The Study of Sigma Functions for Telescopic Curves

(Telescopic 曲線に付随するシグマ関数の研究)

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Abstract

The Weierstrass's elliptic sigma functions are generalized to the case of hyperelliptic curves by Klein and to the case of (n, s) curves by Buchstaber, Enolskii, Leykin. Nakayashiki gave a formula which expresses the sigma functions for (n, s) curves in terms of algebraic integrals and showed the algebraic property of the series expansion of the sigma functions for (n, s) curves by using this formula. In this paper, we consider further generalization of the sigma functions to the case of telescopic curves introduced by Miura. The telescopic curves contain the (n, s) curves and hyperelliptic curves as special cases. The most important contribution of this paper is to construct explicitly a basis of holomorphic one forms and the normalized fundamental form for the telescopic curves. Consequently, we construct sigma functions for the telescopic curves as holomorphic functions with the quasi periodicity and the algebraic property of the series expansion.

Key words: Sigma function, telescopic curve, Schur function, gap sequence.

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1 Introduction

Let E be the elliptic curve defined by $y^2 = 4x^3 - g_2x - g_3$, $(g_2, g_3 \in \mathbb{C})$, α, β a canonical basis of $H_1(E, \mathbb{Z})$, $2\omega_1 := \int_\alpha \frac{dx}{y}, 2\omega_2 := \int_\beta \frac{dx}{y}$, and $\Lambda = \mathbb{Z}(2\omega_1) + \mathbb{Z}(2\omega_2)$. The elliptic sigma function for E is the holomorphic function on $\mathbb{C}$ defined by

$$\sigma(u) = u \prod_{\omega \in \Lambda \backslash \{0\}} \left(1 - \frac{u}{\omega}\right) \exp\left(\frac{u}{\omega} + \frac{u^2}{2\omega^2}\right).$$

The sigma function has the following important properties.

(i) $\sigma(u + 2\omega_i) = -\exp\{2\eta_i(u + \omega_i)\}\,\sigma(u), \quad \eta_i = \sigma'(\omega_i)/\sigma(\omega_i).$ (quasi periodicity)

(ii) $\sigma(u) = u - \frac{g_2}{240}u^5 - \frac{g_3}{840}u^7 - \cdots,$ around $u = 0$. (algebraic property)

The property (ii) means that the sigma function does not depend on the choice of a canonical basis of the homology group (modular invariance) and it can be constructed from the coefficients of the defining equation of the curve. From this, it is known that the sigma functions have many advantages in describing algebro-geometric solutions of integrable systems ([2],[3]). Klein [14],[15] extended the elliptic sigma functions to the case of hyperelliptic curves from this point of view. Buchstaber et al. [3],[4],[6] extended Klein's sigma functions to the case of more general plane algebraic curves called (n, s) curves. Since they were defined, it has been one of the central problem to determine the coefficients of the series expansion of the sigma functions. For elliptic curves, it is well-known that the coefficients can be calculated by using a linear differential equation satisfied by the $\wp$-function. Buchstaber and Leykin [1] determined the coefficients of the series expansion of the sigma functions for hyperelliptic curves of genus 2 by constructing linear differential equations satisfied by the sigma functions. Nakayashiki [21] gave a formula which expresses the sigma functions for (n, s) curves in terms of algebraic integrals. By using this formula, Nakayashiki [21] showed that the first term of the series expansion around the origin is Schur function determined from the gap sequence at infinity and the coefficients of the series expansion become homogeneous polynomials of the coefficients of the defining equation of the curve with respect to certain degree. Also, Nakayashiki [22] determined the series expansion of the sigma functions for (n, s) curves by using the expression of the tau function of the KP-hierarchy in terms of the sigma function. In this paper, we consider further generalization of the sigma functions.

For $m \geq 2$, a sequence of positive integers $(a_1, ..., a_m)$ whose greatest common divisor equals to one is called telescopic if

$$\frac{a_i}{d_i} \in \frac{a_1}{d_{i-1}}\mathbb{Z}_{\geq 0} + \cdots + \frac{a_{i-1}}{d_{i-1}}\mathbb{Z}_{\geq 0}, \quad 2 \leq i \leq m,$$

where d_i is the greatest common divisor of $(a_1, \ldots, a_i)$. For a telescopic sequence $(a_1, \ldots, a_m)$, Miura [19] introduced a nonsingular algebraic curve (telescopic curve) determined by the sequence $(a_1, \ldots, a_m)$. The idea is to express a nonsingular algebraic curve by affine equations of m variables whose orders at infinity are $(a_1, \ldots, a_m)$. The telescopic curves contain the (n, s) curves and hyperelliptic curves as special cases.

The most important contribution of this paper is to construct explicitly a basis of holomorphic one forms and the normalized fundamental form for the telescopic curves. Let X

be a telescopic curve defined by $\{F_2(x_1,\dots,x_m),\dots,F_m(x_1,\dots,x_m)\}$ and g the genus of X. We arrange the monomials $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$, $(\alpha_1,\dots,\alpha_m)\in B(A_m)$, in the ascending order of pole orders at ∞ and denote them by φ_i, $i\geq 1$. Then, we show the following theorems (cf. section 4,6).

Theorem. Let

$$G(x)=\left(\frac{\partial F_i}{\partial x_j}\right)_{2\leq i,j\leq m}\quad\text{and}\quad du_i=-\frac{\varphi_{g+1-i}}{\det G(x)}dx_1.$$

Then, $\{du_i\}_{i=1}^g$ is a basis of holomorphic one forms.

Theorem. Let

$$h_{ij}=\frac{F_i(y_1,\dots,y_{j-1},x_j,x_{j+1},\dots,x_m)-F_i(y_1,\dots,y_{j-1},y_j,x_{j+1},\dots,x_m)}{x_j-y_j},$$

$$H=(h_{ij})_{2\leq i,j\leq m},\ \text{ and }\ \Omega(x,y)=\frac{\det H(x,y)}{(x_1-y_1)\det G(x)}dx_1.$$

Then, we can construct explicitly second kind differentials with a pole only at infinity $\{dr_i\}_{i=1}^g$ such that the algebraic bilinear form

$$\widehat{\omega}(x,y):=d_y\Omega(x,y)+\sum_{i=1}^g du_i(x)dr_i(y)$$

satisfies the following conditions.

- $\widehat{\omega}(x,y)=\widehat{\omega}(y,x)$.
- $\widehat{\omega}(x,y)$ is holomorphic except $\Delta=\{(p,p)\mid p\in X\}$ where it has a double pole.
- For a local coordinate t around $p\in X$, the expansion in $t(x)$ at $t(y)$ is of the form

$$\widehat{\omega}(x,y)=\left(\frac{1}{(t(x)-t(y))^2}+\mathit{regular}\right)dt(x)dt(y).$$

Consequently, when we define sigma functions $\sigma(u)$ for the telescopic curves in terms of Riemann's theta function (cf. section 7), we can show the algebraic property of the series expansion of the sigma functions. Namely, we show the following theorem (cf. section 10).

Theorem. The expansion of $\sigma(u)$ at the origin takes the form

$$\sigma(u)=S_{\mu(A_m)}(T)|_{T_{w_i}=u_i}+\sum b_{k_1,\dots,k_g}u_1^{k_1}\cdots u_g^{k_g},$$

where $S_{\mu(A_m)}(T)$ is Schur function determined from the gap sequence at infinity and $\{b_{k_1,\dots,k_g}\}$ become homogeneous polynomials of the coefficients of the defining equations of the curve with respect to certain degree.

From the above theorem, we find that the sigma functions for telescopic curves also have the modular invariance.

2 Preliminaries

In this section, we review necessary known results following [21].

2.1 Riemann's theta function

For a positive integer g, let τ be a $g \times g$ symmetric matrix whose imaginary part is positive definite. For $a, b \in \mathbb{R}^g$, Riemann's theta function with characteristics a, b is defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z) = \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i\, {}^t(n+a)\tau(n+a) + 2\pi i\, {}^t(n+a)(z+b)\right),$$

where $z \in \mathbb{C}^g$. The theta functions have the following quasi-periodicity:

$$\frac{\theta \begin{bmatrix} a \\ b \end{bmatrix} (z + m_1 + \tau m_2)}{\theta \begin{bmatrix} a \\ b \end{bmatrix} (z)} = \exp\left(2\pi i({}^t a m_1 - {}^t b m_2) - \pi i\, {}^t m_2 \tau m_2 - 2\pi i\, {}^t m_2 z\right).$$

Also, we have

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (-z) = (-1)^{4\,{}^t ab} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z), \quad a, b \in \frac{1}{2}\mathbb{Z}^g. \tag{1}$$

For $\alpha \in \mathbb{C}^g$, we can express $\alpha = \tau\alpha' + \alpha''$ with $\alpha', \alpha'' \in \mathbb{R}^g$ uniquely. For simplicity, we express $\theta \begin{bmatrix} \alpha' \\ \alpha'' \end{bmatrix} (z)$ by $\theta[\alpha](z)$. For $n, m \in \mathbb{Z}^g$, we have

$$\theta \begin{bmatrix} a+n \\ b+m \end{bmatrix} (z) = \exp(2\pi i\, {}^t a m) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z).$$

2.2 Abel-Jacobi's theorem

Let X be a compact Riemann surface of genus g and $\{\alpha_i, \beta_i\}_{i=1}^g$ a canonical basis of the homology group $H_1(X, \mathbb{Z})$, i.e., a basis of $H_1(X, \mathbb{Z})$ such that the intersection numbers satisfy $\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0$ and $\alpha_i \circ \beta_j = \delta_{ij}$ for any i, j, where δ_{ij} is the Kronecker delta. Let $\{dv_i\}_{i=1}^g$ be the basis of holomorphic one forms such that $\int_{\alpha_j} dv_i = \delta_{ij}$. We define the period matrix $\tau := (\int_{\beta_j} dv_i)_{ij}$, then τ is a symmetric matrix whose imaginary part is positive definite. Set $dv := {}^t(dv_1, \ldots, dv_g)$ and $L_\tau := \tau\mathbb{Z}^g + \mathbb{Z}^g$. The Jacobian variety $\mathrm{Jac}(X)$ is defined by $\mathrm{Jac}(X) := \mathbb{C}^g / L_\tau$. Let $\mathrm{Pic}^0(X)$ be the linear equivalent classes of divisors of degree zero. Then, $\mathrm{Jac}(X)$ can be identified with $\mathrm{Pic}^0(X)$ by Abel-Jacobi map:

$$\mathrm{Pic}^0(X) \to \mathrm{Jac}(X), \quad \sum_{i=1}^d p_i - \sum_{i=1}^d q_i \mapsto \sum_{i=1}^d \int_{q_i}^{p_i} dv.$$

2.3 Holomorphic line bundle and flat line bundle

Let $\mathcal{O}^*$ be the sheaf of germs of nowhere-vanishing holomorphic functions and $\mathcal{C}^*$ the sheaf of germs of non-zero constant functions. The elements of the Čech cohomology groups $H^1(X, \mathcal{O}^*)$ and $H^1(X, \mathcal{C}^*)$ are called holomorphic line bundles and flat line bundles on X, respectively. We fix a base point p_0 on X. Let $\pi_1(X, p_0)$ be the fundamental group. The set of flat line bundles $H^1(X, \mathcal{C}^*)$ can be identified with $\mathrm{Hom}(\pi_1(X, p_0), \mathbb{C}^*)$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers (cf. [11] pp.184-189). Let $\phi : H^1(X, \mathcal{C}^*) \to H^1(X, \mathcal{O}^*)$ be the homomorphism induced by the inclusion map $\mathcal{C}^* \to \mathcal{O}^*$. For $\chi \in \mathrm{Hom}(\pi_1(X, p_0), \mathbb{C}^*) \simeq H^1(X, \mathcal{C}^*)$, $\chi \in \mathrm{Ker}(\phi)$ if and only if there exists a holomorphic one form ω such that $\chi(\gamma) = \exp(\int_\gamma \omega)$ for any $\gamma \in \pi_1(X, p_0)$. Let $\mathrm{Pic}(X)$ be the linear equivalent classes of divisors. It is well-known that there exists an isomorphism between $\mathrm{Pic}(X)$ and $H^1(X, \mathcal{O}^*)$. By this isomorphism, an element of $\mathrm{Pic}^0(X)$ corresponds to that of $\mathrm{Im}(\phi) \subset H^1(X, \mathcal{O}^*)$. Therefore, by Abel-Jacobi's theorem, $\mathrm{Jac}(X)$ can be identified with $H^1(X, \mathcal{C}^*)/\mathrm{Ker}(\phi)$. We denote the equivalent class of the flat line bundle corresponding to $\alpha \in \mathrm{Jac}(X)$ by $\mathcal{L}_\alpha$. There exists a unique unitary representation for $\mathcal{L}_\alpha$. Let $\tilde{\alpha} \in \mathbb{C}^g$ be a representative of α. Take $\tilde{\alpha}', \tilde{\alpha}'' \in \mathbb{R}^g$ satisfying $\tilde{\alpha} = \tau\tilde{\alpha}' + \tilde{\alpha}''$. Then, the unitary representation for $\mathcal{L}_\alpha$ is given by

$$\chi(\alpha_j) = \exp(2\pi i \tilde{\alpha}'_j), \quad \chi(\beta_j) = \exp(-2\pi i \tilde{\alpha}''_j).$$

Let $\tilde{X}$ be the universal cover of X. A meromorphic section of the flat line bundle defined by χ is described by a meromorphic function F on $\tilde{X}$ satisfying

$$F(\gamma\tilde{p}) = \chi(\gamma)F(\tilde{p}).$$

2.4 Prime form

Let δ_0 be Riemann divisor for the choice $\{\alpha_i, \beta_i\}$ and L_0 the corresponding holomorphic line bundle of degree $g-1$. For $\alpha \in \mathrm{Jac}(X)$, set $L_\alpha = \mathcal{L}_\alpha \otimes L_0$. There exists a non-singular odd half period α, i.e., $\alpha \in \frac{1}{2}L_\tau/L_\tau$ such that $\theta[\tilde{\alpha}](z)$ is an odd function and $\frac{\partial\theta[\tilde{\alpha}]}{\partial z_i}(0) \neq 0$ for some i. There exists a unique divisor $p_1 + \cdots + p_{g-1}$ such that

$$\alpha = \int_{\delta_0}^{p_1+\cdots+p_{g-1}} dv.$$

From [10] p.10 Corollary 1.4, the divisor of the holomorphic one form

$$\sum_{i=1}^{g} \frac{\partial\theta[\tilde{\alpha}]}{\partial z_i}(0)dv_i$$

is $2\sum_{i=1}^{g-1} p_i$. Since the divisor $\sum_{i=1}^{g-1} p_i$ corresponds to the holomorphic line bundle L_α, there exists a holomorphic section $h_{\tilde{\alpha}}$ of L_α such that

$$h_{\tilde{\alpha}}(p)^2 = \sum_{i=1}^{g} \frac{\partial\theta[\tilde{\alpha}]}{\partial z_i}(0)dv_i(p). \tag{2}$$

Let $\pi : \tilde{X} \to X$ be the projection. We use the same symbol $h_{\tilde{\alpha}}$ for the pull back of $h_{\tilde{\alpha}}$ to $\tilde{X}$. Then, the prime form is defined as

$$E(\tilde{p}_1, \tilde{p}_2) = \frac{\theta[\tilde{\alpha}](\int_{\tilde{p}_1}^{\tilde{p}_2} dv)}{h_{\tilde{\alpha}}(\tilde{p}_1) h_{\tilde{\alpha}}(\tilde{p}_2)}, \quad \tilde{p}_1, \tilde{p}_2 \in \tilde{X}.$$

From [20] p.156, it vanishes to the first order at $\pi(\tilde{p}_1) = \pi(\tilde{p}_2)$ and at no other divisors. Let $\pi_j : X \times X \to X$ be the projection to the j-th component and $I : X \times X \to \mathrm{Jac}(X)$ the map defined by $I(p_1, p_2) = \int_{p_1}^{p_2} dv$. Then, the prime form can be considered as a holomorphic section of the line bundle $\pi_1^* L_0^{-1} \otimes \pi_2^* L_0^{-1} \otimes I^* \Theta$ on $X \times X$, where Θ is the line bundle on $\mathrm{Jac}(X)$ defined by the theta divisor $\Theta = \{z \in \mathrm{Jac}(X) \mid \theta(z) = 0\}$. The prime form has the following properties.

