Title	On almost complex structures on abstract Wiener spaces
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Citation	Osaka Journal of Mathematics. 1996, 33(1), p. 189-206
Version Type	VoR
URL	https://doi.org/10.18910/4608
rights	
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Taniguchi, S.
Osaka J. Math.
33 (1996), 189-206

ON ALMOST COMPLEX STRUCTURES ON ABSTRACT WIENER SPACES

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(Received September 5, 1994)

1. Introduction

An abstract Wiener space is a triplet (B, H, μ) of a real Banach space B, a real Hilbert space H, and a Gaussian measure μ on B, and is called complex if B and H are complex Banach and Hilbert spaces, respectively. For detailed definition, see Section 2. Thinking of H as the tangent space of B and following the notion on finite dimensional manifolds, we define

$$\mathscr{A}\mathscr{C}(H)=\{J: H\to H: J \text{ is an isometry and } J^2=-\mathrm{id}_H\}, \tag{1.1}$$

and call $J\in\mathscr{A}\mathscr{C}(H)$ an almost complex structure on (B, H, μ). A typical example of almost complex structure is the multiplication by $\sqrt{-1}$ in the case where (B, H, μ) is complex. In the present paper, we shall see that each $J\in\mathscr{A}\mathscr{C}(H)$ admits a natural complex abstract Wiener space (B_J, H, μ_J) so that J is realized as the multiplication by $\sqrt{-1}$ under this complex structure. Moreover, studied will be the correspondence between (B_J, H, μ_J) and the original space: if J extends to a closed operator $J': B\to B$, then B_J is imbedded in B continuously and densely, and if it does to a continuous linear operator $\tilde{J}: B\to B$, then $B_J=B$ and the norm n_J is equivalent to the original one $\|\cdot\|_B$, i.e. $C\|\mathbf{z}\|_B\le n_J(\mathbf{z})\le C^{-1}\|\mathbf{z}\|_B$, $\mathbf{z}\in B$, for some $0<C<\infty$. See Theorem 2.6.

An almost complex structure on (B, H, μ) was introduced by Shigekawa [10] as a $J\in\mathscr{A}\mathscr{C}(H)$ extending to an isometry $\tilde{J}: B\to B$, and, with his definition, several complex analytical studies were made on abstract Wiener spaces [2, 8, 12, 13, 14]. The above observation not only asserts that the almost complex structure introduced by Shigekawa is nothing but a complex structure on B, but also enables one to extend some complex analytical observations to general $J\in\mathscr{A}\mathscr{C}(H)$. In fact, for every $J\in\mathscr{A}\mathscr{C}(H)$, one can define a Cauchy-Riemann operator $\bar{\partial}$ on B with the help of the Malliavin calculus. Holomorphic functions on an open set G in B are measurable functions u with $\bar{\partial}u=0$ on G. Assuming that J has a closed extension $J': B\to B$, we shall define skeletons of holomorphic functions on G as

Lebesgue densities at points in $G\cap H$. Then, the holomorphy of skeletons on $G\cap H$ and a uniqueness theorem based on skeletons will be discussed. The investigation is a continuation of those in [12, 14] to general J. See Section 2.

Now arises a natural question whether there is an almost complex structure $J\in\mathcal{A}\mathcal{C}(H)$ which has no extension to a continuous operator of B to B. The answer is yes. In fact, let (B, H, μ) be the abstract Wiener space corresponding to the pinned Brownian motion on $\mathbf{R}^d$. Thinking of B as a based loop group over the abelian Lie group $\mathbf{R}^d$, one can define an almost complex structure $J\in\mathcal{A}\mathcal{C}(H)$ by taking advantage of Fourier series (cf. [9]). We shall see that this J has a closed extension J': $B\to B$, but does not have any extension to a continuous operator of B to B. These will be discussed in Section 3.

On an abstract Wiener space with almost complex structure in the sense of Shigekawa, Sugita [13] introduced a concept of "holomorphically exceptional" which can spell H out of B; a one-point set $\{\mathbf{z}\}$ is holomorphically exceptional or not accordingly as $\mathbf{z}\notin H$ or $\in H$. In the present paper, inspired by his work, we introduce set functions C_p, $p\in(1, \infty)$, on B specifying holomorphically exceptional sets as null sets under a suitable assumption on J, which is weaker than assuming the extendability of J to a closed operator of B to B.

We shall call a holomorphic function on B entire as on $\mathbf{C}$. The set function, moreover, enables one to think of skeletons as restrictions to H of entire functions. In fact, we shall establish an Egorov theorem for entire functions with respect to the set function; every holomorphic function $u\in L^p(B, \mathbf{C}; \mu)$ possesses a μ-version $\tilde{u}$ so that $\tilde{u}$ is continuous on each F_n for some increasing sequence of closed sets in B with $C_p(B\setminus F_n)\to 0$. We shall, further, see that $\tilde{u}$ coincides with the skeleton of u on H. In this sense, the skeleton can be regard as a restriction of u to H, since $H\subset\bigcup_n F_n$. These observations will be given in Section 4. In the same section, also given will be studies of entire functions; an approximation theorem, a uniqueness theorem and a Liouville theorem for holomorphic functions will be seen. The first two theorems are generalizations of those studied by Shigekawa [10] and Fang-Ren [2] in the case where J extends to an isometry of B.

ACKNOWLEDGEMENT. The author is grateful to Professor Hiroshi Sugita for several stimulus discussions.

2. Complex structures

Let (B, H, μ) be an abstract Wiener spece; B is a real separable Banach space with norm $\|\cdot\|_B$, H is a real separable Hilbert space wih norm $\|\cdot\|_H$, which is imbedded continuously and densely in B, and μ is a probability measure on B such that

$$\int_B \exp[\sqrt{-1}\langle \mathbf{z},\ l\rangle]\mu(d\mathbf{z})=\exp[-\|l\|_H^2/4],\ l\in B^*,$$

where $\langle \cdot, \cdot \rangle$ stands for the natural pairing of B and its dual space B^*, and we have used the standard identification of H^* and H so that $B^*\subset H^*=H\subset B$. If B and H are complex Banach and Hilbert spaces, respectively, then the triplet is called a *complex* abstract Wiener space. The space of almost complex structures $\mathscr{A}\mathscr{C}(H)$ on (B, H, μ) is defined by (1.1) following the definition of almost complex structures on finite dimensional manifolds;

$$\mathscr{A}\mathscr{C}(H)=\{J: H\to H: J \text{ is an isometry and } J^2=-\mathrm{id}_H\}.$$

As for assuming for J to be an isometry, it should be recalled that if a linear mapping $K: V\to V$ on finite dimensional real vector space V satisfies that $K^2=-\mathrm{id}_V$, then $\dim V$ is even and there exists a basis $\{e_n, e_n'\}_{n=1}^{\dim V/2}$ such that $Ke_n=e_n'$ and $Ke_n'=-e_n$, and hence one can introduce an inner product on V so that K is an isometry. In this sense, no additional assumption is imposed on $\mathscr{A}\mathscr{C}(H)$ compared with the definition of almost complex structure on finite dimensional manifolds. In [10], for J to be an almost complex structure, it is additionally assumed that J extends to an isometry of B.

Let $J\in\mathscr{A}\mathscr{C}(H)$. For $\eta\in H^*$, define a continuous $\mathbf{R}$-linear mapping $\eta^{(1,0)}$: $H\to\mathbf{C}$ by $\eta^{(1,0)}=\eta-\sqrt{-1}J^*\eta$. Put

$$n_J(\mathbf{h})=\sup\{|{}_H\langle \mathbf{h},\ l^{(1,0)}\rangle_{H^*}|: l\in B^*, \|l\|_{B^*}=1\},$$

where ${}_H\langle\cdot, \cdot\rangle_{H^*}$ stands for the pairing of H and H^*, and extends to that of H and $H^*\oplus\sqrt{-1}H^*$ so that ${}_H\langle \mathbf{h},\ \eta+\sqrt{-1}\eta'\rangle_{H^*}={}_H\langle \mathbf{h},\ \eta\rangle_{H^*}+\sqrt{-1}{}_H\langle \mathbf{h},\ \eta'\rangle_{H^*}$, $\mathbf{h}\in H$, η, $\eta'\in H^*$. By the Hahn-Banach theorem on B, n_J enjoys that

$$\|\mathbf{h}\|_B\leq n_J(\mathbf{h})\leq\|\mathbf{h}\|_B+\|J\mathbf{h}\|_B\leq 2C_{H,B}\|\mathbf{h}\|_H,\ \mathbf{h}\in H, \tag{2.1}$$

where $C_{H,B}$ is a constant such that $\|\mathbf{h}\|_B\leq C_{H,B}\|\mathbf{h}\|_H$, $\mathbf{h}\in H$. Hence n_J is a norm on H. Recall that $\|J\cdot\|_B$ is a measurable norm on H in the sense of Gross [3,6]. Hence, (2.1) yields that n_J is also a measurable norm in the sense of Gross. Let B_J be the completion of H with respect to n_J and denote by the same n_J the extension of n_J to B_J. It also follows from (2.1) that H is imbedded in (B_J, n_J) densely and continuously. By a standard method, one obtains a Gaussian measure μ_J on B_J so that $((B_J, n_J), (H, \|\cdot\|_H), \mu_J)$ is an abstract Wiener space.

