Watanabe Y.
Osaka J. Math.
4 (1967) 227-231

THE DEDEKIND DIFFERENT AND THE HOMOLOGICAL DIFFERENT

YUTAKA WATANABE

(Received August 14, 1967)

We intend, in this paper, to define the Dedekind different of an algebra over a commutative ring and to study the properties of this different. S. Endo and the author [4] defined the reduced trace of a central separable algebra over a commutative ring. Using this reduced trace, we can define as usual the Dedekind different of an algebra. Let R be a commutative ring and A be an R-algebra which is a finitely generated projective R-module. We assume that $\mathfrak{A}=K\otimes A$ is a central separable K-algebra, where K is the total quotient ring of R. Let t be the reduced trace of $\mathfrak{A}$. The two sided A-submodule $C=\{x\in\mathfrak{A}\mid t(xA)\subset R\}$ of $\mathfrak{A}$ is called the *complementary module* of A, and $D=\{x\in\mathfrak{A}\mid xC\subset A\}$ is called the *Dedekind different* of A. D is also a two sided A-module. We shall first show that the reduced trace induces an epimorphism of the Dedekind different to the homological different which was defined in [2]. This fact was shown by Fossum in the case that R is an integrally closed Noetherian domain ([5]). Secondly we shall give a complete generalization of DeMeyer's theorem ([3], Theorem 4), and finally we shall give a generalization of the "different theorem" on maximal orders over Dedekind domains in central simple algebras. Throughout this note we assume that rings have unit elements, that modules are unitary and that algebras are finitely generated as modules.

1. Let R be a commutative ring, A be an R-algebra and A^e be the enveloping algebra of A. $J(A)$ (or briefly, J) denotes the kernel of the canonical A^e-epimorphism φ: $A^e\to A$ given by $\varphi(x\otimes y^0)=xy$, and $N(A)$ (or briefly, N) denotes the right annihilator of $J(A)$ in A^e.

Lemma 1. *Let A be a full matrix algebra of degree n over R. N is an R-free submodule of A^e with basis $\sum_{i=1}^{n}e_{ij}\otimes e^0_{ki}$, $1\leq j, k\leq n$, where e_{ij} denotes the (i, j)-matrix unit.*

Proof. Let $\alpha=\sum_{ijkl}a_{ijkl}(e_{ij}\otimes e^o_{kl})$ be in N. Since $e_{rr}\otimes 1^0-1\otimes e^o_{rr}$ is in J, it is annihilated by α, so we get $\sum_{jkl}a_{rjkl}(e_{rj}\otimes e^o_{kl})=\sum_{ijk}a_{ijkr}(e_{ij}\otimes e^o_{kr})$. Hence, $i\neq l$ implies $a_{ijkl}=0$. So, α is expressed as $\alpha=\sum_{ijk}a_{ijk}(e_{ij}\otimes e^o_{ki})$. Again, by the property of N, α annihilates $e_{rs}\otimes 1^{\circ}-1\otimes e^{\circ}_{rs}$, so we get $\sum a_{sjk}(e_{rj}\otimes e^{\circ}_{ks})=\sum a_{rjk}(e_{rj}\otimes e^{\circ}_{ks})$. Hence we get $a_{sjk}=a_{rjk}$ for all r,s. Therefore α is expressed by the form $\alpha=\sum_{ij} a_{jk}(\sum_i e_{ij}\otimes e^{\circ}_{ki})$. The fact that $\sum_i e_{ij}\otimes e^{\circ}_{ki}$ is in N can be shown straightforward.

Next, we assume that A is a central separable R-algebra.

Lemma 2. *Let A be an R-central separable algebra and t be the reduced trace of A. For any element $\sum x_i\otimes y^{\circ}_i$ in N, the following identities hold :*

$$\sum x_i a y_i = \sum t(ax_i)t(y_i) = \sum t(x_i)t(ay_i)$$

for all $a\in A$.