(i) $E(\tilde{p}_2, \tilde{p}_1) = -E(\tilde{p}_1, \tilde{p}_2)$.

(ii) $E(\tilde{p}_1, \tilde{p}_2) = 0 \Longleftrightarrow \pi(\tilde{p}_1) = \pi(\tilde{p}_2)$.

(iii) If we take a local coordinate t around $\tilde{p} \in \tilde{X}$, then the expansion in $t(\tilde{p}_2)$ at $t(\tilde{p}_1)$ is of the form *

$$E(\tilde{p}_1, \tilde{p}_2)\sqrt{dt(\tilde{p}_1) dt(\tilde{p}_2)} = t(\tilde{p}_2) - t(\tilde{p}_1) + O\left((t(\tilde{p}_2) - t(\tilde{p}_1))^3\right).$$

(iv) For $\tilde{p}_1, \tilde{p}_2 \in \tilde{X}$, consider the function

$$F(\tilde{p}) = \frac{E(\tilde{p}, \tilde{p}_2)}{E(\tilde{p}, \tilde{p}_1)}, \quad \tilde{p} \in \tilde{X}.$$

For $\gamma \in \pi_1(X, p_0)$, we call the image of γ in the homology group $H_1(X, \mathbb{Z})$ the abelian image of γ. If the abelian image of $\gamma \in \pi_1(X, p_0)$ is expressed by $\sum_{i=1}^g m_i \alpha_i + \sum_{i=1}^g n_i \beta_i$, then

$$F(\gamma \tilde{p}) = \exp\left(2\pi i\, {}^t n \int_{\tilde{p}_1}^{\tilde{p}_2} dv\right) F(\tilde{p}),$$

where $n = {}^t(n_1, \ldots, n_g)$.

2.5 Normalized fundamental form

Let K_X be the canonical bundle of X. A section of $\pi_1^* K_X \otimes \pi_2^* K_X$ is called a bilinear form on $X \times X$ and a bilinear form $w(p_1, p_2)$ is called symmetric if $w(p_2, p_1) = w(p_1, p_2)$. Any holomorphic bilinear form $w(p_1, p_2)$ can be written as

$$w(p_1, p_2) = \sum_{i,j=1}^{g} c_{ij}\, dv_i(p_1) dv_j(p_2) \tag{3}$$

with $c_{ij} \in \mathbb{C}$ uniquely. The bilinear form $w(p_1, p_2)$ is symmetric if and only if $c_{ij} = c_{ji}$. Let $\Delta := \{(p, p) \mid p \in X\} \subset X \times X$.

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*$O(t^k)$ denotes a power series of t starting from t^k : $O(t^k) = \sum_{i=k}^{\infty} c_i t^i$ with some constants $\{c_i\}$.

Definition 1 *A meromorphic symmetric bilinear form $\omega(p_1, p_2)$ on $X \times X$ is called a normalized fundamental form if the following conditions are satisfied.*

(i) $\omega(p_1, p_2)$ is holomorphic except Δ where it has a double pole. If we take a local coordinate t around $p \in X$, then the expansion in $t(p_1)$ at $t(p_2)$ is of the form

$$\omega(p_1, p_2) = \left(\frac{1}{(t(p_1) - t(p_2))^2} + regular \right) dt(p_1) dt(p_2). \tag{4}$$

(ii) $\int_{\alpha_i} \omega = 0$, where the integration is with respect to any one of p_1, p_2.

The normalized fundamental form exists and unique. We have the following proposition.

Proposition 2 *[10] For $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \tilde{X}$,*

$$\exp\left(\int_{\tilde{a}}^{\tilde{b}} \int_{\tilde{c}}^{\tilde{d}} \omega \right) = \frac{E(\tilde{b}, \tilde{d}) E(\tilde{a}, \tilde{c})}{E(\tilde{a}, \tilde{d}) E(\tilde{b}, \tilde{c})}.$$

3 Telescopic curves

Here, we give the definition of telescopic curves introduced by Miura [19].

Let $m \geq 2$, $(a_1, ..., a_m)$ a sequence of positive integers such that $\gcd(a_1, ..., a_m) = 1$, and $d_i = \gcd(a_1, ..., a_i)$ for $1 \leq i \leq m$. We call $(a_1, ..., a_m)$ telescopic if

$$\frac{a_i}{d_i} \in \frac{a_1}{d_{i-1}} \mathbb{Z}_{\geq 0} + \cdots + \frac{a_{i-1}}{d_{i-1}} \mathbb{Z}_{\geq 0}, \;\; 2 \leq i \leq m. \tag{5}$$

For a telescopic sequence $A_m = (a_1, ..., a_m)$, let

$$B(A_m) = \{(l_1, ..., l_m) \in \mathbb{Z}_{\geq 0}^m \,|\, 0 \leq l_i < \frac{d_{i-1}}{d_i} \;\text{ for }\; 2 \leq i \leq m\}. \tag{6}$$

Proposition 3 *([19]) For any $x \in a_1 \mathbb{Z}_{\geq 0} + \cdots + a_m \mathbb{Z}_{\geq 0}$, there exists a unique $(k_1, \ldots, k_m) \in B(A_m)$ such that*

$$\sum_{i=1}^{m} a_i k_i = x.$$

For the telescopic sequence $A_m = (a_1, ..., a_m)$, let us define $m-1$ polynomials in m variables $X_1, ..., X_m$ by

$$F_i(X_1, \ldots, X_m) = X_i^{d_{i-1}/d_i} - \prod_{j=1}^{m} X_j^{l_{ij}} - \sum \lambda_{j_1 \cdots j_m}^{(i)} X_1^{j_1} \cdots X_m^{j_m}, \;\; 2 \leq i \leq m, \tag{7}$$

where $(l_{i1}, ..., l_{im})$ is the element of $B(A_m)$ satisfying

$$\sum_{j=1}^{m} a_j l_{ij} = a_i \frac{d_{i-1}}{d_i}, \tag{8}$$

and the sum is over all $(j_1, ..., j_m) \in B(A_m)$ such that

$$\sum_{k=1}^{m} a_k j_k < a_i \frac{d_{i-1}}{d_i}.$$

Let

$$X^{\mathrm{aff}} = \{(z_1, ..., z_m) \in \mathbb{C}^m \mid F_i(z_1, ..., z_m) = 0,\ 2 \le i \le m\}.$$

Then, X^{aff} is an affine algebraic curve (cf. [19]). We assume that X^{aff} is nonsingular. Let X be the compact Riemann surface corresponding to X^{aff}. Then, X is obtained from X^{aff} by adding one point, say ∞ (cf. [19]). The genus of X is given by (cf. [19])

$$g = \frac{1}{2}\left\{1 - a_1 + \sum_{i=2}^{m}\left(\frac{d_{i-1}}{d_i} - 1\right)a_i\right\}. \tag{9}$$

We call X the telescopic curve associated with $(a_1, ..., a_m)$. Let R be the coordinate ring of X^{aff}, x_i the image of X_i for the projection $\mathbb{C}[X_1, \ldots, X_m] \to R$, K the quotient field of R, and $L(k\infty) = \{f \in K \mid \mathrm{div}(f) + k\infty \ge 0\}$. We regard x_i as a meromorphic function on X. For a meromorphic function f on X and $p \in X$, we denote by $\mathrm{ord}_p(f)$ the order of f at p.

Proposition 4 *([19]).*
(i) The set $\{x_1^{\alpha_1} \cdots x_m^{\alpha_m} \mid (\alpha_1, \ldots, \alpha_m) \in B(A_m)\}$ is a basis of R over $\mathbb{C}$.
(ii) $R = \bigcup_{k=0}^{\infty} L(k\infty)$.
(iii) $ord_\infty(x_i) = -a_i$.

We arrange the monomials $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, $(\alpha_1, \ldots, \alpha_m) \in B(A_m)$, in the ascending order of pole orders at ∞ and denote them by φ_i, $i \ge 1$. In particular, $\varphi_1 = 1$. Let $(w_1, ..., w_g)$ be the gap sequence at ∞:

$$\{w_i \,|\, 1 \le i \le g\} = \mathbb{Z}_{\ge 0} \backslash \{\mathrm{ord}_\infty(\varphi_i) \,|\, i \ge 1\}, \ \ (1 = w_1 < \cdots < w_g).$$

Proposition 5 *([13]) $w_g = 2g - 1$.*

Proposition 6 *For the element $(l_{i1}, ..., l_{im}) \in B(A_m)$ satisfying (8), we have $l_{ii} = \cdots = l_{im} = 0$.*

Proof of Proposition 6. Since A_m is telescopic, there exist $k_1, \ldots, k_{i-1} \in \mathbb{Z}_{\ge 0}$ such that $0 \le k_j < d_{j-1}/d_j$ for any $j = 2, \ldots, i-1$ and

$$a_i \frac{d_{i-1}}{d_i} = a_1 k_1 + \cdots + a_{i-1} k_{i-1}.$$

From (6), we have $(k_1, \ldots, k_{i-1}, 0, \ldots, 0) \in B(A_m)$. Since the element of $B(A_m)$ satisfying (8) is unique, we have $(l_{i1}, ..., l_{im}) = (k_1, \ldots, k_{i-1}, 0, \ldots, 0)$.

□

From Proposition 6, the defining equations of telescopic curves have the following forms:

$$F_i(X_1,\ldots,X_m)=X_i^{d_{i-1}/d_i}-\prod_{j=1}^{i-1}X_j^{l_{ij}}-\sum\lambda^{(i)}_{j_1\cdots j_m}X_1^{j_1}\cdots X_m^{j_m}. \tag{10}$$

For the defining equations (10), we assign degrees as

$$\deg X_k=a_k,\quad \deg\lambda^{(i)}_{j_1,\ldots,j_m}=a_id_{i-1}/d_i-\sum_{k=1}^{m}a_kj_k.$$

Example 1. $A_2=(2,3)$.

$$F_2(X_1,X_2)=X_2^2-X_1^3-\lambda^{(2)}_{1,1}X_1X_2-\lambda^{(2)}_{2,0}X_1^2-\lambda^{(2)}_{0,1}X_2-\lambda^{(2)}_{1,0}X_1-\lambda^{(2)}_{0,0},$$

which expresses the elliptic curves.

Example 2. $A_2=(2,2g+1)$.

$$F_2(X_1,X_2)=X_2^2-X_1^{2g+1}-\sum_{i=0}^{g}\lambda^{(2)}_{i,1}X_1^iX_2-\sum_{i=0}^{2g}\lambda^{(2)}_{i,0}X_1^i,$$

which expresses the hyperelliptic curves of genus g.

Example 3. $A_2=(n,s),\quad n,s\in\mathbb{N}_{>0},\quad \gcd\{n,s\}=1$.

$$F_2(X_1,X_2)=X_2^n-X_1^s-\sum_{nj_1+sj_2<ns}\lambda^{(2)}_{j_1,j_2}X_1^{j_1}X_2^{j_2},$$

which expresses the (n,s)-curves (cf. [3] [4] [6]).

Example 4. $A_3=(4,6,5)$.

$$F_2(X_1,X_2,X_3)=X_2^2-X_1^3-\lambda^{(2)}_{0,1,1}X_2X_3-\lambda^{(2)}_{1,1,0}X_1X_2-\lambda^{(2)}_{1,0,1}X_1X_3-\lambda^{(2)}_{2,0,0}X_1^2$$
$$-\lambda^{(2)}_{0,1,0}X_2-\lambda^{(2)}_{0,0,1}X_3-\lambda^{(2)}_{1,0,0}X_1-\lambda^{(2)}_{0,0,0}$$

and

$$F_3(X_1,X_2,X_3)=X_3^2-X_1X_2-\lambda^{(3)}_{1,0,1}X_1X_3-\lambda^{(3)}_{2,0,0}X_1^2-\lambda^{(3)}_{0,1,0}X_2-\lambda^{(3)}_{0,0,1}X_3$$
$$-\lambda^{(3)}_{1,0,0}X_1-\lambda^{(3)}_{0,0,0}.$$

4 Holomorphic one forms for telescopic curves

Let X be a telescopic curve associated with $(a_1, \ldots, a_m)$ and $\Gamma(X, \Omega_X^1)$ the linear space consisting of holomorphic one forms on X. In this section, we construct a basis of $\Gamma(X, \Omega_X^1)$. Let G be the matrix defined by

$$G := \begin{pmatrix} \dfrac{\partial F_2}{\partial X_1} & \cdots & \dfrac{\partial F_2}{\partial X_m} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial F_m}{\partial X_1} & \cdots & \dfrac{\partial F_m}{\partial X_m} \end{pmatrix},$$

G_i the matrix obtained by removing the i-th column from G, and

$$du_i = -\frac{\varphi_{g+1-i}}{\det G_1(x)} dx_1. \tag{11}$$

Then, we have the following theorem.

Theorem 7 *The set $P := \{du_i\}_{i=1}^g$ is a basis of $\Gamma(X, \Omega_X^1)$ over $\mathbb{C}$.*

In order to prove Theorem 7, we need some lemmas.