Now observe that (B_J, n_J) turns to be a complex separable Banach space. To do this, notice that ${}_H\langle J\mathbf{h},\ l^{(1,0)}\rangle_{H^*}=\sqrt{-1}{}_H\langle \mathbf{h},\ l^{(1,0)}\rangle_{H^*}$, and then that

$$n_J(J\mathbf{h})=n_J(\mathbf{h}),\ \mathbf{h}\in H. \tag{2.2}$$

Hence $J: H\to H$ has an extension to a continuous linear operator of B_J to B_J, which will be denoted by J again. In particular, one has

$$J^2\mathbf{x}=-\mathbf{x} \text{ and } n_J(J\mathbf{x})=n_J(\mathbf{x}),\ \mathbf{x}\in B_J.$$

Now, define the multiplication by complex numbers by

$$(a+\sqrt{-1}b)\mathbf{x}=a\mathbf{x}+bJ\mathbf{x},\ a,\ b\in\mathbf{R},\ \mathbf{x}\in B_J. \tag{2.3}$$

It is then easy to check that B_J turns to a complex vector space with the original addition and this multiplication by complex numbers. Since ${}_H\langle\zeta\mathbf{h},\ l^{(1,0)}\rangle_{H^*}=\zeta_H\langle\mathbf{h},\ l^{(1,0)}\rangle_{H^*}$, it holds that $n_J(\zeta\mathbf{h})=|\zeta|n_J(\mathbf{h})$ for $\zeta\in\mathbf{C}$, $\mathbf{h}\in H$. Combined with the continuity of J on B_J, this implies that

$$n_J(\zeta\mathbf{x})=|\zeta|n_J(\mathbf{x}),\ \zeta\in\mathbf{C},\ \mathbf{x}\in B_J.$$

Thus $(B_J,\ n_J)$ is a complex separable Banach space with the complex multiplication given by (2.3).

Mention that $\langle\mathbf{h},\ J\mathbf{h}\rangle_H=\langle J\mathbf{h},\ J^2\mathbf{h}\rangle_H=-\langle J\mathbf{h},\ \mathbf{h}\rangle_H$ and then that $\langle\mathbf{h},\ J\mathbf{h}\rangle_H=0$. Hence

$$\|(a+\sqrt{-1}b)\mathbf{h}\|_H^2=a^2\|\mathbf{h}\|_H^2+b^2\|J\mathbf{h}\|_H^2=|(a+\sqrt{-1}b)|^2\|\mathbf{h}\|_H^2,\ a,\ b\mathbf{R},\ \mathbf{h}\in H,$$

which means that $(H,\ \|\cdot\|_H)$ is a complex separable Hilbert space. It therefore has been seen that $((B_J,\ n_J),\ (H,\ \|\cdot\|_H),\ \mu_J)$ is a complex abstract Wiener space.

By (2.1), there exists a continuous linear mapping $\iota:\ B_J\to B$ such that $\iota(\mathbf{h})=\mathbf{h}$ for any $\mathbf{h}\in H$. It follows from (2.1) and (2.2) that

$$\frac{1}{2}(\|\mathbf{h}\|_B+\|J\mathbf{h}\|_B)\le n_J(\mathbf{h})\le\|\mathbf{h}\|_B+\|J\mathbf{h}\|_B. \tag{2.4}$$

This implies that ι is injective if and only if $J:\ H\to H$ has an extension to a closed operator $J':\ B\to B$. Thus $(B_J,\ n_J)$ is imbedded in $(B,\ \|\cdot\|_B)$ continuously and densely if and only if J extends to a closed operator $J':\ B\to B$. In this case, B_J coincides with the completion of H with respect to the graph norm of J'.

A subset $L\subset B^*$ is said to separate the points of B if, for any $\mathbf{x},\ \mathbf{y}\in B$ with $\mathbf{x}\neq\mathbf{y}$, there is an $l\in L$ so that $\langle\mathbf{x},\ l\rangle\neq\langle\mathbf{y},l\rangle$. A necessary and sufficient condition for $J\in\mathcal{A}\mathcal{C}(H)$ to extend to a close operator $J':\ B\to B$ is that the subspace $L_J\equiv\{l\in B^*:\ Jl\in B^*\}$ separates the points of B, where we have used the identification $B^*\subset H^*=H$ to operate $J:\ H\to H$ on elements in B^*. In fact, the sufficiency can be seen easily. To see the necessity, observe that $J^*=-J$ on H and $(J')^*=J^*$ on $\mathfrak{Dom}((J')^*)$, $J':\ B\to B$ being a closed extension of J, and hence that $\mathfrak{Dom}((J')^*)\subset L_J$. Recall now that the domain $\mathfrak{Dom}(T^*)$ of the adjoint operator T^* of a closed operator $T:\ B\to B$ separates the points of B, which can be seen in the same way as the density of $\mathfrak{Dom}(T^*)$ in B^* is verified when B is reflexive (cf. [1, Theorem III. 21]). Thus L_J separates the points of B.

Now suppose that $J\in\mathcal{A}\mathcal{C}(H)$ admits an extension to a continuous linear mapping $\tilde{J}:\ B\to B$. According to (2.4), this assumption is equivalent to assuming $\sup\{n_J(\mathbf{h})/\|\mathbf{h}\|_B:\ \mathbf{h}\in H,\ \neq0\}<\infty$. In this case, (2.1) reads as

$$\|\mathbf{h}\|_B\le n_J(\mathbf{h})\le(1+\|\tilde{J}\|_{\mathrm{op}})\|\mathbf{h}\|_B,\ \mathbf{h}\in H, \tag{2.5}$$

where $\|\tilde{J}\|_{\mathrm{op}}$ is the operator norm of $\tilde{J}: B \to B$. It then holds that $B=B_J$ as a set. Conversely, if J has an extension to a closed operator $J': B \to B$ and $B=B_J$, then, by (2.4) and the closed graph theorem, J' is a continuous operator, and hence J extends to a continuous linear operator of B to B.

Summarizing up, one has

Theorem 2.6. *Let $J \in \mathcal{AC}(H)$, and define a Banach space (B_J, n_J) and a probability measure μ_J on B_J as above. Then, $((B_J, n_J), (H, \|\cdot\|_H), \mu_J)$ is a complex abstract Wiener space with complex multiplication given in* (2.3) *and $J=\sqrt{-1}\mathrm{id}_H$. Further, the following three conditions are equivalent;*

(a) *(B_J, n_J) is imbedded in $(B, \|\cdot\|_B)$ continuously and densely,*

(b) *J extends to a closed operator $J': B \to B$,*

(c) *the subspace $L_J=\{l \in B^*: Jl \in B^*\}$ of B^* separates the points of B.*

Finally, assume that one of the above conditions is satisfied. Then, $B_J=B$ if and only if J admits an extension to a continuous linear mapping $\tilde{J}: B \to B$. In this case, n_J is equivalent to $\|\cdot\|_B$ in the sense (2.5) *and $((B, n_J), (H, \|\cdot\|_H), \mu)$ is a complex abstract Wiener space.*

As an application of Theorem 2.6, studied will be holomorphic functions on open sets in B and their skeletons. To do this, let $J \in \mathcal{AC}(H)$. We shall first define the Cauchy-Riemann operator $\bar{\partial}$ on B. Set

$$H^{*\mathbf{C}}=\{\eta: H \to \mathbf{C}: \eta \text{ is a continuous } \mathbf{R}\text{-linear mapping}\}=H^* \oplus \sqrt{-1}H^*.$$

In a standard manner, $H^{*\mathbf{C}}$ turns to be a separable complex Hilbert space. Extend the dual operator $J^*: H^* \to H^*$ to $H^{*\mathbf{C}}$ as $J^*(\eta_1+\sqrt{-1}\eta_2)=J^*\eta_1+\sqrt{-1}J^*\eta_2$, and set

$$H^{*(1,0)}=\{\eta \in H^{*\mathbf{C}}: J^*\eta=\sqrt{-1}\eta\} \text{ and } H^{*(1,0)}=\{\eta \in H^{*\mathbf{C}}: J^*\eta=-\sqrt{-1}\eta\}.$$

Obviously $H^{*\mathbf{C}}=H^{*(1,0)} \oplus H^{*(0,1)}$.