Proof. We first assume that R is a quasi-local ring in the sense of [7]. So A has a proper splitting ring S such that $S\otimes A$ is a full matrix algebra over S; $S\otimes A=(S)_d$, where a proper splitting ring means a splitting ring which contains R (See [4]). Since $N(A)$ is contained in $N(S\otimes A)$, and t is R-linear, we have only to show by Lemma 1 that $\sum_i e_{ik}ae_{li}=(\sum_i t(ae_{ik})t(e_{li}))I$, where I denotes the unit matrix. We denote by a_{rs} the (r, s)-component of a matrix a, so $\sum_i e_{ik}ae_{li}=\sum_i a_{kl}e_{ii}=a_{kl}\, I$. While $t(ae_{ik})=a_{ki}$, and $t(e_{li})=\delta_{li}$ (Kronecker's delta). So, the right term is $(\sum_i a_{ki}\delta_{li})I=a_{kl}I=$the left term. In the case that R is global, by the localization argument, there exists $c\notin\mathfrak{m}$ such that $c(\sum x_i ay_i)=c(\sum t(ax_i)t(y_i))$ for any maximal ideal $\mathfrak{m}$ of R, because t induces the reduced trace of $A_{\mathfrak{m}}$. We put $\mathfrak{c}=\{c\in R\,|\, c(\sum x_i ay_i)=c(\sum t(ax_i)t(y_i))\}$. Then $\mathfrak{c}$ is an ideal of R which is not contained in any maximal ideal of R. So 1 is in $\mathfrak{c}$. Therefore the desired equalities hold.

2. Let A be a central projective R-algebra and K be the total quotient ring of R. We assume that $K\otimes A=\mathfrak{A}$ is a central separable K-algebra. Let t be the reduced trace of $\mathfrak{A}$ and let C, D be the complementary module and the Dedekind different of A respectively. Since R is not necessarily integrally closed (i.e. $t(A)$ is not necessarily contained in R), it can not always hold that $C\subset A$ nor $D\subset A$. We proved in [4] that a two sided $\mathfrak{A}$-homomorphism θ: $\mathfrak{A}\to\mathfrak{A}^*=\mathrm{Hom}_K(\mathfrak{A}, K)$ defined by $\theta(x)=xt=tx$ is an isomorphism (i.e. $\mathfrak{A}$ is a symmetric algebra). It immediately follows from the definition that θ induces a two sided A-isomorphism between C and $A^*=\mathrm{Hom}_R(A, R)$. So, C is A-projective if and only if A is a quasi-Frobenius algebra (see the definition in [6]). Clearly, C spans $\mathfrak{A}$ over K, so we get an isomorphism $D\cong\mathrm{Hom}^r_A(C, A)$[1]. So we get

1) Hom^r_A denotes the right A-homomorphisms functor. The converse relation $\mathrm{Hom}^l_A(D, A)\cong C$ holds if A is quasi-Frobenius. For, $\mathrm{Hom}^l_A(D, A)\cong\mathrm{Hom}^l_A(\mathrm{Hom}^r_A(C, A), A)\cong C\otimes_A \mathrm{Hom}^l_A(A, A)\cong C$.

Proposition 3. $\mathrm{Hom}^r_A(A^*, A) \cong D$

We shall describe explicitly this isomorphism. ψ denotes the isomorphism $\mathrm{Hom}^r_{\mathfrak{A}}(\mathfrak{A}^*, \mathfrak{A}) \to \mathfrak{A}$ given by $\psi(f)=f(t)$, $f \in \mathrm{Hom}^r_{\mathfrak{A}}(\mathfrak{A}^*, \mathfrak{A})$, then ψ induces the isomorphisms $\mathrm{Hom}^r_A(A^*, A) \cong D$.

The homomorphism $\tau: A^e \to \mathrm{Hom}_R(A^*, A)$ given by $\tau(x \otimes y^0)(f)=f(x)y$ is an isomorphism because of R-projectivity of A.

