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ON THE PATHWISE UNIQUENESS OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Introduction

In this paper, we shall discuss a problem of the pathwise uniqueness for solutions of one-dimensional stochastic differential equations. Let $a(x)$ and $b(x)$ be bounded Borel measurable functions defined on R . We shall consider the following one-dimensional Itô's stochastic differential equation;

$$(1) \quad dx_t = a(x_t)dB_t + b(x_t)dt.$$

K. Itô [1] proved that, if $a(x)$ and $b(x)$ are Lipschitz continuous, a solution is unique and it can be constructed on a given Brownian motion B_t . On the other hand, if $|a(x)|$ is bounded from below by a positive constant (i.e. uniformly positive), then a solution of (1) exists and it is unique in the law sense. This follows easily from a general result of one-dimensional diffusions (cf. [2]). However, though the distribution of $\{x_t, B_t\}$ is unique, x_t is not always expressed as a measurable function of x_0 and $\{B_s, s \leq t\}$. For example, if $a(x) = \text{sgn } x$, $a(0) = 1$ and $x_0 \equiv 0$, it is not difficult to see that $\sigma\{|x_s|; s \leq t\} = \sigma\{B_s; s \leq t\}$.

Here, we will show that, if $a(x)$ is uniformly positive and of bounded variation on any compact interval, then the pathwise uniqueness holds for (1). This implies, in particular, that x_t is expressed as a measurable function of x_0 and $\{B_s, s \leq t\}$ (cf. [5]). In this direction, M. Motoo (unpublished) already proved that the pathwise uniqueness holds for (1) if $a(x)$ is uniformly positive and Lipschitz continuous and if $b(x)$ is bounded measurable. Also, T. Yamada and S. Watanabe [5] proved the pathwise uniqueness of (1) if $a(x)$ is Hölder continuous of exponent $\frac{1}{2}$ and $b(x)$ is Lipschitz continuous. Our above mentioned result may be interesting in a point that it applies for many discontinuous $a(x)$. It is still an open question whether only the uniform positivity of $a(x)$ implies the pathwise uniqueness.

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A precise meaning of the equation (1) is as follows: $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ stands for a probability space (Ω, \mathcal{F}, P) with an increasing family $\{\mathcal{F}_t\}_{t \in [0, \infty)}$ of sub- σ -algebras of \mathcal{F} .

DEFINITION 1. By a solution of (1), we mean a quadruplet $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and a stochastic process $\mathfrak{X}_t = (x_t, B_t)$ defined on it such that

- (i) with probability one, \mathfrak{X}_t is continuous in t and $B_0 = 0$,
- (ii) \mathfrak{X}_t is an $\{\mathcal{F}_t\}$ -adapted process and B_t is an $\{\mathcal{F}_t\}$ -Brownian motion,
- (iii) \mathfrak{X}_t satisfies

$$x_t = x_0 + \int_0^t a(x_s) dB_s + \int_0^t b(x_s) ds \quad a.s. ,$$

where the integral by dB_s is understood in the sense of the stochastic integral of Itô.

DEFINITION 2. We shall say that the pathwise uniqueness holds for (1), for any two solutions $\mathfrak{X}_t = (x_t, B_t)$, $\mathfrak{Y}_t = (y_t, B'_t)$ defined on a same quadruplet $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $x_0 = y_0$ and $B_t \equiv B'_t$ implies $x_t = y_t$.

REMARK 1. In Definition 2, it is sufficient to assume that $x_0 = y_0 = x$ for some constant $x \in R$.

REMARK 2. A definition of the pathwise uniqueness may be defined in a stronger way as follows; the pathwise uniqueness holds if $\mathfrak{X}_t = (x_t, B_t)$ is a solution on $(\Omega, \mathcal{F}, P; \mathcal{F}_t^1)$ and $\mathfrak{Y}_t = (y_t, B'_t)$ is a solution on $(\Omega, \mathcal{F}, P; \mathcal{F}_t^2)$ (\mathcal{F}_t^1 and \mathcal{F}_t^2 may be different) such that $x_0 = y_0$ and $B_t \equiv B'_t$, then $x_t = y_t$. It is not difficult to show, using a result in [5], that this definition of the pathwise uniqueness is equivalent to Definition 2.