A measurable function $F: B \to \mathbf{C}$ is said to be smooth if there are an $f \in C_0^\infty(\mathbf{R}^n; \mathbf{C})$ and $\eta_1, \cdots, \eta_n \in B^*$ such that

$$F(\mathbf{z})=f(\langle \mathbf{z}, \eta_1 \rangle, \cdots, \langle \mathbf{z}, \eta_n \rangle).$$

The totality of smooth functions are denoted by $\mathcal{F}C_0^\infty(B; \mathbf{C})$. For a separable complex Hilbert space E, $\mathcal{F}C_0^\infty(B; E)$ stands for the space of linear combinations of finite number of measurable functions of the form Fe, $F \in \mathcal{F}C_0^\infty(B; \mathbf{C})$, $e \in E$. Sobolev spaces $\mathbf{D}_p^r(B; E)$, $r \in \mathbf{R}$, $p \in (1, \infty)$, are the completions of $\mathcal{F}C_0^\infty(B; E)$ with respect to the norm

$$\|G\|_{r,p}=\|(I-\mathcal{L})^{r/2}G\|_{L^p(B,E;d\mu)}$$

respectively, where $\mathcal{L}$ denotes the Ornstein-Uhlenbeck operator. Put

$$\mathbf{D}_{1+}^{-\infty}(B\,;\,E)=\bigcup_{r\in(-\infty,\infty)}\bigcup_{p\in(1,\infty)}\mathbf{D}_p^r(B\,;\,E).$$

For $G\in\mathcal{F}C_0^\infty(B\,;\,\mathbf{C})$, the gradient $DG(\mathbf{z})\in H^{*\mathbf{C}}$ is defined by

$$_H\langle\mathbf{h},\ DG(\mathbf{z})\rangle_{H^*}=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\{G(\mathbf{z}+\varepsilon\mathbf{h})-G(\mathbf{z})\}.$$

Since it holds ([11]) that

(2.7) $\operatorname{Sup}\{\|DG\|_{r-1,p}/\|G\|_{r,p}:\ G\in\mathcal{F}C_0^\infty(B\,;\,\mathbf{C}),\ G\neq 0\}<+\infty,\ r\in\mathbf{R},\ p\in(1,\infty),$

D extends to a continuous linear mapping of $\mathbf{D}_p^r(B\,;\,\mathbf{C})$ to $\mathbf{D}_p^{r-1}(B\,;\,H^{*\mathbf{C}})$, and hence to that of $\mathbf{D}_{1+}^{-\infty}(B\,;\,\mathbf{C})$ to $\mathbf{D}_{1+}^{-\infty}(B\,;\,H^{*\mathbf{C}})$, which continues to be denoted by the letter D.

Let $\pi_{0,1}$ be the projection of $H^{*\mathbf{C}}$ onto $H^{(0,1)}$, and set

$$\bar\partial u(\mathbf{z})=\pi_{0,1}(Du(\mathbf{z})),\ \mathbf{z}\in B,\ u\in\mathcal{F}C_0^\infty(B\,;\,\mathbf{C}).$$

By the continuity of D in (2.7), $\bar\partial$ extends to a continuous linear operator of $\mathbf{D}_{1+}^{-\infty}(B\,;\,\mathbf{C})$ to $\mathbf{D}_{1+}^{-\infty}(B\,;\,H^{*(0,1)})$, which is again denoted by $\bar\partial$.

Let G be an open set in B. A$v\in\bigcup_{p>2}L^p(B,\mathbf{C}\,;\,d\mu)$ is said to be holomorphic on G, $\bar\partial v=0$ on G in notation, if $\langle\bar\partial v,\ \varphi\rangle=0$ for any $\varphi\in\mathbf{D}_{\infty-}^1(B\,;\,H^{*\mathbf{C}})\equiv\bigcap_{p\in(1,\infty)}\mathbf{D}_p^1(B\,;\,H^{*\mathbf{C}})$ with $\varphi=\varphi\mathbf{1}_G$ μ-a.e. Set

$$\mathbf{Hol}(G\,;\,B)=\{u\in\bigcup_{p>2}L^p(G,\mathbf{C}\,;\,d\mu):\ \bar\partial(u\mathbf{1}_G)=0\ \text{on}\ G\}.$$

To study the skeleton of $u\in\mathbf{Hol}(G\,;\,B)$, we now assume that there is a closed operator $J'\colon B\to B$ with $J'=J$ on H. By Theorem 2.6, one obtains a complex abstract Wiener space $((B_J,\ n_J),\ H,\ \mu_J)$ so that B_J is included in B continuously and densely. There exists a subset $B_0\subset B_J$ satisfying that

(2.8) $B_0\in\mathcal{B}(B)\cup\mathcal{B}(B_J),$

(2.9) $\{A\cap B_0:\ A\in\mathcal{B}(B)\}=\{A\cap B_0:\ A\in\mathcal{B}(B_J)\}$

(2.10) $\mu(B_0)=\mu_J(B_0)=1$

(2.11) $\mu(A\cap B_0)=\mu_J(A\cap B_0),\ A\in\mathcal{B}(B)\cup\mathcal{B}(B_J),$

where $\mathcal{B}(Y)$ stands for the topological Borel field of a topological space Y. Namely, recall that there is an abstract Wiener space $(B_0,\ H,\ \mu_0)$ such that B_0 is reflexive, and is imbedded in B_J densely and continuously. See [7]. It now suffices to show that, if $(X,\ H,\ \nu)$ is an abstract Wiener spaces such that B_0 is included in X continuously and densely, then $B_0\in\mathcal{B}(X)$, $\mathcal{B}(B_0)=\{A\cap B_0:\ A\in\mathcal{B}(X)\}$, and $\nu(A)=\mu_0(A\cap B_0)$, $A\in\mathcal{B}(X)$. To do this, notice that X^* is dense in B_0^* and B_0^* is separable, because of the reflexivity of B_0. Hence there exists a countable set $\{l_j\}_{j=1}^\infty\subset X^*$ which is dense in B_0^*. One then has that $\|\mathbf{y}\|_{B_0}=\sup\{|\langle\mathbf{y},\ l_j\rangle|/\|l_j\|_{B_0^*}:\ j\in\mathbf{N}\}$, $\mathbf{y}\in B_0$. Set $q(\mathbf{x})=\sup\{|\langle\mathbf{x},\ l_j\rangle|/\|l_j\|_{B_0^*}:\ j\in\mathbf{N}\}$, $\mathbf{x}\in X$. It follows the reflexivity of B_0 that $B_0=\{\mathbf{x}\in X:\ q(\mathbf{z})<\infty\}$. As is easily seen, $B_0\in\mathcal{B}(X)$, $\nu(B_0)$

$=1$, and $\mathcal{B}(B_0)\supset\{A\cap B_0:\ A\in\mathcal{B}(X)\}$. Taking advantage of the density of X^* in B_0^* and a characterization of $\mathcal{B}(B_0)$ by cylindrical sets, one can further show that $\mathcal{B}(B_0)=\{A\cap B_0:\ A\in\mathcal{B}(X)\}$ and $\nu(A)=\mu_0(A\cap B_0), A\in\mathcal{B}(X)$.

Let $u\in\mathbf{Hol}(G;\ B)$. Putting $B_J(\mathbf{x},r)=\{\mathbf{y}\in B_J:\ n_J(\mathbf{y}-\mathbf{x})<r\}$, $\mathbf{x}\in B_J$, and $B_{0,J}(r)=B_0\cap B_J(\mathbf{0},\ r)$, we claim that the Lebesgue density

$$\hat{u}(\mathbf{h})\equiv\lim_{r\downarrow 0}\frac{1}{\mu(B_{0,J}(r))}\int_{B_{0,J}(r)}u(\mathbf{z}+\mathbf{h})d\mu(\mathbf{z}) \tag{2.12}$$

exists for every $\mathbf{h}\in H\cap G$ (we call $\hat{u}$ *the skeleton* of u). It should be mentioned that the definition of the skeleton is independent of a particular choice of B_0 enjoying (2.8)-(2.11). To see the existence of the limit, notice that $G\cap B_J$ is also open in B_J. $\mathbf{Hol}(G\cap B_J;\ B_J)$ is now constructed in exactly the same manner as above, only this time relative to B_J and the complex structure J on B_J. Then, by (2.10) and (2.11), it is easily seen that $u\mathbf{1}_{B_0}\in\mathbf{Hol}(G\cap B_J;\ B_J)$. Let $r(\mathbf{h})=\sup\{t:\ B_J(\mathbf{h},\ t)\subset G\}$. If $r<r(\mathbf{h})$, then $(u\mathbf{1}_{B_0})(\cdot+\mathbf{h})\in\mathbf{Hol}(B_J(\mathbf{0},\ r);\ B_J)$. Apply [14, (3.2)] to $(u\mathbf{1}_{B_0})(\cdot+\mathbf{h})$, and observe that

$$\frac{1}{\mu(B_J(\mathbf{0},\ r))}\int_{B_J(0,r)}(u\mathbf{1}_{B_0})(\mathbf{z}+\mathbf{h})d\mu_J(\mathbf{z})=\frac{1}{\mu(B_J(\mathbf{0},\ s))}\int_{B_J(0,s)}(u\mathbf{1}_{B_0})(\mathbf{z}+\mathbf{h})d\mu_J(\mathbf{z})$$
$$0<s,\ r<r(\mathbf{h}).$$

