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KNOTTED FIXED POINT SETS OF SEMI-FREE S^1-ACTIONS

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1. Introduction

In [2], Browder has shown that there are an infinite number of distinct semi-free S^1-actions on homotopy $(p+2q)$-spheres with S^p as untwisted fixed point set if (a) $p+2q\equiv 1 \bmod 4$, $p>1$ and $q>2$, or if (b) $p+2q=7$, 15 or 31, p: odd, $p>1$ and $q>1$. As open questions, he has posed the followings:

(I) What is the knot type of the fixed point set?

(II) In the cases where his theorem does not construct an infinite number of semi-free S^1-actions, are there in reality only a finite number?

In the present paper, we shall give partial answers on these questions as follows. We shall construct semi-free S^1-actions which have knotted fixed point sets (see Theorem 2.1). As a corollary, we shall have that there are also an infinite number of distinct semi-free S^1-actions on the standard $(p+2q)$-sphere S^{p+2q} with knotted S^p as fixed point set when $p\equiv 3 \bmod 4$ and $4q\leqq p+3$ (see Theorem 2.2).

2. Definitions, notations and statement of results

An action (M, φ, G) is called *semi-free* if it is free outside the fixed point set, i.e., there are two types of orbits, fixed points and G. Let Θ_n be the group of homotopy n-spheres and θ_n be the order of the group Θ_n. Let $\Theta_n(\partial\pi)$ be the subgroup consisting of those homotopy spheres which bound parallelizable manifolds and $\sum_M^n$ be the generator of $\Theta_n(\partial\pi)$ due to Kervaire and Milnor [7] (see also Milnor [9] and Kervaire [5]). D^n and S^{n-1} denote, respectively, the unit disk and the unit sphere in euclidean n-space. When N is a submanifold of M, we shall denote by $\nu(N\subset M)$ the normal bundle of N in M. When a homotopy sphere $\sum^p$ imbedded in $\sum^{p+2q}$ bounds a manifold W^{p+1} in $\sum^{p+2q}$ such that the normal bundle $\nu(W^{p+1}\subset\sum^{p+2q})$ is trivial, we say that $\sum^p$ *bounds a π-submanifold* W^{p+1} in $\sum^{p+2q}$. In [9], Milnor has constructed a manifold W_0^{4k} $(k\geqq 2)$ which satisfies: (1) W_0^{4k} is parallelizable, (2) the index $I(W_0^{4k})$ equals 8, (3) the boundary ∂W_0^{4k} is the homotopy sphere $\sum_M^{4k-1}$ and (4) W_0^{4k} is $(2k-1)$-connected. Let us denote by $W^{4k}(l)$ for $l\in Z$ the manifold obtained by the boundary connected sum

$W_0^{4k} \natural \cdots \natural W_0^{4k}$ of l-copies of the manifold W_0^{4k}. It is clear that the index $I(W^{4k}(l))$ equals $8l$. Then we shall have the following:

Theorem 2.1. *There exists a semi-free S^1-action on a homotopy sphere $\sum^{p+2q}$ with fixed point set $(\prod_{i=2}^{q-1} \theta_{p+2i}) \cdot \sum_M^p$ which bounds a π-submanifold $W^{p+1}(\prod_{i=2}^{q-1} \theta_{p+2i})$ in $\sum^{p+2q}$ for $p \equiv 3 \pmod 4$, $p \geqq 7$ and $q \geqq 2$.*

Theorem 2.2. *There are an infinite number of distinct semi-free S^1-actions on the standard $(p+2q)$-sphere S^{p+2q} with knotted S^p as fixed point set for $p \equiv 3 \pmod 4$, $4q \leqq p+3$ and $q \geqq 2$.*