Lemma 8 *If $\det G_i(p) \neq 0$ for $p = (p_1, \ldots, p_m) \in X^{aff}$ and $1 \leq i \leq m$, then $ord_p(x_i - p_i) = 1$.*

Proof of Lemma 8. Without loss of generality, we assume $i = 1$. Suppose $\mathrm{ord}_p(x_1 - p_1) \geq 2$. Then, there exists k $(2 \leq k \leq m)$ such that $\mathrm{ord}_p(x_k - p_k) = 1$. In fact, if $\mathrm{ord}_p(x_k - p_k) \geq 2$ for any k, then $\mathrm{ord}_p(f) \geq 2$ or $\mathrm{ord}_p(f) = 0$ for any $f \in R$. Then, $\mathrm{ord}_p(g) \geq 2$ or $\mathrm{ord}_p(g) = 0$ for any $g \in R_p$, where R_p is the localization of R at p. This contradicts that R_p is a discrete valuation ring.

There exist $\{\gamma_{ij}, \delta^{(i)}_{j_1,\ldots,j_m}\} \in \mathbb{C}$ such that for $2 \leq i \leq m$

$$F_i(X_1, \ldots, X_m) = \sum_{j=1}^m \gamma_{ij}(X_j - p_j) + \sum_{j_1+\cdots+j_m \geq 2} \delta^{(i)}_{j_1,\ldots,j_m} (X_1 - p_1)^{j_1} \cdots (X_m - p_m)^{j_m},$$

where $\gamma_{ij} = \frac{\partial F_i}{\partial X_j}(p)$. Since $F_i(x_1, \ldots, x_m) = 0$ and $\mathrm{ord}_p(x_1 - p_1) \geq 2$, we have

$$\mathrm{ord}_p \left(\sum_{j=2}^m \gamma_{ij}(x_j - p_j) \right) = \mathrm{ord}_p \left((x_k - p_k)\left(\sum_{j=2}^m \gamma_{ij} \frac{x_j - p_j}{x_k - p_k}\right) \right) \geq 2.$$

Since $\mathrm{ord}_p(x_k - p_k) = 1$, we have $\sum_{j=2}^m \gamma_{ij} b_j = 0$, where $b_j = \left(\frac{x_j - p_j}{x_k - p_k}\right)(p)$. Therefore, we obtain

$$G_1(p) \begin{pmatrix} b_2 \\ \cdot \\ \cdot \\ b_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Since $b_k = 1(\neq 0)$, we have $\det G_1(p) = 0$. This contradicts the assumption of Lemma 8. Therefore, we obtain $\mathrm{ord}_p(x_1 - p_1) = 1$.

$\square$

Lemma 9 *(i) As an element of K, we have* $\det G_1(x) \neq 0$.
(ii) div $\left(\dfrac{dx_1}{\det G_1(x)}\right) = (2g-2)\infty$.

Proof of Lemma 9. Since the differential $d\left(F_i(x_1,\dots,x_m)\right) = 0$ for any i, we have

$$G(x)\begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad .$$

By multiplying some elementary matrices on the left, the above equation becomes

$$\begin{pmatrix} w_2 & z_{22} & z_{23} & \cdots & z_{2m} \\ w_3 & 0 & z_{33} & \cdots & z_{3m} \\ & & \cdots & & \\ w_m & 0 & \cdots & & z_{mm} \end{pmatrix}\begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Since X^{aff} is non-singular, for any $p \in X^{\mathrm{aff}}$ there exists i such that $\det G_i(p) \neq 0$. Therefore, we have $w_m \neq 0$ or $z_{mm} \neq 0$ as elements of K. Since $\mathrm{ord}_\infty(x_j) = -a_j$, we have $x_j \notin \mathbb{C}$, therefore, $dx_j \neq 0$ for any j. Since $w_m dx_1 = z_{mm} dx_m$, we have $w_m \neq 0$ and $z_{mm} \neq 0$. Therefore, by multiplying some elementary matrices on the left, the above equation becomes

$$\begin{pmatrix} w_2' & z_{22} & z_{23} & \cdots & 0 \\ w_3' & 0 & z_{33} & \cdots & 0 \\ & & \cdots & & \\ w_m & 0 & \cdot\cdot & & z_{mm} \end{pmatrix}\begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Similarly, we obtain

$$\begin{pmatrix} w_2'' & z_{22} & 0 & \cdots & 0 \\ w_3'' & 0 & z_{33} & \cdots & 0 \\ & & \cdots & & \\ w_m'' & 0 & \cdots & & z_{mm} \end{pmatrix}\begin{pmatrix} dx_1 \\ \cdot \\ \cdot \\ dx_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

where $w_2'', \dots, w_m'', z_{22}, \dots, z_{mm} \in K$ are non-zero.
Therefore, we obtain $\det G_1(x) = \pm z_{22} \cdots z_{mm} \neq 0$, which complete the proof of (i).

Next, we prove that the one form $dx_1 / \det G_1(x)$ is both holomorphic and non-vanishing on X^{aff}. When $\det G_1(p) \neq 0$ for $p \in X^{\mathrm{aff}}$, from Lemma 8, $dx_1 / \det G_1(x)$ is both holomorphic and non-vanishing at p. Suppose $\det G_1(p) = 0$ for $p \in X^{\mathrm{aff}}$. Since X^{aff} is non-singular, there exists i $(2 \leq i \leq m)$ such that $\det G_i(p) \neq 0$. Since $w_i'' dx_1 + z_{ii} dx_i = 0$, we have

$w_i'' z_{22} \cdots \widehat{z_{ii}} \cdots z_{mm} dx_1 + z_{22} \cdots z_{mm} dx_i = 0$, where $\widehat{z_{ii}}$ denotes to remove z_{ii}. Therefore, we obtain

$$(-1)^{i-2} \det G_i(x) dx_1 + \det G_1(x) dx_i = 0.$$

Since $\det G_1(x) \neq 0$ and $\det G_i(x) \neq 0$, we have

$$\frac{dx_1}{\det G_1(x)} = (-1)^{i-1} \frac{dx_i}{\det G_i(x)}.$$

Therefore, from $\det G_i(p) \neq 0$ and Lemma 8, $dx_1 / \det G_1(x)$ is holomorphic and non-vanishing at p. On the other hand, by Riemann-Roch's theorem, we have

$$\deg \operatorname{div}(dx_1 / \det G_1(x)) = 2g - 2,$$

which complete the proof of (ii).

□

Proof of Theorem 7. From Lemma 9 and Proposition 3, 4, 5, we have $P \subset \Gamma(X, \Omega_X^1)$ and the elements of P are linearly independent. Since $\dim_{\mathbb{C}} \Gamma(X, \Omega_X^1) = g$, P is a basis of $\Gamma(X, \Omega_X^1)$.

□

5 Series expansion of x_i around ∞

In this section, we show the following proposition.

Proposition 10 *(i) It is possible to take a local parameter t around ∞ such that*

$$x_1 = \frac{1}{t^{a_1}}, \quad x_k = \frac{1}{t^{a_k}}(1 + \sum_{l=1}^{\infty} c_{kl} t^l), \quad 2 \leq k \leq m, \tag{12}$$

where c_{kl} belongs to $\mathbb{Q}[\{\lambda_{j_1 \cdots j_m}^{(i)}\}]$ and is homogeneous of degree l if $c_{kl} \neq 0$.

(ii) For the local parameter t of (i), we have, around ∞,

$$\frac{dx_1}{\det G_1(x)} = -t^{2g-2}(1 + \sum_{l=1}^{\infty} c_l' t^l) dt,$$

where c_l' belongs to $\mathbb{Q}[\{\lambda_{j_1 \cdots j_m}^{(i)}\}]$ and is homogeneous of degree l if $c_l' \neq 0$.

(iii) For the local parameter t of (i), we have, around ∞,

$$du_i = \left(t^{w_i - 1} + O(t^{w_i})\right) dt.$$

Proof of Proposition 10. (i) It is possible to take a local parameter t around ∞ such that

$$x_1 = \frac{1}{t_0^{a_1}}.$$

Let $\zeta = \exp(2\pi\sqrt{-1}/a_1)$ and $i \geq 0$. Then, $t_i := \zeta^i t_0$ is also a local parameter around ∞. Let $c_k^{(i)}$ be the coefficient of the series expansion of x_k around ∞ with respect to t_i:

$$x_k = \frac{c_k^{(i)}}{t_i^{a_k}}(1 + O(t_i)), \quad 2 \leq k \leq m.$$

We prove that there exists i such that $c_2^{(i)} = \cdots = c_m^{(i)} = 1$. Let $c^{(i)} = (c_2^{(i)}, \ldots, c_m^{(i)})$ for $0 \leq i < a_1$. First, we show $c^{(i)} \neq c^{(j)}$ for $i \neq j$. Suppose $c^{(i)} = c^{(j)}$. Since $c_k^{(i)} = \zeta^{a_k i} c_k^{(0)}$, we have $\zeta^{a_k(i-j)} = 1$ for any $k = 2, \ldots, m$. From $\gcd(a_1, \ldots, a_m) = 1$ and $0 \leq i, j < a_1$, we have $i = j$.

Let

$$\mathcal{Z} = \{(z_2, \ldots, z_m) \in \mathbb{C}^{m-1} \mid z_2^{d_1/d_2} = 1,\ z_i^{d_{i-1}/d_i} = z_2^{l_{i2}} \cdots z_{i-1}^{l_{ii-1}},\ 3 \leq i \leq m\}.$$

Since $\sharp\mathcal{Z} = (d_1/d_2)\cdots(d_{m-1}/d_m) = (d_1/d_m) = a_1$ and $c^{(i)} \in \mathcal{Z}$ for any $i = 0, \ldots, a_1 - 1$, we have

$$\mathcal{Z} = \{c^{(0)}, \ldots, c^{(a_1-1)}\}.$$

Since $(1, \ldots, 1) \in \mathcal{Z}$, there exists j such that $c^{(j)} = (1, \ldots, 1)$. For $t := t_j$, x_k is expanded as

$$x_1 = \frac{1}{t^{a_1}}, \quad x_k = \frac{1}{t^{a_k}}(1 + \sum_{l=1}^{\infty} c_{kl} t^l), \quad c_{kl} \in \mathbb{C}.$$

Let us prove that c_{kl} belongs to $\mathbb{Q}[\{\lambda_{j_1 \cdots j_m}^{(i)}\}]$ and is homogeneous of degree l if $c_{kl} \neq 0$. We define the order $<$ in the set $\{c_{kl}\}$ so that $c_{k'l'} < c_{kl}$ if

1. $l' < l$ or
2. $l' = l$ and $k' < k$.

We prove the statement by transfinite induction with respect to the well-order $<$.

From (10), we have

$$\left(1 + \sum_{j=1}^{\infty} c_{kj} t^j\right)^{\frac{d_{k-1}}{d_k}} = \prod_{s=2}^{k-1}\left(1 + \sum_{j=1}^{\infty} c_{sj} t^j\right)^{l_{ks}} + \sum \lambda_{j_1 \cdots j_m}^{(k)} t^{\frac{a_k d_{k-1}}{d_k} - \sum_{s=1}^m a_s j_s} \prod_{s=2}^{m}\left(1 + \sum_{j=1}^{\infty} c_{sj} t^j\right)^{j_s}, \tag{13}$$

where we define $\prod_{s=2}^{1}\left(1 + \sum_{j=1}^{\infty} c_{sj} t^j\right)^{l_{2s}} = 1$.

In (13) for $k = 2$, the coefficient of t of the left hand side is $(d_1/d_2)c_{21}$ and that of the right hand side is $\lambda_{j_1, \ldots, j_m}^{(2)}$ with $(j_1, \ldots, j_m)$ satisfying $(a_2 d_1/d_2) - \sum_{s=1}^m a_s j_s = 1$. Therefore, the statement is correct for the minimal element c_{21}. Assume that the statement is correct for $c_{k'l'}$ satisfying $c_{k'l'} < c_{kl}$. The coefficient of t^l of the left hand side of (13) is $(d_{k-1}/d_k)c_{kl} + T$, where T is a sum of $\prod_i c_{kq_i}$ satisfying $\sum_i q_i = l$ and $q_i < l$. The coefficients of t^l of the first and second terms of the right hand side of (13) is a sum of $\prod_i c_{p_i q_i}$ satisfying $2 \leq p_i < k$ and $\sum_i q_i = l$, and a sum of $\lambda_{j_1 \cdots j_m}^{(k)} \prod_i c_{p_i q_i}$ satisfying $\sum_i q_i = l - (a_k d_{k-1}/d_k) + \sum_{s=1}^m a_s j_s$, respectively. Therefore, by the assumption of the induction, we find that c_{kl} belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree l if $c_{kl} \neq 0$.

(ii). From (10) and (i), we have, around ∞,

$$\det G_1(x) = a_1 t^{-\sum_{i=2}^{m}((d_{i-1}/d_i)-1)a_i}(1 + \sum_{l=1}^{\infty} c_l t^l)dt,$$

where c_l belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree l if $c_l \neq 0$. Therefore, from (9), we obtain the assertion.

(iii) Let $w_i^* = \mathrm{ord}_\infty(\varphi_i)$. Note that $\{w_i^*, w_i \mid i = 1, \ldots, g\} = \{0, 1, \ldots, 2g-1\}$. Since $w_g = 2g-1$, we have $2g - 1 - w^*_{g+1-i} = w_i$ for any i. Therefore, from (11), we obtain the assertion.

□

6 Symplectic basis of the first cohomology group of telescopic curves

In this paper, we call a meromorphic differential on X second kind if it is locally exact. Let $H^1(X, \mathbb{C})$ be the space of second kind differentials modulo meromorphic exact forms. We define the intersection form on $H^1(X, \mathbb{C})$ by

$$\eta \circ \eta' = \sum_p \mathrm{Res}(\int^p \eta)\eta'(p).$$

for second kind differentials η, η' (the summation is over all singular points of η and η', and Res means taking a residue at a point). Let $\{\alpha_i, \beta_i\}_{i=1}^g$ be a canonical basis of the homology group $H_1(X, \mathbb{Z})$. Then, we have (Riemann's bilinear relation)

$$\eta \circ \eta' = \frac{1}{2\pi i} \sum_{i=1}^{g} \left(\int_{\alpha_i} \eta \int_{\beta_i} \eta' - \int_{\alpha_i} \eta' \int_{\beta_i} \eta \right). \tag{14}$$

For $2 \leq i \leq m$ and $1 \leq j \leq m$, let

$$h_{ij} = \frac{F_i(Y_1, \ldots, Y_{j-1}, X_j, X_{j+1}, \ldots, X_m) - F_i(Y_1, \ldots, Y_{j-1}, Y_j, X_{j+1}, \ldots, X_m)}{X_j - Y_j}$$

and

$$H = \begin{pmatrix} h_{22} & \ldots & h_{2m} \\ \ldots & \ldots & \ldots \\ h_{m2} & \ldots & h_{mm} \end{pmatrix}.$$

We consider the one form

$$\Omega(x, y) := \frac{\det H(x, y)}{(x_1 - y_1) \det G_1(x)} dx_1$$

and the bilinear form

$$\widehat{\omega}(x,y) := d_y\Omega(x,y) + \sum c_{i_1,\ldots,i_m;j_1,\ldots,j_m} \frac{x_1^{i_1}\cdots x_m^{i_m} y_1^{j_1}\cdots y_m^{j_m}}{\det G_1(x)\det G_1(y)} dx_1 dy_1 \tag{15}$$

on $X\times X$, where $x=(x_1,\ldots,x_m)$, $y=(y_1,\ldots,y_m)$, $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}\in\mathbb{C}$, $(i_1,\ldots,i_m)\in B(A_m)$ satisfying $0\le\sum_{k=1}^m a_k i_k\le 2g-2$, and $(j_1,\ldots,j_m)\in B(A_m)$.