Combined with (2.8)-(2.11), this implies that

$$\frac{1}{\mu(B_{0,J}(r))}\int_{B_{0,J}(r)}u(\mathbf{z}+\mathbf{h})d\mu(\mathbf{z})=\frac{1}{\mu(B_{0,J}(s))}\int_{B_{0,J}(s)}u(\mathbf{z}+\mathbf{h})d\mu(\mathbf{z}), \tag{2.13}$$
$$\mathbf{h}\in G\cap H,\ 0<s,\ r<r(\mathbf{h}),$$

which yields the existence of the limit.

As an application of [14, Corollaries 2.2 and 2.4], one can conclude that

Theorem 2.14. *Suppose that $J\in\mathcal{AC}(H)$ extends to a closed operator J': $B\to B$. Let G be an open subset of B, and $u\in\mathbf{Hol}(G;\ B)$. Define $\hat{u}$ as above. Then the following assertions hold.*

(i) *The mapping $G\cap H\ni\mathbf{h}\to\hat{u}(\mathbf{h})\in\mathbf{C}$ is continuous with respect to the topology inherited from H.*

(ii) *As before, define $\zeta\mathbf{h}=(\mathrm{Re}\,\zeta)\mathbf{h}+(\mathrm{Im}\,\zeta)J\mathbf{h}$, $\zeta\in\mathbf{C}$, $\mathbf{h}\in H$. For any $\mathbf{h}_0,\cdots,\mathbf{h}_n\in G\cap H$, there exists an open set $U\subset\mathbf{C}^n$ so that $\{\mathbf{h}_0+\sum_{i=1}^n\zeta_i\mathbf{h}_i:\ (\zeta_1,\cdots,\zeta_n)\in U\}\subset G\cap H$ and the mapping*

$$U\ni(\zeta_1,\cdots,\zeta_n)\to\hat{u}\left(\mathbf{h}_0+\sum_{i=1}^n\zeta_i\mathbf{h}_i\right)\in\mathbf{C}$$

is holomorphic.

(iii) *If there are $\mathbf{h}\in G\cap H$ and $\delta>0$ such that $\mathbf{k}\in G\cap H$ and $\hat{u}(\mathbf{k})=0$ whenever $\|\mathbf{k}-\mathbf{h}\|_H<\delta$, then $u=0$ μ-a.e. on G.*

3. Abstract Wiener space of Brownian bridge

In this section, we give an example of $J\in\mathcal{A}\mathcal{C}(H)$ possessing no extension to a continuous linear mapping of B to B. Throughout this section, let

$$B_{\mathrm{pin}}=\{\mathbf{z}: [0, 1]\rightarrow\mathbf{R}^{d}: \mathbf{z} \text{ is continuous and } \mathbf{z}(0)=\mathbf{z}(1)=0\},$$

$$H_{\mathrm{pin}}=\left\{\mathbf{h}\in B_{\mathrm{pin}}: \begin{array}{l}\mathbf{h} \text{ is absolutely continuous and possesses}\\ \text{a derivative } \dot{\mathbf{h}} \text{ such that } \int_{0}^{1}|\dot{\mathbf{h}}(t)|^{2}dt<\infty\end{array}\right\},$$

and μ_{pin} be the image measure on B_{pin} of the Brownian bridge on $\mathbf{R}^{d}$ starting at 0 and pinned at 0 at time 1. The norm $\|\cdot\|_{B_{\mathrm{pin}}}$ of B_{pin} and inner product $\langle\cdot, \cdot\rangle_{H_{\mathrm{pin}}}$ on H_{pin} are given by $\|\mathbf{z}\|_{B_{\mathrm{pin}}}=\sup\{|\mathbf{z}(t)|: t\in[0, 1]\}$ and $\langle\mathbf{h}, \mathbf{k}\rangle_{H_{\mathrm{pin}}}=\int_{0}^{1}\langle\dot{\mathbf{h}}(t), \dot{\mathbf{k}}(t)\rangle_{\mathbf{R}^{d}}dt$, respectively.

We now introduce an almost complex structure $J\in\mathcal{A}\mathcal{C}(H_{\mathrm{pin}})$ following [9]. To do this, let $\xi_{i}=(0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0)\in\mathbf{R}^{d}$ and put

$$e_{n,i}^{s}(t)=\frac{\sin(2n\pi t)}{2n\pi}\xi_{i} \text{ and } e_{n,i}^{c}(t)=\frac{\cos(2n\pi t)-1}{2n\pi}\xi_{i} \text{ for } n\in\mathbf{N},\ 1\leq i\leq d.$$

Every $\mathbf{h}\in H_{\mathrm{pin}}$ is expanded as

$$\mathbf{h}=2\sum_{i=d}^{d}\sum_{n>0}\left\{\langle\mathbf{h}, \mathbf{e}_{n,i}^{s}\rangle_{H_{\mathrm{pin}}}\mathbf{e}_{n,i}^{s}+\langle\mathbf{h}, \mathbf{e}_{n,i}^{c}\rangle_{H_{\mathrm{pin}}}\mathbf{e}_{n,i}^{c}\right\} \text{ in } H_{\mathrm{pin}}.$$

One can then define a continuous linear operator $J: H_{\mathrm{pin}}\rightarrow H_{\mathrm{pin}}$ so that

$$J\mathbf{e}_{n,i}^{s}=\mathbf{e}_{n,i}^{c} \text{ and } J\mathbf{e}_{n,i}^{c}=-\mathbf{e}_{n,i}^{s}. \tag{3.1}$$

In particular, one has a series representation of $J\mathbf{h}$ as

$$J\mathbf{h}=2\sum_{i=d}^{d}\sum_{n>0}\left\{\langle\mathbf{h}, \mathbf{e}_{n,i}^{s}\rangle_{H_{\mathrm{pin}}}\mathbf{e}_{n,i}^{c}-\langle\mathbf{h}, \mathbf{e}_{n,i}^{c}\rangle_{H_{\mathrm{pin}}}\mathbf{e}_{n,i}^{s}\right\} \text{ in } H_{\mathrm{pin}}. \tag{3.2}$$

As is easily seen, it holds that

$$J^{2}\mathbf{h}=-\mathbf{h} \text{ and } \langle J\mathbf{h}, J\mathbf{k}\rangle_{H_{\mathrm{pin}}}=\langle\mathbf{h}, \mathbf{k}\rangle_{H_{\mathrm{pin}}},\ \mathbf{h}, \mathbf{k}\in H_{\mathrm{pin}}.$$

Thus $J\in\mathcal{A}\mathcal{C}(H_{\mathrm{pin}})$. The aim of this section is to establish that

Theorem 3.3. *Let $J\in\mathcal{A}\mathcal{C}(H_{\mathrm{pin}})$ be as above.*
(i) *J extends to a closed operator $J': B_{\mathrm{pin}}\rightarrow B_{\mathrm{pin}}$.*
(ii) *J admits no extension to a continuous linear mapping of B to B.*
(iii) *$J(B_{\mathrm{pin}}^{*})\backslash B_{\mathrm{pin}}^{*}\neq\emptyset$; there exists an $l\in B_{\mathrm{pin}}^{*}$ such that $J^{*}l\notin B_{\mathrm{pin}}^{*}$.*

Proof. (i) Define $L\subset H$ by

$$L=\{\mathbf{e}_{n,i}^{s}, \mathbf{e}_{n,i}^{c}: n\in\mathbf{N}, 1\leq i\leq d\}.$$

Then $L\subset B^*$ in the sense that

$$\langle \mathbf{z},\ l\rangle=-\int_0^1\langle \mathbf{z}(t),\ \frac{d^2}{dt^2}l(t)\rangle_{\mathbf{R}^d}dt,\ l\in L,$$

where $\langle\cdot,\cdot\rangle$ stands for the pairing of B_{pin} and B^*_{pin}. It is then an elementary exercise of Fourier series to see that $\mathbf{z}=\mathbf{0}$ if and only if $\langle \mathbf{z},\ l\rangle=0$ for every $l\in L$. In particular, L separate the points of B_{pin}.