3. Proofs of theorems

Proof of Theorem 2.1. As is well-known, the homotopy sphere $\sum_M^p$ can be imbedded in S^{p+2} such that $\sum_M^p$ bounds a π-submanifold W_0^{p+1} of index 8 in S^{p+2} (see Kervaire [6, Theorem 1 of Appendix] and Milnor [9]). Hence, by the natural inclusion $S^{p+2} \subset S^{p+3}$, we can embed $\sum_M^p$ in S^{p+3} such that $\sum_M^p$ bounds a π-submanifold W_0^{p+1} of index 8 in S^{p+3}. Let a be a point of S^2. Then it is easy to prove that there is a diffeomorphism

$$f: \textstyle\sum_M^p \times S^2 \longrightarrow S^p \times S^2$$

such that $f(\sum_M^p \times a)$ bounds the π-submanifold W_0^{p+1} in $D^{p+1} \times S^2$ when we regard $S^p \times S^2$ as $\partial(D^{p+1} \times S^2)$.

Let

$$\xi_N: S^1 \longrightarrow S^{2N+1} \xrightarrow{\pi} CP^N$$

be the classical Hopf bundle. Let $i: S^2 \rightarrow CP^N$ be the inclusion of the 2-skeleton of CP^N, then it is clear that $i^! \xi_N = \xi_1$. Let $p_2: S^p \times S^2 \rightarrow S^2$ and $p_2': \sum_M^p \times S^2 \rightarrow S^2$ be projections. Since CP^N is the $2N$-skeleton of the Eilenberg MacLane complex $K(Z, 2)$, $ip_2 f$ is homotopic to ip_2' for $N > p+2$. Hence there exists a bundle map

$$\tilde{f}: (ip_2')^! \xi_N \longrightarrow (ip_2)^! \xi_N ,$$

i.e., we have a bundle map

$$\tilde{f}: p_2'^! \xi_1 \longrightarrow p_2^! \xi_1 .$$

Thus we obtain the following commutative diagram

$$\begin{array}{ccc}
\sum_M^p \times S^3 & \xrightarrow{\tilde{f}} & S^p \times S^3 \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
\sum_M^p \times S^2 & \xrightarrow{f} & S^p \times S^2
\end{array}$$

where $p: S^p\times S^3\to S^p\times S^2$ (resp. $p': \Sigma_M^p\times S^3\to\Sigma_M^p\times S^2$) denotes the projection of the bundle $p_2{}^!\xi_1$ (resp. $p_2'{}^!\xi_1$). Set $\Sigma^{p+4}=\Sigma_M^p\times D^4\underset{\tilde f}{\cup} D^{p+1}\times S^3$. It is easy to prove that Σ^{p+4} is a homotopy sphere. Let $(\Sigma^{p+4}, \varphi, S^1)$ be the semi-free S^1-action defined by

$$\varphi(g, (x, y)) = (x, gy) \qquad \text{for} \quad x\in\Sigma_M^p,\ y\in D^4$$

and

$$\varphi(g, (x, y)) = (x, gy) \qquad \text{for} \quad x\in D^{p+1},\ y\in S^3\,.$$

Now we prove that the fixed point set $\Sigma_M^p\times\{0\}$ of the action $(\Sigma^{p+4}, \varphi, S^1)$ bounds a π-submanifold W_0 in Σ^{p+4}. Let $\bar p_2: D^{p+1}\times S^2\to S^2$ be the projection and $\bar p: D^{p+1}\times S^3\to D^{p+1}\times S^2$ be the projection of the bundle $\bar p_2{}^!\xi_1$. Since the manifold W_0 is $(p-1)/2$-connected, the restriction of the bundle $\bar p_2{}^!\xi_1$ to W_0 is trivial, i.e., $\bar p^{-1}(W_0)=W_0\times S^1$. It is obvious by definition that $p'^{-1}(\Sigma_M^p\times a)=\Sigma_M^p\times S^1$. Let b be a point of $\pi^{-1}(a)\subset S^3$. It follows from Lemma 2 of Browder [1] (see also Browder and Levine [3]) that the diffeomorphism

$$\tilde f|p'^{-1}(\Sigma_M^p\times a): \Sigma_M^p\times S^1 \longrightarrow f(\Sigma_M^p\times a)\times S^1$$