Then, we have the following theorem.

Theorem 11 *(i) There exists a set of $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ such that $\widehat{\omega}(x,y)=\widehat{\omega}(y,x)$, non-zero $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ is a homogeneous polynomial of $\{\lambda^{(i)}_{l_1,\ldots,l_m}\}$ of degree*

$$2\sum_{k=2}^m \frac{d_{k-1}}{d_k}a_k - \sum_{k=1}^m (i_k+j_k+2)a_k,$$

and $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}=0$ if $2\sum_{k=2}^m \frac{d_{k-1}}{d_k}a_k - \sum_{k=1}^m (i_k+j_k+2)a_k<0$.

For a set of $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ such that $\widehat{\omega}(x,y)=\widehat{\omega}(y,x)$, we have the following properties.

(ii) The bilinear form $\widehat{\omega}$ satisfies the condition (i) of Definition 1.

(iii) For $du_i(x) := (x_1^{k_{i1}}\cdots x_m^{k_{im}}/\det G_1(x))dx_1$, we define

$$dr_i(y) = \sum_{j_1,\ldots,j_m} c_{k_{i1},\ldots,k_{im};j_1,\ldots,j_m} \frac{y_1^{j_1}\cdots y_m^{j_m}}{\det G_1(y)} dy_1.$$

Then, dr_i is a second kind differential which is singular only at ∞, and the set $\{du_i,dr_i\}_{i=1}^g$ is a symplectic basis of $H^1(X,\mathbb{C})$, i.e., $\{du_i,dr_i\}_{i=1}^g$ is a basis of $H^1(X,\mathbb{C})$ such that

$$du_i\circ du_j = dr_i\circ dr_j = 0 \ \text{ and } \ du_i\circ dr_j=\delta_{ij} \quad \text{for each } i,j. \tag{16}$$

Let B be the set of branch points for the map $x_1: X\to\mathbb{P}^1, (x_1,\ldots,x_m)\mapsto[x_1:1]$. Since the ramification index of the map x_1 at ∞ is a_1, we have $\deg x_1 = a_1$ (cf. [24], p.28, Proposition 2.6). For $p\in X$, we set $x_1^{-1}(x_1(p))=\{p^{(0)},p^{(1)},\ldots,p^{(a_1-1)}\}$ with $p=p^{(0)}$, where the same $p^{(i)}$ is listed according to its ramification index.

Lemma 12 *Let U be a domain in $\mathbb{C}$, $f(z_1,z_2)$ a holomorphic function on $U\times U$, and $g(z)=f(z,z)$. If $g\equiv 0$ on U, then there exists a holomorphic function $h(z_1,z_2)$ on $U\times U$ such that $f(z_1,z_2)=(z_1-z_2)h(z_1,z_2)$.*

Proof of Lemma 12. Let $h(z_1,z_2)=f(z_1,z_2)/(z_1-z_2)$. Given z_1, $h(z_1,\cdot\,)$ has a singularity only at z_1, where its singularity is removable. Therefore, $h(z_1,\cdot\,)$ is holomorphic on U. Similarly, $h(\,\cdot,z_2)$ is holomorphic on U. Therefore, h is holomorphic on $U\times U$.

□

Lemma 13 *The one form $\Omega(x,y)$ is holomorphic except $\Delta\cup\{(p^{(i)},p)\mid i\neq 0,\ p\in B \text{ or } p^{(i)}\in B\}\cup X\times\{\infty\}\cup\{\infty\}\times X$.*

Proof of Lemma 13. Since $dx_1/\det G_1(x)$ is holomorphic on X, $\Omega(x,y)$ is holomorphic except $\Delta \cup \{(p^{(i)}, p) \mid p \in X, i \neq 0\} \cup X \times \{\infty\} \cup \{\infty\} \times X$. We prove that $\Omega(x,y)$ is holomorphic on $\{(p^{(i)}, p) \mid i \neq 0, p \notin B, p^{(i)} \notin B\}$. We have

$$F_i(X_1, \dots, X_m) = \sum_{j=1}^{m} h_{ij} \cdot (X_j - Y_j) + F_i(Y_1, \dots, Y_m). \tag{17}$$

Set $X_i = x_i$ and $Y_i = y_i$, then we have

$$\sum_{j=1}^{m} h_{ij}(x,y) \cdot (x_j - y_j) = 0.$$

Take $(p^{(i)}, p) \in X \times X$ such that $i \neq 0, p \notin B$, and $p^{(i)} \notin B$, then we have

$$\begin{pmatrix} h_{21} & \dots & h_{2m} \\ \dots & \dots & \dots \\ h_{m1} & \dots & h_{mm} \end{pmatrix}_{X_j = p_j^{(i)}, Y_j = p_j, 1 \leq j \leq m} \begin{pmatrix} p_1^{(i)} - p_1 \\ \cdot \\ p_m^{(i)} - p_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}.$$

Since $p_1^{(i)} - p_1 = 0$, we have

$$H(p^{(i)}, p) \begin{pmatrix} p_2^{(i)} - p_2 \\ \cdot \\ p_m^{(i)} - p_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \end{pmatrix}.$$

Since $(p_2^{(i)} - p_2, \dots, p_m^{(i)} - p_m) \neq (0, \dots, 0)$, we have $\det H(p^{(i)}, p) = 0$. Since $p \notin B$ and $p^{(i)} \notin B$, we can take (x_1, y_1) as a local coordinate around $(p^{(i)}, p)$. Therefore, from Lemma 12, there exists a holomorphic function $h(x_1, y_1)$ around $(p^{(i)}, p)$ such that $\det H(x,y) = (x_1 - y_1) h(x_1, y_1)$. Therefore, $\Omega(x,y)$ is holomorphic at $(p^{(i)}, p)$.

□

Lemma 14 *Let $p \notin B$, t a local coordinate around p. Then, the expansion of $\Omega(x,y)$ in $t(y)$ at $t(x)$ is of the form*

$$\Omega(x,y) = \left(\frac{-1}{t(y) - t(x)} + regular \right) dt(x).$$

Proof of Lemma 14. Set $Y = y$ in (17), then we have

$$F_i(X_1, \dots, X_m) = \sum_{j=1}^{m} h_{ij}(X_1, \dots, X_m, y_1, \dots, y_m) \cdot (X_j - y_j).$$

Therefore, we obtain

$$\frac{\partial F_i}{\partial X_k}(x_1, \dots, x_m) = \sum_{j=1}^{m} \frac{\partial h_{ij}}{\partial X_k}(x,y) \cdot (x_j - y_j) + h_{ik}(x,y).$$

Set $x = y$, then we have

$$\frac{\partial F_i}{\partial X_k}(x_1, \ldots, x_m) = h_{ik}(x, x).$$

Therefore, we obtain $\det G_1(x) = \det H(x, x)$. On the other hand, since $p \notin B$, we can take (x_1, y_1) as a local coordinate around (p, p). Since $p \notin B$, we have $\det G_1(p) \neq 0$. In fact, if $\det G_1(p) = 0$, then $dx_1/\det G_1(x)$ is not holomorphic at p, which contradicts Lemma 9 (ii). Therefore, $\det H(x, y)/\det G_1(x)$ is holomorphic at (p, p). Therefore, from Lemma 12, there exists a holomorphic function $\tilde{h}(x_1, y_1)$ around (p, p) such that $\det H(x, y)/\det G_1(x) = 1 + (x_1 - y_1)\tilde{h}(x_1, y_1)$. Therefore, we obtain Lemma 14.

□

Lemma 15 *When we express*

$$\det H(X_1, \ldots, X_m, Y_1, \ldots, Y_m) = \sum \epsilon_{l_1,\ldots,l_m,n_1,\ldots,n_m} X_1^{l_1} \cdots X_m^{l_m} Y_1^{n_1} \cdots Y_m^{n_m},$$

we have $\sum_{k=1}^m a_k(l_k + n_k) \le \sum_{k=2}^m a_k\left((d_{k-1}/d_k) - 1\right)$.

Proof of Lemma 15. When we express

$$F_i(X_1, \ldots, X_m) = \sum_{k=0}^{s} \tilde{F}_{ik}^{(j)}(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m) X_j^k,$$

we have $h_{ij} = \sum_{k=1}^s \tilde{F}_{ik}^{(j)}(Y_1, \ldots, Y_{j-1}, X_{j+1}, \ldots, X_m) \sum_{l=0}^{k-1} X_j^l Y_j^{k-l-1}$. Assign degrees as $\deg Y_k = a_k$, then h_{ij} is a homogeneous polynomial of $\{\lambda_{j_1,\ldots,j_m}^{(i)}, X_k, Y_k\}$ of degree $a_i d_{i-1}/d_i - a_j$. Therefore, we obtain Lemma 15.

□

Lemma 16 *The meromorphic bilinear form $d_y\Omega(x, y)$ is holomorphic except $\Delta \cup \{(p^{(i)}, p) \mid i \neq 0,\ p \in B$ or $p^{(i)} \in B\} \cup X \times \{\infty\}$.*

Proof of Lemma 16. It is sufficient to prove that $d_y\Omega(x, y)$ is holomorphic at (∞, y), $y \neq \infty$. From Lemma 15, with respect to x, we obtain

$$\mathrm{ord}_\infty\left(\det H(x, y)\right) \ge -\sum_{k=2}^m a_k\left((d_{k-1}/d_k) - 1\right).$$

If $\mathrm{ord}_\infty\left(\det H(x, y)\right) > -\sum_{k=2}^m a_k\left((d_{k-1}/d_k) - 1\right)$, then from Lemma 9 (ii) and (9) we obtain $\mathrm{ord}_\infty\left(\Omega(x, y)\right) \ge 0$. Therefore, $d_y\Omega(x, y)$ is holomorphic at (∞, y). If $\mathrm{ord}_\infty\left(\det H(x, y)\right) = -\sum_{k=2}^m a_k\left((d_{k-1}/d_k) - 1\right)$, then $\mathrm{ord}_\infty\left(\Omega(x, y)\right) = -1$. Let t be a local coordinate around ∞, then from Lemma 15 there exists a constant e (which does not depend on y) such that

$$\Omega(x, y) = \left(\frac{e}{s} + \text{regular}\right) ds.$$

Therefore, $d_y\Omega(x, y)$ is holomorphic at (∞, y), $y \neq \infty$.

□

Lemma 17 *Let ω be the normalized fundamental form. Then, there exist second kind defferentials $d\widehat{r}_i$ $(1 \le i \le g)$ which are holomorphic except $\{\infty\}$ and satisfy the equation*

$$\omega(x, y) - d_y\Omega(x, y) = \sum_{i=1}^{g} du_i(x) d\widehat{r}_i(y).$$

Proof of Lemma 17. The method of proof is similar to the case of (n, s) curves (cf. [21] Lemma 5). Let us set

$$\omega_1(x, y) = \omega(x, y) - d_y\Omega(x, y).$$

From Lemma 14, 16 and (4), the singularities of ω_1 are contained in $B_2 \cup X \times \{\infty\}$, where $B_2 = \{(p^{(i)}, p) \mid p \in B\backslash\{\infty\}$ or $p^{(i)} \in B\backslash\{\infty\}\}$. Since B_2 is a finite set and $B_2 \cap (X \times \{\infty\}) = \phi$, ω_1 is holomorphic except $X \times \{\infty\}$. Therefore, there exists one forms $d\tilde{r}_i(y)$ on $X\backslash\{\infty\}$ such that

$$\omega_1(x, y) = \sum_{i=1}^{g} du_i(x) d\tilde{r}_i(y)$$

on $X \times (X\backslash\{\infty\})$. Let us take $q_1, \ldots, q_g$ such that $\sum_{i=1}^{g} q_i$ is a general divisor and q_i's are in some small neighborhood of ∞. Take the local coordinate t of (12) around ∞ and write

$$du_i(x) = h_i(t)dt,$$

$$\omega_1(x, y) = K_1(t(x), y)dt(x).$$

Then, we have a set of linear equations

$$\sum_{i=1}^{g} h_i(t(q_j)) d\tilde{r}_i(y) = K_1(t(q_j), y).$$

Since $\sum_{i=1}^{g} q_i$ is a general divisor, we have $\det(h_i(t(q_j))) \neq 0$. Therefore, there exist some constants $c_{ij} \in \mathbb{C}$ such that

$$d\tilde{r}_i(y) = \sum_{j=1}^{g} c_{ij} K_1(t(q_j), y).$$

on $X\backslash\{\infty\}$. Notice that $K_1(t(q_j), y)$ is a second kind differential whose only singularity is ∞. Let us set

$$d\widehat{r}_i(y) = \sum_{j=1}^{g} c_{ij} K_1(t(q_j), y),$$

which is a second kind differential on X singular only at ∞, and set

$$\omega_2(x, y) = \omega_1(x, y) - \sum_{i=1}^{g} du_i(x) d\widehat{r}_i(y).$$

Then, we have $\omega_2 = 0$ on $X \times (X\backslash\{\infty\})$. Therefore, we have $\omega_2 = 0$ on $X \times X$, which proves the lemma.

$\square$

Lemma 18 *Let Q be the linear space consisting of meromorphic differentials on X which are singular only at ∞ and*

$$S = \left\{ \frac{\varphi_i}{\det G_1(x)} dx_1 \mid i \geq 1 \right\}.$$

Then, S is a basis of Q over $\mathbb{C}$.