Since $J\mathbf{e}^c_{n,i}=-\mathbf{e}^s_{n,i}$ and $J^*=-J$ under the identification $H^*=H$, one has that $J^*(L)=L\subset B^*$. Hence, on account of Theorem 2.6, J extends to a closed operator J': $B_{\text{pin}}\to B_{\text{pin}}$.

(ii, iii) The proof is given only in the case where $d=1$, the other cases being dealt with in exactly the same manner.

Let $\mathbf{h}_\alpha(t)=\sin(\alpha t)-t\sin\alpha$, $\alpha\in[0,\ \infty)\backslash 2\pi\mathbf{Z}$. To see the first assertion, it suffices to show that

$$\liminf_{\alpha\to\infty}\|J\mathbf{h}_\alpha\|_{B_{\text{pin}}}=\infty. \tag{3.4}$$

To see this, notice that

$$J\mathbf{h}(t)=\sqrt{-1}\sum_{n\neq 0}(\operatorname{sgn} n)\Big(\int_0^1\mathbf{h}(s)e^{-2\pi ns\sqrt{-1}}ds\Big)(e^{2\pi nt\sqrt{-1}}-1),\ \mathbf{h}\in H_{\text{pin}},$$

where $\operatorname{sgn} n=1$ if $n>0$ and $=-1$ if $n<0$. By a simple computation, one has that

$$\int_0^1\mathbf{h}_\alpha(s)e^{-2\pi ns\sqrt{-1}}ds=\frac{\alpha(1-\cos\alpha)}{\alpha^2-4\pi^2n^2}+\frac{\alpha^2\sin\alpha}{2\pi n(\alpha^2-n^2)\sqrt{-1}},$$

and hence that

$$J\mathbf{h}_\alpha(t)=-2\sum_{n>0}\frac{\alpha(1-\cos\alpha)}{\alpha^2-4\pi^2n^2}\sin(2\pi nt)+\frac{1}{\pi}\sum_{n>0}\frac{\alpha^2\sin\alpha}{n(\alpha^2-4\pi^2n^2)}(\cos(2\pi nt)-1).$$

In particular,

$$J\mathbf{h}_\alpha\Big(\frac{1}{2}\Big)=-\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\alpha^2\sin\alpha}{(2m+1)(\alpha^2-4\pi^2(2m+1)^2)}.$$

Set $\beta_j=(2j+1)+\frac{1}{4}$ and $\alpha_j=2\pi\beta_j$. Then

$$J\mathbf{h}_{\alpha_j}\Big(\frac{1}{2}\Big)=-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\beta_j^2}{(2m+1)(\beta_j^2-(2m+1)^2)} \tag{3.5}$$

By a straightforward computation, one has that

$$\sum_{m=0}^{j}\frac{\beta_j^2}{(2m+1)(\beta_j^2-(2m+1)^2)}\geq\sum_{m=0}^{j}\frac{1}{2m+1}\geq\frac{1}{2}\log(2j+3),$$

and that

$$\sum_{m=j+1}^{\infty}\frac{\beta_j^2}{(2m+1)((2m+1)^2-\beta_j^2)}\leq\frac{\beta_j}{(2j+3)^2-\beta_j^2}+\sum_{m=j+2}^{\infty}\frac{\beta_j}{(2m+1)^2-\beta_j^2}$$
$$\leq\frac{2}{7}+\frac{1}{4}\log\left(\frac{4j+\frac{17}{4}}{\frac{7}{4}}\right).$$

Plugging these into (3.5), one obtains that

$$J\mathbf{h}_{\alpha j}\left(\frac{1}{2}\right)\to-\infty \text{ as } j\to\infty,$$

and which implies that (3.4) holds.

To see the last assertion, define $l\in B_{\mathrm{pin}}^*$ by $\langle \mathbf{z}, l\rangle=\mathbf{z}(1/2)$ and suppose that $J^*l\in B_{\mathrm{pin}}^*$, and hence that there exists a constant C such that $|\langle \mathbf{z}, J^*l\rangle|\leq C\|\mathbf{z}\|_{B_{\mathrm{pin}}}$. Combined with the above observation, this implies that

$$2C\geq C\|\mathbf{h}_{\alpha_j}\|_{B_{\mathrm{pin}}}\geq|{}_H\langle J\mathbf{h}_{\alpha_j}, l\rangle_{H^*}|=\left|J\mathbf{h}_{\alpha_j}\left(\frac{1}{2}\right)\right|\to\infty \text{ as } j\to\infty,$$

which is a contradiction. Thus, $J^*l\notin B_{\mathrm{pin}}^*$. □

As was seen in Section 2, the almost complex structure $J: H_{\mathrm{pin}}\to H_{\mathrm{pin}}$ extends to a continuous linear operator on $(B_{\mathrm{pin}})_J$, say J again, in an abstract manner. In the remainder of this section, given will be an explicit representation of $J\mathbf{x}$, $\mathbf{x}\in(B_{\mathrm{pin}})_J$.

Apply [5, Theorm 4.1] to see that

$$\lim_{N\to\infty}2\sum_{i=1}^{d}\sum_{n=1}^{N}\left\{\langle\mathbf{x}, \mathbf{e}_{n,i}^s\rangle\mathbf{e}_{n,i}^s+\langle\mathbf{x}, \mathbf{e}_{n,i}^c\rangle\mathbf{e}_{n,i}^c\right\}=\mathbf{x} \text{ in } B_{\mathrm{pin}} \text{ for } \mu_{\mathrm{pin}}\text{-a.e. } \mathbf{x}\in B_{\mathrm{pin}},$$

and that there is a measurable $Y: B_{\mathrm{pin}}\to B_{\mathrm{pin}}$ so that

$$\lim_{N\to\infty}2\sum_{i=1}^{d}\sum_{n=1}^{N}\left\{\langle\mathbf{x}, \mathbf{e}_{n,i}^s\rangle\mathbf{e}_{n,i}^c-\langle\mathbf{x}, \mathbf{e}_{n,i}^c\rangle\mathbf{e}_{n,i}^s\right\}=Y(\mathbf{x}) \text{ in } B_{\mathrm{pin}} \text{ for } \mu_{\mathrm{pin}}\text{-a.e. } \mathbf{x}\in B_{\mathrm{pin}}.$$

Let $\bar{J}: B_{\mathrm{pin}}\to B_{\mathrm{pin}}$ be the minimal closed extension of J. On account of the observation made in the paragraph containing (2.4), one sees that

$$(B_{\mathrm{pin}})_J=\mathfrak{Dom}(\bar{J}) \text{ and } J=\bar{J} \text{ on } (B_{\mathrm{pin}})_J. \tag{3.6}$$

Moreover, as observed in (2.8)-(2.11), there exists a Borel set $B_0\in\mathcal{B}(B_{\mathrm{pin}})$ such that $\mu_{\mathrm{pin}}(B_0)=1$ and $B_0\subset(B_{\mathrm{pin}})_J$. Thus, due to (3.6) and the closedness of $\bar{J}$, there exists a $B_0\in\mathcal{B}(B_{\mathrm{pin}})$ such that $\mu_{\mathrm{pin}}(B_0)=1$, $B_0\subset(B_{\mathrm{pin}})_J$, and, for any $\mathbf{x}\in B_0$,

$$\lim_{N\to\infty}2\sum_{i=1}^{d}\sum_{n=1}^{N}\left\{\langle\mathbf{x}, \mathbf{e}_{n,i}^s\rangle\mathbf{e}_{n,i}^s+\langle\mathbf{x}, \mathbf{e}_{n,i}^c\rangle\mathbf{e}_{n,i}^c\right\}=\mathbf{x} \text{ in } B_{\mathrm{pin}},$$

$$\lim_{N\to\infty}2\sum_{i=1}^{d}\sum_{n=1}^{N}\left\{\langle\mathbf{x},\mathbf{e}_{n,i}^s\rangle\mathbf{e}_{n,i}^c-\langle\mathbf{x},\mathbf{e}_{n,i}^c\rangle\mathbf{e}_{n,i}^s\right\}=J\mathbf{x} \text{ in } B_{\mathrm{pin}}.$$

In particular, one has that

$$Jx=2\sum_{i=1}^{d}\sum_{n>0}\left\{\langle \mathbf{x},\mathbf{e}_{n,i}^{s}\rangle\mathbf{e}_{n,i}^{c}-\langle \mathbf{x},\ \mathbf{e}_{n,i}^{c}\rangle\mathbf{e}_{n,i}^{s}\right\}\ \text{for}\ \mathbf{x}\in B_0,$$

where the convergence of series takes place with respect to the norm of B_{pin}.