is pseudo isotopic to a diffeomorphism sending $\Sigma_M^p\times b$ into

$$f(\Sigma_M^p\times a)\times c\ (\subset f(\Sigma_M^p\times a)\times S^1 = p^{-1}(f(\Sigma_M^p\times a)))$$

where c is a point of S^1. Hence $\tilde f(\Sigma_M^p\times b)$ bounds the submanifold W_0 in $\bar p^{-1}(W_0)=W_0\times S^1$. Since the normal bundle of W_0 in $D^{p+1}\times S^3$ is isomorphic to

$$\nu(W_0\subset W_0\times S^1)\oplus\nu(W_0\subset D^{p+1}\times S^2)$$

where $W_0\subset W_0\times S^1$, $W_0\subset D^{p+1}\times S^2$ are the embeddings defined above, W_0 has a normal frame in $D^{p+1}\times S^3$. Let $C:\Sigma_M^p\times I\to\Sigma_M^p\times D^4$ be the embedding defined by $C(x, t)= (x, tb)$ for $x\in\Sigma_M^p$, $t\in I$. By making use of the embedding C and the fact $\Sigma_M^p\times I\cup W_0=W_0$, we have that the fixed point set $\Sigma_M^p\times\{0\}$ bounds a π-submanifold W_0 in $\Sigma^{p+4}=\Sigma_M^p\times D^4\underset{\tilde f}{\cup} D^{p+1}\times S^3$.

Thus we have proved the following step 1 of induction.

Step. 1. There exists a semi-free S^1-action $(\Sigma^{p+4}, \varphi, S^1)$ with fixed point set Σ_M^p which bounds a π-submanifold W_0^{p+1} in Σ^{p+4}.

Step 2. Suppose there exists a semi-free S^1-action $(\Sigma^{p+2q}, \varphi, S^1)$ with fixed point set $(\prod_{i=2}^{q-1}\theta_{p+2i})\cdot\Sigma_M^p$ which bounds a π-submanifold $W^{p+1}(\prod_{i=2}^{q-1}\theta_{p+2i})$ in Σ^{p+2q} for $q\geqq 2$.

Then by the equivariant connected sum

$$(\Sigma^{p+2q}, \varphi, S^1)\#\cdots\#(\Sigma^{p+2q}, \varphi, S^1)$$

of θ_{p+2q}-copies of $(\sum^{p+2q}, \varphi, S^1)$ we have the following

Lemma 3.1. *There exists a semi-free S^1-action (S^{p+2q}, ψ, S^1) with fixed point set $(\prod_{i=2}^{q} \theta_{p+2i})\cdot\sum_M^p$ which bounds a π-submanifold $W^{p+1}(\prod_{i=2}^{q} \theta_{p+2i})$ in S^{p+2q}.*

According to Browder [2] there exists an equivariant diffeomorphism $f: (\prod_{i=2}^{q} \theta_{p+2i})\cdot\sum_M^p \times S^{2q-1} \to S^p \times S^{2q-1}$ such that $((\prod_{i=2}^{q} \theta_{p+2i})\sum_M^p \times D^{2q} \cup_f D^{p+1} \times S^{2q-1}, \bar{\psi}, S^1)$ is equivalent to (S^{p+2q}, ψ, S^1) where the action $\bar{\psi}$ is defined by

$$\bar{\psi}(g, (x, y)) = (x, gy) \qquad \text{for} \quad x \in (\prod_{i=2}^{q} \theta_{p+2i})\sum_M^p, y \in D^{2q}$$

and

$$\bar{\psi}(g, (x, y)) = (x, gy) \qquad \text{for} \quad x \in D^{p+1}, y \in S^{2q-1}.$$

Since $(\prod_{i=2}^{q} \theta_{p+2i})\sum_M^p \times D^{2q} \cup_f D^{p+1} \times S^{2q-1}$ is diffeomorphic to S^{p+2q}, we have the following lemma (c.f. Lemma 4.1 of Kawakubo [4]).