Lemma. Let $(M_t, V_t)_{t \in [0, T]}$ be a pair of continuous real process defined on a probability space (Ω, \mathcal{F}, P) . Suppose that the total variation $|||V(\omega)|||_T$ of $V_t(\omega)$ on $[0, T]$ has a finite expectation. Further, suppose M_t is a martingale satisfying the following conditions;

- (i) $M_0 = 0 \quad a.s. ,$
- (ii) there exist positive constants m_1 and m_2 such that

$$(2) \quad m_1 M_t(\omega) \leq V_t(\omega) \leq m_2 M_t(\omega) \quad a.s. ,$$

for $(t, \omega) \in \{(t, \omega); t \in [0, T] \text{ and } M_t(\omega) \geq 0\}$.

Then, $M_t = 0$ a.s. for $0 \leq t \leq T$.

Proof. For $y \in R$, let $N_1(y, \omega)$ be the number of $t \in [0, T]$ such that $V_t(\omega) = y$. By a theorem of Banach (cf. [4] pp. 280), we have

$$(3) \quad |||V(\omega)|||_T = \int_{-\infty}^{\infty} N_1(y, \omega) dy .$$

Obviously we may assume that $m_1 < m_2$. For $y > 0$, let $N_2(y, \omega)$ be the number of $\left[\frac{1}{m_2}y, \frac{1}{m_1}y\right]$ -downcrossings of $M_t(\omega)$ on $[0, T]$. The condition (2) implies that

$$(4) \quad N_1(y, \omega) \geq N_2(y, \omega) \quad \text{for } y > 0.$$

For $y > 0$, we define a sequence of stopping times $\{T_k\}$ in the following way;

$$\begin{aligned} T_0 &= 0, \\ T_{2n+1} &= \inf \left\{ t \geq T_{2n}; M_t > \frac{1}{m_1}y \right\} \wedge T \quad n = 0, 1, 2, \dots, \\ T_{2n+2} &= \inf \left\{ t \geq T_{2n+1}; M_t < \frac{1}{m_2}y \right\} \wedge T \quad n = 0, 1, 2, \dots. \end{aligned}$$

Then, for $n = 1, 2, \dots$, we can obtain the following inequality;

$$\begin{aligned} &\left(\frac{1}{m_1}y - \frac{1}{m_2}y\right)(N_2(y, \omega) \wedge n + 1) \\ &\geq \sum_{k=1}^n \{M_{T_{2k-1}}(\omega) - M_{T_{2k}}(\omega)\} + \{(M_T(\omega) - \frac{1}{m_1}y) \vee 0\} \chi_{\{N_2(y, \omega) < n\}}. \end{aligned}$$

Taking the expectation, we have

$$E[N_2(y) \wedge n] \geq \frac{E[\{(M_T - \frac{1}{m_1}y) \vee 0\} \chi_{\{N_2(y) < n\}}]}{\left(\frac{1}{m_1} - \frac{1}{m_2}\right)y} - 1.$$

Letting $n \rightarrow \infty$, we have

$$(5) \quad E[N_2(y)] \geq \frac{E[(M_T - \frac{1}{m_1}y) \vee 0]}{\left(\frac{1}{m_1} - \frac{1}{m_2}\right)y} - 1.$$

Now, we assume that $P(M_T \neq 0) > 0$. Then, there exist positive constants ϵ and δ such that

$$(6) \quad E\left[\left(M_T - \frac{1}{m_1}y\right) \vee 0\right] > \epsilon \quad \text{for } 0 < y < \delta.$$

The inequalities (5) and (6) provide us with the equality

- 1) Let x and y be real numbers. $x \wedge y$ means $\min(x, y)$.
- 2) $x \vee y$ means $\max(x, y)$.
- 3) χ_A denotes the indicator function of a set A .

$$\int_0^{\delta} E[N_2(y)] dy = \infty .$$

But this is a contradiction since, by (3) and (4),

$$\int_0^{\delta} E[N_2(y)] dy \leq \int_{-\infty}^{\infty} E[N_1(y)] dy < \infty .$$

Therefore we have

$$P(M_t = 0) = 1 \quad \text{for } 0 \leq t \leq T .$$

This completes the proof.