4. Entire functions

Let $(B,\ H,\ \mu)$ be an abstract Wiener space, $J\in\mathscr{AC}(H)$, and define the Cauchy-Riemann operator $\bar{\partial}$ on B as in Section 2. In the section, studied have been holomorphic functions on open sets in B. In the present section, investigated will be L^p-entire functions, i.e. L^p-functions on B which are holomorphic on whole space B, where $p\in(1,\ \infty)$.

Set

$$\mathscr{H}^p(B)=\{u\in L^p(B,\ \mathbf{C}\ ;\ d\mu):\ \bar{\partial}u=0\ \text{on}\ B\},\ p\in(1,\ \infty).$$

One has that $\bigcup_{p>2}\mathscr{H}^p(B)=\mathbf{Hol}(B\ ;\ B)$. as is well knows, there is a continuous linear mapping $\mathscr{I}:\ H^*\to L^2(B,\ \mathbf{R}\ ;\ d\mu)$ such that $\mathscr{I}(\eta)=\langle\cdot,\ \eta\rangle,\ \eta\in B^*$. For a subset $L\subset H^*$, put

$$\mathscr{P}_h(L)=\left\{q\in\bigcap_{p\in(1,\infty)}L^p(B,\ \mathbf{C}\ ;\ d\mu):\ \begin{array}{c}q=\tilde{q}(\mathscr{I}(\eta_1^{(1,0)}))\ \text{for some}\ n\in\mathbf{N},\\ \text{holomorphic polynomial}\ \tilde{q}\ \text{on}\ \mathbf{C}^n,\\ \text{and}\ \eta_1,\ \cdots,\ \eta_n\in L\end{array}\right\},$$

where $\eta^{(1,0)}=\eta-\sqrt{-1}J^*\eta$ and $\mathscr{I}(\eta^{(1,0)})=\mathscr{I}(\eta)-\sqrt{-1}\mathscr{I}(J^*\eta)$. Mention that

$$\mathscr{P}_h(L)\subset\mathscr{H}^p(B)\ \text{for any}\ p\in(1,\ \infty).$$

One obtains the following approximation theorem ;

Proposition 4.1. *Let $L\subset H^*$ be a dense subspace of H^*. Then, for each $u\in\mathscr{H}^p(B)$, there is a sequence $\{q_n\}\subset\mathscr{P}_h(L)$ so that $q_n\to u$ in $L^p(B,\ \mathbf{C}\ ;\ d\mu)$.*

Proof. Let $L'=\{\eta+J^*\eta':\ \eta,\ \eta'\in L\}$. Obviously $J^*(L)'\subset L'$. Moreover, since $(J^*)^2=-\mathrm{id}_{H^*}$, one has that $(\eta+J^*\eta')^{(1,0)}=\eta^{(1,0)}+\sqrt{-1}(\eta')^{(1,0)}$. Hence $\mathscr{P}_h(L)=\mathscr{P}_h(L')$. Thus, without loss of generality, one may assume that $J^*(L)\subset L$.

Since $\langle\eta,\ J^*\eta\rangle_{H^*}=0$ and $J^*(L)\subset L$, one can choose $\{\eta_j\}_{j=1}^{\infty}\subset L$ so that $\{\eta_j,\ J^*\eta_j\}_{j=1}^{\infty}$ is a CONB of H^*. Let $\mathscr{B}_n$ be the σ-field generated by $\{\mathscr{I}(\eta_j),\ \mathscr{I}(J^*\eta_j)\}_{j=1}^{n}$ and denote by $\mathbf{E}[u|\mathscr{B}_n]$ the conditional expectation of u given $\mathscr{B}_n$. By the martingale convergence theorem, $\mathbf{E}[u|\mathscr{B}_n]$ converges to u in $L^p(B,\mathbf{C}\ ;\ d\mu)$. Thus it suffices to approximate $\mathbf{E}[u|\mathscr{B}_n]$ by elements of $\mathscr{P}_h(L)$.

It is easily seen that $\mathbf{E}[u|\mathscr{B}_n]\in\mathscr{H}^p(B)$, and hence, by the splitting property of μ, $\mathbf{E}[u|\mathscr{B}_n]$ can be thought of a holomorphic function on $\mathbf{C}_n$ which is L^p-integrable with respect to the Gaussian measure $d\mu_{\mathbf{C}^n}(\zeta)\equiv(1/\pi)\exp[-|\zeta|^2]d\zeta$ on

$\mathbf{C}^n$. Then, applying Proposition 4.2 in [10] to the complex abstract Wiener space $(\mathbf{C}^n, \mathbf{C}^n, \mu_{\mathbf{C}^n})$, one obtains a sequence $\{g_k\}$ of holomorphic polynomials on $\mathbf{C}^n$ converging to $\mathbf{E}[u|\mathcal{B}_n]$ in $L^p(\mathbf{C}^n, \mathbf{C}^n; d\mu_{\mathbf{C}^n})$, and which implies the desired assertion. □

For $u \in \mathcal{H}^p(B)$, put

$$\hat{u}(\mathbf{h}) = \int_B u(\mathbf{z}+\mathbf{h})d\mu(\mathbf{z}), \ \mathbf{h} \in H. \tag{4.2}$$

If J admits a closed extension $J': B \to B$ and $u \in \bigcup_{p>2} \mathcal{H}^p(B)$, then, on account of (2.13), $\hat{u}$ is nothing but the skeleton of u introduced in Section 2.

Applying Proposition 4.1, one can now conclude the following analogue of Theorem 2.14. A similar assertion in the case where (B, H, μ) is a complex abstract Wiener space can be found in [2].

Proposition 4.3. *Let $p \in (1, \infty)$ and $u \in \mathcal{H}^p(B)$. Then the mapping $H \ni \mathbf{h} \mapsto \hat{u}(\mathbf{h}) \in \mathbf{C}$ is continuous. Moreover, for any $\mathbf{h}_1, \ldots, \mathbf{h}_n \in H$, the mapping $\mathbf{C}^n \ni (\zeta_1, \ldots, \zeta_n) \mapsto \hat{u}(\sum_{i=1}^n \zeta_i \mathbf{h}_i)$ is holomorphic. Finally, if there are $\mathbf{h} \in H$ and $\delta > 0$ such that $\hat{u}(\mathbf{k}) = 0$ for $\mathbf{k} \in H$ with $\|\mathbf{h}-\mathbf{k}\|_H < \delta$, then $u=0$ μ-a.e.*

Proof. Let $q \in \mathcal{P}_h(H^*)$, i.e. $q = \tilde{q}(\mathcal{I}(\eta_1^{(1,0)}), \ldots, \mathcal{I}(\eta_n^{(1,0)}))$ for some $n \in \mathbf{N}$, holomorphic polynomial $\tilde{p}$ on $\mathbf{C}^n$, and $\eta_1, \ldots, \eta_n \in H^*$. Since

$$q(\cdot+\mathbf{h}) = \tilde{q}(\mathcal{I}(\eta_1^{(1,0)}) + {}_H\langle \mathbf{h}, \eta_1^{(1,0)} \rangle_{H^*}, \ldots, \mathcal{I}(\eta_n^{(1,0)}) + {}_H\langle \mathbf{h}, \eta_n^{(1,0)} \rangle_{H^*}) \ \mu\text{-a.e.},$$

on account of the rotation invariance of the Gaussian measures $\mu_{\mathbf{C}^n}$ on $\mathbf{C}^n$, one has that

$$\hat{q}(\mathbf{h}) = \tilde{q}({}_H\langle \mathbf{h}, \eta_1^{(1,0)} \rangle_{H^*}, \ldots, {}_H\langle \mathbf{h}, \eta_n^{(1,0)} \rangle_{H^*}). \tag{4.4}$$

As an application of the Cameron-Martin formula, one obtains that

$$\sup |\hat{u}(\mathbf{h}) - \hat{v}(\mathbf{h})| \le C_{p,R} \|u-v\|_{L_p(B, \mathbf{C}; d\mu)}, \ u, v \in \mathcal{H}^p(B), \ 0 < R < \infty, \tag{4.5}$$

where $C_{p,R}$ is a constant depending only on p and R. Now the first and the second assertion follows from Proposition 4.1.