Lemma 3.2. *As an equivariant diffeomorphism*

$$f: (\prod_{i=2}^{q} \theta_{p+2i})\sum_M^p \times S^{2q-1} \longrightarrow S^p \times S^{2q-1},$$

we can choose one which can be extended to a diffeomorphism

$$F: (\prod_{i=2}^{q} \theta_{p+2i})\sum_M^p \times D^{2q} \longrightarrow S^p \times D^{2q}.$$

Now we construct an equivariant diffeomorphism

$$\hat{f}: ((\prod_{i=2}^{q} \theta_{p+2i})\cdot\sum_M^p \times S^{2q+1}, \varphi_1, S^1) \longrightarrow (S^p \times S^{2q+1}, \varphi_2, S^1)$$

where the actions φ_1 and φ_2 are the obvious ones.

Let us denote by

$$((\prod_{i=2}^{q} \theta_{p+2i})\cdot\sum_M^p \times S^{2q-1} \times D^2 \cup_{id} (\prod_{i=2}^{q} \theta_{p+2i})\cdot\sum_M^p \times D^{2q} \times S^1, \bar{\varphi}_1, S^1)$$

the differentiable S^1-action defined by

$$\bar{\varphi}_1(g, (x, y, z)) = (x, gy, gz) \qquad \text{for} \quad x \in (\prod_{i=2}^{q} \theta_{p+2i})\cdot\sum_M^p, \quad y \in S^{2q-1}, \quad z \in D^2,$$

and

$$\bar{\varphi}_1(g, (x, y, z)) = (x, gy, gz) \qquad \text{for} \quad x \in (\prod_{i=2}^{g} \theta_{p+2i})\cdot\sum_M^p, \quad y \in D^{2q}, \quad z \in S^1.$$

Let us denote by

$$(S^p\times S^{2q-1}\times D^2 \underset{id}{\cup} S^p\times D^{2q}\times S^1,\ \bar{\varphi}_2,\ S^1)$$

the similar differentiable S^1-action. Since

$$((\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times S^{2q-1}\times D^2 \underset{id}{\cup} (\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times D^{2q}\times S^1,\ \bar{\varphi}_1,\ S^1)$$
$$(\text{resp. } (S^p\times S^{2q-1}\times D^2 \underset{id}{\cup} S^p\times D^{2q}\times S^1,\ \bar{\varphi}_2,\ S^1))$$

is clearly equivalent to

$$((\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times S^{2q+1},\ \varphi_1,\ S^1)$$
$$(\text{resp. } (S^p\times S^{2q+1},\ \varphi_2,\ S^1)),$$

we use them confusedly. Let $F_1\colon (\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times D^{2q}\to S^p$ and $F_2\colon (\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times D^{2q}\to D^{2q}$ be the differentiable maps defined by

$$(F_1(x, y),\ F_2(x, y)) = F(x, y) \qquad \text{for} \quad x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p,$$
$$y\in D^{2q},$$

then we construct an equivariant diffeomorphism

$$\hat{f}\colon (\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times S^{2q+1}\longrightarrow S^p\times S^{2q+1}$$

by

$$\hat{f}\,|\,(\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times S^{2q-1}\times D^2 = f\times id$$

and

$$\hat{f}(x, y, z) = \Big(F_1(x, z^{-1}y),\ zF_2(x, z^{-1}y),\ z\Big)$$
$$\text{for} \quad x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p,\quad y\in D^{2q},\quad z\in S^1.$$

Lemma 3.3. *$\hat{f}$ is well-defined and an equivariant diffeomorphism.*

Proof of Lemma 3.3. First we shall prove that $\hat{f}$ is well-defined. Let $f_1\colon (\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times S^{2q-1}\to S^p$ and $f_2\colon (\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p\times S^{2q-1}\to S^{2q-1}$ be differentiable maps defined by

$$(f_1(x, y), f_2(x, y)) = f(x, y) \qquad \text{for} \quad x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\Sigma_M^p,\quad y\in S^{2q-1}.$$

Since f is equivariant, $f_1(x, gy)=f_1(x, y)$ and $f_2(x, gy)=gf_2(x, y)$ for $x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p$, $y\in S^{2q-1}$. Hence, for $x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p$, $y\in\partial D^{2q}=S^{2q-1}$, $z\in S^1$, we have that $F_1(x, z^{-1}y)=f_1(x, z^{-1}y)=f_1(x, y)$ and $zF_2(x, z^{-1}y)=zf_2(x, z^{-1}y)=f_2(x, y)$, i.e., $\hat{f}$ is well-defined. If we take F carefully, $\hat{f}$ becomes a differentiable map.