Proof of Lemma 18. For $\eta \in Q$, we consider the meromorphic function $\eta / \frac{dx_1}{\det G_1(x)}$. From Lemma 9 (ii), it may have a pole only at ∞. From Proposition 4, $\eta / \frac{dx_1}{\det G_1(x)}$ is a linear combination of φ_i and the elements of S are linearly independent. □

Proof of Theorem 11. (i) We have

$$d_y \Omega(x,y) = \frac{\{\sum_{k=1}^m (-1)^{k+1}(x_1 - y_1)\frac{\partial \det H}{\partial Y_k}(x,y) \det G_k(y)\} + \det G_1(y) \det H(x,y)}{(x_1 - y_1)^2 \det G_1(x) \det G_1(y)} dx_1 dy_1.$$

Then, $\det G_k$, $\det H$, and $(\partial \det H / \partial Y_k)$ are homogeneous polynomials of $\{\lambda^{(i)}_{j_1,\ldots,j_m}, X_j, Y_j\}$ of degree $\sum_{i=2}^m \frac{d_{i-1}}{d_i} a_i - \sum_{i \neq k} a_i$, $\sum_{i=2}^m (\frac{d_{i-1}}{d_i} - 1) a_i$, and $\{\sum_{i=2}^m (\frac{d_{i-1}}{d_i} - 1) a_i\} - a_k$, respectively. Let us write

$$d_y \Omega(x,y) = \frac{\sum q_{i_1,\ldots,i_m;j_1,\ldots,j_m} x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m}}{(x_1 - y_1)^2 \det G_1(x) \det G_1(y)} dx_1 dy_1,$$

where $(i_1, \ldots, i_m), (j_1, \ldots, j_m) \in B(A_m)$, and $q_{i_1,\ldots,i_m;j_1,\ldots,j_m} \in \mathbb{C}$. Then, $q_{i_1,\ldots,i_m;j_1,\ldots,j_m} \in \mathbb{Z}[\{\lambda^{(i)}_{l_1,\ldots,l_m}\}]$ and $q_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ is homogeneous of degree $2\sum_{k=2}^m (\frac{d_{k-1}}{d_k} - 1) a_k - \sum_{k=1}^m (i_k + j_k) a_k$. Note that if $(n_1, \ldots, n_m) \in B(A_m)$, then $(n_1 + n, n_2, \ldots, n_m) \in B(A_m)$ for $n \in \mathbb{Z}_{\geq 0}$. Therefore, we obtain

$$\sum c_{i_1,\ldots,i_m;j_1,\ldots,j_m} \frac{x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m}}{\det G_1(x) \det G_1(y)}$$

$$= \frac{\sum (c_{i_1-2,\ldots,i_m;j_1,\ldots,j_m} - 2c_{i_1-1,\ldots,i_m;j_1-1,\ldots,j_m} + c_{i_1,\ldots,i_m;j_1-2,\ldots,j_m}) x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m}}{(x_1 - y_1)^2 \det G_1(x) \det G_1(y)},$$

where $(i_1, \ldots, i_m), (j_1, \ldots, j_m) \in B(A_m)$. Therefore, $\widehat{\omega}(x,y) = \widehat{\omega}(y,x)$ is equivalent to

$$c_{i_1-2,\ldots,i_m;j_1,\ldots,j_m} - 2c_{i_1-1,\ldots,i_m;j_1-1,\ldots,j_m} + c_{i_1,\ldots,i_m;j_1-2,\ldots,j_m} - c_{j_1-2,\ldots,j_m;i_1,\ldots,i_m}$$

$$+ 2c_{j_1-1,\ldots,j_m;i_1-1,\ldots,i_m} - c_{j_1,\ldots,j_m;i_1-2,\ldots,i_m} = q_{j_1,\ldots,j_m;i_1,\ldots,i_m} - q_{i_1,\ldots,i_m;j_1,\ldots,j_m}.$$

By Lemma 17,18, the system of the above linear equations has a solution. Moreover, it has a solution such that each $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ is a linear combination of $q_{i'_1,\ldots,i'_m;j'_1,\ldots,j'_m}$ satisfying $i'_1 + j'_1 = i_1 + j_1 + 2$, $(i'_k, j'_k) = (i_k, j_k)$ or $(i'_k, j'_k) = (j_k, i_k)$ for $k = 2, \ldots, m$. In particular, one can take $c_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ such that $c_{i_1,\ldots,i_m;j_1,\ldots,j_m} = 0$ if $2\sum_{k=2}^m \frac{d_{k-1}}{d_k} a_k - \sum_{k=1}^m (i_k + j_k + 2) a_k < 0$ and

$$\deg c_{i_1,\ldots,i_m;j_1,\ldots,j_m} = 2\sum_{k=2}^m \frac{d_{k-1}}{d_k} a_k - \sum_{k=1}^m (i_k + j_k + 2) a_k$$

if $c_{i_1,\dots,i_m;j_1,\dots,j_m} \neq 0$.

$\square$

(ii) From Lemma 17, $d_y\Omega(x,y)$ is holomorphic except $\Delta \cup X \times \{\infty\}$ and so is $\widehat{\omega}$. Since $\widehat{\omega}(x,y) = \widehat{\omega}(y,x)$, $\widehat{\omega}$ is holomorphic except Δ. From the definition of dr_i, we obtain

$$\widehat{\omega} - \omega = \sum_{i=1}^{g} du_i(x)(dr_i(y) - d\widehat{r}_i(y)).$$

On the other hand, $\widehat{\omega} - \omega$ is holomorphic except Δ and $\sum_{i=1}^{g} du_i(x)(dr_i(y) - d\widehat{r}_i(y))$ is holomorphic except $X \times \{\infty\}$. Therefore, $\widehat{\omega} - \omega$ is holomorphic except $\{\infty\} \times \{\infty\}$. Therefore, $\widehat{\omega} - \omega$ and $dr_i - d\widehat{r}_i$ are holomorphic on $X \times X$ and X, respectively, which completes the proof.

$\square$

In order to prove (iii), we need some lemmas.

Lemma 19 *([21]) Let ω_1 and ω_2 be meromorphic symmetric bilinear form satisfying the condition (i) of Definition 1. Then, there exist some constants $c_{ij} \in \mathbb{C}$ such that $c_{ij} = c_{ji}$ and*

$$\omega_1(x,y) - \omega_2(x,y) = \sum_{i,j=1}^{g} c_{ij} du_i(x) du_j(y). \tag{18}$$

Proof of Lemma 19. The left hand side of (18) is holomorphic symmetric bilinear form. Therefore, from (3), we obtain the assertion.

$\square$

We define the period matrices by

$$2\omega_1 = \left(\int_{\alpha_j} du_i\right),\ 2\omega_2 = \left(\int_{\beta_j} du_i\right),\ -2\eta_1 = \left(\int_{\alpha_j} dr_i\right),\ -2\eta_2 = \left(\int_{\beta_j} dr_i\right).$$

Then, ω_1 is invertible. Set $\tau = \omega_1^{-1}\omega_2$, then τ is symmetric and $\mathrm{Im}\ \tau > 0$.

Lemma 20 *We have*

$$\omega(x,y) = \widehat{\omega}(x,y) + {}^t du(x)\eta_1\omega_1^{-1} du(y),$$

where $du = {}^t(du_1, \dots, du_g)$. In particular, $\eta_1\omega_1^{-1}$ is symmetric.

Proof of Lemma 20. The method of proof is similar to the case of (n,s) curves (cf. [21] Lemma 8). By Lemma 19, there exists a constant symmetric $g \times g$ matrix $C = (c_{ij})$ such that

$$\omega(x,y) - \widehat{\omega}(x,y) = {}^t du(x) C du(y).$$

Since $\int_{\alpha_k} \omega(x,y) = 0$ and $\int_{\alpha_k} d_y\Omega(x,y) = 0$, we have

$$\sum_{i=1}^{g} du_i(x)(\eta_1)_{ik} = \sum_{i,j=1}^{g} c_{ij} du_i(x)(\omega_1)_{jk}.$$

Since $\{du_i\}_{i=1}^g$ are linearly independent, we have

$$(\eta_1)_{ik} = \sum_{j=1}^{g} c_{ij}(\omega_1)_{jk}.$$

Therefore, we have

$$C = \eta_1\omega_1^{-1}.$$

□

Proof of Theorem 11 (iii). The method of proof is similar to the case of (n,s)-curves (cf. [21] Proposition 3). The one form dr_i is a second kind differential which is singular only at ∞. In fact, $dr_i - d\widehat{r}_i$ is holomorphic one form as is just proved in the proof of Theorem 11 (ii) and $d\widehat{r}_i$ is a second kind differential which is singular only at ∞ from Lemma 17. The relation $du_i \circ du_j = 0$ is obvious. Let us prove $du_i \circ dr_j = \delta_{ij}$. By Theorem 11 (ii), we have

$$\widehat{\omega}(x,y) \circ du_j(y) = \mathrm{Res}_{y=x}(\int^y \widehat{\omega})du_j(y) = -du_j(x). \tag{19}$$

On the other hand, from (14), we have

$$\widehat{\omega}(x,y) \circ du_j(y) = \left(d_y\Omega(x,y) + \sum_{i=1}^{g} du_i(x)dr_i(y)\right) \circ du_j(y) = \sum_{i=1}^{g} du_i(x)(dr_i \circ du_j). \tag{20}$$

Since $\{du_i\}_{i=1}^g$ are linearly independent, we have $du_i \circ dr_j = \delta_{ij}$. Next, let us prove $dr_i \circ dr_j = 0$. Similarly to (20), we have

$$\widehat{\omega}(x,y) \circ dr_j(y) = \sum_{i=1}^{g} du_i(x)(dr_i \circ dr_j). \tag{21}$$

From Lemma 20 and $du_i \circ dr_j = \delta_{ij}$, we have

$$\widehat{\omega}(x,y) \circ dr_j(y) = \omega(x,y) \circ dr_j(y) - \sum_{i=1}^{g} du_i(x)(\eta_1\omega_1^{-1})_{ij}. \tag{22}$$

From (14) and $\int_{\alpha_k} \omega = 0$, we have

$$2\pi i\; \omega(x,y) \circ dr_j(y) = \sum_{k=1}^{g} \left(\int_{\alpha_k} \omega \int_{\beta_k} dr_j - \int_{\alpha_k} dr_j \int_{\beta_k} \omega\right) = 2\sum_{k=1}^{g} (\eta_1)_{jk} \int_{\beta_k} \omega. \tag{23}$$

Lemma 21 *We have*

$$\int_{\beta_k} \omega = \pi i \sum_{i=1}^{g} (\omega_1^{-1})_{ki} du_i(x).$$

Proof of Lemma 21. Similarly to (19), we have

$$\omega(x,y)\circ du_i(y) = -du_i(x).$$

Similarly to (23), we have

$$2\pi i\ \omega(x,y)\circ du_i(y) = -2\sum_{k=1}^{g}(\omega_1)_{ik}\int_{\beta_k}\omega.$$

From these, we obtain the assertion.

$\square$

By Lemma 21, we obtain

$$\omega(x,y)\circ dr_j(y) = \sum_{i=1}^{g}(\eta_1\omega_1^{-1})_{ji}du_i(x).$$

From (22) and Lemma 20, we have

$$\widehat{\omega}(x,y)\circ dr_j(y) = \sum_{i=1}^{g}du_i(x)\left((\eta_1\omega_1^{-1})_{ji}-(\eta_1\omega_1^{-1})_{ij}\right)=0.$$

From (21), we obtain $dr_i\circ dr_j=0$.

From (16), we find that $\{du_i, dr_i\}_{i=1}^g$ are linearly independnet. Since $\dim_{\mathbb{C}}H^1(X,\mathbb{C})=2g$ (cf. [16], pp. 29-31, Theorem 8.1,8.2), $\{du_i, dr_i\}_{i=1}^g$ are a basis of $H^1(X,\mathbb{C})$.

$\square$

7 Sigma functions for telescopic curves

In this section, we define the sigma function for the telescopic curve X.

From (16) and (14), for the matrix

$$M := \begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix},$$

we have

$$M\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}{}^tM = -\frac{\pi i}{2}\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix},$$

where I_g denotes the unit matrix of degree g. Since $\eta_1\omega_1^{-1}$ is symmetric, we obtain the following proposition.

Proposition 22 *(generalized Legendre relation)*

$${}^tM\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}M = -\frac{\pi i}{2}\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}.$$

Let $\delta = \tau\delta' + \delta''$ be the Riemann's constant of X with respect to our choice $(\infty, \{\alpha_i, \beta_i\}_{i=1}^g)$. Since the divisor of the holomorphic one form du_g is $(2g-2)\infty$, the Riemann's constant δ becomes a half period. Then, the sigma funtion $\sigma(u)$ associated with X is defined as follows.

Definition 23 *For $u \in \mathbb{C}^g$, we define*

$$\widehat{\sigma}(u) = \exp\left(\frac{1}{2}\,{}^t u \eta_1 \omega_1^{-1} u\right) \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} ((2\omega_1)^{-1}u, \tau).$$

By Proposition 22, we obtain the following proposition.

Proposition 24 *For any $m_1, m_2 \in \mathbb{Z}^g$ and $u \in \mathbb{C}^g$, we have*

$$\widehat{\sigma}(u + 2\omega_1 m_1 + 2\omega_2 m_2)/\widehat{\sigma}(u) = \exp\left(\pi i\left({}^t m_1 m_2 + 2\,{}^t\delta' m_1 - 2\,{}^t\delta'' m_2\right)\right)$$
$$\times \exp\left({}^t(2\eta_1 m_1 + 2\eta_2 m_2)(u + \omega_1 m_1 + \omega_2 m_2)\right).$$

8 Algebraic expression of sigma functions

8.1 Algebraic expression of prime form

Let $\tilde{X}$ be the universal cover of the telescopic curve X and $\pi : \tilde{X} \to X$ the projection. Hereafter, for $\tilde{p} \in \tilde{X}$, we denote $\pi(\tilde{p})$ by p. Let $\{dv_i\}_{i=1}^g$ be the basis of holomorphic one forms such that $\int_{\alpha_j} dv_i = \delta_{ij}$.

Lemma 25 *For $p_1, p_2 \in X$, we have*

$$\sum_{i=0}^{a_1-1} \int_{p_1^{(i)}}^{p_2^{(i)}} dv \in \mathbb{Z}^g + \tau\mathbb{Z}^g.$$

Proof of Lemma 25. We define the meromorphic function f on X by

$$f(z) = \begin{cases} (x_1(z) - x_1(p_2))/(x_1(z) - x_1(p_1)) & \text{if } p_1 \neq \infty, p_2 \neq \infty \\ 1/(x_1(z) - x_1(p_1)) & \text{if } p_1 \neq \infty, p_2 = \infty \\ x_1(z) - x_1(p_2) & \text{if } p_1 = \infty, p_2 \neq \infty \\ 1 & \text{if } p_1 = \infty, p_2 = \infty \end{cases}.$$

Since $\mathrm{div}(f) = \sum_{i=0}^{a_1-1} p_2^{(i)} - \sum_{i=0}^{a_1-1} p_1^{(i)}$, $\sum_{i=0}^{a_1-1} p_2^{(i)} - \sum_{i=0}^{a_1-1} p_1^{(i)}$ is a principal divisor. Therefore, by the Abel-Jacobi's theorem, we obtain the assertion.