Proposition 4. *τ induces an isomorphism between $N(A)$ and* $\mathrm{Hom}^r_A(A^*, A)$

Proof. By the same argument of the discussion at the top of p. 757 of [1], we can show that $\tau(N)$ is contained in $\mathrm{Hom}^r_A(A^*, A)$. Conversely, if $\tau(\sum x_i \otimes y_i^0)$ is an A-homomorphism, we get $\tau(\sum zx_i \otimes y_i^0)(f)=\sum f(zx_i)y_i=\sum (fz)(x_i)y_i= \tau(\sum x_i \otimes y_i^0)(fz)=(\tau(\sum x_i \otimes y_i^0)(f))z=\sum f(x_i)y_iz=\tau(\sum x_i \otimes (y_iz)^0)(f)$ for any $f \in A^*$. Since τ is monomorphic, $\sum x_i \otimes y_i^0$ is in N.

Corollary 5. $\psi\tau(N(A))=D$

We define the map η: $A^e \to \mathrm{Hom}_R(A, A)$ as follows: $\eta(x \otimes y^0)(z)=xzy$. Since A is central, $\eta(N)$ is contained in $\mathrm{Hom}_R(A, R)$.

Proposition 6. *The map θ gives an isomorphism between D and $\eta(N)$*

Proof. τ induces an isomorphism: $N(\mathfrak{A}) \to \mathrm{Hom}^r_{\mathfrak{A}}(\mathfrak{A}^*, \mathfrak{A})$ and η induces an isomorphism: $N(\mathfrak{A}) \to \mathfrak{A}^*$ (because $\mathfrak{A}$ is central separable (see [2])) and we denote also by τ, η these induced isomorphisms. To prove this proposition, we have only to show that the following diagram is commutative, since $N(A) \subset N(\mathfrak{A})$ and $\psi\tau(N(A))=D$.

$$\begin{array}{ccc} N(\mathfrak{A}) & \xrightarrow{\tau} & \mathrm{Hom}^r_{\mathfrak{A}}(\mathfrak{A}^*, \mathfrak{A}) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \psi} \\ \mathfrak{A}^* & \xrightarrow{\theta} & \mathfrak{A} \end{array}$$

For any $\sum x_i \otimes y_i^0 \in N(\mathfrak{A})$, we get

$$\begin{aligned} \theta\psi\tau(\textstyle\sum x_i \otimes y_i^0)(z) &= \theta(\textstyle\sum t(x_i)y_i)(z) = \textstyle\sum t(t(x_i)y_iz) \\ &= \textstyle\sum t(t(zx_i)y_i) = \textstyle\sum t(zx_i)t(y_i), \end{aligned}$$

while $\eta\sum (x_i \otimes y_i^0)(z)=\sum x_izy_i$. So the fact that $\theta\psi\tau=\eta$ follows immediately from Lemma 2.

We denote by H the homological different of A, i.e. $H=\varphi(N(A))$ (see [1], [2]). H is an ideal of R since A is central.

Theorem 7. *The reduced trace t induces an epimorphism*: $D \to H$.

Proof. We consider the following commutative diagram

$$\begin{tikzcd} N(A) \arrow[rr, "\eta"] \arrow[dr, "\varphi"'] & & A^* \arrow[dl, "\varphi_0"] \\ & R & \end{tikzcd}$$

where φ_0 is defined as follows: $\varphi_0(\omega)=\omega(1)$. By the diagram above, we get $\varphi_0(\eta(N))=H$. Se we get the epimorphism $\varphi_0\theta\colon D\to H$ by Proposition 6. While $\varphi_0\theta(d)=\varphi_0(dt)=dt(1)=t(d)$ for $d\in D$, so $\varphi_0\theta=t$ on D.