Secondly, we shall prove that $\hat{f}$ is equivariant. Obviously $\hat{f}|(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times S^{2q-1}\times D^2$ is equivariant. For $x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p$, $y\in D^{2q}$, $z\in S^1$,

$$\begin{aligned}
&\hat{f}(\varphi_1(g, (x, y, z)))\\
&\quad=\hat{f}(x, gy, gz)\\
&\quad=(F_1(x, (gz)^{-1}gy), gzF_2(x, (gz)^{-1}gy), gz)\\
&\quad=(F_1(x, z^{-1}y), gzF_2(x, z^{-1}y), gz)\\
&\quad=\varphi_2(g, (F_1(x, z^{-1}y), zF_2(x, z^{-1}y), z))\\
&\quad=\varphi_2(g, \hat{f}(x, y, z))\,,
\end{aligned}$$

i.e., $\hat{f}$ is equivariant.

Thirdly, we shall prove that $\hat{f}$ is a diffeomorphism. For this purpose, we show that $\hat{f}$ has a differentiable inverse map. Let $\bar{F}_1\colon S^p\times D^{2q}\to(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p$ and $\bar{F}_2\colon S^p\times D^{2q}\to D^{2q}$ be the differentiable maps defined by

$$(\bar{F}_1(x, y), \bar{F}_2(x, y))=F^{-1}(x, y)\qquad\text{for}\quad x\in S^p,\ y\in D^{2q}\,.$$

Define a differentiable map

$$\hat{\hat{f}}\colon S^p\times S^{2q+1}\longrightarrow(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times S^{2q+1}$$

by

$$\hat{\hat{f}}|S^p\times S^{2q-1}\times D^2=f^{-1}\times id$$

and

$$\hat{\hat{f}}(x, y, z)=(\bar{F}_1(x, z^{-1}y), z\bar{F}_2(x, z^{-1}y), z)$$

$$\text{for}\quad x\in S^p,\quad y\in D^{2q},\quad z\in S^1\,.$$

It is easy to prove by the same way as in the case of $\hat{f}$ that $\hat{\hat{f}}$ is well-defined and a differentiable map., It is clear that

$$\hat{\hat{f}}\circ\hat{f}|(\prod_{i=2}^{q}\theta_{p+2i})\sum_M^p\times S^{2q-1}\times D^2=id\,.$$

For $x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p$, $y\in D^{2q}$, $z\in S^1$,

$$
\begin{aligned}
&\hat{\hat{f}}\circ\hat{f}(x, y, z)\\
&\quad=\hat{\hat{f}}(F_1(x, z^{-1}y), zF_2(x, z^{-1}y), z)\\
&\quad=(\bar{F}_1(F_1(x, z^{-1}y), z^{-1}(zF_2(x, z^{-1}y))), z\bar{F}_2(F_1(x, z^{-1}y), z^{-1}(zF_2(x, z^{-1}y))), z)\\
&\quad=(\bar{F}_1(F_1(x, z^{-1}y), F_2(x, z^{-1}y)), z\bar{F}_2(F_1(x, z^{-1}y), F_2(x, z^{-1}y)), z)\\
&\quad=(x, z(z^{-1}y), z)\\
&\quad=(x, y, z)\,,
\end{aligned}
$$

i.e., $\hat{\hat{f}}\circ\hat{f}=$identity.

Similarly we can prove that $\hat{f}\circ\hat{\hat{f}}=$identity. Hence f is a diffeomorphism. This completes the proof of Lemma 3.3.