□

For $\tilde{p}_1, \tilde{p}_2 \in \tilde{X}$, take $\tilde{p}_1^{(i)}, \tilde{p}_2^{(i)} \in \tilde{X}$, $0 \le i < a_1$, $\tilde{p}_1^{(0)} = \tilde{p}_1$, $\tilde{p}_2^{(0)} = \tilde{p}_2$ such that $\pi(\tilde{p}_1^{(i)}) = p_1^{(i)}, \pi(\tilde{p}_2^{(i)}) = p_2^{(i)}$ and

$$\sum_{i=0}^{a_1-1} \int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}} dv = 0. \tag{24}$$

Proposition 26 *[10] For $\tilde{p}_1, \tilde{p}_2 \in \tilde{X}$, we have*

$$E(\tilde{p}_1,\tilde{p}_2)^2 = \frac{(x_1(p_2)-x_1(p_1))^2}{dx_1(p_1)dx_1(p_2)} \exp\left(\sum_{i=1}^{a_1-1}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\int_{\tilde{p}_1}^{\tilde{p}_2}\omega\right).$$

Proof of Proposition 26. For the sake to be complete and self-contained, we give a proof of this proposition. The method of the proof is similar to [21].

Lemma 27 *We have*

$$\frac{(x_1(w)-x_1(p_2))\,(x_1(z)-x_1(p_1))}{(x_1(w)-x_1(p_1))\,(x_1(z)-x_1(p_2))} = \exp\left(\sum_{i=0}^{a_1-1}\int_{\tilde{z}}^{\tilde{w}}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\omega\right).$$

Proof of Lemma 27. By Proposition 2,

$$\exp\left(\sum_{i=0}^{a_1-1}\int_{\tilde{z}}^{\tilde{w}}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\omega\right) = \prod_{i=0}^{a_1-1}\frac{E(\tilde{w},\tilde{p}_2^{(i)})E(\tilde{z},\tilde{p}_1^{(i)})}{E(\tilde{w},\tilde{p}_1^{(i)})E(\tilde{z},\tilde{p}_2^{(i)})}.$$

Let us consider the right hand side of this equation as a function of $\tilde{w}$ and denote it by $F(\tilde{w})$. By the property (iv) of the prime form, if the abelian image of $\gamma \in \pi_1(X,\infty)$ is $\sum_{i=1}^g m_i\alpha_i + \sum_{i=1}^g n_i\beta_i$, then

$$F(\gamma\tilde{w}) = F(\tilde{w})\exp\left(2\pi i\sum_{j=0}^{a_1-1}{}^t n\int_{\tilde{p}_1^{(j)}}^{\tilde{p}_2^{(j)}}dv\right).$$

From (24), $F(\tilde{w})$ is $\pi_1(X,\infty)$-invariant and can be considered as a meromorphic function on X. By comparing zeros and poles, there exists a constant C such that

$$F(\tilde{w}) = C\frac{x_1(w)-x_1(p_2)}{x_1(w)-x_1(p_1)}.$$

Since $F(\tilde{z})=1$, we have

$$C = \frac{x_1(z)-x_1(p_1)}{x_1(z)-x_1(p_2)},$$

which proves the lemma.

□

Proof of Proposition 26. In Lemma 27, take the limit $\tilde{z}\to\tilde{p}_1, \tilde{w}\to\tilde{p}_2$ and use

$$\lim_{\tilde{w}\to\tilde{q}}\frac{x_1(w)-x_1(q)}{E(\tilde{w},\tilde{q})} = -dx_1(q),$$

$$\exp\left(\int_{\tilde{z}}^{\tilde{w}}\int_{\tilde{p}_1}^{\tilde{p}_2}\omega\right) = \frac{E(\tilde{w},\tilde{p}_2)E(\tilde{z},\tilde{p}_1)}{E(\tilde{w},\tilde{p}_1)E(\tilde{z},\tilde{p}_2)}.$$

Then, we obtain the desired result.

□

8.2 Prime function

Since du_g has a zero of order $2g-2$ at ∞, we can define, as in the case of (n,s) curves, the prime function $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ by

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)=-E(\tilde{p}_1,\tilde{p}_2)\sqrt{du_g(p_1)}\sqrt{du_g(p_2)}\exp\left(\frac{1}{2}\int_{\tilde{p}_1}^{\tilde{p}_2}{}^t du\cdot\eta_1\omega_1^{-1}\cdot\int_{\tilde{p}_1}^{\tilde{p}_2}du\right),$$

where $\sqrt{du_g}$ is the holomorphic section of the line bundle on X defined by the divisor $(g-1)\infty$ satisfying

$$(\sqrt{du_g})^2=du_g,$$

$$\sqrt{du_g}=t^{g-1}(1+O(t))\sqrt{dt}, \tag{25}$$

(t is the local parameter (12) around ∞).

Since

$$\delta=\int_{(g-1)\infty}^{\delta_0}dv=\int_{\delta_0}^{(g-1)\infty}dv \quad \text{in} \quad \mathrm{Jac}(X),$$

$\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ can be considered as a holomorphic section of the line bundle $\pi_1^*\mathcal{L}_\delta\otimes\pi_2^*\mathcal{L}_\delta\otimes I^*\Theta$ on $X\times X$. From (24), Proposition 26, and Lemma 20, we have

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)^2=\frac{(x_1(p_2)-x_1(p_1))^2}{\det G_1(p_1)\det G_1(p_2)}\exp\left(\sum_{i=1}^{a_1-1}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\int_{\tilde{p}_1}^{\tilde{p}_2}\widehat{\omega}\right). \tag{26}$$

Fix $\tilde{\infty}\in\tilde{X}$ such that $\pi(\tilde{\infty})=\infty$. We define $\tilde{E}(\tilde{\infty},\tilde{p})$, as in the case of (n,s) curves, in the following manner. Take the local coordinate t (12) and the local frame $\sqrt{dt}$ as above and define

$$E(\tilde{\infty},\tilde{p}_2)=E(\tilde{p}_1,\tilde{p}_2)\sqrt{dt(p_1)}|_{t(p_1)=0},$$

$$\tilde{E}(\tilde{\infty},\tilde{p})=E(\tilde{\infty},\tilde{p})\sqrt{du_g(p)}\exp\left(\frac{1}{2}\int_{\tilde{\infty}}^{\tilde{p}}{}^t du\cdot\eta_1\omega_1^{-1}\cdot\int_{\tilde{\infty}}^{\tilde{p}}du\right).$$

Notice that $E(\tilde{\infty},\tilde{p})$ and $\tilde{E}(\tilde{\infty},\tilde{p})$ can be considered as holomorphic sections of $L_0^{-1}\otimes I_1^*\Theta$ and $\mathcal{L}_\delta\otimes I_1^*\Theta$, respectively, where $I_1(p)=\int_\infty^p dv$. From (25) and the property (iii) of the prime form, we have

$$-\tilde{E}(\tilde{p}_1,\tilde{p}_2)=\tilde{E}(\tilde{\infty},\tilde{p}_2)t(p_1)^{g-1}+O(t(p_1)^g). \tag{27}$$

From the properties (i)(ii)(iii) of the prime form, we obtain the following proposition.

Proposition 28 *(i)* $\tilde{E}(\tilde{p}_2,\tilde{p}_1)=-\tilde{E}(\tilde{p}_1,\tilde{p}_2)$.

(ii) As a section of a line bundle on $X\times X$, the zero divisor of $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ is

$$\Delta+(g-1)(\{\infty\}\times X+X\times\{\infty\}).$$

(iii) As a section of a line bundle on X, the zero divisor of $\tilde{E}(\tilde{\infty},\tilde{p})$ is $g\infty$.

Proposition 29 *Let the abelian image of $\gamma \in \pi_1(X,\infty)$ be $\sum_{i=1}^g m_i\alpha_i + \sum_{i=1}^g n_i\beta_i$. Then, we have*

(i) $\tilde{E}(\tilde{p}_1, \gamma\tilde{p}_2)/\tilde{E}(\tilde{p}_1,\tilde{p}_2) = (-1)^{{}^t mn + 2({}^t\delta' m - {}^t\delta'' n)} \times \exp\left({}^t(2\eta_1 m + 2\eta_2 n)(\int_{\tilde{p}_1}^{\tilde{p}_2} du + \omega_1 m + \omega_2 n)\right)$.

(ii) $\tilde{E}(\tilde{\infty}, \gamma\tilde{p})/\tilde{E}(\tilde{\infty},\tilde{p}) = (-1)^{{}^t mn + 2({}^t\delta' m - {}^t\delta'' n)} \times \exp\left({}^t(2\eta_1 m + 2\eta_2 n)(\int_{\tilde{\infty}}^{\tilde{p}} du + \omega_1 m + \omega_2 n)\right)$.

Proof of Proposition 29. The method of proof is similar to the case of (n,s)-curves (cf. [21] Proposition 8).

(i) Let

$$F_1(\tilde{p}_1,\tilde{p}_2) = \frac{\sqrt{du_g(p_1)}\sqrt{du_g(p_2)}}{h_{\tilde{\alpha}}(p_1)h_{\tilde{\alpha}}(p_2)},$$

which is a section of the bundle $\pi_1^*\mathcal{L}_{\delta-\alpha} \otimes \pi_2^*\mathcal{L}_{\delta-\alpha}$. Then, we have

$$F_1(\tilde{p}_1,\gamma\tilde{p}_2) = \chi(\gamma)F_1(\tilde{p}_1,\tilde{p}_2), \quad \gamma \in \pi_1(X,\infty),$$

where $\chi \in \mathrm{Hom}(\pi_1(X,p_0),\mathbb{C}^*)$ corresponding to $\mathcal{L}_{\delta-\alpha}$. Since $du_g/h_{\tilde{\alpha}}^2$ is a function on X, we have $\chi(\gamma)^2 = 1$, i.e., χ is a unitary representation. Therefore, if the abelian image of γ is $\sum_{i=1}^g m_i\alpha_i + \sum_{i=1}^g n_i\beta_i$, we have

$$\chi(\gamma) = \exp\left(2\pi i({}^t(\delta'-\alpha')m - {}^t(\delta''-\alpha'')n)\right).$$

Let

$$F_2(\tilde{p}_1,\tilde{p}_2) = \theta[\tilde{\alpha}](\int_{\tilde{p}_1}^{\tilde{p}_2} dv).$$

Then, we have

$$\frac{F_2(\tilde{p}_1,\gamma\tilde{p}_2)}{F_2(\tilde{p}_1,\tilde{p}_2)} = \exp\left\{2\pi i({}^t\alpha' m - {}^t\alpha'' n) - \pi i\, {}^t n\tau n - 2\pi i\, {}^t n \int_{\tilde{p}_1}^{\tilde{p}_2} dv\right\}.$$

Let

$$F_3(\tilde{p}_1,\tilde{p}_2) = \exp\left(\frac{1}{2}\int_{\tilde{p}_1}^{\tilde{p}_2} {}^t du \cdot \eta_1\omega_1^{-1} \cdot \int_{\tilde{p}_1}^{\tilde{p}_2} du\right).$$

Then, we have

$$\frac{F_3(\tilde{p}_1,\gamma\tilde{p}_2)}{F_3(\tilde{p}_1,\tilde{p}_2)} = \exp\left({}^t(2\eta_1 m + 2\eta_2 n)(\int_{\tilde{p}_1}^{\tilde{p}_2} du + \omega_1 m + \omega_2 n) + \pi i\, {}^t mn\right)$$
$$\times \exp\left(\pi i\, {}^t n\tau n + 2\pi i\, {}^t n \int_{\tilde{p}_1}^{\tilde{p}_2} dv\right).$$

Here, we use Proposition 22 and the relation

$$dv = (2\omega_1)^{-1} du.$$

Therefore, we obtain the desired result.

(ii) We can prove the statement in a manner similar to (i).

□

8.3 Algebraic expression of sigma functions

We have the following theorem.

Theorem 30 *There exists $c \in \mathbb{C}$ such that for $N \geq g$*

$$\widehat{\sigma}(\sum_{i=1}^{N} \int_{\tilde{\infty}}^{\tilde{p}_i} du) = c \cdot \frac{\prod_{i=1}^{N} \tilde{E}(\tilde{\infty}, \tilde{p}_i)^N}{\prod_{1 \leq i < j \leq N} \tilde{E}(\tilde{p}_i, \tilde{p}_j)} \det(\varphi_i(p_j))_{1 \leq i,j \leq N}. \tag{28}$$

Proof of Theorem 30. The method of proof is similar to the case of (n, s)-curves (cf. [21] Theorem 1). Let

$$G(\tilde{p}_1, \ldots, \tilde{p}_N) = \frac{\prod_{i=1}^{N} \tilde{E}(\tilde{\infty}, \tilde{p}_i)^N}{\prod_{1 \leq i < j \leq N} \tilde{E}(\tilde{p}_i, \tilde{p}_j)} \det(\varphi_i(p_j))_{1 \leq i,j \leq N} / \widehat{\sigma}(\sum_{i=1}^{N} \int_{\tilde{\infty}}^{\tilde{p}_i} du).$$

Then, G is a symmetric function of $\tilde{p}_1, \ldots, \tilde{p}_N$. From Proposition 28, 29, one can check the following properties.

(i) $G(\gamma\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_N) = G(\tilde{p}_1, \ldots, \tilde{p}_N)$ for any $\gamma \in \pi_1(X, \infty)$.

(ii) The right hand side of (28) is holomorphic.

Let us consider G as a function of $\tilde{p}_1, \ldots, \tilde{p}_g$. By (i), G can be considered as a meromorphic function on the g-th symmetric product $S^g X$ and therefore on the Jacobian $J(X) = \mathbb{C}^g/(2\omega_1\mathbb{Z}^g + 2\omega_2\mathbb{Z}^g)$. By (ii), as a meromorphic function on $J(X)$, G has poles only on $\{\widehat{\sigma}(u) = 0\}$ of order at most one. Therefore, it is a constant which means that it is independent of $\tilde{p}_i$, $1 \leq i \leq g$. Since G is symmetric, it is independent of $\tilde{p}_i$, $1 \leq i \leq N$. Therefore, there exists $c \in \mathbb{C}$ such that the equation (28) is satisfied. If $N > g$, by setting $\tilde{p}_N = \tilde{\infty}$ and (27), one can check

$$\widehat{\sigma}(\sum_{i=1}^{N-1} \int_{\tilde{\infty}}^{\tilde{p}_i} du) = c \cdot \frac{\prod_{i=1}^{N-1} \tilde{E}(\tilde{\infty}, \tilde{p}_i)^{N-1}}{\prod_{1 \leq i < j \leq N-1} \tilde{E}(\tilde{p}_i, \tilde{p}_j)} \det(\varphi_i(p_j))_{1 \leq i,j \leq N-1}.$$

Therefore, the constant c does not depend on N.