3. We shall give a generalization of the DeMeyer's theorem (see Theorem 4 in [3]).

Theorem 8. *Let R, K, A, $\mathfrak{A}$ be as above. The following conditions are equivalent*:

1) $At=A^*$
2) $C=A$
3) $D=A$
4) *A is R-separable.*

Proof. 1)$\Rightarrow$2). Since $At=A^*$, θ induces an isomorphism: $A\to A^*$. It implies $C=A$.
2)$\Rightarrow$3). Trivial.
3)$\Rightarrow$4). θ sends D to $\eta(N)$ and C to A^*. Since $\eta(N)$ is contained in A^*, D is always contained in C. By the definition, DC is in A, so 3) implies that $C=AC\subset A=D\subset C$ i.e.$C=D$. Then we have $\eta(N)=A^*$. Using the commutative diagram in the proof of Theorem 7, $\varphi(N)=\varphi_0\eta(N)=\varphi_0(A^*)$. Since A is R-completely faithful, we get $\varphi_0(A^*)=R$. Therefore, A is separable by Proposition 1.1 of [2].
4)$\Rightarrow$1) See [4] § 4.

4. Let R, K, A, $\mathfrak{A}$ be as in the previous section. Let $\mathfrak{M}$ be a two sided maximal ideal of A. We put $\mathfrak{m}=\mathfrak{M}\cap R$. Clearly $\mathfrak{m}$ is a maximal ideal of R. If 1) $\mathfrak{M}=\mathfrak{m}A$ and 2) $A/\mathfrak{M}$ is a separable algebra over a field $R/\mathfrak{m}$, we say that $\mathfrak{M}$ is *unramified*. If $\mathfrak{M}$ is not unramified, we say that $\mathfrak{M}$ is *ramified*. In the case that R is a quasi-local ring, any maximal ideal $\mathfrak{M}$ of A is unramified if and only if A is R-separable by [4], (1.1). Clearly $\mathfrak{M}_{\mathfrak{m}}$ is a maximal two sided ideal of $A_{\mathfrak{m}}$ and $\mathfrak{M}$ is unramified if and only if $\mathfrak{M}_{\mathfrak{m}}$ is an unramified maximal ideal of $A_{\mathfrak{m}}$, and $\mathfrak{M}\not\supset D$ if and only if $\mathfrak{M}_{\mathfrak{m}}\not\supset D_{\mathfrak{m}}$.

Theorem 9. *If $\mathfrak{M}$ is unramified, then $\mathfrak{M}$ does not contain the Dedekind different D.*

Proof. As remarked above, we may assume that R is quasi-local ring. Since A is separable, D coincides with A by Theorem 8. So, $\mathfrak{M}\not\supset D$.

To prove the other half part of the "different theorem", we shall add further assumptions, We did not succeed in omitting these assumptions,

Theorem 10. *Suppose that $t(A)$ is contained in R, and that $A/\mathfrak{m}A$ is a primary algebra over $R/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R. Then a maximal ideal $\mathfrak{M}$ of A is unramified whenever $\mathfrak{M}$ does not contain D.*

Proof. Again, we assume that R is a quasi-local ring. Since $A/\mathfrak{m}A$ is primary, $\mathfrak{M} \not\supset D$ implies $A=D$. By Theorem 8, A is a separable R-algebra. So $\mathfrak{M}$ is unramified.

REMARK. If A is a maximal order over a Dedekind domain in a central simple algebra, the assumptions in Theorem 10 are satsified.

TOKYO UNIVERSITY OF EDUCATION

References

[1] M. Auslander and D.A. Buchsbaum: *On ramification theory in Noetherian rings,* Amer. J. Math. **81** (1959), 749–765.

[2] M. Auslander and O. Goldman: *The Brauer group of a commutative ring,* Trans. Amer. Math. Soc. **97** (1960), 367–409.

[3] F.R. DeMeyer: *The trace map and separable algebras,* Osaka J. Math. **3** (1966), 7–11.

[4] S. Endo and Y. Watanabe: *On separable algebras over a commutative ring,* Osaka J. Math. **4** (1967), 233-242.

[5] R.M. Fossum: *The noetherian different of projective orders,* Bull. Amer. Math. Soc. **72** (1966), 898–900.

[6] B. Müller: *Quasi-Frobenius-Erweiterungen,* Math. Z. **85** (1964), 345–368.

[7] M. Nagata: Local Rings, Interscience, 1962.