REMARK 3. In the above lemma, we may suppose the following condition instead of (2); there exist positive constants m_1 and m_2 such that

$$(7) \quad m_1 |M_t| \leq |V_t| \leq m_2 |M_t| \quad \text{a.s. for } 0 \leq t \leq T .$$

Now we will state our main result.

Theorem. *Let $a(x)$ and $b(x)$ be bounded Borel measurable. Suppose $a(x)$ is of bounded variation on any compact interval. Further, suppose there exists a constant $c > 0$ such that*

$$(8) \quad a(x) \geq c \quad \text{for } x \in R .$$

Then, the pathwise uniqueness holds for (1).

Proof. We assume that $|a(x)| \leq M$ and $|b(x)| \leq M$ for $x \in R$. Let $\mathfrak{X}_t = (x_t, B_t)$ and $\mathfrak{Y}_t = (y_t, B_t)$ be solutions of (1) such that $x_0 = y_0$ is a constant. For $N > |x_0|$, we define that

$$\begin{aligned} \tau_N &= \begin{cases} \inf \{t \geq 0; |x_t| = N\} \\ \infty \end{cases} & \text{if } \{ \} = \phi , \\ \eta_N &= \begin{cases} \inf \{t \geq 0; |y_t| = N\} \\ \infty \end{cases} & \text{if } \{ \} = \phi , \\ \gamma_N &= \tau_N \wedge \eta_N . \end{aligned}$$

Let, for $x \in R$,

$$f(x) = -2 \int_0^x \frac{b(y)}{a^2(y)} dy , \quad \varphi(x) = \int_0^x \exp [f(y)] dy .$$

By the time substitution and Cameron-Martin's formula (cf. [3]), there exists a constant $K_1 > 0$ depending only on c, M, N and t such that

$$(9) \quad E \left[\int_0^{t \wedge \gamma_N} g(x_s) ds \right] \leq K_1 \|g\|_{L^1([-N, N])} \quad \text{for } g \in L^1([-N, N]) .$$

Since $\varphi'(x)$ is absolutely continuous and $\varphi''(x)$ is locally integrable, the inequality (9) assures us that Itô's formula applies to φ and we have

$$\varphi(x_{t \wedge \gamma_N}) = \varphi(x_0) + \int_0^{t \wedge \gamma_N} \varphi'(a(x_s)) dB_s \quad a.s. .$$

Since φ is a homeomorphism R onto $I = (\varphi(-\infty), \varphi(\infty))$, we can define that

$$\sigma(x) = \varphi' a \circ \varphi^{-1}(x), \quad h(x) = \int_0^x \frac{1}{\sigma(y)} dy \quad \text{for } x \in I.$$

Obviously σ is of bounded variation on any compact interval of I . Let $\|\sigma\|_N$ be the total variation of σ on $[\varphi(-N), \varphi(N)]$.

We can take an approximate sequence $\{\sigma_n(x)\}_{n=1,2,\dots}$ such that

- (i) $\sigma_n(x) \in C^1(R)$ and $c \exp\left[-\frac{2MN}{c^2}\right] \leq \sigma_n(x) \leq M \exp\left[\frac{2MN}{c^2}\right]$
for $x \in R$,
- (ii) $\|\sigma - \sigma_n\|_{L^1([\varphi(-N), \varphi(N)])} \leq \frac{1}{n!}$ and $\|\sigma'_n\|_{L^1([\varphi(-N), \varphi(N)])} \leq \|\sigma\|_N$.