□

Let $\sigma(u) = c^{-1}\widehat{\sigma}(u)$. Then, from Theorem 30, we have

$$\sigma(\sum_{i=1}^{N} \int_{\tilde{\infty}}^{\tilde{p}_i} du) = \frac{\prod_{i=1}^{N} \tilde{E}(\tilde{\infty}, \tilde{p}_i)^N}{\prod_{1 \leq i < j \leq N} \tilde{E}(\tilde{p}_i, \tilde{p}_j)} \det(\varphi_i(p_j))_{1 \leq i,j \leq N}. \tag{29}$$

9 Schur functions

In this section, we give the definition of Schur functions following [21]. For details, see [17]. For $n \geq 0$, let $p_n(T)$ be the polynomial of $T_1, T_2, \ldots$ defined by

$$\exp(\sum_{n=1}^{\infty} T_n k^n) = \sum_{n=0}^{\infty} p_n(T) k^n,$$

where k is a variable.

Example. $p_0 = 1,\ \ p_1 = T_1,\ \ p_2 = T_2 + \dfrac{T_1^2}{2},\ \ p_3 = T_3 + T_1 T_2 + \dfrac{T_1^3}{6}.$

For $n < 0$, we define $p_n(T) = 0$. A sequence of non-negative integers $\mu = (\mu_1, \ldots, \mu_l)$ is called a partition if $\mu_1 \geq \cdots \geq \mu_l$. We set $|\mu| = \mu_1 + \cdots + \mu_l$, which is called the weight of μ. Let $\mu' = (\mu'_1, \ldots, \mu'_{l'}),\ l' = \mu_1$, with

$$\mu'_i = \sharp\{j \mid \mu_j \geq i\}.$$

The sequence μ' is called the conjugate of μ. For a partition $\mu = (\mu_1, \ldots, \mu_l)$, the Schur function $S_\mu(T)$ is defined by

$$S_\mu(T) = \det(p_{\mu_i - i + j}(T))_{1 \leq i,j \leq l}.$$

We have

$$S_{(\mu, 0^r)}(T) = S_\mu(T),$$

where $(\mu, 0^r) = (\mu_1, \ldots, \mu_l, 0, \ldots, 0)$ for $r \in \mathbb{Z}_{\geq 0}$.

Example. $S_{(1)}(T) = T_1,\ \ S_{(2,1)}(T) = -T_3 + \dfrac{T_1^3}{3},$

$S_{(3,2,1)}(T) = T_1 T_5 - T_3^2 - \dfrac{1}{3} T_1^3 T_3 + \dfrac{1}{45} T_1^6,\ \ S_{(3,1,1)}(T) = T_5 - T_1 T_2^2 + \dfrac{1}{20} T_1^5.$

We prescribe the degree $-i$ to T_i:

$$\deg T_i = -i.$$

The following results are well-known.

Lemma 31 *([9])*

(i) $S_\mu(T)$ *is a homogeneous polynomial of degree* $-|\mu|$.

(ii) $S_\mu(-T) = (-1)^{|\mu|} S_{\mu'}(T)$.

For a partition $\mu = (\mu_1, \ldots, \mu_l)$, we define the symmetric polynomial of $t_1, t_2, \ldots, t_l$ by

$$s_\mu(t) = \frac{\det(t_j^{\mu_i + l - i})_{1 \leq i,j \leq l}}{\prod_{1 \leq i < j \leq l}(t_i - t_j)},$$

which we also call the Schur function. Two Schur functions are related by

$$S_\mu(T) = s_\mu(t), \ \text{ if } \ T_i = \frac{\sum_{j=1}^{l} t_j^i}{i}.$$

We define the partition associated with the telescopic curve X by

$$\mu(A_m) = (w_g, \ldots, w_1) - (g-1, \ldots, 1, 0).$$

Then, we have the following proposition.

Proposition 32 *(i)* $S_{\mu(A_m)}(T)$ *is a polynomial of the variables* $T_{w_1}, \ldots, T_{w_g}$.

(ii) $\mu(A_m)' = \mu(A_m)$.

(iii)
$$|\mu(A_m)| = \frac{C_m}{12a_1}\sum_{i=2}^{m} a_i^2\left(\frac{d_{i-1}}{d_i}-1\right)\left(2\frac{d_{i-1}}{d_i}-1\right)$$
$$+\frac{C_m}{4a_1}\sum_{2\le i<j\le m} a_ia_j\left(\frac{d_{i-1}}{d_i}-1\right)\left(\frac{d_{j-1}}{d_j}-1\right)$$
$$-\frac{C_m}{4}\sum_{i=2}^{m} a_i\left(\frac{d_{i-1}}{d_i}-1\right)+\frac{a_1^2-1}{12}+\frac{1}{8}-\frac{1}{8}\left\{-a_1+\sum_{i=2}^{m}\left(\frac{d_{i-1}}{d_i}-1\right)a_i\right\}^2,$$
where $C_m = \prod_{i=2}^{m}(d_{i-1}/d_i)$.

Proof of Proposition 32. We can prove (i) (ii) in a similar manner to the case of (n,s) curves (cf. [4]), because they were proved in [4] by the property $w_g = 2g-1$, which is satisfied also for telescopic curves. Let us prove (iii). By definition, we have

$$|\mu(A_m)| = \sum_{i=1}^{g} w_i - \frac{g(g-1)}{2}. \tag{30}$$

Lemma 33 *We have*

$\{w_1, \ldots, w_g\}$

$= \{x \in \mathbb{Z}_{>0} \mid x = -a_1k_1 + a_2k_2 + \cdots + a_mk_m, k_1 > 0, 0 \le k_i < d_{i-1}/d_i, i = 2, \ldots, m\}$.

In particular, the expression $w_j = -a_1k_1 + a_2k_2 + \cdots + a_mk_m$, $k_1 > 0, 0 \le k_i < d_{i-1}/d_i$ *is unique.*

Proof of Lemma 33. Since $\gcd\{a_1, \ldots, a_m\} = 1$, for any $x \in \mathbb{Z}_{\ge 0}$, there exist $k_1, \ldots, k_m \in \mathbb{Z}$ such that $x = a_1k_1 + \cdots + a_mk_m$. In particular, from (5), one can take k_i such that $0 \le k_i < d_{i-1}/d_i$ for $i = 2, \ldots, m$. For $u = (0, u_2, \ldots, u_m)$ and $v = (0, v_2, \ldots, v_m) \in \mathbb{Z}_{\ge 0}^m$ satisfying $u \ne v$ and $0 \le u_i, v_i < d_{i-1}/d_i$, from Proposition 3, we have $\sum_{i=2}^m a_iu_i \not\equiv \sum_{i=2}^m a_iv_i$ mod a_1. Therefore, for $x \in \mathbb{Z}_{\ge 0}$ the expression $x = a_1k_1 + \cdots + a_mk_m$, $0 \le k_i < d_{i-1}/d_i$, $i = 2, \ldots, m$ is unique. Therefore, from Proposition 3, we have $\{w_1, \ldots, w_g\} = \{x \in \mathbb{Z}_{>0} \mid x = -a_1k_1 + a_2k_2 + \cdots + a_mk_m, k_1 > 0, 0 \le k_i < d_{i-1}/d_i, i = 2, \ldots, m\}$.

□

Let $L_{k_2,\ldots,k_m} = (a_2k_2 + \cdots + a_mk_m)/a_1$, $M_{k_2,\ldots,k_m} = [L_{k_2,\ldots,k_m}]$, and $\epsilon_{k_2,\ldots,k_m} = L_{k_2,\ldots,k_m} - M_{k_2,\ldots,k_m}$. Then, from Lemma 33, we have

$$\sum_{i=1}^{g} w_i = \sum_{0\le k_i<d_{i-1}/d_i, i=2,\ldots,m}\ \sum_{k_1=1}^{M_{k_2,\ldots,k_m}} (-a_1k_1 + a_2k_2 + \cdots + a_mk_m)$$
$$= \sum_{0\le k_i<d_{i-1}/d_i, i=2,\ldots,m}\left\{-a_1\frac{M_{k_2,\ldots,k_m}^2 + M_{k_2,\ldots,k_m}}{2} + a_1L_{k_2,\ldots,k_m}M_{k_2,\ldots,k_m}\right\}$$

$$= \sum_{0\le k_i<d_{i-1}/d_i, i=2,\dots,m} \left\{\frac{a_1}{2}L_{k_2,\dots,k_m}^2 - \frac{a_1}{2}\epsilon_{k_2,\dots,k_m}^2 - \frac{a_1}{2}L_{k_2,\dots,k_m} + \frac{a_1}{2}\epsilon_{k_2,\dots,k_m}\right\}.$$

From Proposition 3, for $(k_2,\dots,k_m) \neq (k_2',\dots,k_m')$ satisfying $0 \le k_i, k_i' < d_{i-1}/d_i$, we have $\sum_{i=2}^m a_i k_i \not\equiv \sum_{i=2}^m a_i k_i' \bmod a_1$. Therefore, we have $\{\epsilon_{k_2,\dots,k_m}\}_{0\le k_i<d_{i-1}/d_i} = \{0, 1/a_1, \dots, (a_1-1)/a_1\}$. Therefore, by calculation, we have

$$\sum_{i=1}^g w_i = \frac{C_m}{12a_1}\sum_{i=2}^m a_i^2\left(\frac{d_{i-1}}{d_i}-1\right)\left(2\frac{d_{i-1}}{d_i}-1\right) + \frac{C_m}{4a_1}\sum_{2\le i<j\le m} a_i a_j\left(\frac{d_{i-1}}{d_i}-1\right)\left(\frac{d_{j-1}}{d_j}-1\right)$$

$$-\frac{C_m}{4}\sum_{i=2}^m a_i\left(\frac{d_{i-1}}{d_i}-1\right) + \frac{a_1^2-1}{12}.$$

Therefore, from (9) and (30), we obtain (iii).

□

10 Series expansion of sigma functions

In this section, we determine the series expansion of the sigma functions for telescopic curves. We have the following theorem.

Theorem 34 *(i) The expansion of $\sigma(u)$ at the origin takes the form*

$$\sigma(u) = S_{\mu(A_m)}(T)|_{T_{w_i}=u_i} + \sum_{w_1k_1+\cdots+w_gk_g>|\mu(A_m)|} b_{k_1,\dots,k_g} u_1^{k_1}\cdots u_g^{k_g},$$

where $b_{k_1,\dots,k_g}$ belongs to $\mathbb{Q}[\{\lambda_{j_1\cdots j_m}^{(i)}\}]$ and is homogeneous of degree $\sum_{i=1}^g w_i k_i - |\mu(A_m)|$ if $b_{k_1,\dots,k_g} \neq 0$.

(ii) $\sigma(-u) = (-1)^{|\mu(A_m)|}\sigma(u)$.

The method of proof of Theorem 34 is similar to the case of (n,s) curves (cf. [21]). However, for the sake to be complete and self-contained, we give a proof.

Lemma 35 *Let $t_x = t(x)$, $t_y = t(y)$, and*

$$\widehat{\omega}(x,y) = \left(\frac{1}{(t_x-t_y)^2} + \sum_{k,l=0}^{\infty} a_{kl} t_x^k t_y^l\right) dt_x dt_y.$$

Then, $a_{kl} = a_{lk}$, a_{kl} belongs to $\mathbb{Q}[\{\lambda_{j_1\cdots j_m}^{(i)}\}]$ and is homogeneous of degree $k+l+2$ if $a_{kl} \neq 0$.

Proof of Lemma 35. The method of proof is similar to the case of (n,s) curves (cf. [21], Lemma 15 (iii)).

From Theorem 11 (ii),

$$\widehat{\omega}-\frac{dt_x dt_y}{(t_x-t_y)^2}$$

is holomorphic near $\{\infty\}\times\{\infty\}$. Therefore, one can expand as

$$\widehat{\omega}(x,y)-\frac{dt_x dt_y}{(t_x-t_y)^2}=\left(\sum_{k,l=0}^{\infty}a_{kl}t_x^k t_y^l\right)dt_x dt_y.$$

Let us prove that a_{kl} belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $k+l+2$ if $a_{kl}\neq 0$. We have

$$\widehat{\omega}(x,y)=\frac{\sum p_{i_1,\ldots,i_m;j_1,\ldots,j_m}x_1^{i_1}\cdots x_m^{i_m}y_1^{j_1}\cdots y_m^{j_m}}{(x_1-y_1)^2}\frac{dx_1}{\det G_1(x)}\frac{dy_1}{\det G_1(y)},$$

where $p_{i_1,\ldots,i_m;j_1,\ldots,j_m}$ belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $2\sum_{k=2}^{m}(\frac{d_{k-1}}{d_k}-1)a_k-\sum_{k=1}^{m}(i_k+j_k)a_k$ if $p_{i_1,\ldots,i_m;j_1,\ldots,j_m}\neq 0$. From Proposition 10, one can expand as

$$\left(\widehat{\omega}(x,y)-\frac{dt_x dt_y}{(t_x-t_y)^2}\right)(t_x^{a_1}-t_y^{a_1})^2=\widehat{\omega}(x,y)(t_x^{a_1}-t_y^{a_1})^2-\left(\sum_{i=0}^{a_1-1}t_x^{a_1-1-i}t_y^{i}\right)^2 dt_x dt_y$$

$$=\left(\sum_{k,l=0}^{\infty}p_{kl}t_x^k t_y^l\right)dt_x dt_y,$$

where p_{kl} belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $k+l+2-2a_1$ if $p_{kl}\neq 0$. Therefore, we have

$$(t_x^{a_1}-t_y^{a_1})^2\sum_{k,l=0}^{\infty}a_{kl}t_x^k t_y^l=\sum_{k,l=0}^{\infty}p_{kl}t_x^k t_y^l.$$

By comparing the coefficient of $t_x^{k+2a_1}t_y^l$ in the above equation, we have

$$a_{kl}=2a_{k+a_1,l-a_1}-a_{k+2a_1,l-2a_1}+p_{k+2a_1,l}, \tag{31}$$

where we set $a_{ij}=0$ if $j<0$.

If $0\leq l<a_1$, from $a_{kl}=p_{k+2a_1,l}$, we find that a_{kl} belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $k+l+2$. Suppose that $a_{k'l'}$ belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $k'+l'+2$ if $l'<l$. Then, from (31), we find that a_{kl} belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $k+l+2$. Therefore, by induction, we obtain the assertion. $\square$

Lemma 36 *(i) The expansion of $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ near $(\tilde{\infty},\tilde{\infty})$ is of the form*

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)=(t_x-t_y)(t_xt_y)^{g-1}\left(1+\sum_{k+l\geq 1}q_{kl}t_x^kt_y^l\right),$$

where q_{kl} belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\dots j_m}\}]$ and is homogeneous of degree $k+l$ if $q_{kl}\neq 0$.