Recall $v \in L^p(B, \mathbf{C}; d\mu)$ vanishes μ-a.e. if and only if $\int_B v(\mathbf{z}+\mathbf{h})d\mu(\mathbf{z}) = 0$ for every $\mathbf{h} \in H$. Hence, the last assertion is an immediate consequence of the definition of $\hat{u}$ and the second assertion. □

As an application, one has a Liouville theorem on B;

Corollary 4.6. *Let $u \in \mathcal{H}^p(B)$ and suppose that*

$$\lim_{R\to\infty}\frac{1}{\log R}\sup_{\|\mathbf{h}\|_H\le R}\int_B |u(\mathbf{z}+\mathbf{h})|\,d\mu(\mathbf{z})=0.$$

Then $u=$constant *μ-a.e. In particular, if* $u\in\mathscr{H}^p(B)$ *is bounded, then u is a constant function.*

Proof. Notice that $\|\zeta\mathbf{h}\|_H=|\zeta|\|\mathbf{h}\|_H$ for any $\zeta\in\mathbf{C}$ and $\mathbf{h}\in H$. Fix an arbitrary $\mathbf{h}\in H$, $\neq 0$. Then, by Proposition 4.3 and the assumption, a function $v(\zeta)=\hat{u}(\zeta\mathbf{h})$ is holomorphic on $\mathbf{C}$, and moreover, enjoys that

$$\lim_{R\to\infty}\frac{1}{\log R}\sup_{|\zeta|\le R}|v(\zeta)|=0.$$

As is well known, this implies that $v=$constant. Then, by the second assertion of Proposition 4.3 and the uniqueness theorem on $\mathbf{C}^n$, one obtains that $\hat{u}=$constant on H. Apply the third assertion of the proposition, and conclude that $u=$ constant μ-a.e. □

If J extends to a closed operator of B to B, then, by Proposition 4.1 and (4.5), (2.13) remains valid for $u\in\mathscr{H}^p(B)$, $1<p<\infty$, and $0<s<r\le\infty$. In particular, in this case, $\hat{u}$ is the restriction of u to H whenever u is continuous on B. Arises a question if one may regard $\hat{u}$ as a restriction of u to H in general. To answer to the question affirmatively, we shall introduce a set function and establish an Egorov theorem with respect to the set function. It should be mentioned that the desired answer does not follow from the Egorov theorem with respectto the measure μ, since $\mu(H)=0$.

Set

$$\mathscr{M}_J=\{M: M \text{ is a dense subspace of } H^* \text{ with } M\cup J^*(M)\subset B^*\}.$$

and, in the sequel, assume that

$$\mathscr{M}_J\neq\emptyset. \tag{4.7}$$

This hypothesis is satisfied if J extends to a closed operator J': $B\to B$. In fact, $\mathfrak{Dom}((J')^*)$ separates the points of B, in particular, those of H. Thus $\mathfrak{Dom}((J')^*)$ is dense in H^*. Since $J^*=(J')^*$ on $\mathfrak{Dom}((J')^*)$, $\mathfrak{Dom}((J')^*)\in\mathscr{M}_J$.

Define a dense subspace M_J of H^* by

$$\begin{aligned} M_J&=\text{ the subspace of } H^* \text{ spanned by } \bigcup_{M\in\mathscr{M}_J} M\\ &=\left\{\sum^{\text{finite}} a_j l_j:\ a_j\in\mathbf{R},\ l_j\in\bigcup_{M\in\mathscr{M}_J} M\right\}.\end{aligned}$$

It is easily seen that

$$M_J\in\mathscr{M}_J \text{ and } J^*(M_J)\subset M_J\subset B^*.$$

In general, $M_J \subsetneqq B^*$. For example, let J be the almost complex structure on the abstract Wiener space of Brownian bridge introduced in Section 3. Then, by Theorem 3.3 and the remark after (4.7), $\mathcal{M}_J \neq \emptyset$ but $M_J \subsetneqq B^*$.

If $l \in M_J$, then l, $J^* l \in B^*$ and $\mathcal{J}(l^{(1,0)})(\mathbf{z}) = \langle \mathbf{z}, l^{(1,0)} \rangle$, $\mathbf{z} \in B$. Hence exery $q \in \mathcal{P}_h(M_J)$ is defined on B without any μ-null exceptional set, and the mapping $B \ni \mathbf{z} \to q(\mathbf{z}) \in \mathbf{C}$ is continuous. For $1 < p < \infty$, define

$$\mathcal{G}_p = \left\{ \{q_n\}_{n=1}^\infty \subset \mathcal{P}_h(M_J) : \sum_{n=1}^\infty \|q_n\|_{L_p(B,\mathbf{C};d\mu)} = 1 \right\},$$

$$C_p(A) = \inf_{\{q_n\} \in \mathcal{G}_p} \sup \left\{ \exp\left[-\sum_{n=1}^\infty |q_n(\mathbf{z})| \right] : \mathbf{z} \in A \right\}, \ \emptyset \neq A \subset B, \text{ and } C_p(\emptyset) = 0.$$

Proposition 4.8. *Suppose that* (4.7) *is fulfilled.*
(i) *It holds that*

$$0 \leq C_p(A) \leq C_p(A') \text{ for } A \subset A' \subset B.$$

(ii) *For any sequences* $\{\varepsilon_k\}$ *and* $\{a_k\}$ *of positive numbers with* $\sum_k a_k = 1$,

$$C_p\left(\bigcup_{k=1}^\infty A_k \right) \leq \sum_{k=1}^\infty (C_p(A_k) + \varepsilon_k)^{a_k}.$$

In particular, if $n \in \mathbf{N}$, $b_1 + \cdots + b_n = 1$, *then*

$$C_p\left(\bigcup_{k=1}^n A_k \right) \leq \sum_{k=1}^n (C_p(A_k))^{b_k}$$

(iii) $C_p(A) = 0$ *if and only* $A \subset \{\mathbf{z} : \sum_{n=1}^\infty |q_n(\mathbf{z})| = \infty\}$ *for some* $\{q_n\} \in \mathcal{G}_p$. *Especially, a Borel set* $A \subset B$ *is* μ*-null if* $C_p(A) = 0$.
(iv) *For* $\mathbf{z} \in B$, $C_p(\{\mathbf{z}\}) = 0$ *if and only if* $\mathbf{z} \notin H$.

Proof. (i) The assertion follows from the definition of C_p immediately.
(ii) For each $\varepsilon_k > 0$, choose $\{q_n^{(k)}\}_{n=1}^\infty \in \mathcal{G}_p$ so that

$$\sup \left\{ \exp\left[-\sum_{n=1}^\infty |q_n^{(k)}(\mathbf{z})| \right\} : \mathbf{z} \in A_k \right\} \leq C_p(A_k) + \varepsilon_k.$$

Renumber $\{a_k q_n^{(k)} : n, k \in \mathbf{N}\}$ to obtain $\{q'_n\}$. Then $\{q'_n\} \in \mathcal{G}_p$ and it holds that

$$\sum_{n=1}^\infty |q'_n(\mathbf{z})| \geq a_k \sum_{n=1}^\infty |q_n^{(k)}| \geq -a_k \log[C_p(A_k) + \varepsilon_k] \text{ for } \mathbf{z} \in A_k.$$

Hence one has that

$$C_p\left(\bigcup_{k=1}^\infty A_k \right) \leq \sup \left\{ \exp\left[-\sum_{n=1}^\infty |q'_n(\mathbf{z})| \right] : \mathbf{z} \in \bigcup_{k=1}^\infty A_k \right\} \leq \sum_{k=1}^\infty \exp[a_k \log(C_p(A_k) + \varepsilon_k)],$$

which means that the former inequality holds.

To see the latter inequality, for $\delta > 0$ and $m \in \mathbf{N}$, set $a_k(m) = (1 - 1/m) b_k$, $k \leq n$, $= 2^{n-k}/m$, $k > n$, $\varepsilon_k(m, \delta) = (2^{-k}\delta)^{1/a_k(m)}$, and $A_k = \emptyset$, k$> n$. Apply then the former inequality, and let $\delta\downarrow 0$ and $m\uparrow\infty$.

(iii) The "if" part is an immediate consequence of the definition of C_p. To see the "only if" part, suppose that $A\subset B$ enjoys $C_p(A)=0$. For each $k\in\mathbf{N}$, take a sequence $\{q_n^{(k)}\}_{n=1}^\infty\in\mathcal{G}_p$ such that

$$\sup\left\{\exp\left[-\sum_{n=+1}^\infty|q_n^{(k)}(\mathbf{z})|\right]:\ \mathbf{z}\in A\right\}\le\exp[-k],\ i.e.,\ \inf\left\{\sum_{n=1}^\infty|q_n^{(k)}(\mathbf{z})|:\ \mathbf{z}\in A\right\}\ge 2k.$$

Renumber $\{(6/\pi^2)k^{-2}q_n^{(k)}\}$ to have $\{q'_n\}$. Obviously $\{q'_n\}\in\mathcal{G}_p$. Further, one has that

$$\sum_{n=1}^\infty|q'_n(\mathbf{z})|=\sum_{k=1}^\infty\frac{6}{\pi^2}\frac{1}{k^2}\sum_{n=1}^\infty|q_n^{(k)}(\mathbf{z})|\ge\frac{6}{\pi^2}\sum_{k=1}^\infty\frac{1}{k}=\infty,\ \mathbf{z}\in A,$$

and hence $A\subset\{\mathbf{z}:\ \sum_n|q'_n(\mathbf{z})|=\infty\}$.