Let

$$h_n(x) = \int_0^x \frac{1}{\sigma_n(y)} dy \quad \text{for } x \in I.$$

Since $h_n(x) \in C^2(R)$, we can apply Itô's formula to h_n and have

$$\begin{aligned} h_n(\varphi(x_{t \wedge \gamma_N})) &= h_n(\varphi(x_0)) + \int_0^{t \wedge \gamma_N} \frac{\sigma(\varphi(x_s))}{\sigma_n(\varphi(x_s))} dB_s \\ &\quad - \frac{1}{2} \int_0^{t \wedge \gamma_N} \frac{\sigma'_n(\varphi(x_s)) \sigma^2(\varphi(x_s))}{\sigma_n^2(\varphi(x_s))} ds . \\ &= h_n(\varphi(x_0)) + L_t^n + W_t^n . \end{aligned}$$

It follows from (ii) that there exists a constant $K_2 > 0$ depending only on c, M and N such that

$$|h(x) - h_n(x)| \leq K_2 \frac{1}{n!} \quad \text{for } x \in [\varphi(-N), \varphi(N)] .$$

From this, we see that $h_n(\varphi(x_{t \wedge \gamma_N}))$ converges almost surely to $h(\varphi(x_{t \wedge \gamma_N}))$. There exists a constant $K_3 > 0$ depending only on c, M, N , and t such that

$$E[(L_t^n - B_{t \wedge \gamma_N})^2] \leq K_3 \frac{1}{n!} .$$

Therefore L_t^n converges almost surely to $B_{t \wedge \gamma_N}$. Let

$$W_t = h(\varphi(x_{t \wedge \gamma_N})) - h(\varphi(x_0)) - B_{t \wedge \gamma_N} .$$

From the above results, W_t^n converges almost surely to W_t .

It is easy to see that there exists a constant $K_4 > 0$ depending only on $c, M,$ and N such that

$$E[|||W^n|||_t] \leq K_4 E\left[\int_0^{t \wedge \gamma_N} |\sigma'_n(\varphi(x_s))| ds\right],$$

where $|||W^n|||_t$ is the total variation of W^n_s on $[0, t]$. Using the time substitution, we easily see that there exists a constant $K_5 > 0$ depending only on $c, M, N,$ and t such that

$$\begin{aligned} E\left[\int_0^{t \wedge \gamma_N} |\sigma'_n(\varphi(x_s))| ds\right] &\leq K_5 \|\sigma'_n\|_{L^1(\varphi(-N), \varphi(N))} \\ &\leq K_5 \|\sigma\|_N. \end{aligned}$$

Hence it holds that

$$E[|||W|||_t] \leq K_4 K_5 \|\sigma\|_N,$$

where $|||W|||_t$ is the total variation of W_s on $[0, t]$.

From the definition of $h(x)$, there exists positive constants m_1 and m_2 such that

$$m_1(x - y) \leq h(x) - h(y) \leq m_2(x - y) \quad \text{for } y \leq x \text{ and } x, y \in [\varphi(-N), \varphi(N)].$$

Let

$$\begin{aligned} M_t &= \int_0^{t \wedge \gamma_N} (\sigma(\varphi(x_s)) - \sigma(\varphi(y_s))) dB_s, \\ V_t &= h(\varphi(x_{t \wedge \gamma_N})) - h(\varphi(y_{t \wedge \gamma_N})). \end{aligned}$$

We can apply Lemma to (M_t, V_t) and it follows that

$$P(\varphi(x_{t \wedge \gamma_N}) = \varphi(y_{t \wedge \gamma_N})) = 1.$$

Therefore we have

$$P(x_{t \wedge \gamma_N} = y_{t \wedge \gamma_N}) = 1.$$

Since $\lim_{N \rightarrow \infty} \gamma_N = \infty$ a.s., we obtain that $P(x_t = y_t) = 1$ and the proof is complete.

REMARK 4. In Theorem, if $a(x)$ is continuous, we may assume that $a(x)$ is positive instead of (8).

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