(ii) The expansion of $\tilde{E}(\tilde{\infty},\tilde{p})$ near $\tilde{\infty}$ is of the form

$$\tilde{E}(\tilde{\infty},\tilde{p})=t^g\left(1+\sum_{k=1}^{\infty}q_{0k}t^k\right),$$

where q_{0k} is the same as that in (i).

Proof of Lemma 36. The method of proof is similar to the case of (n,s) curves (cf. [21], Lemma 16).

(i) From the property (iii) of the prime form and (25), we have the expansion of the form

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)=(t_x-t_y)(t_xt_y)^{g-1}(1+\sum_{k+l\geq 1}q_{kl}t_x^kt_y^l).$$

In order to prove that q_{kl} has the required properties, we use (26). The right hand side of (26) is calculated in the following way.

Let $\zeta=\exp(2\pi\sqrt{-1}/a_1)$. Since $x_1(p^{(i)})=x_1(p)=1/t^{a_1}$, we take $t^{(i)}=\zeta^{-i}t$ as a local parameter of $p^{(i)}$ by rearranging i of $p^{(i)}$ if necessary. Using these local parameters, from Lemma 35, we obtain

$$\exp\left(\sum_{i=1}^{a_1-1}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\int_{\tilde{p}_1}^{\tilde{p}_2}\widehat{\omega}\right)$$

$$=\prod_{i=1}^{a_1-1}\frac{(t_y^{(i)}-t_y)(t_x^{(i)}-t_x)}{(t_y^{(i)}-t_x)(t_x^{(i)}-t_y)}\exp\left(\sum_{i=1}^{a_1-1}\sum_{k,l=0}^{\infty}a_{kl}\frac{(t_y^{k+1}-t_x^{k+1})((t_y^{(i)})^{l+1}-(t_x^{(i)})^{l+1})}{(k+1)(l+1)}\right)$$

$$=\left(\prod_{i=1}^{a_1-1}\{-\zeta^i(\zeta^{-i}-1)^2\}\right)\frac{(t_xt_y)^{a_1-1}}{\prod_{i=1}^{a_1-1}(t_x-t_y^{(i)})^2}\exp\left(\sum_{k,l=0}^{\infty}\left(\sum_{i=1}^{a_1-1}\zeta^{-i(l+1)}\right)a_{kl}\frac{(t_y^{k+1}-t_x^{k+1})(t_y^{l+1}-t_x^{l+1})}{(k+1)(l+1)}\right).$$

Claim.

$$\prod_{i=1}^{a_1-1}\{-\zeta^i(\zeta^{-i}-1)^2\}=a_1^2,\qquad \sum_{i=1}^{a_1-1}\zeta^{-i(l+1)}=\begin{cases}a_1-1 & if\ \ a_1|l+1\\ -1 & if\ \ a_1\not|l+1\end{cases}.$$

Proof of Claim. We have

$$\prod_{i=1}^{a_1-1}(-\zeta^i)=(-1)^{a_1-1}\zeta^{(a_1-1)a_1/2}=1.$$

On the other hand, we have

$$(z-\zeta^{-1})\cdots(z-\zeta^{-(a_1-1)}) = \frac{z^{a_1}-1}{z-1} = z^{a_1-1}+\cdots+z+1.$$

Set $z=1$, then we have

$$\prod_{i=1}^{a_1-1}(\zeta^{-i}-1)^2 = a_1^2.$$

Therefore, we obtain

$$\prod_{i=1}^{a_1-1}\{-\zeta^i(\zeta^{-i}-1)^2\} = a_1^2.$$

We have

$$\sum_{i=1}^{a_1-1}\zeta^{-i(l+1)} = \sum_{i=1}^{a_1-1}(\zeta^{l+1})^{-i} = \sum_{i=1}^{a_1-1}(\zeta^{l+1})^{i} = \begin{cases} a_1-1 & if\ \ a_1|l+1 \\ -1 & if\ \ a_1 \not| l+1 \end{cases}.$$

□

On the other hand, from Proposition 10, we have

$$\frac{(x_1(p_2)-x_1(p_1))^2}{\det G_1(p_1)\det G_1(p_2)} = \frac{1}{a_1^2}(t_xt_y)^{2g-a_1-1}(t_x^{a_1}-t_y^{a_1})^2\left(1+\sum_{i=1}^{\infty}c_i't_x^i\right)\left(1+\sum_{i=1}^{\infty}c_i't_y^i\right),$$

where c_i' is that in Proposition 10 (ii). The assertions for q_{kl} follows from these expressions and Proposition 10.

(ii) The assertions follows from (i) and the definition of $\tilde{E}(\tilde{\infty},\tilde{p})$.

□

Proof of Theorem 34. The method of proof is similar to the case of (n,s) curves. (cf. [21], p.204, Proof of Theorem 3).

(i): Let $t_i = t(p_i)$. From Lemma 36, we have

$$\frac{\prod_{i=1}^N \tilde{E}(\tilde{\infty},\tilde{p}_i)^N}{\prod_{i<j}\tilde{E}(\tilde{p}_i,\tilde{p}_j)} = \frac{(\prod_{i=1}^N t_i)^{N+g-1}}{\prod_{i<j}(t_i-t_j)}(1+\sum_{k_1+\cdots+k_N\geq 1}\tilde{c}_{k_1,\ldots,k_N}t_1^{k_1}\cdots t_N^{k_N}),$$

where $\tilde{c}_{k_1,\ldots,k_N}\in\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and $\deg\tilde{c}_{k_1,\ldots,k_N} = \sum_{i=1}^N k_i$. From Proposition 10 (i), for $N>g$, we have

$$\begin{aligned}&(\varphi_1(t),\ldots,\varphi_N(t))\\ &= (1,\frac{1}{t^{w_2^*}}(1+O(t)),\ldots,\frac{1}{t^{w_g^*}}(1+O(t)),\frac{1}{t^{2g}}(1+O(t)),\ldots,\frac{1}{t^{N+g-1}}(1+O(t))),\end{aligned}$$

where all $O(t)$ parts are series in t with the coefficients in $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and are homogeneous of degree 0 with respect to $\{t,\lambda^{(i)}_{j_1\cdots j_m}\}$, where we define $\deg t=-1$. We have

$$(N+g-1,\ldots,N+g-1)+(0,-w_2^*,\ldots,-w_g^*,-2g,\ldots,-(N+g-1))$$

$$= (\mu(A_m)_1, \ldots, \mu(A_m)_g, 0, \ldots, 0) + (N-1, N-2, \ldots, 1, 0).$$

Let us denote the partition $(\mu(A_m), 0^{N-g})$ by $\mu^{(N)}(A_m)$. Then, we have

$$\frac{(\prod_{i=1}^N t_i)^{N+g-1}}{\prod_{i<j}(t_i - t_j)} \det(\varphi_i(t_j))_{1\le i,j\le N} = s_{\mu^{(N)}(A_m)}(t_1, \ldots, t_N) + \sum \widehat{c}_{k_1,\ldots,k_N} t_1^{k_1} \cdots t_N^{k_N}, \quad (32)$$

where $\widehat{c}_{k_1,\ldots,k_N} \in \mathbb{Q}[\{\lambda^{(i)}_{j_1\ldots j_m}\}]$, $\deg \widehat{c}_{k_1,\ldots,k_N} = -|\mu(A_m)| + \sum_{i=1}^N k_i$, and the summation is taken for k_i's satisfying $\sum_{i=1}^N k_i > |\mu(A_m)|$. From Proposition 10, we have

$$\int_{\tilde{\infty}}^{\tilde{p}} du_i = \frac{t^{w_i}}{w_i} + \sum_{j=1}^{\infty} c_{ij} t^{j+w_i}, \quad c_{ij} \in \mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}], \quad \deg c_{ij} = j.$$

Let

$$T_k = T_k(t_1, \ldots, t_N) = \frac{\sum_{j=1}^N t_j^k}{k}.$$

Then, $T_1, \ldots, T_N$ are algebraically independent and become a generator of the ring of symmetric polynomials of $t_1, \ldots, t_N$ with the coefficients in $\mathbb{Q}$,

$$\mathbb{Q}[t_1, \ldots, t_N]^{S_N} = \mathbb{Q}[T_1, \ldots, T_N].$$

Moereover, if we prescribe degree for t_i and T_i by

$$\deg t_i = -1, \quad \deg T_i = -i,$$

a symmetric homogeneous polynomial of $t_1, \ldots, t_N$ of degree k can be uniquely written as a homogeneous polynomial of $T_1, \ldots, T_N$ of degree k.

We have

$$\begin{aligned} u_i = \sum_{k=1}^N \int_{\tilde{\infty}}^{\tilde{p}_k} du_i &= T_{w_i} + \sum_{j=1}^{\infty} (j + w_i) c_{ij} T_{j+w_i} \\ &= T_{w_i} + \sum_{\sum jk_j > w_i} \tilde{c}^{(i)}_{k_1,\ldots,k_N} T_1^{k_1} \cdots T_N^{k_N}, \end{aligned} \quad (33)$$

where $\tilde{c}^{(i)}_{k_1,\ldots,k_N} \in \mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$, $\deg \tilde{c}^{(i)}_{k_1,\ldots,k_N} = -w_i + \sum jk_j$, and the second expression is unique.

Let us take $N \ge w_g$. Then, $T_{w_1}, \ldots, T_{w_g}$ are algebraically independent. Let

$$\sigma(u) = \sigma(u_1, \ldots, u_g) = \sum b_{k_1,\ldots,k_g} u_1^{k_1} \cdots u_g^{k_g}, \quad b_{k_1,\ldots,k_g} \in \mathbb{C},$$

be the series expansion around the origin. From (29) (32) (33) and the fact that $T_{w_1}, \ldots, T_{w_g}$ are algebraically independent, we have

$$b_{k_1,\ldots,k_g} = 0$$

if $w_1k_1+\cdots+w_gk_g<|\mu(A_m)|$. Let

$$G(u_1,\dots,u_g)=\sum_{w_1k_1+\cdots+w_gk_g=\mu(A_m)} b_{k_1,\dots,k_g}u_1^{k_1}\cdots u_g^{k_g}.$$

Then, we have

$$G\left(\frac{\sum_{j=1}^N t_j^{w_1}}{w_1},\dots,\frac{\sum_{j=1}^N t_j^{w_g}}{w_g}\right)=s_{\mu^{(N)}(A_m)}(t_1,\dots,t_N).$$

Therefore, we have

$$G(T_{w_1},\dots,T_{w_g})=S_{\mu(A_m)}(T).$$

Since $T_{w_1},\dots,T_{w_g}$ are algebraically independent, $G(u_1,\dots,u_g)=S_{\mu(A_m)}(T)|_{T_{w_i}=u_i}$.

Take $k>|\mu(A_m)|$. Suppose that $b_{k_1,\dots,k_g}$ belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $\sum_{i=1}^g w_ik_i-|\mu(A_m)|$ if $w_1k_1+\cdots+w_gk_g<k$. Take $(k_1,\dots,k_g)$ satisfying $w_1k_1+\cdots+w_gk_g=k$. Let ϵ be the coefficient of $T_{w_1}^{k_1}\cdots T_{w_g}^{k_g}$ in

$$\sum_{w_1l_1+\cdots+w_gl_g<k} b_{l_1,\dots,l_g}(T_{w_1}+\sum_{\sum js_j>w_1}\tilde{c}^{(1)}_{s_1,\dots,s_N}T_1^{s_1}\cdots T_N^{s_N})^{l_1}\cdots(T_{w_g}+\sum_{\sum js_j>w_g}\tilde{c}^{(g)}_{s_1,\dots,s_N}T_1^{s_1}\cdots T_N^{s_N})^{l_g}.$$

Then, ϵ belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $\sum_{i=1}^g w_ik_i-|\mu(A_m)|$. Express the right hand side of (32) by $T_1,\dots,T_N$ and let δ be the the coefficient of $T_{w_1}^{k_1}\cdots T_{w_g}^{k_g}$. Then, δ belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $\sum_{i=1}^g w_ik_i-|\mu(A_m)|$. From (29), we have $b_{k_1,\dots,k_g}+\epsilon=\delta$. Therefore, $b_{k_1,\dots,k_g}$ belongs to $\mathbb{Q}[\{\lambda^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree $\sum_{i=1}^g w_ik_i-|\mu(A_m)|$.

(iii) Since Riemann's constant $\tau\delta'+\delta''$ is a half period, from (1) and Definition 23, $\sigma(u)$ is even or odd. From Lemma 31 and Proposition 32, we obtain the desired assertion.

□

11 Example: (4,6,5) curve

In this section, as an example, we consider the sigma function for a special case of (4,6,5) curves. Let X be the $(4,6,5)$ curve defined by

$$x_2^2=x_1^3+1,\quad x_3^2=x_1x_2.$$

One can check that X is nonsingular. The genus of X is 4. The holomorphic one forms are given by

$$du_1(x) = -\frac{x_2}{4x_2x_3}dx_1, \quad du_2(x) = -\frac{x_3}{4x_2x_3}dx_1,$$
$$du_3(x) = -\frac{x_1}{4x_2x_3}dx_1, \quad du_4(x) = -\frac{1}{4x_2x_3}dx_1.$$

An algebraic symmetric bilinear form $\widehat{\omega}(x,y)$ and the second kind differentials $\{dr_i\}_{i=1}^4$ are given by

$$\widehat{\omega}(x,y) = \frac{x_1^2y_1^2y_2 + x_1^2x_2y_1^2 + 3x_1^3y_1y_2 + 3x_1x_2y_1^3 + 2x_1^2x_3y_1y_3 + 2x_1x_3y_1^2y_3}{16x_2x_3y_2y_3(x_1-y_1)^2}dx_1dy_1$$
$$+\frac{4x_2x_3y_2y_3 + 4x_3y_3 + 2x_1x_2 + 2y_1y_2 + 2x_1y_2 + 2x_2y_1}{16x_2x_3y_2y_3(x_1-y_1)^2}dx_1dy_1,$$

and

$$dr_1(y) = -\frac{y_1^2}{4y_2y_3}dy_1, \quad dr_2(y) = -\frac{2y_1y_3}{4y_2y_3}dy_1,$$
$$dr_3(y) = -\frac{3y_1y_2}{4y_2y_3}dy_1, \quad dr_4(y) = -\frac{7y_1^2y_2}{4y_2y_3}dy_1.$$

The sigma function for X can be expanded around the origin as

$$\sigma(u) = u_1u_3^2 - \frac{1}{3}u_1^3u_2^2 - \frac{1}{4}u_1^4u_3 + \frac{3}{2}u_2^2u_3 - \frac{251}{252}u_1^7 + \cdots.$$

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