathbb{Z}(n \neq 0)$. If the total space E admits a complex structure, E is an algebraic surface of general type and satisfies the Noether condition, the Noether inequality and the Bogomolov-Miyaoka-Yau inequality (cf. [2]). But E cannot be a geometric 4-manifold in the sense of Thurston [20], in particular, a compact Kähler surface covered by the unit ball in $\mathbb{C}^2$.

Let $\Gamma(h,n)$ be the fundamental group of the total space of $\xi = \hat{\xi}(h,n)(h \geq 3, n \geq 1)$ constructed in the first half of the proof of Theorem 1.2. Computing an invariant defined by Johnson [11], we obtain the following result.

Corollary 4.5. *The family $\{\Gamma(h,n)\}_{h\geq 3, n\geq 1}$ contains infinitely many commensurability classes of discrete groups. In particular, $\{\Gamma(h,n)\}_{n\geq 1}$ is a family of infinitely many non-commensurable discrete groups for each $h(\geq 3)$.*

Proof. The commensurability invariant $\gamma(\Gamma)$ [11] for $\Gamma = \Gamma(h,n)$ is

$$\gamma(\Gamma(h,n)) = \frac{n}{(111n-1)(h-1)} \quad (h \geq 3, n \geq 1),$$

which runs over infinitely many rational numbers. □

REMARK 4.6. Although the author attempted to show that the value $g(h,n)$ does not depend on the genus h (≥ 3) of fiber Σ_h for each $n \in \mathbb{Z}$ $(n \neq 0)$, it was not achieved because of some serious transformation problems on words in free generators.

References

[1] M.F. Atiyah: *The signature of fibre-bundles*, Global Analysis, Papers in Honor of K. Kodaira, Tokyo Univ. Press, (1969), 73–84

[2] W. Barth, C. Peters and A. Van de Ven: Compact Complex Surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, Bd 4, Springer-Verlag, Berlin-Heidelbelg-New York-Tokyo, 1984.

[3] J. Birman and H. Hilden: *On mapping class groups of closed surfaces as covering spaces*, Advances in the Theory of Riemann Surfaces, Ann. Math. Stud. **66**, Princeton Univ. Press, (1971), 81–115.

[4] K.S. Brown: Cohomology of Groups, Graduate Texts in Math. **87**, Springer-Verlag, 1982.

[5] S.S. Chern, F. Hirzebruch and J.P. Serre: *On index of a fibred manifold*, Proc. Amer. Math. Soc. **8** (1957), 587–596.

[6] C.J. Earle and J. Eells: *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc. **73** (1967), 557–559.

[7] A.M. Gaglione and H.V. Waldinger: *The commutator collection process*, Contemp. Math. **109** (1990), 43–58.

[8] A. Hatcher and W. Thurston: *A presentation for the mapping class group of a closed oriented surface*, Topology, **19** (1980), 221–237.

[9] F. Hirzebruch: *The signature of ramified coverings*, Global Analysis, Papers in Honor of K. Kodaira, Tokyo Univ. Press, (1969), 253–265.

[10] S. Humphries: *Generators for the mapping class group of a closed orientable surface*, Topology of Low-dimensional Manifolds, Lecture Notes in Mathematics **722**, Springer Berlin, (1979), 44–47.

[11] F.E.A. Johnson: *A rational invariant for certain infinite discrete groups*, Math. Proc. Camb. Phil. Soc. **113** (1993), 473–478.

[12] K. Kodaira: *A certain type of irregular algebraic surfaces*, J. Anal. Math. **19** (1967), 207–215.

[13] W.B.R. Lickorish: *A finite set of generators for the homotopy group of 2-manifold*, Proc. Camb. Phil. Soc. **60** (1964), 769–778.

[14] W. Lück: *L^2-Betti numbers of mapping tori and groups*, Topology, **33** (1994), 203–214.

[15] W. Magnus, A. Karrass and D. Solitar: Combinatorial Group Theory, Interscience Publ, 1966.

[16] W. Meyer: *Die Signatur von lokalen Koeffizientensystemen und Faserbündeln*, Bonner Mathematische Schriften, **53** (1972).

[17] W. Meyer: *Die Signatur von Flächenbündeln*, Math. Ann. **201** (1973), 239–264.

[18] S. Morita: *Characteristic classes of surface bundles*, Invent. Math. **90** (1987), 551–577.

[19] B. Wajnryb: *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45** (1989), 157–174.

[20] C.T.C. Wall: *Geometric structures on complex analytic surfaces*, Topology, **25** (1986), 119–153.

Department of Mathematics
Osaka University
Toyonaka Osaka
560–0043, Japan
email: endo@math.sci.osaka-u.ac.jp