Since $\mu(\{\mathbf{z}:\ \sum_n|q_n(\mathbf{z})|=\infty\})=0$ for $\{q_n\}\in\mathcal{G}_p$, the second assertion follows from the first one.

(iv) It was seen in [13] that $\mathbf{z}\in B\backslash H$ if and only if $\mathbf{z}\in\{\mathbf{x}:\ \sum_n|q_n(\mathbf{x})|=\infty\}$ for some $\{q_n\}\in\mathcal{G}_p$, under the assumption that J extends to an isometry of B to B. The argument there still works under the assumption (4.7), once one notice that

$$\sup\{|\langle\mathbf{z},\ l^{(1,0)}\rangle|:\ l\in M_J\text{ and }\|l\|_{H^*}=1\}=\infty\text{ for any }\mathbf{z}\in B\backslash H.$$

Hence the above equivalence remains valid under (4.7), and the desired assertion follows from (iii). □

A measurable function $f:\ B\to\mathbf{R}$ is said to be C_p-quasi continuous if, for every $\varepsilon>0$, there exists an open set G such that $C_p(G)<\varepsilon$ and $f|_{B\backslash G}:\ B\backslash G\to\mathbf{R}$ is continuous. For measurable $v:\ B\to\mathbf{R}$, its C_p-quasi continuous version $\tilde{v}$ is a C_p-quasi continuous function $\tilde{v}$ with $v=\tilde{v}\,\mu$-a.e. If $\tilde{v}$ is a C_p-quasi continuous version of v and $\mathbf{h}\in H$, then, on account of Proposition 4.8 (i, iv), there is a closed set F such that $\mathbf{h}\in F$ and f is continuous on F. In this sense, measurable functions possessing a C_p-quasi continuous version can be restricted to H.

One has the following Egorov theorem for holomorphic functions with respect to C_p.

Theorem 4.9. *Assume* (4.7). *Then, every $u\in\mathcal{H}^p(B)$ admits a C_p-quasi continuous version $\tilde{u}$. Further, there exists a sequence $\{q_n\}\subset\mathcal{P}_h(M_J)$ and a subset $N\subset B$ such that $\|q_n-\tilde{u}\|_{L^p(B,\mathbf{C};d\mu)}\to 0$, $C_p(N)=0$, and $\lim_n q_n(\mathbf{z})=\tilde{u}(\mathbf{z})$ for $\mathbf{z}\notin N$. Finally, $\tilde{u}$ coincides on H with the skeleton $\hat{u}$ given in* (4.2); $\tilde{u}(\mathbf{h})=\hat{u}(\mathbf{h})$, $\mathbf{h}\in H$.

The second assertion says that $\tilde{u}$ is p-regular in the sense of Sugita [13].

Proof. Let $u\in\mathcal{H}^p(B)$. According to Proposition 4.1, there exists a sequence $\{q_n\}\subset\mathcal{P}_h(M_J)$ such that

(4.10) $$\lim_{n\to\infty}\|q_n-u\|_{L^p(B,\mathbf{C};d\mu)}=0 \text{ and } \sum_{n=1}^{\infty}\|q_{n+1}-q_n\|_{L^p(B,\mathbf{C};d\mu)}<\infty.$$

Define

$$N=\left\{\mathbf{z}\in B:\ \sum_{n=1}^{\infty}|q_{n+1}(\mathbf{z})-q_n(\mathbf{z})|=\infty\right\},$$
$$\tilde{u}(\mathbf{z})=\begin{cases}\lim_{n\to\infty}q_n(\mathbf{z}) & \text{if } \mathbf{z}\notin N,\\ 0 & \text{if } \mathbf{z}\in N.\end{cases}$$

Due to (4.10) and Proposition 4.8 (iii), one has that

(4.11) $$C_p(N)=0 \text{ and } \tilde{u}=u\ \mu\text{-a.e. on } B.$$

In order to see the first and second assertions, it now suffices to show the C_p-quasi continuity of $\tilde{u}$. To do this, let

$$g_n^{(m)}=\left(\sum_{n=m}^{\infty}|q_{n+1}-q_n\|_{L^p(B;\mathbf{C},\mu)}\right)^{-1}(q_{n+1}-q_n)$$

and notice that $\{g_n^{(m)}\}_{n=1}^{\infty}\in\mathcal{G}_p$, $m\in\mathbf{N}$. Then, by the very definition of C_p, one obtains that, for any $m\in\mathbf{N}$ and $\varepsilon>0$,

(4.12) $$\begin{aligned}&C_p\left(\left\{\mathbf{z}\in B:\ \sum_{n=m}^{\infty}|q_{n+1}(\mathbf{z})-q_n(\mathbf{z})|>\varepsilon\right\}\right)\\ &\le\sup\left\{\exp\left[-\sum_{n=1}^{\infty}|g_n^{(m)}(\mathbf{z})|\right]:\ \mathbf{z}\in B,\ \sum_{n=m}^{\infty}|q_{n+1}(\mathbf{z})-q_n(\mathbf{z})|>\varepsilon\right\}\\ &\le\exp\left[-\left(\sum_{n=m}^{\infty}\|q_{n+1}-q_n\|_{L^p(B;\mathbf{C},\mu)}\right)^{-1}\varepsilon\right].\end{aligned}$$

For $k\in\mathbf{N}$, choose $m_k\in\mathbf{N}$ so that

(4.13) $$\sum_{n=m_k}^{\infty}\|q_{n+1}-q_n\|_{L^p(B;\mathbf{C},\mu)}\le(k4^k)^{-1},$$

and define a subset $A_k\subset B$ by

$$A_k=\left\{\mathbf{z}\in B:\ \sum_{n=m_k}^{\infty}|q_{n=m_k}(\mathbf{z})-q_n(\mathbf{z})|>2^{-k}\right\}.$$

Then A_k is open in B and open sets G_j, $j\in\mathbf{N}$, are given by

$$G_j=\bigcup_{k=j}^{\infty}A_k,\ j\in\mathbf{N}.$$

By (4.12) and (4.13), it holds that

$$C_p(A_k)\le\exp[-k2^k],\ k\in\mathbf{N}.$$

Apply Proposition 4.8 (ii) with $a_k=1/2^k$, $\varepsilon_k=\exp[-k2^k]$ and conclude that

$$C_p(G_j) \leq \sum_{k=j}^{\infty}(2\exp[-k2^k])^{1/2^k} \leq \frac{1}{e-1}e^{-j} \text{ for every } j \in \mathbf{N}. \tag{4.14}$$

Note that

$$N \subset \bigcap_{k=1}^{\infty} A_k,$$

and hence that $\tilde{u}(\mathbf{z}) = \lim_n q_n(\mathbf{z})$ for any $\mathbf{z} \in B \backslash G_j$ and $j \in \mathbf{N}$. Further, it holds that

$$\sup_{z \in B \backslash G_j} |\tilde{u}(\mathbf{z}) - q_{m_i}(\mathbf{z})| \leq \sup_{z \in B \backslash G_j} \sum_{n=m_i}^{\infty} |q_{n+1}(\mathbf{z}) - q_n(\mathbf{z})| \leq 2^{-i},$$
$$\text{for every } i, j \in \mathbf{N} \text{ with } i > j.$$

This implies that, on each closed set $B \backslash G_j$, $j \in \mathbf{N}$, $\{q_{m_i}\}_{i=1}^{\infty}$ converges to $\tilde{u}$ uniformly, and hence $\tilde{u}$ is continuous on it. Thus, by (4.14), $\tilde{u}$ is C_p-quasi continuous. The first and second assertions has been verified.

To see the last assertion, apply the second assertion and choose $\{q_n\} \subset \mathcal{P}_h(M_J)$ and $N \subset B$ so that $\|q_n - u\|_{L^p(B,\mathbf{C};d\mu)}$, $C_p(N) = 0$, and $\lim_n q_n(\mathbf{z}) = \tilde{u}(\mathbf{z})$, $\mathbf{z} \notin N$. It follows from (4.4) that $q_n(\mathbf{h}) = \hat{q}_n(\mathbf{h})$, $\mathbf{h} \in H$. By (4.5) and Proposition 4.8 (iv), one obtains that $\tilde{u}(\mathbf{h}) = \hat{u}(\mathbf{h})$ □

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Math. Soc. Japan **47** (1995), 655-670.

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