Set $\sum^{p+2q+2}=(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times D^{2q+2}\underset{\hat{f}}{\cup} D^{p+1}\times S^{2q+1}.$ It is easy to prove that $\sum^{p+2q+2}$ is a homotopy sphere. Then we construct a semi-free S^1-action $(\sum^{p+2q+2}, \hat{\phi}\ S^1)$ by

$$\hat{\phi}(g, (x, y))=(x, gy) \qquad \text{for} \quad x\in(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p,\ y\in D^{2q+2}$$

and

$$\hat{\phi}(g, (x, y))=(x, gy) \qquad \text{for} \quad x\in D^{p+1},\ y\in S^{2q+1}\,.$$

Since $\hat{f}$ is equivariant with respect to $\hat{\phi}$, the above action is well-defined. Regarding S^{p+2q} as $(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times D^{2q}\underset{f}{\cup} D^{p+1}\times S^{2q-1}$ and $\sum^{p+2q+2}$ as $(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times D^{2q}\times D^2\underset{f}{\cup}(D^{p+1}\times S^{2q-1}\times D^2\underset{id}{\cup} D^{p+1}\times D^{2q}\times S^1)$,

we obtain an embedding $e\colon S^{p+2q}\to\sum^{p+2q+2}$ by identifying

$$(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times D^{2q} \quad \text{with} \quad (\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p\times D^{2q}\times\{0\}$$

and

$$D^{p+1}\times S^{2q-1} \quad \text{with} \quad D^{p+1}\times S^{2q-1}\times\{0\}\,.$$

It is clear that the embedding e is well-defined and equivariant with respect to ψ and $\bar{\varphi}$ by definition, i.e., (S^{p+2q}, ψ, S^1) is an invariant submanifold of $(\sum^{p+2q+2}, \hat{\phi}, S^1)$. Since S^{p+2q} is $(p+2q-1)$-connected, $\nu(e(S^{p+2q})\subset\sum^{p+2q+2})$ is trivial and since the normal bundle of $e(W^{p+1}(\prod_{i=2}^{q}\theta_{p+2i}))$ in $\sum^{p+2q+2}$ is isomorphic to $\nu(W^{p+1}(\prod_{i=2}^{q}\theta_{p+2i})\subset S^{p+2q})\oplus\nu(e(S^{p+2q})\subset\sum^{p+2q+2})\,|\,e(W^{p+1}(\prod_{i=2}^{q}\theta_{p+2i}))$, the normal bundle $\nu(e(W^{p+1}(\prod_{i=2}^{q}\theta_{p+2i}))\subset\sum^{p+2q+2})$ is trivial. Thus we have proved that there exists a semi-free S^1-action $(\sum^{p+2q+2}, \hat{\phi}, S^1)$ with fixed point set $(\prod_{i=2}^{q}\theta_{p+2i})\cdot\sum_M^p$ which bounds a π-submanifold $W^{p+1}(\prod_{i=2}^{q}\theta_{p+2i})$ in $\sum^{p+2q+2}$, completing the induction.

This makes the proof of Theorem 2.1 complete.

Proof of Theorem 2.2. It follows from Theorem 2.1 that there exists a semi-free S^1-action (S^{p+2q}, φ, S^1) with fixed point set the natural sphere S^p which bounds a π-submanifold of non zero index constructed by the equivariant connected sum operation with itself. Denote by $l(S^{p+2q}, \varphi, S^1)$ the action induced by the equivariant connected sum

$$(S^{p+2q}, \varphi, S^1)\# \cdots \#(S^{p+2q}, \varphi, S^1)$$

of l-copies of (S^{p+2q}, φ, S^1). Because of the difference of the indices of the π-submanifolds bounded by the fixed point sets, $l(S^{p+2q}, \varphi, S^1)$ is not equivalent to $m(S^{p+2q}, \varphi, S^1)$ for $l \neq m$ (see Levine [8 Theorem 6.7]). This completes the proof of Theorem 2.2.

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